On the common right factors of meromorphic functions

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Abstract: We consider the class B_1 of entire functions of the the form

$$f = \sum a_i e^{g_i}$$

1 Introduction and Main Results.

The subject of factorization under the the composition of meromorphic functions is clearly related to the dynamics of rational or entire functions.

We assume that the reader is familiar with some basic results and notations of Nevanlinna Theory of entire and meromorphic functions (see Hayman [**?**]), which will be the main tool of our investigations.

In [**?**], H. Urabe proved the following theorem.

Theorem . Let $P_i(i = 1, 2)$ and Q be non-constant polynomials with $\deg P_1 < \deg P_2$ such that $P_1(z) - P_2(z)$ has only simple zeros. Let

$$F(z) = \frac{p_1(z) + Q(e^z)}{p_2(z) + Q(e^z)} \ .$$

Then F is *prime*.

Therefore, it is natural to consider the *primeness* and *pseudo-primeness* of the functions of the general form $(p_1 + p_2 F)/(p_3 + p_4 F)$, where F is a transcendental meromorphic function and the p_i's are polynomials. In this paper, common right factors of functions $p_1 + p_2 F$ and $p_3 + p_4 F$ (in the sense of composition) will first be investigated. Then we shall also prove that the quotient $(p_1 + p_2 F)/(p_3 + p_4 F)$ is $pseudo - prime$ under some further restrictions on F and the p_i's as follows.

Theorem 1 . *Let g be a common right factor of $p_1 + p_2 F$ and $p_3 + p_4 F$, where the p_i's are polynomials, $p_2, p_4 \not\equiv 0$ and F is a transcendental meromorphic function. If $\{p_1 p_4 - p_2 p_3, p_2, p_4\}$ is a linear independent set over $\mathbf{C}$, then g must be a polynomial and is a right factor of p_4/p_2.*

Theorem 2 . Let p_i's be polynomials for $i = 1, \ldots, 4$, $p_2, p_4 \not\equiv 0$ and F be a transcendental meromorphic function with at most a finite number of poles. If p_2/p_4 is not a constant, then $(p_1 + p_2 F)/(p_3 + p_4 F)$ is pseudo-prime.

Remark: If p_2/p_4 is equal to a non-zero constant, then $(p_1 + p_2 F)/(p_3 + p_4 F)$ may not be $pseudo-prime$. For example, let f be a transcendental meromorphic function such that $F = f(e^z)$ is of finite order. Then $(1 + F)/(2 + F) = [(1 + f)/(2 + f)]oe^z$, which is not $pseudo-prime$. However, we can still draw the same conclusion under some further restrictions on F and the p_i's.

Theorem 3 . Let F be a periodic entire function of finite order, p_1, p_2 and p be three polynomials. Let $Q(z)$ be the canonical product of the common roots of $p_1 + pF$ and $p_2 + cpF$, where c is a non-zero constant. If Q/p is not a constant, then $(p_1 + pF)/(p_2 + cpF)$ is pseudo-prime.

Corollary 1 . Let p_1, p_2 be two polynomials and not both of them are constants. Suppose that H is a periodic entire function of finite order. Then $F = p_1 + p_2 H$ is pseudo-prime.

The proof of Corollary 1. If p_2 is a constant, it reduce to a result of F. Gross [?]. Suppose that p_2 is not a constant. Consder $1 + 1/F = (p_1 + 1 + p_2 H)/(p_1 + p_2 H)$ and we apply Theorem 3 to $1 + 1/F$ and conclude that $1 + 1/F$ is pseudo-prime and hence is F. As another application of Theorem 3, we can confirm the following result which was conjectured by J.H.Zheng in [?].

Theorem 4 . Let H be a periodic entire function of finite order. Then for any non-constant rational function $R(z)$, $F(z) = R(z)H(z)$ is pseudo-prime.

It is natural to conjecture the following :
Conjecture : For any non-constant rational function $R(z)$ and periodic entire function H, $R(z)H(z)$ is pseudo-prime.

