Recent progress in unique factorization of entire functions

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Abstract

We shall survey the recent developments in the unique factorizations of polynomials, rational functions and transcendental entire functions.

1. Introduction.

The factorization theory of meromorphic functions of one complex variable is to study how a given meromorphic function can be factorized into other simplier meromorphic functions in the sense of composition. For an introduction to this subject, the readers are refered to the books [11], [26], [6] and the survey articles [12], [25], [27], [28], [29], [30]. In the past years, the main focus of this subject is to use Nevanilnna Theory and classical complex function theory to prove specific classes of meromorphic functions to be prime, pseudo-prime or uniquely factorizable. It is because there were no powerful tool to tackle general problem. Recently, two important papers ([15] and [8]) appearred. These papers contain new methods which can be used to tackle some general factorization problems and obtain some general results. In this article, we shall survey those results concerning unique factorization of entire functions.

2. Unique factorization of polynomials and rational functions.

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The whole field of factorization of meromorphic functions started from a fundmental paper of J.F. Ritt in 1922. In that paper Ritt introduced the concept of prime polynomials, i.e. those polynomials which cannot be expressed as a composition of two non-linear polynomials. The concept of prime polynomials is very similar to that of prime numbers and we can also define prime entire functions in a similar way. It is clear that every nonconstant polynomial can be expressed as a composition of finitely many prime polynomials. Ritt then considered the following problem. Let $p(z)$ be a nonconstant polynomial such that $p(z)$ has two factorizations $p = p_1 \circ \cdots \circ p_m$ and $p = q_1 \circ \cdots \circ q_n$, where all the polynomials p_i, q_j are prime. What is the relationship between these two factorizations? Ritt solved this problem by proving the following two remarkable theorems.

Theorem 1 *([22]). Let p be a nonconstant polynomial. Suppose that p has two prime factorizations $p = p_1 \circ \cdots \circ p_m$ and $p = q_1 \circ \cdots \circ q_n$, where all the polynomials p_i, q_j are prime. Then $m = n$ and $(\deg q_1, \ldots, \deg q_m)$ is obtained from $(\deg p_1, \ldots, \deg p_m)$ by a permutation of its components. Moreover, one can pass from one factorization to the other one by altering two adjacent polynomials in the ways mentioned in Theorem 2.*

Theorem 2 *([22]). Let p_j, p_{j+1}, q_j and q_{j+1} be prime polynomials such that $p_j \circ p_{j+1} = q_j \circ q_{j+1}$. Then the factorization $q_j \circ q_{j+1}$ can be obtained from the factorization $p_j \circ p_{j+1}$ by a sequence of applications of any of the following three types (or their inverses).*

1) replace p_j and p_{j+1} by $p_j \circ l$ and $l^{-1} \circ p_{j+1}$, where l is any linear polynomial, or

2) replace p_j and p_{j+1} by p_{j+1} and p_j, where p_j and p_{j+1} are Tchebychev polynomials, or

3) replace z^k and $z^r g(z^k)$ (which are p_j and p_{j+1} respectively) by $z^r g(z)^k$ and z^k, where r and k are integers, and g is a polynomial.

Ritt's first result says that the number of prime polynomial factors in a factorization, as well as the set of the degrees of the prime factors, are two invariants of prime factorizations of a polynomial. In a recent joint work with A.F. Beardon, we find out a third invariant of prime factorizations of a polynomial. Let $\Gamma(p)$ be the set of linear transformations γ such that $p \circ \gamma = p$. Since polynomial p cannot be periodic, $\Gamma(p)$ is a finite rotation group.

Let $|\Gamma(p)|$ denote the number of elements in $\Gamma(p)$. Then we obtain the third invariant as follows.

Theorem 3 *([4]). Let $p = p_1 \circ \cdots \circ p_m = q_1 \circ \cdots \circ q_m$ be two prime factorizations of p. Then $(|\Gamma(q_1)|, \ldots, |\Gamma(q_m)|)$ is obtained from $(|\Gamma(p_1)|, \ldots, |\Gamma(p_m)|)$ by a permutation of its components.*

Ritt's results about polynomial factorizations are rather complete. However, these results do not indicate when two factorizations of p are equivalent. Two factorizations of p are equivalent if one can be obtained from the other only by applying operation (1). In the same paper, We also obtain a sufficient condition for two factorizations of a polynomial to be equivalent.

