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CONSTRUCTIVE APPROXIMATION

The Critical Values of a Polynomial

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Abstract. It has been known for a long time that any real sequence $y_1, \dots, y_{n-1}$ is the sequence of critical values of some real polynomial. Here we show that any complex sequence $w_1, \dots, w_{n-1}$ is the sequence of critical values of some complex polynomial.

1. The Problem

The critical points of a polynomial p are the solutions z_j of $p'(z) = 0$, and the critical values of p are the numbers $p(z_j)$. If $y_1, \dots, y_n$ are given real numbers, then a necessary and sufficient condition for there to exist a real polynomial p of degree $n+1$ with real critical points x_j satisfying $x_1 < \cdots < x_n$ and critical values $y_j = p(x_j)$, $j = 1, \dots, n$, is that the numbers $(-1)^k(y_k - y_{k+1})$, $k = 1, \dots, n-1$, are all nonnegative or all nonpositive (see [3], [7], and also [8]). As any real sequence $y_1, \dots, y_n$ can be reordered so as to satisfy this last criterion, it follows that any real sequence $y_1, \dots, y_n$ is the sequence of critical values of some real polynomial. In an editorial footnote to [3], A. W. Goodman asked whether every complex sequence $w_1, \dots, w_n$ is the sequence of critical values of some complex polynomial of degree $n+1$. We shall show that it is.

Theorem 1.1. *Let $w_1, \dots, w_n$ be any sequence of complex numbers. Then there is a polynomial p of degree $n+1$ whose critical values are $w_1, \dots, w_n$.*

Clearly w is a critical value of a polynomial p if and only if the equations $p(z) - w = 0$ and $p'(z) = 0$ have a common solution, and this is so if and only if their resultant $\mathrm{Res}(p - w, p')$ is zero (see, e.g., [6, p. 52]). Although this gives (explicitly) a polynomial whose roots are the critical values of p (and whose coefficients are polynomials in the coefficients of p), we are unable to use this to good effect.

Given a polynomial p, the polynomial $z \mapsto p(az+b)$ has the same critical values as p and the same ramification over these critical values. Thus, by passing to $p(az+b)$ for some suitable a and b, we may confine our attention to polynomials p that are normalized by the conditions $p(0) = 0$ and p' is monic. Given any vector $u = (u_1, \dots, u_n)$ in $\mathbf{C}^n$,

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the unique polynomial p_u normalized in this way, and with critical points u, is

$$p_u(z) = \int_0^z (w - u_1) \cdots (w - u_n)\, dw, \tag{1.1}$$

and the sequence of critical values of p_u is given by $v = (v_1, \dots, v_n)$, where $v_j = p_u(u_j)$.

We now define the map $\theta : \mathbf{C}^n \to \mathbf{C}^n$ by $\theta(u) = v$; explicitly,

$$\theta(u_1, \dots, u_n) = (p_u(u_1), \dots, p_u(u_n)) = (v_1, \dots, v_n),$$

and this takes critical points to critical values (via normalized polynomials). As each v_j is (obviously) a homogeneous polynomial in the u_i, θ is holomorphic in $\mathbf{C}^n$, and in Section 2 we use some elementary results on several complex variables to show that $\theta(\mathbf{C}^n) = \mathbf{C}^n$; this will prove Theorem 1.1. This proof has its origins in [3] (where only real polynomials are considered), and our aim is to give as simple a proof as possible. Our second result, which we prove in Section 3, describes the action of θ in greater detail.

Theorem 1.2. *The map $\theta : \mathbf{C}^n \to \mathbf{C}^n$ is holomorphic, open, closed, proper, and finite. Further, for each v in $\mathbf{C}^n$ there are exactly $(n+1)^n$ points u such that $\theta(u) = v$.*

In Section 4 we illustrate Theorems 1 and 2 with a brief discussion of cubic polynomials, and in Section 5 we give a brief discussion of the polynomials p_u in the context of the equivalence classes of topologically equivalent polynomials introduced by Arnold in [1].

It is also possible to construct a polynomial with given critical values by a topological argument. This approach uses ideas introduced by Hurwitz and developed, for example, by Patterson [10]. Suppose that we wish to construct a polynomial with critical values at the points $w_1, w_2, \dots, w_k$ in the Riemann sphere $\mathbf{C}_\infty$. We first construct a compact topological surface S and a map $f : S \to \mathbf{C}_\infty$ which is a local homeomorphism at each point except those in $f^{-1}(w_j)$ for $j = 1, 2, \dots, k$. Then we can put a unique Riemann surface structure on S so that f is analytic. If the Euler characteristic $\chi(S)$ is 2, then S is homeomorphic to a sphere and hence the Riemann Mapping Theorem shows that S is conformally equivalent to $\mathbf{C}_\infty$. Consequently, there is an analytic map $f : \mathbf{C}_\infty \to \mathbf{C}_\infty$ with critical values at $w_1, w_2, \dots, w_k$, and this map f must then be a rational function. If f has degree d, and ∞ is a critical point of order d, then f must be a polynomial. We give the details of this argument in Section 6.