2 Some Lemmas.

Lemma 1 [?]. Let f be a meromorphic function. Then for all irreducible rational functions in f,
$$R(z, f) = \frac{\sum_{i=0}^{p} a_i(z) f^i}{\sum_{i=0}^{q} b_i(z) f^i}$$
with meromorphic coefficients $a_i(z), b_i(z)$ and $a_p(z), b_q(z) \not\equiv 0$ such that
$T(r, a_i) = o(T(r, f))$, $T(r, b_j) = o(T(r, f))$ $i = 0, \ldots, p$, $j = 0, \ldots, q$. Then we have $T(r, R(z, f)) = max(p, q)T(r, f) + o(T(r, f))$.

Remark : In the sequel, for any function g satisfying $T(r, g) = o(T(r, f))$ as $r \to \infty, r \notin E$ (a set of finite linear measure) will be called a small function of f.

Lemma 2 *[?]. Let f be a transcendental meromorphic function, g be a transcendental entire function and R be a nonconstant rational function. Then the function $f(g(z)) - R(z)$ has infinitely many zeros.*

Lemma 3 *[?]. Let f be a meromorphic function and g be a transcendental entire function. If $\rho_{f(g)} < \infty$, then $\rho_f = 0$.*

Lemma 4 *[?]. Let f be a meromorphic function and g be an entire function such that $\rho_{f(g)} < \infty$. Let $f = f_1/f_2$, where f_1, f_2 are the canonical products formed by zeros and poles of f respectively, then for $\epsilon > 0$ and arbitrary large r,*

$$\log M(r, f_1(g)) + \log M(r, f_2(g)) < r^k$$

where $k = (1 + \epsilon)(\rho_{f(g)} + \epsilon) + \rho_g + \epsilon$. In particular $\rho_{f_i(g)} < \infty$ for $i = 1, 2$.

In order to investigate the common right factors of functions $p_1 + p_2 F$ and $p_3 + p_4 F$, we prove the following lemma which is of some interests.

Lemma 5 . *Suppose F and G are two transcendental meromorphic functions which satisfy the equation*

$$\sum_{i,j} a_{ij}(z) F^i(z) G^j(z) \equiv 0 ,$$

where $a_{ij}(z)$'s are linearly independent polynomials over $\mathbf{C}$. If g is a common right factor of F and G, then g must be a polynomial.

Proof. Suppose that g is transcendental and we write $F = f_1(g)$ and $G = f_2(g)$, f_1, f_2 are meromorphic functions. Then the above equation becomes $\sum_{i,j} a_{ij}(z) f_1^i(g(z)) f_2^j(g(z)) \equiv 0$. Since g is transcendental and $a_{ij}(z)$ are polynomials, $\sum_{i,j} T(r, a_{ij}) = o(T(r, g))$. Then we can apply the well known result of N.Steinmetz ([9] or [5]) and obtain a set of polynomials $P_{ij}(z)$, not all identically zero such that $\sum_{i,j} a_{ij}(z) P_{ij}(g(z)) \equiv 0$. By rearranging the expression according to the power of g, we have

$$\sum_k c_k(z) g^k(z) \equiv 0,$$

where $c_k(z)$ is a linear combination of the $a_{ij}(z)$'s over $\mathbf{C}$. By Lemma 1, we must have $c_k(z) \equiv 0$ for all the k's. Since at least one of the $P_{ij}(z)$ is not identically zero, one of the $c_k(z)$ is a non-trivial linear combination of the $a_{ij}(z)$'s over $\mathbf{C}$. This contradicts our hypothesis on the $a_{ij}(z)$'s.

Remark : The assumption of linear independence on the $a_{ij}(z)$'s is essential. For example,

$$\cos^2 z + \sin^2 z = 1$$

but $\cos z$ and $\sin z$ have the non-polynomial right common factor e^z.

