Smale's Mean Value conjecture and the hyperbolic metric

A.F. Beardon, D. Minda and T.W. Ng

1. Smale's Mean Value conjecture

Let P be any polynomial; then z is a critical point of P if and only if $P'(z) = 0$, and w is a critical value of P if and only if $w = P(z)$ for some critical point z of P. A nonconstant linear polynomial has no critical points, so throughout the paper we shall be assuming that P has degree d, where $d \geq 2$. We begin with a result and conjecture of Smale.

Theorem 1.1 [5]. *Let P be a non-linear polynomial with critical points z_j. If z is not a critical point of P then*

$$\min_j \left| \frac{P(z) - P(z_j)}{z - z_j} \right| \leq 4|P'(z)|. \tag{1.1}$$

Smale proved this in 1981 ([5],p.33), and then asked whether one can replace the factor 4 in the upper bound in (1.1) by 1, or even possibly by $(d-1)/d$. He repeated this problem in [6] (p.289, although not in his list of major problems). The number $(d-1)/d$ would, if true, be the best possible bound here as it is attained (for any nonzero λ) when $P(z) = z^d - \lambda z$ and $z = 0$ in (1.1). The conjecture has been verified for $d = 2, 3, 4$, and also in some other special circumstances (see [4] p.159, [7] and [8]) but the general case remains open.

It is convenient to use the notation

$$S(P, z) = \min_j \left| \frac{P(z) - P(z_j)}{z - z_j} \right| \frac{1}{|P'(z)|},$$

where P has critical points $z_1, \ldots, z_{d-1}$. By expressing P as a Taylor series about z_j, we see that $\big(P(z) - P(z_j)\big)/\big((z - z_j)P'(z)\big)$ has a removable singularity at z_j with absolute value at most $1/2$ there. Here, we shall prove the following two results.

Theorem 1.2. *Let P be a non-linear polynomial of degree d. Then for all z,*

$$S(P, z) \leq \frac{4}{4^{1/(d-1)}}. \tag{1.2}$$

The factor 4 in (1.1) has been replaced by $4^{(d-2)/(d-1)}$ in (1.2), and when P has degree five (the smallest degree for which the conjecture is still open), this factor is approximately 2.83; in all cases it is strictly less than 4.

Theorem 1.3. *Let P be a non-linear polynomial.*
(i) *Let C be the convex hull of the critical values of P. If $P(z) \notin C$ then*

$$S(P,z) \leq 3.079\cdots. \tag{1.3}$$

(ii) *Let D be the smallest closed disc that contains all critical values of P, and let ζ and r be the radius, and the center, respectively, of D. If $P(z) \notin D$ then*

$$S(P,z) \leq \frac{(r+|P(z)-\zeta|)\sqrt{r^2+|P(z)-\zeta|^2}}{|P(z)-\zeta|(|P(z)-\zeta|-r)}. \tag{1.4}$$

The right-hand side of (1.4) is a decreasing function of $|P(z)-\zeta|$, and it tends to 1 as $z \to \infty$. It shows, for example, that if $|P(z)-\zeta| > 5r$ then $S(P,z) \leq 1.5297\ldots$.

Our proofs depend on the use of the hyperbolic metric which does not seem to have been applied in this context before. Smale's proof of Theorem 1.1 uses Koebe's 1/4-Theorem. In [7] (p.439) Tischler refers to Smale's proof and asks "whether the inverse branches of polynomials satisfy a stronger version of the Koebe 1/4-Theorem". Now Koebe's Theorem is equivalent to a statement about the hyperbolic metric, and by using this (a more flexible tool than Koebe's Theorem) we obtain Theorem 1.2. Section 2 contains some elementary remarks, and in Section 3 we discuss the hyperbolic metric. The proof of Theorem 1.3 is simpler than that of Theorem 1.2, so we prove Theorem 1.3 in Section 4 and Theorem 1.2 in Section 5. In Section 6 we broaden the discussion and try to place Smale's conjecture in the context of a certain class of entire functions (that includes all polynomials). We shall see that the conclusion of Theorem 1.2 remains valid in this larger class, and that Smale's theorem and conjecture are closely related to the problem of comparing certain conformal metrics in the complex plane.