2. The Proof of Theorem 1.1

We begin with a straightforward, but important, property of the map θ.

Lemma 2.1. *With θ as above, $\theta(u) = 0$ if and only if $u = 0$.*

Proof. For any $a = (a_1, \dots, a_n)$ in $\mathbf{C}^n$ we write $\|a\| = (|a_1|^2 + \cdots + |a_n|^2)^{1/2}$. If $u = 0$, then $p_u(z) = z^{n+1}/(n+1)$ so that for each j, $v_j = p(0) = 0$; thus, $\theta(u) = 0$.

Conversely, suppose that $\theta(u) = 0$, and that the distinct u_j are written as $a_1, \dots, a_s$, where there are m_1 of the u_i corresponding to a_1, and so on. Then

$$p_u'(z) = (z - a_1)^{m_1} \cdots (z - a_s)^{m_s},$$

where the a_j are distinct and $m_1 + \cdots + m_s = n$. This shows that $p_u^{(k)}(a_j) = 0$ for $k = 1, \dots, m_j$, and as $p_u(a_j) = 0$, we see that $(z - a_j)^{m_j+1}$ divides $p_u(z)$. This implies that $\deg(p_u) \geq \deg(p_u') + s$ and from this we see that $s = 1$ and $p_u'(z) = (z - a_1)^n$. Integrating this from 0 to z gives $p_u(z)$, and noting that $p_u(a_1) = 0$ (because $\theta(u) = 0$) we find that $a_1 = 0$; thus $u_1 = \cdots = u_n = a_1 = 0$ as required. ∎

If $p_u(u_1), \dots, p_u(u_n)$ are nonzero then so are $u_1, \dots, u_n$ (because $p_u(0) = 0$) and, clearly, if $p_u(u_1), \dots, p_u(u_n)$ are distinct then so too are $u_1, \dots, u_n$. This means that if we define V by

(2.1) $$V = \{(z_1, \dots, z_n) \in \mathbf{C}^n : z_1, \dots, z_n \text{ nonzero and distinct}\},$$

then $\theta^{-1}(V) \subset V$. In some sense, the simplest part of the action of θ is its restriction to $\theta^{-1}(V)$; that is, the map $\theta : \theta^{-1}(V) \to V$.

We come now to discuss the topological properties of the map θ, but first we need to describe some general results. Let $f : X \to Y$ be a continuous map between topological spaces. Then f is *open* if f maps every open set to an open set, and *closed* if it maps every closed set to a closed set. Next, we say that f is a *covering map* if each point x in X has a neighborhood N such that the restriction of f to N is a homeomorphism (this definition is that used in Riemann surface theory and differs from that commonly used in topology). Clearly, any covering map is an open map.

We shall also need the notion of a proper map, and of fibers. We say that f is *proper* if $f^{-1}(K)$ is compact whenever K is. Finally, a *fiber* (of f) is a set of the form $f^{-1}(y)$, where $y \in f(X)$, and we say that f is *finite* if each fiber is a finite set. It is known that if $f : X \to Y$ is continuous, and if X and Y are locally compact Hausdorff topological spaces, then f is proper if and only if f is closed and each fiber $f^{-1}(y)$ is compact [9, p. 76]. For the convenience of the reader we shall now prove that part of this that is crucial to our argument. Throughout, we use $\bar{A}$ and A° to denote the closure and interior, respectively, of a set A.

Lemma 2.2. *Suppose that X and Y are locally compact Hausdorff topological spaces, and that $f : X \to Y$ is continuous and proper. Then f is closed.*

Proof. Let E be any closed set in X, and suppose that $z \in \overline{f(E)}$. It suffices to show that $z \in f(E)$. As Y is locally compact, there exists a compact neighborhood U of z. Now take any open set W containing z; then $W \cap U^\circ$ is an open neighborhood of z and so contains a point of $f(E)$. This means that W meets $U \cap f(E)$, and we deduce that

(2.2) $$z \in \overline{U \cap f(E)}.$$

Next, it is immediate that

(2.3) $$U \cap f(E) = f(f^{-1}(U) \cap E).$$

As f is proper, $f^{-1}(U)$ is compact. As X is Hausdorff, $f^{-1}(U)$ is closed, and it follows from this that $f^{-1}(U) \cap E$ is compact and, hence, from (2.3), that $U \cap f(E)$ is compact in Y. As Y is Hausdorff, $U \cap f(E)$ is closed in Y so, from (2.2), $z \in U \cap f(E)$. In particular, $z \in f(E)$ as required.

We now return to discuss the topological properties of the map θ.