The following lemma is implicitly contained in the proof of the result obtained by F. Gross and C.F. Osgood in [?], which is a key factor used in the proof of Theorem 3. By observing the fact that $\sum_{i=1}^n T(r, h_i) = o(T(r, g))$, $\sum_{i=1}^n T(r, \psi_i) = o(T(r, g))$ allow us to choose $M = 1$ in [6,p.291]. and together with the lemma in [6,p.288], we can obtain the following result.

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Lemma 6 . *Suppose that g is a nonconstant entire function, and that $\psi_i \not\equiv 0$ for $1 \leq i \leq n$, are entire functions and $n \geq 2$ is a natural number. Also, $f_i \not\equiv 0$ and $h_i \not\equiv 0 (1 \leq i \leq n)$ are meromorphic functions, f_i's are analytic at $z = 0$, $f_1(0) \neq 0$. Suppose further that $\sum_{i=1}^{n} T(r, h_i) = o(T(r,g))$ and $\sum_{i=1}^{n} T(r, \psi_i) = o(T(r,g))$. If*

$$\sum_{i=1}^{n} f_i(g\psi_i)h_i(z) = 0,$$

then there exist n functions $P_i(z, w)$ not all zero, where each $P_i(z, w)$ is a polynomial of w with coefficients are functions of $\psi_1, \psi_2, \psi_3, \ldots, \psi_n$, such that

$$\sum_{i=1}^{n} P_i(z, g)h_i(z) = 0,$$

where $deg P_i(z, w)$ (in w) $\leq (n - 1)$ for $i = 1, \ldots, n$, $P_1 \not\equiv 0$ and $P_1 f_i(\psi_i g) - P_i f_1(\psi_1 g)$ vanishes to at least the order n at $g = 0$.

3 Proofs of Theorems.

The proof of theorem 1.   Let $p_1 + p_2 F = f_1(g)$ and $p_3 + p_4 F = f_2(g)$, where f_1, f_2 are meromorphic functions. Then by eliminating F, we have

$$p_1(z)p_4(z) - p_2(z)p_3(z) = p_4(z)f_1(g(z)) - p_2(z)f_2(g(z)) . \tag{1}$$

By applying Lemma 5, one can conclude that g is a non-constant polynomial. Therefore, f_1 and f_2 are transcendental for F is transcendental. Since p_i's are polynomials, we still have $\sum_{i=1}^{4} T(r, p_i) \leq O(T(r,g))$. Now by the well known result of N.Steinmetz ([9] or [5]), there exist not all identically zero polynomials $A(z), B(z)$ and $C(z)$ such that

$$(p_1 p_4 - p_2 p_3)(z)C(g(z)) = p_4(z)A(g(z)) - p_2(z)B(g(z)) . \tag{2}$$

By a simple elimination, we conclude from (1) and (2) that

$$p_4(z)(f_1 C - A)(g(z)) = p_2(z)(f_2 C - B)(g(z)) .$$

The fact that f_1, f_2 are transcendental and one of A, B or $C \not\equiv 0$ ensure that $(f_1 C - A)(g(z))$ and $(f_2 C - B)(g(z))$ are not identically zero. Hence g is a right factor of p_4/p_2 .

The proof of theorem 2.   Suppose that $(p_1 + p_2 F)/(p_3 + p_4 F)$ is not $pseudo - prime$. Then $(p_1 + p_2 F)/(p_3 + p_4 F) = f(g)$ where f is a transcendental meromorphic function and g is a transcendental entire function. Therefore,

$$\begin{aligned} f(g) - \frac{p_2}{p_4} &= \frac{p_1 + p_2 F}{p_3 + p_4 F} - \frac{p_2}{p_4} \\ &= \frac{p_1 p_4 - p_2 p_3}{p_4(p_3 + p_4 F)} \end{aligned}$$

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which has only a finite number of zeros. This contradicts Lemma 2.