2. The normalized problem

It is clear that if $\alpha(z) = az+b$, where $a \neq 0$, then $S(\alpha \circ P, z) = S(P,z)$, where $\circ$ denotes the composition of functions (although we often omit this symbol and write, for example, αP). Likewise, as the critical points of $P\circ\alpha$ are $\alpha^{-1}(z_j)$, we readily find that $S\big(P\circ\alpha,\alpha^{-1}(z)\big) = S(P,z)$. Suppose now that we are interested in the value $S(P,\zeta)$. We let $\beta(z) = z+\zeta$, $\alpha(z) = a\big(z-P(\zeta)\big)$ and $Q = \alpha\circ P\circ\beta$; then $S(Q,0) = S(P,\zeta)$, and $Q(0)=0$. Moreover, by choosing a appropriately, we may assume that if w_j are the critical *values* of Q, then $\min_j |w_j| = 1$. To summarize, to show that $S(P,z)\leq K$, it suffices to assume that

$$P(0)=0, \quad P'(0)\neq 0, \quad \min_{1\leq j\leq d-1} |P(z_j)| = 1, \tag{2.1}$$

where P has critical points z_j, and then show that

$$\min\{|P(z_j)|/|z_j| : j=1,\dots,d-1\} \leq K\,|P'(0)|.$$

Note that because of (2.1), this inequality will hold if, *for all* j,

$$1\leq K|z_j|\,|P'(0)|. \tag{2.2}$$

3. The hyperbolic metric of a simply connected domain

Any simply connected proper subdomain Ω of $\mathbb{C}$ supports a hyperbolic metric $\lambda_\Omega(z)\,|dz|$, and if f is a conformal map of a simply connected domain Σ onto Ω, then the two metrics are related by the formula

$$\lambda_\Omega\big(f(z)\big)|f'(z)| = \lambda_\Sigma(z). \tag{3.1}$$

In the particular case of the unit disc $\mathbb{D}$, $\lambda_{\mathbb{D}}(z) = 2/(1-|z|^2)$ (and this and the Riemann map of Ω onto $\mathbb{D}$ defines λ_Ω in Ω). For the upper half-plane $\mathbb{H}$, $\lambda_{\mathbb{H}}(z) = 1/\mathrm{Im}(z)$. Now for any simply connected domain Σ with boundary $\partial\Sigma$,

$$\lambda_\Sigma(z)\geq \frac{1}{2\mathrm{dist}(z,\partial\Sigma)}; \tag{3.2}$$

this is a consequence of the Koebe 1/4-Theorem (see [3], p.45).

Suppose now that P satisfies (2.1), and let Ω be any simply connected subdomain of $\mathbb{C}$ that contains $P(0)$ $(=0)$ but not any critical value $P(z_j)$ of P. Then we can define a unique single-valued branch of the inverse P^{-1} on Ω (by the condition $P^{-1}(0)=0$) and if we let $\Sigma = P^{-1}(\Omega)$, then P is a conformal map of Σ onto Ω, and (3.1) holds with $f=P$. If we now put $z=0$ in (3.1), and then use (3.2) we obtain $1\leq 2\mathrm{dist}(0,\partial\Sigma)\lambda_\Omega(0)|P'(0)|$.

However, as $0 \in \Sigma$, and as each critical point z_j lies outside Σ, we see that $|z_j| \geq \mathrm{dist}(0, \partial\Sigma)$ for every j, and hence

$$1 \leq 2|z_j|\,|P'(0)|\lambda_\Omega(0). \tag{3.3}$$

If we now compare this with (2.2) we obtain a variant of Theorem 1.1 in which the factor in the upper bound in (1.1) is replaced by $2\lambda_\Omega(0)$. If we let $\Omega = \mathbb{D}$ (as we may since (2.1) holds), and note that $\lambda_{\mathbb{D}}(0) = 2$, we obtain Theorem 1.1. This is Smale's proof written in the context of the hyperbolic metric, and our proof of Theorem 1.2 simply depends on making a better choice of Ω.

