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ON A NEW SINGULAR DIRECTION OF MEROMORPHIC FUNCTIONS

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In this paper, by using Ahlfors' theory of covering surfaces, we establish the existence of a new singular direction for a meromorphic functions f, namely a T direction for f, for which the Nevanlinna characteristic function $T(r,f)$ is used as a comparison function.

1. INTRODUCTION AND RESULTS

Let $f(z)$ be a transcendental meromorphic function defined on the whole complex plane. The singular direction for f is one of the main objects studied in the theory of value distribution of meromorphic functions. Several types of singular directions have been introduced in the literature. Their existence and some connections between them have also been established. Here we shall give a brief history of this research area and refer the readers to [1] or [2] for a detailed survey. The study of singular directions for meromorphic functions was started by Julia in 1919. In [5], Julia introduced the concept of Julia direction for a meromorphic function f, that is, a ray $\arg z = \theta$ having the following property: for any ε $(0 < \varepsilon < \pi)$ and for all a, with at most two exceptions, on the Riemann sphere $\mathbb{C}_\infty$,

$$\lim_{r\to+\infty} n(r, \Omega_\varepsilon, f = a) = +\infty,$$

where $\Omega_\varepsilon = \{z : \theta - \varepsilon < \arg z < \theta + \varepsilon\}$ and $n(r, \Omega_\varepsilon, f = a)$ is the number of the solutions of $f(z) = a$ in $\Omega_\varepsilon \cap \{|z| < r\}$, counting multiplicities. In the same paper, Julia showed that every transcendental meromorphic function with an asymptotic value has a Julia direction. Therefore, every transcendental entire function has at least one Julia direction and this is a refinement of Picard's theorem. In order to have a similar refinement for Borel's theorem, a more refined notion of Borel directions was introduced by Valiron [9] in 1928. A ray $\arg z = \theta$ is called a Borel direction of order ρ for f if for every ε $(0 < \varepsilon < \pi)$,

$$\limsup_{r\to+\infty} \frac{\log n(r, \Omega_\varepsilon, f = a)}{\log r} \geqslant \rho \tag{1}$$

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for all a on $\mathbb{C}_\infty$ with at most two exceptions. Note that this definition meaningfully characterises the growth of f only when $0 < \rho < \infty$. In this case it is well known that f must have at least one Borel direction ([**10**]). To take into account the $\rho = 0$ or $\rho = \infty$ case, the Nevanlinna characteristic $T(r, f)$ is a more appropriate comparison function than $\log r$. With this in mind Zheng [**14**] introduced a new singular direction, namely the T direction for f.

Definition: A ray $\arg z = \theta$ is called a T direction for a meromorphic function $f(z)$, if for any ε $(0 < \varepsilon < \pi)$,

$$\limsup_{r\to+\infty} \frac{\overline{N}(r, \Omega_\varepsilon, f = a)}{T(r, f)} > 0 \tag{2}$$

for all a on $\mathbb{C}_\infty$ with at most two exceptions.

Here $\overline{N}(r, \Omega_\varepsilon, f = a)$ is the counting function of distinct a-points of $f(z)$ in $\Omega_\varepsilon \cap \{|z| < r\}$ defined by the formula

$$\overline{N}(r, \Omega_\varepsilon, f = a) = \int_0^r \frac{\overline{n}(t, \Omega_\varepsilon, f = a)}{t}\, dt$$

and $\overline{n}(r, \Omega_\varepsilon, f = a)$ is the number of the distinct solutions of $f(z) = a$ in $\Omega_\varepsilon \cap \{|z| < r\}$.

We believe that it is more natural to use the Nevanlinna characteristic function $T(r, f)$ as a comparison function, because in general $T(r, f)$ is the most basic function that one uses to describe the growth of meromorphic functions.

It was conjectured by Zheng [**14**] that a meromorphic function $f(z)$ has at least one T direction, if

$$\limsup_{r\to+\infty} \frac{T(r, f)}{(\log r)^2} = +\infty. \tag{3}$$

With some additional assumptions, Zheng [**14**] proved the existence of a T direction and in fact, obtained something more.