Lemma 2.3. *The map $\theta : \mathbf{C}^n \to \mathbf{C}^n$ is proper.*

Lemma 2.4. *The map $\theta : \theta^{-1}(V) \to V$ is a surjective covering map.*

Lemmas 2.3 and 2.4 lead easily to Theorem 1.1. By Lemma 2.4, $\theta(\mathbf{C}^n)$ contains V. By Lemmas 2.2 and 2.3, $\theta(\mathbf{C}^n)$ is a closed subset of $\mathbf{C}^n$, and so contains the closure of V, namely, $\mathbf{C}^n$. Thus $\theta(\mathbf{C}^n) = \mathbf{C}^n$, and this proves Theorem 1.1 subject to proving Lemmas 2.3 and 2.4.

Proof of Lemma 2.3. Let $K = \{u \in \mathbf{C}^n : \|u\| = 1\}$. As θ is continuous in $\mathbf{C}^n$, the infimum η of $\|\theta(u)\|$ over the compact set K is attained at some point of K, and $\eta \neq 0$ for, otherwise (by Lemma 2.1), we would have 0 in K. We deduce that $\eta > 0$.

Now take any nonzero u in $\mathbf{C}^n$ and let $\lambda = \|u\|^{-1}$. Then $\lambda u \in K$ so that $\|\theta(\lambda u)\| \geq \eta$. Finally, let q be the polynomial constructed as in (1.1) but using λu instead of u, and let $v' = \theta(\lambda u)$. A simple change of variable gives $q(z) = \lambda^{n+1} p_u(z/\lambda)$ so that $v'_j = q(\lambda u_j) = \lambda^{n+1} v_j$. This shows that $\theta(\lambda u) = \lambda^{n+1}\theta(u)$ and, hence, that $\|\theta(u)\| \geq \eta\|u\|^{n+1}$. In fact, we have the double inequality

$$\eta\|u\|^{n+1} \leq \|\theta(u)\| \leq 2^n\|u\|^{n+1}, \tag{2.4}$$

for the upper bound of $\|\theta(u)\|$ here is a trivial consequence of the definition (1.1) of p_u (because when we evaluate v_j using (1.1), we can integrate along the segment from 0 to u_j). Finally, if E is a compact subset of $\mathbf{C}^n$, then E is bounded and closed. As θ is continuous, $\theta^{-1}(E)$ is closed and, from the lower bound in (2.4), it is also bounded. Thus θ is proper, and this completes the proof of Lemma 2.3. ■

Proof of Lemma 2.4. We prove first that the Jacobian J_θ is nonzero at each point of V. As $\theta^{-1}(V) \subset V$, this will show that $J_\theta \neq 0$ on $\theta^{-1}(V)$. Let the Jacobian matrix of θ be (θ_{ij}). Then

$$\theta_{ij} = \frac{\partial v_i}{\partial u_j} = \frac{\partial}{\partial u_j}\left(\int_0^{u_i} (w-u_1)\cdots(w-u_n)\,dw\right) = -\int_0^{u_i} \frac{p'_u(w)}{w-u_j}\,dw;$$

this holds even when $i = j$ because (under appropriate hypotheses)

$$\frac{d}{dx}\int_0^x f(x,t)\,dt = \int_0^x \frac{\partial f(x,t)}{\partial x}\,dt + f(x,x).$$

Now $\det(\theta_{ij}) \neq 0$ at a point u in V if the columns of (θ_{ij}) are linearly independent there, so suppose that for $i = 1, \ldots, n$, $\sum_j \mu_j \theta_{ij} = 0$, where the μ_j are some constants. Then

$$(2.5)\quad 0 = \sum_{j=1}^{n} \mu_j \int_0^{u_i} \frac{p_u'(w)}{w - u_j}\, dw = \int_0^{u_i} \left(\sum_{j=1}^{n} \mu_j \frac{p_u'(w)}{w - u_j} \right) dw = \int_0^{u_i} f(w)\, dw,$$

say, where

$$(2.6)\quad f(z) = \sum_{j=1}^{n} \mu_j \frac{(z - u_1) \cdots (z - u_n)}{(z - u_j)},$$

a polynomial of degree $n - 1$. Now let

$$F(z) = \int_0^z f(w)\, dw.$$

Then $F : \mathbf{C} \to \mathbf{C}$ is a polynomial of degree n with $F(0) = 0$, and (2.5) implies that $F(u_j) = 0$ for each j. As $u \in V$, the u_j are distinct and nonzero so that F has $n + 1$ zeros (namely, $0, u_1, \ldots, u_n$). We deduce that F is zero throughout $\mathbf{C}$, hence, so too is its derivative, namely, f. If we let $z \to u_i$ in (2.6), and recall that the u_j are distinct, we see that $\mu_i = 0$. As this holds for all i, we see that $\det(\theta_{ij}) \neq 0$ in V.

The Inverse Mapping Theorem (see, e.g., [5, p. 121] or [4, p. 17]) shows that if D is an open subset of $\mathbf{C}^n$, and if $f : D \to \mathbf{C}^n$ is holomorphic with Jacobian J_f nonzero at a point u^* in D, then the restriction of f to some open neighborhood of u^* is a homeomorphism. As θ is continuous $\theta^{-1}(V)$ is open, and it follows from this that $\theta : \theta^{-1}(V) \to V$ is a covering map.