We now illustrate these ideas by finding the hyperbolic metrics (which we shall need later) of three specific regions.

Lemma 3.1. *Let* $\Pi = \{x + iy : x < 0\} \cup \{z : |z| < 1\}$. *Then* $\lambda_\Pi(0) = 8/3\sqrt{3}$.

Proof The map $g(z) = -i(z+i)/(z-i)$ maps Π conformally onto $\{x+iy : x < 0 \text{ or } \ y > 0\}$ (the complement of the fourth quadrant), and $h(z) = z^{2/3}$ maps this region onto the upper half-plane $\mathbb{H}$. Let $F = h \circ g$; then

$$\lambda_\Pi(0) = \lambda_{\mathbb{H}}\big(F(0)\big)|F'(0)| = \lambda_{\mathbb{H}}(e^{i\pi/3})\,|g'(0)|\,|h'(i)|,$$

and a simple computation gives the result.

Lemma 3.2. *Let* $\Lambda = \{x + iy : x > 0\} \cap \{z : |z| > 1\}$. *Then*

$$\lambda_\Lambda(z) = \frac{|z^2 + 1|}{(|z|^2 - 1)\mathrm{Re}\,(z)}.$$

Proof It is easy to see that if $Q = \{z = x + iy : x > 0 \text{ and } y > 0\}$, then

$$\lambda_Q(z) = \frac{2|z|}{\mathrm{Im}\,(z^2)} = \frac{|z|}{xy}.$$

As $f(z) = (z + i)/(z - i)$ is a conformal map of Λ onto Q, we see that

$$\lambda_\Lambda(z) = \lambda_Q(f(z))|f'(z)| = \lambda_Q\left(\frac{z+i}{z-i}\right)\frac{2}{|z-i|^2}$$

and a calculation gives the result.

Lemma 3.3. *Suppose that* $\Omega = \mathbb{C}\backslash\left[\overline{\mathbb{D}} \cup (-\infty, 0)\right]$, *and that* $x > 1$. *Then*

$$\lambda_\Omega(x) = \frac{x+1}{2x(x-1)}.$$

Proof The principal branch $g(z)$ of $\sqrt{z}$ on Ω maps Ω conformally onto Λ, so that

$$\lambda_\Omega(z) = \lambda_\Lambda(g(z))|g'(z)| = \lambda_\Lambda(\sqrt{z})\frac{1}{2|z|^{1/2}} = \frac{|z+1|}{2|z|^{1/2}(|z|-1)\mathrm{Re}\,(\sqrt{z})}.$$

The result follows immediately from this.

Finally, we shall need the results in Lemmas 3.1, 3.2 and 3.3 after the sets there have been rescaled by a positive factor r. These results are found by applying the map $z \mapsto rz$, and they are as follows :

(a) if $\Pi_r = \{x+iy : x < 0\} \cup \{z : |z| < r\}$ then $\lambda_{\Pi_r}(0) = 8/(3\sqrt{3}r)$;

(b) if $\Lambda_r = \{x+iy : x > 0\} \cap \{z : |z| > r\}$ then

$$\lambda_{\Lambda_r}(z) = \frac{|z^2+r^2|}{(|z|^2-r^2)\mathrm{Re}\,(z)};$$

(c) if $\Omega_r = \mathbb{C}\backslash\left[\{z : |z| \le r\} \cup (-\infty, 0)\right]$, and $x > r$ then

$$\lambda_{\Omega_r}(x) = \frac{x+r}{2x(x-r)}.$$

We shall reserve the notation Π_r, Λ_r and Ω_r for these sets for the rest of this paper.