Theorem 4 *([4]). Let $p = p_1 \circ \cdots \circ p_m = q_1 \circ \cdots \circ q_m$ be two factorizations of p. Suppose that $(\deg p_1, \ldots, \deg p_m) = (\deg q_1, \ldots, \deg q_m)$, then the two factorizations are equivalent.*

We remark that the proof of Ritt's results involved considerations of the monodromy group and the Riemann surface of the inverse of a polynomial. However, there exists some algebraic proofs which also work for algebraically closed field of characrestic zero (see [16]).

In [23] and [24], Ritt also tried to consider the factorization of rational functions. However, he were not able to provide a complete result in this case. It is because the unique factorization problem of rational functions is much more difficult. It was mentioned by Ritt [22] and only proved by W. Bergweiler [5] in 1993 that there exists a rational function with two prime factorizations which consist of two and three prime factors respectively. Whether similar example exists for transcendental entire functions remains open. As far as I know, there is essentially no progress for the rational case over the past years.

2. Unique factorization of entire functions.

It is natural to ask whether we can obtain results similar to that of Ritt for entire functions. This question is very difficult because the factorization of a transcendental entire functions can be very complicated. For example, the number of prime factors of a given polynomial or rational function is bounded, while this is not the case for transcendental entie functions. In fact, we have the following result.

Theorem 5 *There exists a sequence of positive real number* $\{c_n\}_{n\in\mathbb{N}}$ *such that the sequence of functons* $F_n(z) = (c_n e^z + z) \circ \cdots \circ (c_1 e^z + z)$ *converges uniformly on compact subsets to an entire function* $F(z)$. *Furthermore, for each* $n \in \mathbb{N}$, $F(z) = H_n \circ (c_n e^z + z) \circ \cdots \circ (c_1 e^z + z)$ *for some entire function* H_n. *Hence, there is no uniform bound on the number of prime factors* $c_n e^z + z$ *in different factorizations of* F.

Proof of Theorem 5 We define the c_i inductively. Take $c_1 = 1$ and suppose $c_1, \ldots c_k$ has been defined. Define $c_{k+1} = \{2^k \max_{|z|\leq k} |e^{F_k(z)}|\}^{-1}$. Now for each disk $|z| \leq R$, for all $k \geq R$, we have $|F_{k+1}(z) - F_k(z)| = |c_{k+1}e^{F_k(z)} + F_k(z) - F_k(z)| \leq |c_{k+1}e^{F_k(z)}| \leq 2^{-k}$ on $|z| \leq R$. It follows that $\{F_n\}$ is a Cauchy sequence in the space of analytic functions on $|z| \leq R$. Hence, $\{F_n\}$ converges to an entire function F uniformly on compact subsets. For each $c_n e^z + z$, it is obvious that it is increasing on the real axis and $c_n e^n + n > n$ for all $n \in \mathbb{N}$. This implies that $F(n) > n$. So F is unbounded and hence nonconstant. For each $n \in \mathbb{N}$ and $m \geq n+1$, define $H_{n,m}(z) = (c_m e^z + z) \circ \cdots \circ (c_{n+1}e^z + z)$. Then by similar arguments, we can show that $\{H_{n,m}\}_{m\in\mathbb{N}}$ converges to a nonconstant entire function H_n as m trends to infinity. Clearly, $F(z) = H_n \circ (c_n e^z + z) \circ \cdots \circ (c_1 e^z + z)$. Note that each $c_k e^z + z$ is prime (see [6], p.118) and we are done.

Ritt's first theorem (Theorem 1) is a global result about prime factorizations of a given polynomial. It gives some global invariants about prime factorizations of a polynomial. While Ritt's second theorem (Theorem 2) is a local result, it says that there are essentially only three possible cases can occur for prime polynomials a, b, c and d to satisfy $a \circ b = c \circ d$. Theorem 5 shows that it should be very difficult to extend Ritt's first theorem to transcendental entire functions. Therefore, as a first step to build a factorization theory for entire functions, one may try to extend Ritt's second result to entire functions first. Until now, we do not know whether if a, b, c and d are prime entire functions such that $a \circ b = c \circ d$, then there are only a fintite number of ways such that a, b, c and d are related. The case 2 in Theorem 2 is about permutable polynomials and we still do not have a complete classification of permutable entire functions. Therefore, it is important to investigate permutable entire functions and we shall talk about this topic in the next section. It is clearly that case 3 can occur for entire functions. For example $a(z) = z^2, b(z) = ze^{z^2}, c(z) = ze^{2z}$ and $d(z) = z^2$. Therefore not all the factorizations $a \circ b = c \circ d$ are equivalent (i.e. a, b, c, d are related as in case 1) So it would be nice to have some general results which guarantee that two factorizations are equivalent. In a joint work [19] with C.C. Yang, we obtained the following results. The functions f, g, p and q considered below are all entire and nonlinear.