It remains to show that $\theta : \theta^{-1}(V) \to V$ is surjective. Let $W = \theta(\theta^{-1}(V))$. As $\theta^{-1}(V)$ is open, $\theta^{-1}(V) \subset V$, and the restriction of θ to V is an open map, so W is an open subset of V. We shall now show that W is relatively closed in V. Suppose, then, that $w_1, w_2, \ldots$ are points in W that converge to a point w^* in V. We may assume that the w_j lie in some compact neighborhood K of w^*, where $K \subset V$, and so, by Lemma 2.3, there are points z_j lying in the compact set $\theta^{-1}(K)$ with $\theta(z_j) = w_j$. By passing to a subsequence, we may assume that $z_j \to z^*$, say, and then by continuity, $\theta(z^*) = w^*$. As $w^* \in V$, so $z^* \in \theta^{-1}(V)$, so that $w^* \in W$ as required. As W is relatively open and relatively closed in V, and as V is arcwise connected (this is clear), we deduce that $W = V$ or $W = \emptyset$. However, $(1, \omega, \ldots, \omega^{n-1})$, where $\omega = \exp(2\pi i/n)$, is in $\theta^{-1}(V)$, so that $W \neq \emptyset$. We deduce that $W = V$ and this shows that $\theta : \theta^{-1}(V) \to V$ is surjective. This completes the proof of Lemma 2.4 and, therefore, also the proof of Theorem 1.1.■

3. The Proof of Theorem 1.2

A subset W of $\mathbf{C}^n$ is an *analytic subset* of $\mathbf{C}^n$ if there are a positive finite number of complex valued functions $f_1, \ldots, f_k$, holomorphic in $\mathbf{C}^n$, such that

$$W = \{z \in \mathbf{C}^n : f_1(z) = \cdots = f_k(z) = 0\}.$$

The union of a finite number of analytic sets is an analytic set so, for example, the complement $C(V)$ of V given in (2.1) is an analytic set. Indeed, if we define the coordinate functions π_j by $(z_1, \ldots, z_n) \mapsto z_j$, then $\mathcal{C}(V)$ is the union of the set of zeros of each of the functions π_i and $\pi_i - \pi_j$, where $i \neq j$. It now follows that $\theta^{-1}(\mathcal{C}(V))$ is an analytic set and, hence, that its complement, namely, $\theta^{-1}(V)$, is arcwise connected and dense in $\mathbf{C}^n$ (for details, see [2, pp. 14–15]). An easy argument using Lemma 2.4 and the connectedness of $\theta^{-1}(V)$ now shows that the cardinality of each fiber $\theta^{-1}(v)$ is independent of v in V, and this is the motivation for Theorem 1.2.

We have already seen that θ is holomorphic, proper, and closed; we shall now show that it is finite. Let $\theta = (\theta_1, \ldots, \theta_n)$, so that each θ_j is a homogeneous polynomial of degree $n+1$ in the variables u_i, and choose any v in $\mathbf{C}^n$. As

$$\theta^{-1}(v) = \{u \in \mathbf{C}^n : \theta_1(u) - v_1 = \cdots = \theta_n(u) - v_n = 0\},$$

we see that $\theta^{-1}(v)$ is an analytic set. It is also compact (as θ is proper), and as every compact analytic set is finite (see [2, p. 31] and [9, p. 235]), we deduce that θ is a finite map.

Now not every holomorphic nonconstant map from $\mathbf{C}^n$ to itself is open (e.g., the map $(z, w) \mapsto (z, zw)$ is not). However, it is known that if $f : \mathbf{C}^n \to \mathbf{C}^n$ is holomorphic, and if z is an isolated point of a fiber $f^{-1}(v)$, then f maps each open neighborhood of z onto an open neighborhood of v (see, e.g., [5, Theorem 11, p. 121]). It follows that any finite holomorphic map is open; in particular, θ is an open map. Note that this argument does not require the Jacobian to be nonzero so it applies equally well to points in $\mathcal{C}(V)$.