Theorem A. ([**14**]) *Let $f(z)$ be a transcendental meromorphic function with finite lower order μ such that for some $a \in \mathbb{C}\setminus\{0\}$ and an integer $p \geqslant 0$, $\delta = \delta(a, f^{(p)}) > 0$. Then in any angular domain $\Omega = \{z : \alpha < \arg z < \beta\}$ with*

$$\beta - \alpha > \max\left\{\frac{\pi}{\mu}, 2\pi - \frac{4}{\mu}\arcsin\sqrt{\frac{\delta}{2}}\right\},$$

there exists a ray $\arg z = \theta$ such that for any ε $(0 < \varepsilon < \pi)$ and $d > 1$,

$$\limsup_{r\to+\infty} \frac{\overline{N}(r, \Omega_\varepsilon, f = 0) + \overline{N}(r, \Omega_\varepsilon, f = \infty)}{T(dr, f)} > 0.$$

The main purpose of this paper is to confirm the conjecture mentioned above by proving the following theorem.

THEOREM. *If $f(z)$ is a meromorphic function defined on the whole complex plane and satisfies* (3), *then $f(z)$ has at least one T direction.*

It is clear that a T direction must be a Julia direction since

$$\overline{N}(r, \Omega_\varepsilon, f = a) \leqslant \overline{n}(r, \Omega_\varepsilon, f = a) \cdot \log r$$

and

$$\limsup_{r \to +\infty} \frac{\log r}{T(r, f)} = 0$$

by the fact that f is transcendental. Ostrowski [7] gave a simple example of a transcendental meromorphic function f such that $T(r, f) = O\big((\log r)^2\big)$ and f has no Julia direction. Therefore this example shows that the growth condition (3) is sharp.

2. Proof of the theorem

We shall prove the Theorem by using Ahlfors–Shimizu characteristic $T(r, \Omega)$ of a meromorphic function on an angular domain Ω. First of all, we recall its definition. For $\Omega = \{z : \alpha < \arg z < \beta\}$, define

$$S(r, \Omega) = \frac{1}{\pi} \int_\alpha^\beta \int_0^r \Big(\frac{|f'(te^{i\phi})|}{1 + |f(te^{i\phi})|^2}\Big)^2 t dt d\phi$$

and

$$T(r, \Omega) = \frac{1}{\pi} \int_0^r \frac{S(t, \Omega)}{t} dt.$$

We write the Ahlfors–Shimizu characteristic of $f(z)$ in the whole complex plane as $T_a(r, f)$. Then from [3, Theorem 1.4], we have

$$|T(r, f) - T_a(r, f) - \log^+ |f(0)|| \leqslant \frac{1}{2} \log 2. \tag{4}$$

In order to prove the Theorem, we shall need several lemmas.

LEMMA 1. *Let $T(r)$ be a continuous non-decreasing function of r. Then*

$$T\Big(r + \frac{r}{(\log T(r))^3}\Big) \leqslant eT(r),\ r \notin F, \tag{5}$$

where F is a set of positive numbers with finite logarithmic measure.

Lemma 1 follows from [6, Lemma 9] by noting that $1 + x < e^x$ when $x > 0$ and that $T(r)$ is non-decreasing. By using the method in [12], we can obtain the following lemma.

Lemma 2. *Let $f(z)$ be a meromorphic function and define*

$$L(r,\phi)=\frac{1}{2\pi}\int_0^r \frac{|f'(te^{i\phi})|}{1+|f(te^{i\phi})|^2}\,dt.$$

Suppose that

$$T(r,f)>(\log r)^p,\ p>3. \tag{6}$$

Then for any θ_1 and θ_2, $0\leqslant\theta_1<\theta_2<2\pi$, there exists a sequence $\{r_j\}$ and a $\phi\in(\theta_1,\theta_2)$ satisfying

$$L(r_j,\phi)=o\Big(\frac{T(r_j,f)}{\log r_j}\Big), \tag{7}$$

where $r_j\to+\infty$ as $j\to+\infty$, and $\{r_j\}$ depends only on $f(z)$ and $\theta_2-\theta_1$.