4. The proof of Theorem 1.3

First, we take any point ζ that is not a critical point of P, and for which $P(\zeta) \notin C$, and prove (1.3) with z replaced by ζ. Let Q be the polynomial defined by

$$Q(z) = P(z+\zeta) - P(\zeta).$$

Then $Q(0) = 0$ and the critical points of Q are the points w_j, where $w_j = z_j - \zeta$. It follows that the critical values of Q lie in a half-plane bounded by a line through the origin, and we may assume that this half-plane is the right half-plane given by $x \ge 0$. As $Q(0) = 0$, $Q(0)$ is not a critical value of P and so the nearest critical value to 0 is, say a distance r away from the origin. It follows that we can define a single-valued branch of Q^{-1} on the region Π_r defined above and so (exactly as before) for all j we have

$$1 \leq 2\lambda_{\Pi_r}(0)|w_j|\,|Q'(0)|.$$

This gives, for some j,

$$|Q(w_j)| = r \leq \frac{16}{3\sqrt{3}}|w_j|\,|Q'(0)|,$$

which, in turn, yields

$$S(P,\zeta) = S(Q,0) \leq \frac{16}{3\sqrt{3}} = 3.0792\cdots.$$

This completes the proof of Theorem 1.3(i).

Next, we prove Theorem 1.3(ii). By replacing $P(z)$ by $P(z)-\zeta$, we may suppose that $\zeta = 0$, so that the smallest closed disc that contains all critical values of P is of the form $\{z : |z| \leq r\}$. Thus we want to prove that (1.4) holds when $|P(z)| > r$.

Suppose now that $|P(z)| > r$; in fact, there is no harm in actually assuming that $P(z) > r$. There is a single-valued branch g of P^{-1} defined on Ω_r, and if we let $\Sigma = g(\Omega_r)$ we see that P is a conformal mapping of Σ onto Ω_r. Thus

$$\lambda_\Sigma(z) = \lambda_{\Omega_r}(P(z))|P'(z)| \leq \frac{(P(z)+r)|P'(z)|}{2P(z)(P(z)-r)}.$$

Because Σ is simply connected, and none of the critical points z_j of P lie in Σ, we have

$$\frac{1}{2|z-z_j|} \leq \frac{1}{2\mathrm{dist}(z,\partial\Sigma)} \leq \lambda_\Sigma(z),$$

and consequently,

$$\frac{1}{|z-z_j|} \leq \frac{(P(z)+r)|P'(z)|}{P(z)(P(z)-r)}.$$

Because $\{z : |z| = r\}$ is the circumcircle for the finite set of critical values of P, there must be a critical value, say $P(c)$, of P lying on the closed semicircle $\{re^{i\theta} : -\pi/2 \leq \theta \leq \pi/2\}$. Then $|P(z)-P(c)|^2 \leq r^2 + |P(z)|^2$ so that

$$S(P,z) \leq \frac{|P(z)-P(c)|}{|z-c||P'(z)|} \leq \frac{(r+|P(z)|)\sqrt{r^2+|P(z)|^2}}{|P(z)|(|P(z)|-r)}.$$

This completes the proof of Theorem 1.3.

5. The proof of Theorem 1.2

In order to prove Theorem 1.2 we note from (3.3) that it suffices to take any P that satisfies (2.1), and then construct a simply connected domain Ω that contains 0 but not any critical value of P, and which is such that $\lambda_\Omega(0) \leq 2/4^{1/(d-1)}$. We shall now show how to do this.

Consider the $d-1$ critical values $P(z_j)$ of P; from (2.1) these all lie outside the unit disc $\mathbb{D}$. Let R_j be the ray of the form $\{re^{i\theta} : r \geq 1\}$ that passes through $P(z_j)$ and take Ω to be the complement of $\cup_j R_j$ in $\mathbb{C}$. It may happen that two or more of the $P(z_j)$ lie on the same ray R_k; in this case we draw additional rays so that in all cases, $\Omega = \mathbb{C}\backslash(R_1 \cup \cdots \cup R_{d-1})$, where here the R_j are disjoint. Let Ω_0 be the domain of this type (with exactly $d-1$ radial slits) in which the R_j emanate from the $(d-1)$-th roots of unity. We shall complete the proof of Theorem 1.2 by proving the following result which shows that Ω_0 (with the symmetrically placed slits) is the extremal case among all radially slit domains. For brevity, we let $n = d-1$.