It remains to show that for each v in $\mathbf{C}^n$ there are (with an appropriate count of multiplicities) exactly $(n+1)^n$ points u such that $\theta(u) = v$. The appropriate count of points in $\theta^{-1}(v)$ that takes into account multiple zeros is given by Bézout's theorem as stated in [9, pp. 431–432]. We first write any point in $\mathbf{C}^{n+1}$ as $(u_0, u_1, \ldots, u_n)$, and then embed $\mathbf{C}^n$ in $\mathbf{C}^{n+1}$ by identifying $(u_1, \ldots, u_n)$ with $(1, u_1, \ldots, u_n)$. Given v in $\mathbf{C}^n$ we now wish to count the number of projective solutions of the n homogeneous equations (each of degree $n+1$)

$$p_u(u_i) - v_i u_0^{n+1} = 0, \qquad i = 1, \ldots, n.$$

The number of projective solutions of these equations is finite (because θ is proper), and so Bézout's theorem is applicable; this says that there are exactly $(n+1)^n$ solutions with due count of multiplicities, and this completes the proof of Theorem 1.2. ∎

4. Cubic Polynomials

We illustrate Theorems 1 and 2 with a brief discussion of the map $\theta : \mathbf{C}^n \to \mathbf{C}^n$ when $n = 2$ (and p_u is a cubic polynomial). A straightforward calculation shows that

$$\theta(u_1, u_2) = \tfrac{1}{6}(u_1^2(3u_2 - u_1), u_2^2(3u_1 - u_2)), \tag{4.1}$$

and that

$$\theta'(u_1, u_2) = \begin{pmatrix} u_1u_2 - u_1^2/2 & u_1^2/2 \\ u_2^2/2 & u_1u_2 - u_2^2/2 \end{pmatrix},$$

so that

$$J_\theta(u_1, u_2) = \det \theta'(u_1, u_2) = -\tfrac{1}{2}u_1u_2(u_1 - u_2)^2.$$

Thus, in this case $\theta^{-1}(V) = V$, and $J_\theta(u) \neq 0$ if and only if $u \in V$. The Jacobian J_θ has rank 1 at points where $u_1 = u_2 \neq 0$ and is the zero matrix at the point 0 in $\mathbf{C}^2$.

According to Theorem 1.2, for each general point (v_1, v_2), there are exactly nine vectors (u_1, u_2) such that $(p_u(u_1), p_u(u_2)) = (v_1, v_2)$ (it is easy to see this directly). For example, consider u in $\theta^{-1}(\frac{1}{6}, \frac{14}{81})$ and, for brevity, write $(x, y) = (u_1, u_2)$. Then, from (4.1), we have

$$x^2(3y - x) = 1, \qquad y^2(3x - y) = \tfrac{28}{27}, \tag{4.2}$$

so that $y = (x^3 + 1)/3x^2$, and substituting this expression in the second equation we obtain

$$(X - 1)(8X^2 - 5X + 1) = 0, \qquad X = x^3.$$

This shows that there are exactly nine values of x for which $\theta(x, y) = (\frac{1}{6}, \frac{14}{81})$ and for each such value of x there is a unique value of y determined by (4.2).

Let us now find the cardinality of the fibers $\theta^{-1}(v_1, v_2)$ at points (v_1, v_2) in $\mathcal{C}(V)$. First, suppose that $\theta(x, y) = (v, v)$, where $v \neq 0$. Then $x^2(3y - x) = v = y^2(3x - y)$, so that $x = y$ and $2x^3 = v$, and the fiber $\theta^{-1}(v, v)$ contains exactly three points. Next, if $(x, y) \in \theta^{-1}(\alpha, 0)$, then $y^2(3x - y) = 0$ and this gives rise to exactly six points (x, y). Finally, $\theta^{-1}(0, 0)$ contains only one point (see the proof of Lemma 2.3).

5. Topologically Inequivalent Polynomials

In [1] Arnold defines two polynomials p and q to be *topologically equivalent* if there is a homeomorphism h of the Riemann sphere onto itself such that $p \circ h = q$. Now such a homeomorphism h must be of the form $h(z) = az + b$ (for, except at the critical values of p, h must, by continuity, be locally a branch of $p^{-1} \circ q$, and isolated singularities are removable for univalent maps). It follows that any polynomial is equivalent to one of the polynomials p_u defined in Section 1. Topologically equivalent polynomials have the same critical values and also the same generators of their monodromy groups.

Suppose now that u and w are in $\mathbf{C}^n$ and that p_u and p_w are topologically equivalent, and of degree $n + 1$. Then, for some complex a and b, $p_u(az + b) = p_w(z)$. As p'_u and p'_w are monic, this shows that $a^{n+1} = 1$. Also, as $p_u(b) = p_w(0) = 0$, there are exactly $n + 1$ choices for b. Thus each topological equivalence class contains exactly $(n + 1)^2$ polynomials of the form p_u. It follows from Theorem 1.2 that, given any v in $\mathbf{C}^n$, there are exactly $(n + 1)^{n-2}$ topologically inequivalent polynomials p_u with v as their vector of critical values, and as this is indeed the number of topological equivalence classes of inequivalent polynomials given in [1]. In particular, each equivalence class is represented by some p_u.

Finally, we have considered vectors u of critical values so as to handle multiple roots properly, and this means that we have distinguished between two vectors of critical values (one obtained by permuting the components of the other) which yield the same polynomial p_u. To remove this unnatural distinction we could work with quotient spaces with respect to the transformation group induced on $\mathbf{C}^n$ by permuting the components of each vector in the natural way.