Proof: For brevity, we shall denote $T(r,f)$ by $T(r)$. For $r>e$, let

$$X(r)=T(r)^{1/2-\varepsilon},$$

where ε $(0<\varepsilon<1/2)$ is chosen so that

$$X(r)=o\Big(\frac{T(r)}{(\log T(r)\cdot\log r)^3}\Big)^{1/2}.$$

By (6) such a choice is possible. Let

$$E(r)=\{\phi\in(\theta_1,\theta_2):\pi^{-1/2}S^{1/2}(r,\Omega)X(r)(\log r)^{1/2}\leqslant L(r,\phi)-L(1,\phi)\}, \tag{8}$$

where $\Omega=\{z:\theta_1<\arg z<\theta_2\}$. We shall estimate the linear measure $m\big(E(r)\big)$ of $E(r)$. By Schwarz inequality, we have

$$\begin{aligned}\pi^{-1/2}S^{1/2}(r,\Omega)X(r)(\log r)^{1/2}m\big(E(r)\big)&\leqslant\int_{\theta_1}^{\theta_2}(L(r,\phi)-L(1,\phi))d\phi\\&=\frac{1}{\pi}\int_{\theta_1}^{\theta_2}\int_1^r\Big(\frac{|f'(te^{i\phi})|}{1+|f(te^{i\phi})|^2}\sqrt{t}\Big)\frac{1}{\sqrt{t}}\,dtd\phi\\&\leqslant\frac{1}{\pi}\Big\{\int_{\theta_1}^{\theta_2}\int_1^r\Big(\frac{|f'(te^{i\phi})|}{1+|f(te^{i\phi})|^2}\Big)^2 tdtd\phi\cdot\int_{\theta_1}^{\theta_2}\int_1^r\frac{1}{t}\,dtd\phi\Big\}^{1/2}\\&=\pi^{-1/2}\sqrt{\theta_2-\theta_1}S^{1/2}(r,\Omega)(\log r)^{1/2}.\end{aligned}$$

Then $m\big(E(r)\big)\leqslant\sqrt{\theta_2-\theta_1}X^{-1}(r)$. Since $X(r)$ is continuous and tends to $+\infty$ when r trends to $+\infty$, we can select an increasing sequence $\{r_j\}$, $r_j\to+\infty$ as $j\to+\infty$, such that

$$r_j\notin F \text{ and } X(r_j)\geqslant\frac{d}{\sqrt{\theta_2-\theta_1}}j^2,\ j=1,2,\dots, \tag{9}$$

where F is from Lemma 1 and $d = 2\sum_{j=1}^{+\infty} 1/j^2$. Thus we have the following estimation

$$\begin{aligned} m\Big(\bigcup_{j=1}^{+\infty} E(r_j)\Big) &\leqslant \sum_{j=1}^{+\infty} m(E(r_j)) \\ &\leqslant \sqrt{\theta_2-\theta_1}\sum_{j=1}^{+\infty}\Big(\frac{d}{\sqrt{\theta_2-\theta_1}}\Big)^{-1} j^{-2} \\ &< \frac{\theta_2-\theta_1}{2}. \end{aligned} \tag{10}$$

This implies that $(\theta_1,\theta_2)\setminus \bigcup_{j=1}^{+\infty} E(r_j) \neq \emptyset$ and we can therefore choose a $\phi \in (\theta_1,\theta_2)\setminus \bigcup_{j=1}^{+\infty} E(r_j)$ so that

$$L(r_j,\phi) - L(1,\phi) \leqslant \pi^{-1/2} S^{1/2}(r_j,\Omega)(\log r_j)^{1/2} X(r_j),\ j = 1,2,\ldots. \tag{11}$$

Now we shall estimate $S(r_j,\Omega)$ in terms of $T(r_j,f)$ by using Lemma 1. When $T(r_j) > e$,

$$\begin{aligned} S(r_j,\Omega) &\leqslant \Big[\log\Big(1+\frac{1}{(\log T(r_j))^3}\Big)\Big]^{-1} \int_{r_j}^{r_j + r_j/(\log T(r_j))^3} \frac{S(t,\Omega)}{t}\,dt \\ &\leqslant 4\big(\log T(r_j)\big)^3 T\Big(r_j + \frac{r_j}{(\log T(r_j))^3}\Big) \\ &\leqslant 4eT(r_j)\big(\log T(r_j)\big)^3. \end{aligned} \tag{12}$$

Applying the above inequality to (11), we obtain

$$\begin{aligned} L(r_j,\phi) - L(1,\phi) &\leqslant \pi^{-1/2}(4e)^{1/2} T^{1/2}(r_j)\big(\log T(r_j)\big)^{3/2}(\log r_j)^{1/2} X(r_j) \\ &= T^{1/2}(r_j)\big(\log T(r_j)\big)^{3/2}(\log r_j)^{1/2}\cdot o\Big(\frac{T(r_j)}{(\log T(r_j))^3(\log r_j)^3}\Big)^{1/2} \\ &= o\Big(\frac{T(r_j)}{\log r_j}\Big). \end{aligned} \tag{13}$$

for each j with $T(r_j) > e$ and we are done. □

By using the Ahlfors' theory of covering surfaces and Lemma 2, we can establish the following.