Theorem 5.1. *For any domain Ω of the form $\mathbb{C}\backslash(R_1 \cup \cdots \cup R_n)$, where the rays R_j are distinct, we have*

$$\lambda_\Omega(0) \leq \lambda_{\Omega_0}(0) = 2/4^{1/n}. \tag{5.1}$$

Proof First we prove that $\lambda_{\Omega_0}(0) = 2/4^{1/n}$. We know that F given by $F(z) = -4z/(1-z)^2$ maps $\mathbb{D}$ conformally onto $\mathbb{C}\backslash[1,+\infty)$, that $z \mapsto z^n$ is a n-fold covering map of $\mathbb{D}$ onto itself that is branched only at the origin, and that $z \mapsto z^n$ is a n-fold covering map of Ω_0 onto $\mathbb{C}\backslash[1,+\infty)$ that is branched only at the origin. Now let $\mathbb{D}^* = \mathbb{D}\backslash\{0\}$, and note that any two unbranched n-fold coverings of $\mathbb{D}^*$ differ from each other by a homeomorphism (this is Corollary 4.22 on p.70 of [9]). In our case (above) the homeomorphism is necessarily analytic, and this shows that the conformal map $F : \mathbb{D} \to \mathbb{C}\backslash[1,+\infty)$ lifts to a conformal map G defined by

$$G(z) = \big[F(z^n)\big]^{1/n} = (-4)^{1/n} z + \cdots$$

(for some choice of the n-th root) of $\mathbb{D}$ onto Ω_0. We deduce that

$$2 = \lambda_{\mathbb{D}}(0) = \lambda_{\Omega_0}(0)|G'(0)| = 4^{1/n}\lambda_{\Omega_0}(0)$$

as required. We remark that the existence of G can also be derived from elementary complex analytic arguments, and we can even give an explicit formula for G^{-1}. Indeed, the composition of the maps $z \mapsto 1/z$, $z \mapsto z^{n/2}$, $z \mapsto z + \sqrt{z^2-1}$, $z \mapsto z^{2/n}$ and $z \mapsto 1/z$ (taken in this order) maps the sector $\{z : -2\pi/n < \arg z < 0\}$ conformally onto the sector $\{z : -2\pi/n < \arg z < 0, \quad |z| < 1\}$, and this can be extended (by repeated applications of the Reflection Principle) to give the desired mapping.

It remains to prove that $\lambda_\Omega(0) \leq \lambda_{\Omega_0}(0)$, and our proof of this is based on the following result of Dubinin [1], p.270.

Theorem 5.2. *Suppose that* $0 < r < 1$, *and let* $\zeta_1, \ldots, \zeta_n$ *be* n *distinct points on the unit circle. Let* D *denote the region obtained by removing from* $\mathbb{D}$ *the* n *radial slits* $\{t\zeta_j : r \leq t \leq 1\}$, $j = 1, \ldots, n$, *and let* D_0 *be this region in the case when the* ζ_j *are the* n*-th roots of unity. Let* f *be the unique conformal map of* D *onto* $\mathbb{D}$ *with* $f(0) = 0$ *and* $f'(0) > 0$, *and let* f_0 *be the corresponding map for* D_0. *Then* $f'(0) \leq f_0'(0)$.

We need to convert this into information about hyperbolic metrics, and as $\lambda_{\mathbb{D}}\big(f(0)\big)|f'(0)| = \lambda_D(0)$ and similarly for f_0, we see that

$$\lambda_D(0) \leq \lambda_{D_0}(0). \tag{5.2}$$

Next, we need to consider the limit of a sequence of domains described in Theorem 5.2. For $R > 1$ we now let

$$D(R) = \{z \in \mathbb{C} : |z| < R\} \backslash \bigcup_{j=1}^{n} R_j,$$

where R_j is the radial slit $\{t\zeta_j : 1 \leq t \leq R\}$ in the disc $\{|z| < R\}$. Similarly, we let $D_0(R)$ be the corresponding slit disc when the ζ_j are the n-th roots of unity. As the inequality (5.2) is true for any r in $(0,1)$ (where r is as described in Theorem 5.2) we can take $r = 1/R$ and apply the map $z \mapsto Rz$ to obtain the corresponding inequality between the hyperbolic metrics of $D(R)$ and $D_0(R)$. We now appeal to the following result (Theorem 1, [2]).