The proof of theorem 3. We may assume $c = 1$. Suppose $(p_1 + pF)/(p_2 + pF)$ is not *pseudo − prime*. Then $(p_1 + pF)/(p_2 + pF) = H(K) = H_1(K)/H_2(K)$, where H is transcendental meromorphic, K is transcendental entire and H_1, H_2 are the canonical products formed by zeros and poles of H respectively. Note that F is of finite order, then so is $(p_1 + pF)/(p_2 + pF)$. Hence $\rho_H = \rho_{H_1} = \rho_{H_2} = 0$ by Lemma 3 and 4. Since H_1 and H_2 have no common zeros, so do $H_1(K)$ and $H_2(K)$. Therefore, the zeros of $H_1(K)$ and $H_2(K)$ are zeros and poles of $(p_1 + pF)/(p_2 + pF)$ respectively. Hence, we have

$$p_1 + pF = H_1(K)e^R Q \tag{3}$$
$$p_2 + pF = H_2(K)e^R Q, \tag{4}$$

where R is an entire function and Q is the canonical product of the common roots of $p_1 + pF$ and $p_2 + pF$. Clearly, Q is a polynomial and divides $p_1 - p_2$ in the product sense. By Lemma 4, the order of $H_1(K)$ is finite. Therefore from (3) , e^R is of finite order and hence R must be a polynomial. Subtracting (4) from (3) and then dividing both sides by e^R, we get

$$\frac{p_1 - p_2}{Q}e^{-R} = (H_1 - H_2)oK .$$

Clearly R can't be a constant, otherwise K will be a polynomial. Now if $H_1 - H_2$ has more than one distinct zeros, then as K is transcendental entire, by Picard's theorem , $(H_1 - H_2)oK$ would have infinitely many zeros which is impossible for the number of zeros of $(p_1 - p_2)/Q$ is finite.

Since the order of $H_1 - H_2$ is zero, we must have $H_1 - H_2 = (z - a)^n$ and then $K(z) = a + P(z)e^{-R(z)/n}$, where a is a complex number and $P(z)^n = (p_1 - p_2)/Q$ for some natural number n. By considering $H_i(z+a)$ instead of $H_i(z)$, we may assume $a = 0$. Since H_1 and H_2 have no common zeros, we may also assume $H_1(0) \neq 0$. Now

$$p_1(z) + p(z)F(z) = H_1(P(z)e^{-R(z)/n})e^{R(z)}Q(z) .$$

Eliminating F by using the fact that F is periodic with period 1(say), we get

$$H_1(P(z+1)e^{-R(z+1)/n})e^{R(z+1)}Q(z+1)p(z) - H_1(P(z)e^{-R(z)/n})e^{R(z)}Q(z)p(z+1)$$
$$+p_1(z)p(z+1) - p_1(z+1)p(z) \equiv 0 .$$

If we let $S(z) = R(z + 1) - R(z)$ and multiply both sides by $e^{-R(z)}$, then

$$H_1(P(z+1)e^{-R(z+1)/n})e^{S(z)}Q(z+1)p(z) - H_1(P(z)e^{-R(z)/n})Q(z)p(z+1)$$
$$+(p_1(z)p(z+1) - p_1(z+1)p(z))e^{-R(z)} \equiv 0 .$$

Set $f_1 = H_1, f_2 = H_1, f_3 = z^n$, $h_1(z) = e^{S(z)}Q(z+1)p(z)$, $h_2(z) = -Q(z)p(z+1)$ and $h_3(z) = p_1(z)p(z+1) - p_1(z+1)p(z)$. Also let $g(z) = e^{-R(z)/n}$, $\psi_1(z) = P(z+1)e^{-R(z+1)/n}/e^{-R(z)/n}$, $\psi_2(z) = P(z), \psi_3(z) = 1$.