Theorem 6 *([19]). Let f, p be two prime entire functions and f be transcendental. Suppose that $f \circ p = g \circ q$ and both p, q are polynomials with* $\deg p \neq 3, 6$. *Then $f = g \circ L$ and $p = L^{-1} \circ q$, where L is a linear polynomial.*

Theorem 7 *([19]). Let f, p be two prime entire functions and f be transcendental. Suppose that $p \circ f = q \circ g$ and both p, q are polynomials. Then $p = q \circ L^{-1}$ and $f = L \circ g$, where L is a linear polynomial.*

Theorem 8 *([19]. Let f, p be two non-periodic prime entire functions and p be a polynomial. Suppose that $p \circ f = q \circ g$ and both f, g are transcendental. Then $p = q \circ L^{-1}$ and $f = L \circ g$, where L is a linear polynomial.*

It is conjectured that the condition that $\deg p \neq 3, 6$ in Theorem 6 can be removed. Theorem 6, 7 and 8 due with the relationships between polynomials p and q, transcendental functions f and g when we have factorizations of the form $p \circ f = q \circ g$ or $f \circ p = g \circ q$. It is natural to investigate the case $f \circ p = q \circ g$.

Theorem 9 *([19]). Let f and g be two transcendental entire functions, p and q be two non-linear polynomials with degree n and m respectively. If $f \circ p = q \circ g$ and p is not a right factor of g, then deg $p \leq$ deg q. In particular, the conclusion is true when g is prime.*

Remark. Let $f(z) = e^z$, $g(z) = e^{\frac{z^3}{2}}$, $p(z) = z^3$ and $q(z) = z^2$. Then $f \circ p = q \circ g$ and $\deg p > \deg q$. Therefore, the condition that p is not a right factor of g is essential.

The proof of Theorem 6 is based on a method developed by S.A. Lysenko in [15] which depends on some fundamental results of local holomorphic dynamics. While the proof of

Theorem 7, 8, 9 depends on the following result of H.Grauert ([10]) on complex analytic equivalence relations.

Theorem 10 *([10]). Let R be any equivalence relation on $\mathbb{C}$ whose graph $G = \{(x, y) \in \mathbb{C} \times \mathbb{C} | xRy\}$ is a complex analytic subset of $\mathbb{C} \times \mathbb{C}$ containing no vertical or horizontal lines (i.e subsets of the form $\{x\} \times \mathbb{C}$ or $\mathbb{C} \times \{y\}$). Suppose that G is of pure dimension one (i.e. G is everywhere of the same dimension one). Then, there exists a holomorphic map h from $\mathbb{C}$ onto a Riemann surface S such that xRy if and only if $h(x) = h(y)$.*

Definition 1 . *Let $F(z)$ be an nonconstant entire function. An entire function $g(z)$ is a generalized right factor of F (denoted by $g \leq F$) if there exists a function f, which is analytic on the image of g, such that $F = f \circ g$. If $h \leq f$ and $h \leq g$, we say that h is a generalized common right factor of f and g.*

Using the above theorem, we can prove the following lemma. This lemma was extracted from the proof of Theorem 1.1 in a paper of A. Eremenko and L. Rubel [8]. A quite detailed proof can also be found in [18].

Lemma 1 . *Let f, g be two entire functions. For $i = 1, ...k$, $k \geq 2$, let $S_i = \{z_{in}\}_{n \in \mathbb{N}}$ be a sequence of distinct complex numbers with limit point z_i. Suppose that all the limit points z_i are distinct and for all $n \in \mathbb{N}$,*

$$\begin{cases} f(z_{1n}) = f(z_{2n}) = \cdots = f(z_{kn}) \\ g(z_{1n}) = g(z_{2n}) = \cdots = g(z_{kn}). \end{cases}$$

Then there exists an entire function h (which depends on f and g only and is independent of k and S_i's) satisfying $h \leq f$, $h \leq g$ and $h(z_1) = h(z_i)$ for all $2 \leq i \leq k$.