Lemma 5.3. *Suppose that* D_n *is an increasing sequence of domains whose union is* D, *and suppose that* $\mathbb{C}\backslash D$ *contains at least two points. Let* λ_n *be the hyperbolic metric on* D_n, *and let* λ *be the hyperbolic metric on* D. *Then for any* z *in* D, $\lambda_n(z)$ *is defined for all sufficiently large* n, *and* $\lambda_n(z) \to \lambda(z)$.

Finally, let

$$\Omega = \mathbb{C} \setminus \bigcup_{j=1}^{n} \{t\zeta_j : t \geq 1\},$$

and let Ω_0 be the corresponding region when the ζ_j are the n-th roots of unity. Then, appealing to Lemma 5.3 and the inequality corresponding to (5.2) but applied to $D(R)$ and $D_0(R)$, we see that

$$\lambda_\Omega(0) \leq \lambda_{\Omega_0}(0),$$

and this completes the proof of Theorems 5.1 and 1.2.

6. Equivalence of conformal metrics

Suppose that f is holomorphic in a domain Ω in $\mathbb{C}$ and let a be a point in Ω. Let $\delta(a,f)$ be the radius of the largest open disc, say $\Delta(a,f)$, with centre a on which f is injective. Clearly, every critical point z_j of f lies outside this disc, and so satisfies $|z_j - a| \geq \delta(a,f)$, but note that there need not be any critical points on $\partial\Delta(a,f)$. If we now remove the critical points from Ω to form

$$\Omega^* = \{z \in \Omega : f'(z) \neq 0\},$$

then $z \mapsto \delta(z,f)$ is positive and continuous on Ω^* and so defines a metric $|dz|/\delta(z,f)$ there.

We shall now carry out a similar construction in the range of f. Suppose that $a \in \Omega^*$. Then $f'(a) \neq 0$ and so we can define a local inverse function f^{-1} at $f(a)$ by insisting that $f^{-1}\big(f(a)\big) = a$. Now let $\sigma(a,f)$ be the radius of the largest open disc, say $\Sigma(a,f)$, with centre $f(a)$ on which this branch of f^{-1} is freely continuable with values in Ω. We now define a second metric on Ω^* namely $|f'(z)|\,|dz|/\sigma(z,f)$. Our first result shows that these two metrics are equivalent to each other with constants independent of the region.

Theorem 6.1. *If f is holomorphic on Ω, then, for all $z \in \Omega^*$,*

$$\frac{|f'(z)|}{4\sigma(z,f)} \leq \frac{1}{\delta(z,f)} \leq \frac{4|f'(z)|}{\sigma(z,f)}. \tag{6.1}$$

Further, each inequality here is attained for some f and Ω.

Proof We shall prove (6.1) with z replaced by a. For brevity, we shall write $b = f(a)$, $\Delta = \Delta(a,f)$, and similarly for δ, Σ and σ.

As the inverse of the conformal map $f : \Delta \to f(\Delta)$ is single-valued on the disc with centre b and radius $\operatorname{dist}(b, \partial f(\Delta))$, we see that $\sigma \geq \operatorname{dist}\big(b, \partial f(\Delta)\big)$. Thus

$$\frac{|f'(a)|}{2\sigma(a,f)} \leq \frac{|f'(a)|}{2\operatorname{dist}\big(b, \partial f(\Delta)\big)} \leq \lambda_{f(\Delta)}\big(b\big)|f'(a)| = \lambda_\Delta(a) = \frac{2}{\delta(a,f)},$$

and this is the first inequality in (6.1).