Lemma 3. *Let $f(z)$ be a meromorphic function satisfying* (6) *and $a_1, a_2, \cdots, a_q$ be q distinct points on the Riemann sphere $\mathbb{C}_\infty$. For any given $\phi \in [0,2\pi)$ and any small $\varepsilon > 0$, there exists a sequence $\{r_j\}$, $\alpha \in (\phi-\varepsilon, \phi-\varepsilon/2)$ and $\beta \in (\phi+\varepsilon/2, \phi+\varepsilon)$ such that*

$$(q-2)T(r_j,\Omega) \leqslant \sum_{v=1}^{q} \overline{N}(r_j,\Omega, f=a_v) + o\big(T(r_j)\big), \tag{14}$$

where $\Omega = \Omega(\alpha, \beta) = \{z : \alpha < \arg z < \beta\}$, $r_j \to +\infty$ as $j \to +\infty$, and $\{r_j\}$ depends only on $f(z)$ and ε.

PROOF: By applying Lemma 2, there exists a common sequence $\{r_j\}$, $\alpha \in (\phi - \varepsilon, \phi - \varepsilon/2)$ and $\beta \in (\phi + \varepsilon/2, \phi + \varepsilon)$ such that

$$L(r_j, \alpha) = o\Big(\frac{T(r_j)}{\log r_j}\Big) \text{ and } L(r_j, \beta) = o\Big(\frac{T(r_j)}{\log r_j}\Big), \tag{15}$$

where the sequence $\{r_j\}$ is selected by means of (9) and depends only on $f(z)$ and $\varepsilon/2$, $r_j \to +\infty$ as $j \to +\infty$.

Let $z = \phi(\omega)$ be a conformal mapping from the unit disk Δ onto $\Omega_t = \Omega \cap \{z : |z| < t\}$. Set $g(\omega) = f(\phi(\omega))$. By reducing the islands $D_1, D_2, \ldots, D_q$ to $a_1, a_2, \ldots, a_q$ in Ahlfors' covering theorem (see [**3**, Theorem 5.5], we have that

$$(q-2)S(1, g) \leqslant \sum_{v=1}^{q} \overline{n}(1, g = a_v) + hL(1, g),$$

where h is a constant depending only on $\{a_1, a_2, \ldots, a_q\}$. But

$$\begin{aligned}
S(1, g) &= \frac{1}{\pi} \iint\limits_{\Delta} \Big(\frac{|g'(\omega)|}{1 + |g(\omega)|^2}\Big)^2 dudv \\
&= \frac{1}{\pi} \iint\limits_{\Delta} \Big(\frac{|f'(\phi(\omega))|}{1 + |f(\phi(\omega))|^2}\Big)^2 |\phi'(\omega)|^2 dudv \\
&= \frac{1}{\pi} \iint\limits_{\Omega_t} \Big(\frac{|f'(z)|}{1 + |f(z)|^2}\Big)^2 dxdy \\
&= S(t, \Omega),
\end{aligned}$$

$$\overline{n}(1, g = a_v) = \overline{n}(t, \Omega, f = a_v),$$

$$L(1, g) = L(t, \alpha, \beta) + L(t, \alpha) + L(t, \beta),$$

where $\omega = u + iv$, $z = x + iy$ and

$$L(t, \alpha, \beta) = \frac{1}{\pi} \int_{\alpha}^{\beta} \frac{|f'(te^{i\theta})|}{1 + |f(te^{i\theta})|^2} td\theta.$$

So we can immediately deduce that

$$(q-2)S(t, \Omega) \leqslant \sum_{v=1}^{q} \overline{n}(t, \Omega, f = a_v) + h\big[L(t, \alpha, \beta) + L(t, \alpha) + L(t, \beta)\big], \tag{16}$$

where h is a constant depending only on $\{a_1, a_2, \ldots, a_q\}$ and

$$L(t, \alpha, \beta) = \frac{1}{\pi} \int_{\alpha}^{\beta} \frac{|f'(te^{i\theta})|}{1 + |f(te^{i\theta})|^2} td\theta.$$

