An Example Concerning Infinite Factorizations of Transcendental Entire Functions

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Abstract. A non-linear entire function is called prime if it cannot be expressed as a composition of two non-linear entire functions. In this paper, we construct an entire function which is a composition of infinitely many prime transcendental entire functions.

Let $\mathcal{P}, \mathcal{R}$ and $\mathcal{T}$ be the set of non-constant polynomials, non-constant rational functions and transcendental entire functions respectively. Throughtout this paper X will denote one of $\mathcal{P}, \mathcal{R}, \mathcal{T}$. Let $f, f_1, \ldots, f_n$ be functions in X such that $f = f_1 \circ \cdots f_n$. An expression like this is called a factorization of f through X. The functions f_i are factors of f. Notice that a factorization through X means every factor belongs to X while in the literatures, the meaning of factorization does not have this restriction. For example, the factorization $z^2 \circ ze^{z^2}$ is not a factorization through $\mathcal{T}$ of $z^2e^{2z^2}$. The simplest class of functions (in the sense of composition) is the so-called prime functions. A non-linear meromorphic function is prime if it cannot be expressed as a composition of two non-Möbius meromorphic functions. If every factor in a factorization through X is prime, we called it a prime factorization through X. Examples of prime functions are rational functions of prime degrees, $e^z + z$, ze^{z^n}, $\sin ze^{\cos z}$ etc (see [2] for more examples). In fact there are plenty of prime entire functions as Y. Noda proved in [3] showing that for any transcendental entire function f, $f(z) + az$ is prime for all $a \in \mathbb{C}\backslash E_f$ where E_f is some countable set. The study of prime factorizations through $\mathcal{P}$ started from J.F. Ritt's paper ([4]) in 1922. His results are rather complete. However, very little is known about prime factorizations through $\mathcal{R}$ and $\mathcal{T}$. The following results are known.

(1) For $f \in \mathcal{P}$, there is an uniform bound for the numbers of prime factors in different

1991 *Mathematics Subject Classification Primary* 30D15.

The author thanks the Croucher Foundation for their support during the preparation of this paper.

factorizations of f through $\mathcal{P}$. Clearly $\deg f$ will be such a bound.

(2) The number of prime factors of $f \in \mathcal{P}$ in each prime factorization through $\mathcal{P}$ is the same.

(1) is clearly also true for rational functions, while (2) is no longer true because W. Bergweiler constructed in [1] a rational function with two prime factorization through $\mathcal{R}$ which consist of two and three prime factors respectively. Whether a similar example exists for the case $\mathcal{T}$ remains open. However, (1) is not true for transcendental entire functions because of the following result.

THEOREM 1 *There exists a sequence of positive real number* $\{c_n\}_{n\in\mathbb{N}}$ *such that the sequence of functons* $F_n(z) = (c_n e^z + z) \circ \cdots \circ (c_1 e^z + z)$ *converges uniformly on compact subsets to an entire function* $F(z)$*. Furthermore, for each* $n \in \mathbb{N}$, $F(z) = H_n \circ (c_n e^z + z) \circ \cdots \circ (c_1 e^z + z)$ *for some entire function* H_n*. Hence, there is no uniform bound on the number of prime factors* $c_n e^z + z$ *in different decompositions of* F *through* $\mathcal{T}$

Proof of Theorem 1 We define the c_i inductively. Take $c_1 = 1$ and suppose $c_1, \ldots c_k$ has been defined. Define $c_{k+1} = \{2^k \max_{|z|\le k} |e^{F_k(z)}|\}^{-1}$. Now for each disk $|z| \le R$, for all $k \ge R$, we have $|F_{k+1}(z) - F_k(z)| = |c_{k+1}e^{F_k(z)} + F_k(z) - F_k(z)| \le |c_{k+1}e^{F_k(z)}| \le 2^{-k}$ on $|z| \le R$. It follows that $\{F_n\}$ is a Cauchy sequence in the space of analytic functions on $|z| \le R$. Hence, $\{F_n\}$ converges to an entire function F uniformly on compact subsets. For each $c_n e^z + z$, it is obvious that it is increasing on the real axis and $c_n e^n + n > n$ for all $n \in \mathbb{N}$. This implies that $F(n) > n$. So F is unbounded and hence non-constant. For each $n \in \mathbb{N}$ and $m \ge n+1$, define $H_{n,m}(z) = (c_m e^z + z) \circ \cdots \circ (c_{n+1}e^z + z)$. Then by similar arguments, we can show that $\{H_{n,m}\}_{m\in\mathbb{N}}$ converges to a non-constant entire function H_n as m trends to infinity. Clearly, $F(z) = H_n \circ (c_n e^z + z) \circ \cdots \circ (c_1 e^z + z)$. Note that each $c_k e^z + z$ is prime (see [2], p.118) and we are done.

It is clear that the same arguments work even if we replace each e^z in $F_n(z) = (c_n e^z + z) \circ \cdots \circ (c_1 e^z + z)$ by some other entire functions (not necessarily the same in each factor), provided that we can make sure that the limiting function is not constant. For example if we consider $G_n(z) = (c_n z^2 + z) \circ \cdots \circ (c_1 z^2 + z)$, then the limiting function G will be non-constant. Note that G is a composition of infinitely many polynomials while itself is transcendental. Since the complex dynamics of polynomials and transcendental entire function are quite different (for example, some transcendental entire functions have wandering domains while no polynomial has a wandering domain), it would be interesting to investigate the complex dynamics of functions like G.

References

[1] W. Bergweiler, An example concerning factorization of rational functions, Expo. Math. **11** (1993), 281-283.

[2] C.T. Chuang and C.C. Yang, Fix-points and factorization of meromorphic functions, World Scientific, Singapore, 1990.

[3] Y. Noda, On factorization of entire functions, Kodai Math. J., **4** (1981), 480-494.

[4] J.F. Ritt, Prime and composite polynomials, Trans. Amer. Math. Soc. **23** (1922), 51-66.

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