The second inequality follows by considering the conformal map $f^{-1} : \Sigma \to f^{-1}(\Sigma)$, which gives

$$\lambda_{f^{-1}(\Sigma)}\big(f^{-1}(b)\big)\big|(f^{-1})'(b)\big| = \lambda_\Sigma(b) = \frac{2}{\sigma(a,f)},$$

together with $|(f^{-1})'(b)| = 1/|f'(a)|$, and

$$\lambda_{f^{-1}(\Sigma)}\big(f^{-1}(b)\big) = \lambda_{f^{-1}(\Sigma)}(a) \geq \frac{1}{2\operatorname{dist}\big(a, \partial f^{-1}(\Sigma)\big)} \geq \frac{1}{2\delta(a,f)}.$$

To show that the lower bound is sharp, note that the function $f(z) = (1+z)^2/(1-z)^2$ is a conformal map of $\mathbb{D}$ onto $\mathbb{C}\backslash(-\infty, 0]$. It is easy to see that $\delta(0,f) = 1 = \sigma(0,f)$, and $f'(0) = 4$, and this demonstrates that the lower bound is sharp. The sharpness of the upper bound follows by considering f^{-1}.

We shall now show that Theorem 6.1 contains Smale's Theorem 1.1.

Corollary 6.2. *Suppose that P is a nonconstant polynomial and that z is not a critical point of P. Then there is a critical point c of P such that $P(c) \in \partial\Sigma\big(P(z), P\big)$, and $|P(z) - P(c)| \leq 4|P'(z)||z - c|$. In particular, $S(P,z) \leq 4$.*

Proof Theorem 6.1 gives

$$\sigma(z,P) \leq 4|P'(z)|\delta(z,P) \leq 4|P'(z)|\,|z - z_j|$$

for every j, as no critical point of P can lie in $\Delta(z,P)$. Now for polynomials (but not, in general, for entire functions) there must be a critical value, say $P(c)$, where c is some critical point of P, on $\partial\Sigma\big(P(z), P\big)$. Thus

$$|P(z) - P(c)| = \sigma(z,P) \leq 4|P'(z)|\,|z - c|.$$

The proof is complete.

Let us now consider a class of entire functions that properly includes nonconstant polynomials. Nonconstant polynomials are n-sheeted branched coverings of $\mathbb{C}$ onto itself, and we shall now consider the more general class of unlimited branched coverings of $\mathbb{C}$ onto itself. In addition to containing all nonconstant polynomials, this class contains such functions as $P(\cos z)$ and $P(\sin z)$, for any nonconstant polynomial P, and $\cos\sqrt{z}$. With this class we can obtain an improvement in the upper bound in Theorem 6.1.

Theorem 6.3. *Suppose that f is an unlimited branched covering of $\mathbb{C}$ onto itself with $d-1$ critical values, where $d \geq 2$. If z is not a critical point of f, then*

$$\frac{1}{\delta(z,f)} \leq 4^{(d-2)/(d-1)} \frac{|f'(z)|}{\sigma(z,f)}.$$

The proof is exactly as for the proof of Theorem 1.2. If we repeat the proof of Corollary 6.2 we see that Theorem 6.3 contains Theorem 1.2 as a special case.

References

[1] Dubinin, V.N., On the change in harmonic measure under symmetrization, *Mat.Sb* 124 (1984), 272-279, English trans. in *Math. USSR Sbornik* 52 (1985), 267-273.

[2] Hejhal, D.A., Universal covering maps for variable regions, *Math. Zeit.* 137 (1974), 7-20.

[3] Kra, I., *Automorphic forms and Kleinian groups*, Benjamin, 1972.

[4] Shub, M. and Smale, S., Computational complexity: on the geometry of polynomials and a theory of cost, *SIAM J. Comput.* 15 (1986), 145-161.

[5] Smale, S., The fundamental theorem of algebra and complexity theory, *Bull. Amer. Math. Soc.* 4 (1981), 1-36.

[6] Smale, S., *Mathematical Problems for the Next Century, Mathematics: frontiers and perspectives*, eds. Arnold, V., Atiyah, M., Lax, P. and Mazur, B., Amer. Math. Soc., 2000.

[7] Tischler, D., Critical points and values of complex polynomials, *J. of Complexity*, 5 (1989), 438-456.

[8] Tischler, D., Perturbations of Critical Fixed Points of Analytic Maps, *Astérisque* 222 (1994), 407-422.

[9] Völklein, H., *Groups as Galois groups*, Cambridge Univ. Press, 1996.