6. A Topological Construction

Consider first a nonconstant analytic map $f : S \to \mathbf{C}_\infty$ from a compact Riemann surface S into the Riemann sphere. As f has only finitely many critical points, it has only finitely many critical values, and we let $Q = \{w_1, w_2, \dots, w_K\}$ be the set of its critical values, where the w_j are distinct. Now f has degree N, say, where $N \geq 1$, and for each critical value w_k, the inverse image $f^{-1}(w_k)$ has less than N points, say $N - \delta(w_k)$ points. We call $\delta(w_k)$ the *deficiency* of f at w_k. Since f has degree N, we have

$$\sum_{z\in f^{-1}(w_k)} \deg(f; z) = N, \qquad \sum_{z\in f^{-1}(w_k)} [\deg(f; z) - 1] = \delta(w_k).$$

The Riemann–Hurwitz formula shows that

$$\chi(S) = N\chi(\mathbf{C}_\infty) - \sum_{z\in\mathbf{C}_\infty} (\deg(f; z) - 1),$$

and as $\chi(\mathbf{C}_\infty) = 2$, this gives

$$\sum_k \delta(w_k) = 2N - \chi(S). \tag{6.1}$$

For each $w \in \mathbf{C}_\infty\backslash Q$, the inverse image $f^{-1}(w)$ consists of exactly N distinct points, and the restriction of f to $S\backslash f^{-1}(Q)$ is an N-fold regular covering map; thus each curve in $\mathbf{C}_\infty\backslash Q$ lifts to N curves in $S\backslash f^{-1}(Q)$. Choose a base point b in $\mathbf{C}_\infty\backslash Q$. Then the set $B = f^{-1}(b)$ has exactly N points, say $b_1, b_2, \dots, b_N$. Any curve γ in $\mathbf{C}_\infty\backslash Q$, which begins and ends at b, has a unique lift Γ_n which begins at b_n and ends at another point $b_{\sigma(n)}$ of B, and the map $n \mapsto \sigma(n)$ is in the group S_N of all permutations of $\{1, 2, \dots, N\}$. The monodromy theorem shows that the permutation σ depends only on the homotopy class of γ in $\mathbf{C}_\infty\backslash Q$, and the group of such permutations σ is the *monodromy group* of f.

For each k, choose a curve α_k from b to the critical value w_k. Consider now the path γ_k which follows α_k until just before w_k, then follows a small circle winding positively once around w_k, and then returns to b along α_k. The corresponding permutation σ_k has a cycle of order $\deg(f; z)$ for each value of z in $f^{-1}(w_k)$, and these cycles are mutually disjoint. If we replace α_k by any other path from b to w_k, then σ_k changes to a conjugate permutation and so the conjugacy class $[\sigma_k]$ of σ_k is completely determined by the critical value w_k. We will say that σ_k is a permutation *associated* with the critical value w_k.

Recall that the conjugacy classes in the symmetric group S_N are determined by the sizes of the orbits or, equivalently, by the lengths of the disjoint cycles of the permutations. Thus specifying the orders of all the critical points z which map to w_k is equivalent to specifying the conjugacy class of the permutation σ_k. When the permutation σ_k has orbits of lengths $d_1, d_2, \dots, d_M$, the deficiency $\delta(w_k)$ is $\sum_m (d_m - 1)$. We will also write this as $\delta([\sigma_k])$ and, as $\sum_m d_m = N$, we see that

$$\delta([\sigma_k]) = N - (\text{number of orbits of } \sigma_k).$$

The permutations $\sigma_1, \sigma_2, \dots, \sigma_K$ have to satisfy a compatibility condition. If we reorder the points $w_1, w_2, \dots, w_K$ appropriately, the product path $\gamma_K \cdot \gamma_{K-1} \cdot \dots \cdot \gamma_2 \cdot \gamma_1$, which follows first γ_1, then γ_2, and so on up to γ_K, will be null-homotopic in $\mathbf{C}_\infty \backslash Q$. Therefore,

$$\sigma_K \sigma_{K-1} \cdots \sigma_2 \sigma_1 = I, \tag{6.2}$$

where I is the identity permutation in S_N.

We will now show that the existence of permutations satisfying (6.2) is also sufficient for there to be an analytic function with prescribed critical values.