Example : Let $f(z) = \cos z$ and $g(z) = \sin z$. Let $z_{1n} = \frac{1}{n}$, $z_{2n} = 2\pi + \frac{1}{n}$ and $z_{3n} = -2\pi + \frac{1}{n}$. Then $\lim_{n\to\infty} z_{1n} = 0$, $\lim_{n\to\infty} z_{2n} = 2\pi$, $\lim_{n\to\infty} z_{3n} = -2\pi$. For all $n \in \mathbb{N}$,

$$\begin{cases} f(z_{1n}) = f(z_{2n}) = f(z_{3n}) \\ g(z_{1n}) = g(z_{2n}) = g(z_{3n}). \end{cases}$$

Note that there exists an entire function $h(z) = e^z$ satisfying $h \le f$, $h \le g$ and $h(0) = h(-2\pi) = h(2\pi)$.

In many situations, Lemma 1 is not so easy to use because of the difficulties in finding the sequences required in the lemma. The following Common Right Factor Theorem is much easier to use and I beleive that it will be very useful in tackling factorization problems of entire functions.

Theorem 11 *(Common Right Factor Theorem) ([17]). Let f and g be two entire functions and $z_1, \dots, z_k$ be $k \ge 2$ distinct complex numbers such that*

$$\begin{cases} f(z_1) = f(z_2) = \cdots = f(z_k) = A \\ g(z_1) = g(z_2) = \cdots = g(z_k) = B. \end{cases}$$

Suppose that there exist nonconstant functions f_1 and g_1 such that $f_1 \circ f \equiv g_1 \circ g$ on $\cup_{i=1}^k U_i$, where U_i is some open neighborhood containing z_i. If f_1 is analytic in a neighborhood of A and the order of f_1 at A is $K < k$, then there exists an entire function h (which depends on f and g only and is independent of k and z_i) with $h \le f$, $h \le g$. Moreover, among the z_is, there exist at least $m = [\frac{k-1}{K}] + 1$ distint points $z_{n_1}, \dots z_{n_m}$ such that $h(z_{n_1}) = \cdots = h(z_{n_m})$.

We immediately have the following

Corollary 1 *([17]). Let f and g be two entire functions and $\{z_n\}_{n \in \mathbb{N}}$ be an infinite sequence of distinct complex numbers such that for all $n \in \mathbb{N}$, $f(z_n) = A$ and $g(z_n) = B$. Suppose that there exist nonconstant functions f_1 and g_1 such that $f_1 \circ f \equiv g_1 \circ g$ on $\cup_{i=1}^{\infty} U_i$, where U_i is some open neighborhood containing z_i. If f_1 is analytic in a neighborhood of A, then there exists a transcendental entire function h with $h \le f$, $h \le g$.*

Remark : In Theorem 11, the condition that $k > K$ is essential. Let $f(z) = z^2$, $g(z) = e^{iz}$, $f_1(z) = \cos\sqrt{z}$ and $g_1(z) = \frac{1}{2}(z + z^{-1})$. Then $\cos z = f_1 \circ f(z) = g_1 \circ g(z)$. Although $f(-\pi) = f(\pi) = \pi^2$ and $g(-\pi) = g(\pi)$, f and g do not have a nonlinear generalized common right factor. Note that in this case, the order K of f_1 at π^2 is exactly two.

Theorem 8, 9 follows easily from the Theorem 11 and Corollary 1.

3. Permutable entire functions.

As we mentioned before, to build a general theory for the factorization of entire functions, it is important to have a good understanding of permutable entire functions. The investigations of permutable functions were started by Julia [14] and Fatou [9]. Using methods from complex dynamics, they proved that for permutable nonlinear polynomials f and g, there exist natural numbers m, n such that (up to a conjugacy of linear maps) either i) $f^m(z) = g^n(z)$; ii) $f(z) = z^m$ and $g(z) = z^n$ or iii) $f(z) = T_m(z)$ and $g(z) = T_n(z)$, where T_k is the Tchebycheff polynomial determined by the equation $\cos kw = T_k(\cos w)$.

