AS1/MATH6101/TWN/ML/07-08

THE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH6101: Intermediate Complex Analysis

Assignment 1

1. Suppose that $\{f_n\}$ is a uniformly bounded family of holomorphic functions on a domain Ω. Let $\{z_k\}_{k=1}^{\infty} \subseteq \Omega$ and suppose that $z_k \to z_0 \in \Omega$. Assume that $\lim_{n\to\infty} f_n(z_k)$ exists for each $k = 1, 2, \ldots$. Prove that the full sequence $\{f_n\}$ converges uniformly on compact subsets of Ω.

2. Show that the functions z^n, n a nonnegative integer, form a normal family in $|z| < 1$, also in $|z| > 1$, but not in any region that contains a point on the unit circle.

3. Show that if $\mathcal{F}_1, \cdots, \mathcal{F}_k$ are normal families of meromorphic functions in a domain Ω, then so is their union. Show that the corresponding statement for infinite families is false.

4. Let $\Omega \subseteq \mathbb{C}$ be a bounded domain and let $\{f_j\}$ be a sequence of holomorphic functions on Ω. Assume that
$$\int_{\Omega} |f_j(z)|^2 dx\, dy < C < \infty,$$
where C does not depend on j. Prove that $\{f_j\}$ is a normal family in Ω.

5. Prove that in any region Ω the family of analytic functions with positive real part is normal.

6. Let Ω be the half strip $\{x + iy \in \mathbb{C} : a < x < b,\ y > 0\}$. Suppose that f is analytic and bounded in Ω, and that for some c in (a, b), $f(c + iy) \to A$ as $y \to +\infty$. Then for every c in (a, b), $f(c + iy) \to A$ as $y \to +\infty$.

7. If $f(z)$ is analytic in the whole plane, show that the family formed by all functions $f(kz)$ with constant k is normal in the annulus $r_1 < |z| < r_2$ if and only if f is a polynomial.

8. Suppose that $\Omega \underset{\neq}{\subset} \mathbb{C}$ is a simply connected domain, and $\zeta \in \Omega$. Show that if $f : \Omega \to \Omega$ is analytic and $f(\zeta) = \zeta$, then there is a compact subset $K \neq \{\xi\}$ of Ω such that $f(K) \subset K$.