Dividing both sides of (16) by t and then integrating both sides from 1 to r_j, we obtain the following inequality

$$(q-2)T(r_j,\Omega) \leqslant \sum_{v=1}^{q} \overline{N}(r_j,\Omega,f=a_v) + h\int_1^{r_j} \frac{L(t,\alpha,\beta)}{t}\,dt + o\big(T(r_j)\big). \tag{17}$$

Here we have used the equalities (15) and the fact that $L(t,\alpha)$ and $L(t,\beta)$ are both increasing functions of t.

Now we need to estimate the following:

$$\begin{aligned}
\int_1^{r_j} \frac{L(t,\alpha,\beta)}{t}\,dt &= \frac{1}{\pi}\int_1^{r_j}\int_\alpha^\beta \frac{|f'(te^{i\theta})|}{1+|f(te^{i\theta})|^2}\,d\theta\,dt \\
&= \frac{1}{\pi}\int_1^{r_j}\int_\alpha^\beta \Big(\frac{|f'(te^{i\theta})|}{1+|f(te^{i\theta})|^2}\sqrt{t}\Big)\frac{1}{\sqrt{t}}\,d\theta\,dt \\
&\leqslant \frac{1}{\pi}\Big\{\int_1^{r_j}\int_\alpha^\beta \Big(\frac{|f'(te^{i\theta})|}{1+|f(te^{i\theta})|^2}\Big)^2 t\,d\theta\,dt \cdot \int_1^{r_j}\int_\alpha^\beta \frac{dt\,d\theta}{t}\Big\}^{1/2} \\
&= \frac{1}{\sqrt{\pi}} S^{1/2}(r_j,\Omega)\sqrt{\beta-\alpha}(\log r_j)^{1/2}.
\end{aligned} \tag{18}$$

Applying (12) to the above inequality shows that

$$\int_1^{r_j} \frac{L(t,\alpha,\beta)}{t}\,dt \leqslant \sqrt{\frac{4e(\beta-\alpha)}{\pi}}\,T^{1/2}(r_j)\big(\log T(r_j)\big)^{3/2}(\log r_j)^{1/2} = o\big(T(r_j)\big).$$

Combining the above inequality with (17) implies the desired inequality (14). □

We also need the following lemma in [8, Theorem VII.3].

Lemma 4. *Let $f(z)$ be meromorphic on the whole complex plane. Then for any three distinct points a_1, a_2 and a_3 on $\mathbb{C}_\infty$ and any small $\varepsilon > 0$, we have*

$$T(r,\Omega_\varepsilon) \leqslant 3\sum_{v=1}^{3} \overline{N}(2r,\Omega,f=a_v) + O\big((\log r)^2\big),$$

where $\Omega = \{z : \alpha < \arg z < \beta\}$ and $\Omega_\varepsilon = \{z : \alpha+\varepsilon < \arg z < \beta-\varepsilon\}$.

Now we are in the position to prove the Theorem.

Proof of the theorem: Suppose the theorem does not hold. Then for any $\phi \in [0,2\pi)$, we have a $\varepsilon_\phi > 0$ and three distinct points a_ϕ, b_ϕ and c_ϕ on $\mathbb{C}_\infty$ such that

$$\overline{N}(r,\Omega_\phi,f=a_\phi) + \overline{N}(r,\Omega_\phi,f=b_\phi) + \overline{N}(r,\Omega_\phi,f=c_\phi) = o\big(T(r)\big), \tag{19}$$

where $\Omega_\phi = \{z : \phi-\varepsilon_\phi < \arg z < \phi+\varepsilon_\phi\}$.

If we identify $[0,2\pi)$ with the unit circle and $(\phi-\varepsilon_\phi/2,\phi+\varepsilon_\phi/2)$ with the corresponding open arc on the unit circle, then since the unit circle is compact and

$$[0,2\pi)\subseteq\bigcup_{\phi\in[0,2\pi)}(\phi-\varepsilon_\phi/2,\phi+\varepsilon_\phi/2),$$

we can choose finitely many $(\phi-\varepsilon_\phi/2,\phi+\varepsilon_\phi/2)$, say $(\phi_1-\varepsilon_{\phi_1}/2,\phi_1+\varepsilon_{\phi_1}/2)$, $(\phi_2-\varepsilon_{\phi_2}/2,\phi_2+\varepsilon_{\phi_2}/2)$, $\ldots$, $(\phi_q-\varepsilon_{\phi_q}/2,\phi_q+\varepsilon_{\phi_q}/2)$, to cover $[0,2\pi)$.