Then the above equation becomes

$$f_1(\psi_1 g)h_1 + f_2(\psi_2 g)h_2 + f_3(\psi_3 g)h_3 \equiv 0 .$$

Since $deg(R(z+1) - R(z)) \le deg R(z)$, $T(r, \psi_1) = o(T(r, g))$ and $T(r, h_1) = o(T(r, g))$. It is then easy to check that $\sum_{i=1}^{3} T(r, h_i) = o(T(r, g))$ and $\sum_{i=1}^{3} T(r, \psi_i) = o(T(r, g))$. Therefore by Lemma 6, there exist polynomials $P_1(z, w)$, $P_2(z, w)$ and $P_3(z, w)$ of $degree P_i$ (in w) $\le 3 - 1 = 2$, with coefficients being functions of ψ_i such that,

$$P_1(z, g)h_1 + P_2(z, g)h_2 + P_3(z, g)h_3 \equiv 0, \tag{5}$$

where the P_i's satisfy the conditions that

$$P_1 f_i(\psi_i g) - P_i f_1(\psi_1 g) \tag{6}$$

vanishes to at least order 3 at $g = 0$ for $i = 2, 3$ and $P_1 \not\equiv 0$. We rewrite (5) as

$$P_1(z, g)e^{S(z)} Q(z+1)p(z) - P_2(z, g)Q(z)p(z+1) + P_3(z, g)h_3 \equiv 0 . \tag{7}$$

Let $P_i(z, g) = \sum_{k=0}^{2} c_{ik}(z)g^k$ and $H_1(z) = \sum_{j=0}^{\infty} a_j z^j$. Then it follows from the fact that $P_1 f_2(\psi_2 g) - P_2 f_1(\psi_1 g)$ vanishes to at least order 3 at $g = 0$, we have for $m = 0, 1, 2$,

$$c_{10} a_m \psi_2^m + c_{11} a_{m-1} \psi_2^{m-1} + \cdots + c_{1m} a_0$$
$$= c_{20} a_m \psi_1^m + c_{21} a_{m-1} \psi_1^{m-1} + \cdots + c_{2m} a_0 . \tag{8}$$

On the other hand, from $P_1 f_3(\psi_2 g) - P_3 f_1(\psi_1 g) \equiv P_1 g^n - P_3 f_1(\psi_1 g)$ which vanishes to at least order 3 at $g = 0$ gives immediately that $c_{30} \equiv 0$ as $n \ge 1$.

Now rearrange (7) according to the powers of g, we get a polynomial equation in g with small functions of g as the coefficients. By Lemma 1, these coefficients must be identically zero and we have for $m = 0, 1, 2$,

$$c_{1m}(z)e^{S(z)} Q(z+1)p(z) - c_{2m}(z)Q(z)p(z+1) + c_{3m}(z)h_3 \equiv 0 . \tag{9}$$

Since $c_{30} \equiv 0$, for $m = 0$,

$$c_{10} e^{S(z)} Q(z+1)p(z) - c_{20} Q(z)p(z+1) \equiv 0 . \tag{10}$$

From (8), we have $c_{10} a_0 = c_{20} a_0$. Note that $a_0 = f_1(0) = H_1(0) \ne 0$, therefore

$$c_{10}(e^{S(z)} Q(z+1)p(z) - Q(z)p(z+1)) \equiv 0 . \tag{11}$$

If $e^{S(z)} \equiv 1$ and $c_{10} \not\equiv 0$, then $Q(z+1)/p(z+1) \equiv Q(z)/p(z)$ which implies that $Q(z)/p(z)$ is a constant and this contradicts our assumption. Therefore either $e^{S(z)} \not\equiv 1$ or $c_{10} \equiv 0$. If $e^{S(z)} \not\equiv 1$, then $(e^{S(z)} Q(z)p(z+1) - Q(z+1)p(z)) \not\equiv 0$ and therefore $c_{10} \equiv 0$.

Hence $c_{10} = c_{20} = c_{30} = 0$. $c_{10} = c_{30} = 0$ and the coefficient of g in $P_1 g^n - P_3 f_1(\psi_1 g)$ is identically zero implies that $c_{31} \equiv 0$.

