AS/2905/WKC/02

UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
Queueing Theory and Simulation
Assignment 2

Due Date: 14 March 2014 (5:00pm) [3%]

1. There are $n-1$ customers waiting for service and their service times are given in the following order $\{1, 2, 3, \cdots, n-1\}$. Show that the average waiting time before service for the customers is $(n^2-2n)/6$ when the service discipline is FCFS and is $(n^2-2n)/3$ when the service discipline is LCFS.

   [Hint: Note that $1+2+\cdots+r = \frac{r(r+1)}{2}$ and $1^2+2^2+\cdots r^2 = \frac{r(r+1)(2r+1)}{6}$.]

2. Let $X_1, X_2, \cdots, X_n$ be mutually independent identically distributed exponential r.v. with mean μ^{-1}. Define

$$R = \max(X_1, X_2, \cdots, X_n) - \min(X_1, X_2, \cdots, X_n)$$

   By using the Markov property show that $P\{R \le x\} = (1-e^{-\mu x})^{n-1}$.

3. Customers arrive according to a Poisson process with rate λ at a single server with n waiting positions. Customers who find all waiting positions occupied are cleared. All other customers wait as long as necessary for service. The mean service time is μ^{-1}, and $\rho = \frac{\lambda}{\mu}$. Making no assumption about the form of the service time distribution function, show that

$$\rho = \frac{1-p_0}{1-p_{n+1}}.$$

   Here p_i is the steady state probability that there are i customers in the system. [Hint: The mean number of customers joining the system per mean service time is equal to the mean number of customers completing service per mean service time].

4. A company has 3 telephone lines. Incoming calls are generated by the customers according to a Poisson process with a mean rate of 20 calls per hour. Calls that find all telephone lines busy are rejected and those calls that are able to get through will be served for a length of time that is exponentially distributed with mean 4 minutes.

   (a) Let the number of busy lines be the state of the system. Write down the Markov chain and the generator matrix for this system. Hence solve the steady state probability distribution for the system.

   (b) Find the proportion of calls that are rejected.

   (c) Find the mean number of calls that the company serves per hour.

   (d) Find the probability that an incoming call finds exactly one of the lines busy.

   (e) Now suppose the mean service time is 2 minutes, what would be the change of the proportion of the rejected calls?

5. There is a ticket selling office in a theater. Its opening hour is from 8 a.m. to 4 p.m. Suppose that both the arrival of customers and the service time follow the exponential distribution, and the mean interval time between customers' arrival is 2.5 minutes and the mean service time for each customer is 1.5 minutes.

   (a) Find the probability that an arrived customer does not need to wait in the queue.

   (b) Find the average number of customers in the queueing system.

   (c) Find the average time of the customers stay in the system.

   (d) Find the average number of customers waiting in the queue.

   (e) Find the average waiting time in the queue.

   (f) Find the probability that there are more than 4 customers in the system.

   (g) Find the probability that customers stay in the system for more than 15 minutes.

   (h) Find the average number of hours that there is no one in the system for a six working day's week.

6. The cashier in a supermarket can serve 30 people per hour on average, and the arrival rate of customers is 25 people per hour on average.

   (a) Find the mean length of queue when there is one or more customers.

   (b) If we want the average length of the queue to be decreased by one, what can we do?

7. An company offers services that can be modelled as an s-server Erlang loss system (M/M/s/0 queue). Suppose the arrival rate is 2 customers per hour and the average service time is 1 hour. The entrepreneur earns \$2.50 for each customer served and the company's operating cost is \$1.00 per server per hour (whether the server is busy or idle).

   (a) Write down the expected hourly net profit $C(s)$ in terms of s.

   (b) Show that $\lim_{s \to \infty} \frac{C(s)}{s} = -1$ and interprets this result.

   (c) If the maximum number of servers available is 5, what is the optimal number of servers which maximizes the expected hourly net profit ? What is the expected hourly net profit earned when the optimal number of servers is provided?

8. Customers arrive at a 3-server Erlang delay system (M/M/3/∞ queue) at the rate of 1 customer per minute with service in order of arrival (first-come-first-served). The mean service time is 2.5 minutes.

   (a) Find the steady state probability that there are i customers in the system.

   (b) Find the probability that all servers are busy as observed by an outside observer.

   (c) Find the average number of busy servers.

   (d) Find the average number of customers in the system.

   (e) Find the mean sojourn time (waiting plus service times) of a customer in the system by using the Little's queueing formula.

   (f) Find the mean number of customers waiting in the queue and the mean waiting time of customers.

9. Consider an Erlang loss system with two servers. The arrival rate is 1 and mean service time is μ_i^{-1} for server $i (i = 1, 2)$. When the system is idle, an arrived customer will visit the first server. Denote the states of the system as $(0, 0), (1, 0), (0, 1), (1, 1)$, where $(0, 0)$ denotes the state that both servers are idle, $(1, 0)$ denotes the state that the first server is busy and the second server is idle, $(0, 1)$ denotes the state that the first server is idle and the second server is busy and $(1, 1)$ denotes the state that both servers are busy.

(a) We adopt the order of states: $(0, 0), (1, 0), (0, 1), (1, 1)$, write down the generator matrix for this Markov process.

(b) Find the proportion of customers loss $B(\mu_1, \mu_2)$ in terms of μ_1 and μ_2

(c) Show that it is better to put the faster server as the first server, in order to achieve lower proportion of customers loss.