We shall consider two different cases.

(I) The lower order of $f(z)$ is positive or infinite. Then for all sufficiently large $r>0$, $T(r,f)>(\log r)^4$. By Lemma 3, for ϕ_v $(v=1,2,\ldots,q)$, there exists a common sequence $\{r_j\}$, $\alpha_{\phi_v}\in(\phi_v-\varepsilon_{\phi_v},\phi_v-\varepsilon_{\phi_v}/2)$ and $\beta_{\phi_v}\in(\phi_v+\varepsilon_{\phi_v}/2,\phi_v+\varepsilon_{\phi_v})$ such that

$$\begin{aligned}T\big(r_j,\Omega(\alpha_{\phi_v},\beta_{\phi_v})\big)\leqslant{}&\overline{N}\big(r_j,\Omega(\alpha_{\phi_v},\beta_{\phi_v}),f=a_{\phi_v}\big)+\overline{N}\big(r_j,\Omega(\alpha_{\phi_v},\beta_{\phi_v}),f=b_{\phi_v}\big)\\&+\overline{N}\big(r_j,\Omega(\alpha_{\phi_v},\beta_{\phi_v}),f=c_{\phi_v}\big)+o\big(T(r_j)\big),\end{aligned}$$

where the sequence $\{r_j\}$ is selected by means of (9) and depends only on $f(z)$ and $\min\{\varepsilon_{\phi_v}/2: v=1,2,\ldots,q\}$, $r_j\to+\infty$ as $j\to+\infty$. Then by (19), we can further obtain that for $v=1,2,\ldots,q$,

$$\begin{aligned}T\big(r_j,\Omega(\alpha_{\phi_v},\beta_{\phi_v})\big)&\leqslant\overline{N}\big(r_j,\Omega(\alpha_{\phi_v},\beta_{\phi_v}),f=a_{\phi_v}\big)+\overline{N}\big(r_j,\Omega(\alpha_{\phi_v},\beta_{\phi_v}),f=b_{\phi_v}\big)\\&\qquad+\overline{N}\big(r_j,\Omega(\alpha_{\phi_v},\beta_{\phi_v}),f=c_{\phi_v}\big)+o\big(T(r_j)\big)\\&\leqslant\overline{N}(r_j,\Omega_{\phi_v},f=a_{\phi_v})+\overline{N}(r_j,\Omega_{\phi_v},f=b_{\phi_v})\\&\qquad+\overline{N}(r_j,\Omega_{\phi_v},f=c_{\phi_v})+o\big(T(r_j)\big)\\&=o\big(T(r_j)\big).\end{aligned}\tag{20}$$

Thus by noting that $(\alpha_{\phi_1},\beta_{\phi_1}),(\alpha_{\phi_2},\beta_{\phi_2}),\ldots,(\alpha_{\phi_q},\beta_{\phi_q})$ can also cover $[0,2\pi)$, it is easy to see that

$$T_a(r_j,f)\leqslant\sum_{v=1}^{q}T\big(r_j,\Omega(\alpha_{\phi_v},\beta_{\phi_v})\big)=o\big(T(r_j)\big)$$

from (20). This contradicts (4).

(II) The lower order of $f(z)$ is zero. Noting that $(\phi_1-\varepsilon_{\phi_1}/2,\phi_1+\varepsilon_{\phi_1}/2)$, $(\phi_2-\varepsilon_{\phi_2}/2,\phi_2+\varepsilon_{\phi_2}/2)$, $\ldots$, $(\phi_q-\varepsilon_{\phi_q}/2,\phi_q+\varepsilon_{\phi_q}/2)$ cover $[0,2\pi)$, from Lemma 4 and (19), we have

$$\begin{aligned}T_a(r,f)&\leqslant\sum_{v=1}^{q}T\big(r,\Omega(\phi_v-\varepsilon_{\phi_v}/2,\phi_v+\varepsilon_{\phi_v}/2)\big)\\&\leqslant3\sum_{v=1}^{q}\big[\overline{N}(2r,\Omega_{\phi_v},f=a_{\phi_v})+\overline{N}(2r,\Omega_{\phi_v},f=b_{\phi_v})\\&\qquad+\overline{N}(2r,\Omega_{\phi_v},f=c_{\phi_v})\big]+O\big((\log r)^2\big)\\&=o\big(T(2r)\big)+O\big((\log r)^2\big).\end{aligned}\tag{21}$$

We need to treat two subcases below.