Lemma 6.1. *Let $w_1, w_2, \ldots, w_K$ be distinct points in $\mathbf{C}_\infty$, and let $\sigma_1, \sigma_2, \ldots, \sigma_K$ be permutations in S_N that satisfy $\sigma_K \sigma_{K-1} \cdots \sigma_2 \sigma_1 = I$. Then there is a compact Riemann surface S (not necessarily connected), and an analytic map $f : S \to \mathbf{C}_\infty$ which has:*

(a) *critical values precisely at the points $w_1, w_2, \ldots, w_K$; and*
(b) *σ_k as its permutation associated with the critical value w_k ($k = 1, \ldots, K$).*

Proof. Let $Q = \{w_1, w_2, \ldots, w_K\}$. Choose a base point $b \in \mathbf{C}_\infty \backslash Q$ and paths $\alpha_1, \ldots, \alpha_K$ as above; the paths α_k can and will be chosen so that they meet only at b, and that their complement Δ in $\mathbf{C}_\infty$ is simply connected. Label the two sides of the path α_k as α_k^- and α_k^+ in such a way that the sides occur in the order

$$\alpha_1^+, \alpha_2^-, \alpha_2^+, \alpha_3^-, \ldots, \alpha_K^+, \alpha_1^-$$

going positively around b. Now take N copies of Δ, which we label $\Delta(1), \Delta(2), \ldots, \Delta(N)$, and construct a surface S by joining each edge $\alpha_k^-(n)$ of $\Delta(n)$ to the edge $\alpha_k^+(\sigma(n))$ of $\Delta(\sigma(n))$. This clearly gives a topological surface except possibly at the points of S corresponding to b. However, the condition $\sigma_K \sigma_{K-1} \cdots \sigma_2 \sigma_1 = I$ ensures that there are exactly N points in S corresponding to b and that S is a surface even at these points. It is also clear that S is compact.

Let $f : S \to \mathbf{C}_\infty$ be the map which sends each point of $\Delta(n)$ to the corresponding point in Δ. Then f is a local homeomorphism except at the finitely many points of $f^{-1}(Q)$. Consequently, there is an unique Riemann surface structure on $S \backslash f^{-1}(Q)$ which makes f analytic. The points of $f^{-1}(Q)$ are removable singularities for f, so we now have a Riemann surface structure on all of S with respect to which $f : S \to \mathbf{C}_\infty$ is analytic. It is clear from the construction of S that the given permutations σ_k are the permutations of the points in $f^{-1}(b)$ obtained by lifting the paths γ_j. Therefore they are associated with the critical values w_k as required. The proof of Lemma 6.1 is complete.

Lemma 6.1 reduces the construction of analytic maps with prescribed critical values, and prescribed branchings at these values, to the construction of permutations satisfying the identity $\sigma_K \sigma_{K-1} \cdots \sigma_2 \sigma_1 = I$. In order to obtain a rational map (or a polynomial), we must ensure that the Riemann surface S constructed in the proposition is connected and a sphere. Clearly S will be connected precisely when the permutations $\sigma_1, \sigma_2, \ldots, \sigma_K$ are transitive on $\{1, 2, \ldots, N\}$, and once this is known, we find that S is a compact, connected Riemann surface. In this case S is then determined up to homeomorphism by its Euler characteristic, and this can be calculated from (6.1).

Suppose that we are given points $w_1, w_2, \ldots, w_K$ in $\mathbf{C}_\infty$ and conjugacy classes $C_1, C_2, \ldots, C_K$ in S_N. In order for there to be a rational function $f : \mathbf{C}_\infty \to \mathbf{C}_\infty$ of degree N with critical values at $w_1, w_2, \ldots, w_K$ and associated permutations $\sigma_k \in C_k$, (6.1), namely, $\sum \delta(C_k) = 2N - 2$, must hold. However, this condition is not sufficient. For example, consider the three conjugacy classes

$$C_1 = C_2 = \{(1,2)(3,4),\ (1,3)(2,4),\ (1,4)(2,3)\} \qquad \text{and} \qquad C_3 = \{\text{3-cycles}\}$$

in the group S_4. If $\sigma_1 \in C_1$ and $\sigma_2 \in C_2$, then both lie in the subgroup $V = \{I,\ (1,2)(3,4),\ (1,3)(2,4),\ (1,4)(2,3)\}$ of S_4. Hence $\sigma_3 = (\sigma_2\sigma_1)^{-1}$ must lie in V and so cannot be a 3-cycle. This shows that we cannot always construct a rational function with prescribed critical values w_k and prescribed orders at the points of $f^{-1}(w_k)$.

We shall now show that, by contrast, the conditions specified above are sufficient to construct polynomials with prescribed critical values and branchings. Note first that when we consider a polynomial of degree N as a rational function $f : \mathbf{C}_\infty \to \mathbf{C}_\infty$, it has a critical point of order N at ∞, and it fixes ∞. We may take ∞ as w_K; then $f^{-1}(w_K) = \{\infty\}$ and the permutation σ_K is an N-cycle in S_N. Therefore the permutations $\sigma_1, \sigma_2, \ldots, \sigma_{K-1}$ must be such that $\sigma_{K-1}\sigma_{K-2}\cdots\sigma_2\sigma_1$ is an N-cycle in S_N. In particular, $\delta(w_K) = N - 1$, so that

$$\sum_{k=1}^{K-1} \delta(w_k) = N - 1.$$

Conversely, we have the following result:

Theorem 6.2. *Let $w_1, w_2, \ldots, w_{K-1}$ be distinct points of $\mathbf{C}$ and let $C_1, C_2, \ldots, C_{K-1}$ be conjugacy classes in S_N, not equal to $\{I\}$, that satisfy*

$$\sum_{k=1}^{K-1} \delta(C_k) = N - 1.$$

Then there is a polynomial f of degree N, with $\{w_1, w_2, \ldots, w_{K-1}\}$ as its set of critical values, and such that for each k, the permutation σ_k associated with the critical value w_k lies in C_k.