The rational case was first solved completely by J.F. Ritt [23] in 1923. However, Ritt did not use methods from complex dynamics. A proof of Ritt's result in the spirit of the ideas of Julia and Fatou was provided by A. E. Eremenko [7] in 1989. In 1958-59, I.N. Baker [1] and V.G. Iyer [13] started the investigations of permutable entire functions. They both proved that if a nonconstant polynomial f is permutable with a transcendental entire function g, then $f(z) = e^{2m/k\pi i}z + b$ for some $m, k \in \mathbb{N}$ and complex number b. It follows from this result, as well as Julia and Fatou's results, that one remains to consider permutable transcendental entire functions. Let $m, n \in \mathbb{N}$ and h be a trancendental entire function. Suppose that $az+b$ and $cz+d$ permute with h^m and h^n respectively. If $az+b$ also permutes with $cz + d$, then $f = ah^m + b$ permutes with $g = ch^n + d$. Up to a conjugacy of linear maps, any known examples of permutable transcendental entire functions are of this form. Note that a and c must be a p-th root and q-th root of unity for some $p, q \in \mathbb{N}$. If both $a, b \neq 1$, then it is easy to check that $f^p = h^{mp}$ and $g^q = h^{nq}$ so that $f^{npq} = h^{mnpq} = g^{mpq}$. This is case i) above.

In [1], I.N. Baker characterized all nonlinear entire functions permute with the expontential function and proved the following result.

Theorem 12 *([1]). Let g be a nonlinear entire function permutable with $f(z) = ae^{bz} + c$*

($ab \neq 0, a, b, c \in \mathbb{C}$), then $g = f^n$.

This result shows that there are only countablely infinite nonlinear entire functions which permute with $f(z) = e^z$. This is in fact true for general f (see [3]). Besides Baker's result, T. Kobayashi, H. Urabe, C.C. Yang and J.H. Zheng also obtained some results about permutable entire functions. Unlike Baker's result, they only considered finite order entire functions which permute with a given entire function. It is because of the lack of a powerful tool to handle the general case. By using the Common Right Factor Theorem, the following general results are obtained. Before stating our main result, we recall that an entire function F is prime (left-prime) in entire sense if whenever $F(z) = f(g(z))$ for some entire functions f, g, then either f or g is linear (f is linear whenever g is transcendental). We also denote the n-th iterate of f by f^n.

Theorem 13 *([17]). Let q be a nonconstant entire function and p be a polynomial with at least two distinct zeros. Suppose that $f(z) = p(z)e^{q(z)}$ is prime in entire sense. Then any nonlinear entire function g permutes with f is of the form $g(z) = af^n(z) + b$, where a is a k-th root of unity and $b \in \mathbb{C}$.*

It is known that if q is a polynomail and p and q do not have a nonlinear common right factor, then $f(z) = p(z)e^{q(z)}$ is prime in entire sense.

Theorem 14 *([17]). Let f be a transcendental entire function which satisfies the following conditions.*

A1) f is not of the form $H \circ Q$, where H is periodic entire and Q is a polynomial.

A2) f is left-prime in entire sense.

A3) f' has at least two distinct zeros.

A4) There exists a natural number N such that for any complex number c, the simultaneous equations $f(z) = c, f'(z) = 0$ has at most N solutions.

A5) The orders of zeros of f' are bounded by M for some $M \in \mathbb{N}$.

Let g be a nonlinear entire function permutes with f, then $g(z) = af^n(z) + b$, where a is a k-th root of unity and $b \in \mathbb{C}$.

The conditions A4 and A2 are related. For example, M. Ozawa [21] proved that if f is of finite oreder and for any $c \in \mathbb{C}$, the simultaneous equations $f(z) = c, f'(z) = 0$ has finite number of solutions, then f is left-prime in entire sense. Other similar results can also be found in [21] and [20]. It can be verified that for any nonconstant polynomial p, $e^z + p(z)$ and $\sin z + p(z)$ satisfy conditions A1-A5. The conditions A1-A4 are not restrictive. In certain sense, almost all entire functions satisfy these conditions as can be seen from the following result of Y. Noda [20].

Theorem 15 *([20]) Let f be a transcendental entire function. There exists a countable set $E_f \subset \mathbb{C}$ such that for all $a \notin E_f$, $f_a(z) = f(z) + az$ satisfies the following conditions.*

B1) f_a is nonperiodic.

B2) f_a is prime in entire sense.

B3) f'_a has infinitely many zeros.

B4) For any complex number c, the simultaneous equations $f_a(z) = c, f'_a(z) = 0$ has at most one solution.

Clearly, Bi implies Ai for $i = 2, 3, 4$. This is not difficult to check that conditions B1 and B2 together imply condition A1. If we can remove the condition A5 from Theorem 14, then together with Theorem 15, we can say that in certain sense, for almost all entire functions, only the iterates of a function can permute with the function itself. We shall sketch the proof of Theorem 13.