(II.1) The order ρ of $f(z)$ is positive. Then there exists a Polya sequence $\{r_n\}$ (see [**11**]) such that

$$T(2r_n) \leqslant 2^{\rho+1}T(r_n) \text{ and } \lim_{n\to+\infty} \frac{\log T(r_n)}{\log r_n} > 0.$$

From the latter inequality, it is easy to see that $(\log r_n)^2 = o\big(T(r_n)\big)$. Applying this and the former inequality to (21), we can deduce that

$$T_a(r_n, f) = o\big(T(r_n)\big).$$

This also contradicts (4).

(II.2) The order ρ of $f(z)$ is zero. Let $F = \{r > 0 : T(2r) \leqslant 2T(r)\}$. From [**4**, Lemma 4], we know that

$$\underline{\log\operatorname{dens}} F = 1, \tag{22}$$

where the lower logarithmic density $\underline{\log\operatorname{dens}} F$ of F is defined as $\underline{\log\operatorname{dens}} F = \liminf_{r\to+\infty}\big(\int_{F\cap[1,r]} dt/t\big)/\log r$.

From the condition (3), there exists an increasing sequence $\widetilde{r}_n$ such that $\widetilde{r}_n \to +\infty$ and

$$(\log \widetilde{r}_n)^2 = o\big(T(\widetilde{r}_n)\big). \tag{23}$$

We can assert that for sufficiently large n,

$$F \cap (\widetilde{r}_n, \widetilde{r}_n^2) \neq \emptyset. \tag{24}$$

In fact, if this is not the case, there exists a subsequence $\{\widetilde{r}_{n_k}\}$ of $\{\widetilde{r}_n\}$ such that $\widetilde{r}_{n_k} \to +\infty$ as $k \to +\infty$ and $F \cap (\widetilde{r}_{n_k}, \widetilde{r}_{n_k}^2) = \emptyset$ for any k. Then $F \cap \Big(\bigcup_{k=1}^{+\infty} (\widetilde{r}_{n_k}, \widetilde{r}_{n_k}^2)\Big) = \emptyset$. Furthermore, we can have

$$\begin{aligned}
1 &= \limsup_{r\to+\infty}\left(\int_{[1,r]} \frac{dt}{t}\right)\Big/\log r\\
&\geqslant \limsup_{r\to+\infty}\left(\int_{(F\cup(\bigcup_{k=1}^{+\infty}(\widetilde{r}_{n_k},\widetilde{r}_{n_k}^2)))\cap[1,r]} \frac{dt}{t}\right)\Big/\log r\\
&= \limsup_{r\to+\infty}\left\{\left(\int_{F\cap[1,r]} \frac{dt}{t}\right)\Big/\log r + \left(\int_{(\bigcup_{k=1}^{+\infty}(\widetilde{r}_{n_k},\widetilde{r}_{n_k}^2))\cap[1,r]} \frac{dt}{t}\right)\Big/\log r\right\}\\
&\geqslant \liminf_{r\to+\infty}\left(\int_{F\cap[1,r]} \frac{dt}{t}\right)\Big/\log r + \limsup_{r\to+\infty}\left(\int_{(\bigcup_{k=1}^{+\infty}(\widetilde{r}_{n_k},\widetilde{r}_{n_k}^2))\cap[1,r]} \frac{dt}{t}\right)\Big/\log r\\
&= \underline{\log\operatorname{dens}} F + \overline{\log\operatorname{dens}}\left(\bigcup_{k=1}^{+\infty}(\widetilde{r}_{n_k}, \widetilde{r}_{n_k}^2)\right),
\end{aligned}$$

where the upper logarithmic density $\overline{\log \operatorname{dens}} A$ of a set A is defined as

$$\overline{\log \operatorname{dens}} A = \limsup_{r \to +\infty} \left(\int_{A \cap [1,r]} \frac{dt}{t} \right) \Big/ \log r.$$

But by a simple calculation, we can obtain that

$$\overline{\log \operatorname{dens}} \left(\bigcup_{k=1}^{+\infty} (\widetilde{r}_{n_k}, \widetilde{r}_{n_k}^2) \right) \geqslant \frac{1}{2}.$$

So combining (22), we deduce that $1 \geqslant 1 + 1/2$ which is impossible.