Proof. Our aim is to construct permutations σ_k and apply Lemma 6.1. We will proceed by induction on N. When $N = 1$, each permutation is the identity and the result is trivial. Now suppose it is true for permutations in S_{N-1}. We will regard S_{N-1} as the subgroup of S_N which fixes the point N.

Consider, for the moment, any permutation τ in S_N which has no fixed points. As all orbits under τ must have length at least two, there are at most $\frac{1}{2}N$ orbits; thus, $\delta([\tau]) \geq \frac{1}{2}N$. As

$$N > N - 1 = \sum_{k=1}^{K-1} \delta(C_k),$$

it follows that at most one of the conjugacy classes C_k can contain permutations with no fixed points. If there is such a conjugacy class (and there need not be) we may assume that it is C_1. It follows that in all cases, for $k = 2, 3, \ldots, K-1$, each of the permutations in C_k has a fixed point and so the conjugacy class

$$C'_k = C_k \cap S_{N-1}$$

in S_{N-1} is nonempty. Now choose any permutation $\sigma_1 \in C_1$ which moves N and let $\sigma_1' = \sigma_1 \rho$, where ρ is the transposition that interchanges N and $\sigma_1^{-1}(N)$. As σ_1' fixes N it lies in some conjugacy class C_1' of S_{N-1}.

We now apply the inductive hypothesis to the conjugacy classes C_k', and as

$$\delta(C_1') = \delta(C_1) - 1 \qquad \text{and} \qquad \delta(C_k') = \delta(C_k) \quad \text{for } k = 2, 3, \ldots, K-1,$$

there are permutations $\sigma_k' \in C_k'$ for which $\sigma_{K-1}' \cdots \sigma_2' \sigma_1'$ is an $(N-1)$-cycle. By conjugating in S_{N-1} we may assume that this $(N-1)$-cycle is $(1, 2, \ldots, N-2, N-1)$. Then we can set $\sigma_k = \sigma_k'$ for $k = 2, 3, \ldots, K-1$ and get

$$\sigma_{K-1} \cdots \sigma_2 \sigma_1 = (\sigma_{K-1}' \cdots \sigma_2' \sigma_1')\rho = (1, 2, \ldots, N-2, N-1)\rho$$

which is an N-cycle which we call σ_K^{-1}.

Lemma 6.1 now shows that there is analytic map $f : S \to \mathbf{C}_\infty$ from some compact Riemann surface S with critical points at $w_1, w_2, \ldots w_{K-1}$ and $w_K = \infty$ and associated permutations $\sigma_1, \sigma_2, \ldots, \sigma_{K-1}$ and σ_K. Since σ_K is an N-cycle, the permutations $\sigma_1, \sigma_2, \ldots, \sigma_{K-1}, \sigma_K$ certainly act transitively on $\{1, 2, \ldots, N\}$; hence, S is connected. As the Euler characteristic of S is 2 it is homeomorphic to a sphere. By the Riemann Mapping Theorem, S is then conformally equivalent to a sphere, and we have now constructed the desired polynomial map $f : \mathbf{C}_\infty \to \mathbf{C}_\infty$.

It is also possible to adapt the above argument to deal with maps from the unit disk $\mathbf{D}$ to itself. Because the disk is not compact, we shall consider only proper analytic maps $f : \mathbf{D} \to \mathbf{D}$. These are the Blaschke products:

$$f(z) = \omega \prod_{n=1}^{N} \left(\frac{z - a_n}{1 - \overline{a_n} z} \right)$$

where $|\omega| = 1$ and $a_1, a_2, \ldots, a_N \in \mathbf{D}$. The topological arguments in Lemma 6.1 and Theorem 1.2 now apply, essentially unchanged, to show that for points $w_1, w_2, \ldots, w_{K-1} \in \mathbf{D}$ and conjugacy classes $C_1, C_2, \ldots, C_{K-1}$ in S_N that satisfy $\sum_k \delta(C_k) = N - 1$ there is a Blaschke product of degree N with critical values at $w_1, w_2, \ldots, w_{K-1}$ and the permutations associated with w_k lying in C_k.

Finally, although the corresponding result fails for the Riemann sphere, as we saw above, we can always construct rational maps with prescribed critical values w_k and prescribed values for $\delta(w_k)$ even though we cannot specify the conjugacy classes of the permutations σ_k. We can do this, for example, by choosing each σ_k to be a cycle of length $\delta(w_k) + 1$ (the cycles required can be written explicitly); alternatively, the method used earlier in the paper can be adapted to prove it.

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