Proof of Theorem 13. Note that g is transcendental as it is nonlinear. Since p has at least two distinct zeros, $p(g(z))$ has infinitely many zeros by Little Picard Theorem. It follows from $p(g(z))e^{q(g(z))} = g(p(z)e^{q(z)})$ that $g(p(z)e^{q(z)})$ also has infinite number of zeros. Therefore there exists a zero b of g such that $p(z)e^{q(z)} - b$ has infinitely many zeros $\{z_i\}_{i \in \mathbb{N}}$. Note that the z_is are zeros of $p(g(z))e^{q(g(z))} = g(p(z)e^{q(z)})$. Hence, $p(g(z_i)) = 0$ for all $i \in \mathbb{N}$. Therefore we can find an infinite subsequence $\{z_{n_i}\}_{i \in \mathbb{N}}$ such that $f(z_{n_i}) = b$ and $g(z_{n_i}) = a$ for some zero a of p. By Corollary 1, there exists a transcendental entire function h with $h \leq f$ and $h \leq g$. Since f is nonperiodic and prime, it can be shown that the image of h is the whole complex plane and $h = L \circ f$, where L is a linear polynomail. Hence $g = g_1 \circ f$ for some entire function g_1. From $f \circ g = g \circ f$, we have

$f \circ g_1 \circ f = g_1 \circ f \circ f$. Note that the image of f equals that of h. Therefore, $f \circ g_1 \equiv g_1 \circ f$ on $\mathbb{C}$. If g_1 is nonlinear, by repeating the same arguments, we can find an entire g_2 such that $g_1 = g_2 \circ f$ and $g = g_2 \circ f^2$. Inductively, we have $g = g_m \circ f^m$ provided that g_{m-1} is nonlinear. It is possible to show that there exists some m such that g_m is linear and we are done (see [17] for the details).

4. Some general uniquely factorizable entire functions.

The Common Right Factor Theorem is also useful to show certain general class of entire functions are uniquely factorizable. Let f and g be nonlinear entire functions. Recall that the entire function $f \circ g$ is uniquely factorizable if whenever $f \circ g = f_1 \circ g_1$ for some nonlinear entire functions f_1 and g_1, then $f = f_1 \circ L$ and $g = L^{-1} \circ g_1$ for some linear function L.

Theorem 16 . *Let $f(z), g(z)$ be two prime entire functions. Suppose that f is of the form $(z-a)e^{\alpha(z)}$, where $\alpha(z)$ is an entire function and $a \in \mathbb{C}$. If $g(z)$ is nonperiodic and has infinite number of zeros, then $f(g(z))$ is uniquely factorizable.*

Proof of Theorem 16. Suppose that $f \circ g = f_1 \circ g_1$ where f_1, g_1 are nonlinear entire functions. Then it follows from the assumption of the theorem that $f_1 \circ g_1$ has infinite number of zeros. Therefore, f_1 must has at least one zero. If f_1 has only finitely many zeros, $a_1, ..., a_n$, then there exists some a_j such that $g_1(z) = a_j$ has infinitely many roots, for otherwise $f_1 \circ g_1$ will only have finite number of zeros. If f_1 has infinite number of zeros $\{a_i\}_{i \in \mathbb{N}}$, then there exists some a_j such that $g_1(z) = a_j$ has at least two distinct roots because g_1 is nonlinear. In any case, we can get some a_j and two distinct z_1, z_2 such that $f_1(a_j) = 0$ and $g_1(z_1) = g_1(z_2) = a_j$. Note that z_1, z_2 are zeros of $(g(z)-a)e^{\alpha(g(z))}$. Hence, $g(z_1) = g(z_2) = a$. By Theorem 11, there exists a nonlinear entire function h with $h \leq g$ and $h \leq g_1$. Hence, $g = h_1 \circ h$ and $g_1 = h_2 \circ h$, where h_1, h_2 are analytic on the image of h. Use the fact that g is prime and nonperiodic, one can prove that the image of h is $\mathbb{C}$. This implies that both h_1, h_2 are entire. g is prime and h is nonlinear, so h_1 must be linear. It follows that $g_1 = h_2 \circ h_1^{-1} \circ g$. From $f \circ g = f_1 \circ g_1$, we get $f = f_1 \circ h_2 \circ h_1^{-1}$.

The fact that f is prime and f_1 is nonlinear will force $L = h_2 \circ h_1^{-1}$ be to linear and we are done.

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