From (24), for sufficiently large n, we can select a sequence $\{r_n\}$ such that $r_n \in F \cap (\widetilde{r}_n, \widetilde{r}_n^2)$, that is, $r_n \in F$ and $r_n \in (\widetilde{r}_n, \widetilde{r}_n^2)$. Then $T(2r_n) \leqslant 2T(r_n)$ and

$$(\log r_n)^2 \leqslant 4(\log \widetilde{r}_n)^2 = o\big(T(\widetilde{r}_n)\big) = o\big(T(r_n)\big)$$

by the definition of F and (23). Applying these to (21), we have

$$T_a(r_n, f) = o\big(T(r_n)\big).$$

This contradicts (4) and the Theorem follows. $\square$

3. Remarks

Recalling Hayman's inequality on the estimation of $T(r, f)$ in terms of only two counting functions for the solutions of $f(z) = a$ and $f^{(k)} = b$, a new singular direction can be introduced. A ray $\arg z = \theta$ is called a Hayman T direction for a meromorphic function $f(z)$, if for any ε $(0 < \varepsilon < \pi)$ and for all complex numbers a and b $(b \neq 0)$, we have

$$\limsup_{r \to +\infty} \frac{N(r, \Omega_\varepsilon, f = a) + N(r, \Omega_\varepsilon, f^{(k)} = b)}{T(r, f)} > 0,$$

where $\Omega_\varepsilon = \{z : \ \theta - \varepsilon < \arg z < \theta + \varepsilon\}$.

We do not know whether a meromorphic function satisfying (3) must have a Hayman T direction. But from the work of Zhang and Yang [**13**] we think it possible that such a function may have a Hayman T direction and a T direction may also be a Hayman T direction.

References

[1] C.T. Chuang, *Singular directions of meromorphic functions*, (in Chinese) (Science Press, China, 1982).

[2] A.A. Gonchar, V.P. Havin and N.K. Nikolski, *Complex analysis I*, Encyclopaedia of Mathematical Sciences **85** (Springer-Verlag, Berlin, 1997).

[3] W.K. Hayman, *Meromorphic functions* (Clarendon Press, Oxford, 1964).

[4] W.K. Hayman, 'On the characteristic of functions meromorphic in the plane and of their integrals', *Proc. London Math. Soc. (3)* **14** (1965), 93–128.

[5] G. Julia, 'Sur quelques proprietes nouvelles des fonctions entieres ou meromorphes', *Ann. Ecole Norm. Sup.* **36** (1919), 93–125; **37** (1920), 165–218.

[6] A.C. Offord, 'Lacunary entire functions', *Math. Proc. Camb. Philos. Soc.* **114** (1993), 67–83.

[7] A. Ostrowski, 'Ueber Folgen analytischer Funktionen und einige Verscharfungen des Picardschen Satzes', *Math. Zeit.* **24** (1926), 215–258.

[8] M. Tsuji, *Potential theory in modern function theory* (Maruzen Co. Ltd., Tokyo, 1959).

[9] G. Valiron, 'Recherches sur le theoreme de M. Borel dans la theorie des fonctions meromorphes', *Acta Math.* **52** (1928), 67–92.

[10] G. Valiron, 'Directions de Borel des fonctions meromorphes', *Mem. Sci. Math. Fasc.* **89** (1938).

[11] L. Yang, 'Borel directions of meromorphic functions in an angular domain', *Sci. Sinica* (1979), 149–163.

[12] Q.D. Zhang, 'On Nevanlinna direction and Borel direction of meromorphic functions', (in Chinese), *Acta Math. Sinica* **29** (1986), 550–554.

[13] Q.D. Zhang and L. Yang, 'New singular direction of meromorphic functions', (in Chinese), *Sci. Sinica Ser. A* **11** (1983), 982–994.

[14] J.H. Zheng, 'On transcendental meromorphic functions with radially distributed values', *Sci. Sinica Ser A* (to appear).

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