From (8) again we have $c_{11} a_0 = c_{21} a_0$, from this together with $c_{31} \equiv 0$, (9) becomes

$$c_{11}(e^{S(z)} Q(z+1)p(z) - Q(z)p(z+1)) \equiv 0 . \tag{12}$$

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This allows us to repeat the above argument to conclude that $c_{11} = c_{21} = 0$. Then repeating the argument once more, we have $c_{1i} \equiv 0$ for $i = 0, 1, 2$. This contradicts the fact that $P_1(z) \not\equiv 0$.

The proof of Theorem 4. Let $R(z) = p_1(z)/p_2(z)$, where p_1 and p_2 do not have any common zero. Consider $1 + 1/F = (p_2 + p_1 H)/(p_1 H)$. Let $Q(z)$ be the canonical product of the common roots of $p_2 + p_1 H$ and $p_1 H$. Then clearly $Q(z)$ divides p_2 in product sense. If one of p_1 and Q is non-constant, then Q/p_1 is non-constant as p_1 and p_2 do not have any common zero. Now by Theorem 3, we conclude that $1 + 1/F$ is pseudo-prime and so is F.

If both p_1 and Q are constants (say $p_1 = 1$), then H and $p_2 + H$ do not have any common zero and therefore $H(z)/p_2(z)$ has finitely many poles.

Suppose that $H(z)/p_2(z) = f_1(g)$, where f_1 is a transcendental meromorphic function and g is a transcendental entire function. Using the fact that $H(z)/p_2(z)$ has finitely many poles, we can conclude that $g(z) = a + P(z)e^{-K(z)/n}$ and $f_1(z) = f(z)/(z - a)^n$ where $P(z), K(z)$ are polynomials, $K(z)$ is non-constant and f is a transcendental entire function of zero order with $f(a) \neq 0$. By considering $f(z + a)$ instead of $f(z)$, we may assume $a = 0$. Note that $P(z)^n = p_2(z)$ where n is an natural number. Hence, we have $H(z) = f(w)o(P(z)e^{-K(z)/n})e^{K(z)}$

The fact that $H(z)$ is periodic (say with period 1) gives

$$f(\psi(z)g(z))e^{\beta(z)} - f(g(z)) = 0, \tag{13}$$

where $\beta(z) = K(z + 1) - K(z)$ and $\psi(z) = P(z + 1)/P(z)e^{-\beta(z)/n}$ is an meromorphic function with $T(r, \psi) = o(T(r, g))$. By applying Lemma 6, there exist two polynomials of w, $P_1(z, w) \not\equiv 0$, $P_2(z, w)$, with $degree P_i$ (in w) $\leq 2 - 1 = 1$ and cofficients being functions of ψ such that

$$P_1(z, g(z))e^{\beta(z)} - P_2(z, g(z)) = 0$$

and then,

$$\frac{P_2(z, g)}{P_1(z, g)} = e^{\beta(z)}.$$

By Lemma 1, this is impossible if one of $degree P_i$ (in w) $\neq 0$. Therefore, both of $degree P_i$ (in w) $= 0$ and hence $P_i(z, g) = c_i(z)$ where $c_i(z)$ is a function of ψ. Using the fact that $P_1 f(g) - P_2 f(\psi g)$ vanishes to at least order 2 at $g=0$ and by argruments similar to the proof of Theorem 3, we can conclude that $c_1 \equiv c_2$ and hence $e^{\beta(z)} \equiv 1$. Then Equation (13) becomes

$$f(\psi(z)g(z)) = f(g(z)), \tag{14}$$

where $\psi(z) = P(z + 1)/P(z)$.

Now we can choose a constant A such that the equation $\psi(z)g(z) = A$ has infinitely many roots, say $\{a_i\}$. Note that $g(a_i) = A/\psi(a_i)$ which converges to A as i trends to

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infinity and from (14), $f(g(a_i)) = f(A)$. By the uniqueness theorem, we have $f(z) \equiv f(A)$ which is impossible. Finally, we conclude that $R(z)H(z)$ is $pesudo-prime$.

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