Department of Mathematics
The University of Hong Kong

2905/3905 Queueing Theory and Simulation

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Reference Books:

1. S .M. Ross (2000) *Introduction to Probability Models* , (7th Edition) San Diego, Calif. : Academic Press.

   [HKU Library Call Number : 519.2 R82 i]

2. R. B. Cooper (1981) *Introduction to Queueing Theory*, (2nd Edition), London: Arnold.

   [HKU Library Call Number : 519.82 C77]

Method of Assessment: Final Examination and Tests and Assignments.

Grading: Final Examination 50% and Tests 40% and Assignments 10%.

2905 Queueing Theory and Simulation

PART I: STOCHASTIC PROCESS AND PROBABILITY THEORY

1 Revisions on Probability Theory and Markov Chain

In many situations, we are interested in some numerical values that are associated with the outcome of an experiment rather than the actual outcomes themselves.

• For example, in an experiment of throwing two dice, we may be interested in the sum of the numbers (X) shown on the dice, say $X = 5$. Thus we are interested in a function which maps the outcome onto some points or an interval on the real line. In this example, the outcomes are $\{2, 3\}$, $\{3, 2\}$, $\{1, 4\}$ and $\{4, 1\}$, and the point on the real line is 5.

• This mapping that assigns a real value to each outcome in the sample space is called a **random variable** (r.v.), i.e. $X : \Omega \to \mathbf{R}$ is a r.v. from the sample space Ω to the set of real numbers $\mathbf{R}$. We usually write $\{X \leq x\}$ to denote the event $\{\omega \in \Omega, X(\omega) \leq x\}$.

• If X can assume at most a finite or a countable infinite number of possible values, then it is called a **discrete** r.v. If X can assume any value in an interval or union of intervals, and the probability that $\{X = x\}$ is equal to 0 for any $x \in \mathbf{R}$ then it is called a **continuous** r.v.

• The **cumulative probability distribution function**, or just the **Cumulative Distribution Function (CDF)** for short is defined as $F_X : \mathbf{R} \to [0,1]$ such that

$$\boxed{F_X(x) = P\{X \le x\} \quad \text{where} \quad P\{X \le x\}}$$

denotes the probability of the event $\{X \le x\}$.

• For simplicity, we may also write $F_X(x)$ as $F(x)$. For a continuous r.v., the CDF is a continuous function. For a discrete r.v., its CDF is a step function, and it is more convenient to consider the **probability (mass) function** which is defined as follows. If X is a discrete r.v. assuming the values $A = \{x_1, x_2, \ldots\}$. Then $p(x_i) = P\{X = x_i\}$ is called the **Probability Density Function (PDF)**. Clearly

$$\boxed{p(x_i) > 0, \quad \sum_{x_i \in A} p(x_i) = 1 \quad \text{and} \quad F(x) = \sum_{x_i \le x} p(x_i).}$$

The PDF for a continuous r.v. is defined as the function $f(t)$ such that

$$\boxed{F(x) = \int_{-\infty}^{x} f(t)dt.}$$

Here $F(x)$ is the CDF. All the continuous r.v. considered in this course have distribution functions that are differentiable except at a finite number of points and their density functions satisfies

$$\boxed{f(x) = \frac{d}{dx} F(x).}$$

We now list some important discrete and continuous r.v.

1.1 Examples of Discrete Random Variables

(i) **Bernoulli r.v.:** A Bernoulli r.v. X has two possible outcomes, say

$$X = 1 \quad \text{and} \quad X = 0$$

(very often it is called success and failure, respectively) occurring with probabilities

$$\boxed{P\{X = 1\} = p \quad \text{and} \quad P\{X = 0\} = q,}$$

where $p + q = 1$. Note that

$$\boxed{E(X) = p \quad \text{and} \quad \mathrm{Var}(X) = pq.}$$

(ii) **Geometric r.v.:** In a sequence of Bernoulli trials, the r.v. X that counts the number of failures preceding the first success is called a geometric r.v. with the probability function

$$\boxed{P\{X = k\} = (1 - p)^k p \; ; \quad k = 0, 1, 2, \dots .}$$

Note that

$$\boxed{E(X) = \frac{q}{p} \quad \text{and} \quad \mathrm{Var}(X) = \frac{q}{p^2}}$$

where $q = 1 - p$.

(iii) **Binomial r.v.:** If a Bernoulli trial is repeated n times then the r.v. X that counts the number of successes in the n trials is called a binomial r.v. with parameter n and p. The probability density function is given by

$$P\{X = k\} = \frac{n!}{(n-k)!k!} p^k q^{n-k}, \quad k = 0, 1, 2, \ldots, n \quad \text{and} \quad q = 1 - p.$$

We note that $E(X) = np$ and $\text{Var}(X) = npq$.

(iv) **Poisson r.v.:** A Poisson r.v. X with parameter λ has the probability function

$$\boxed{P\{X = k\} = \frac{\lambda^k}{k!} e^{-\lambda}; \quad k = 0, 1, 2, \ldots.}$$

We note that

$$\boxed{E(X) = \text{Var}(X) = \lambda.}$$

One may derive the Poisson distribution from the binomial distribution by letting $\lambda = np$ and $n \to \infty$. We derive the relationship as follows:

$$\boxed{\begin{aligned} P\{X = k\} &= \frac{n!}{(n-k)!k!} p^k q^{n-k} \\ &= \frac{1}{k!}[p(n-k+1)][p(n-k+2)] \ldots [p(n)](1-p)^{n-k} \\ &= \frac{1}{k!}[\frac{(n-k+1)\lambda}{n}][\frac{(n-k+2)\lambda}{n}] \ldots [\lambda](1 - \frac{\lambda}{n})^{n-k} \\ &= \frac{\lambda^k}{k!} e^{-\lambda} \quad \text{as } n \to \infty. \end{aligned}}$$

1.2 Examples of Continuous Random Variables

(i) **Uniform r.v.:** A continuous r.v. X with its probabilities distributed uniformly over an interval (a, b) is said to be a uniform r.v.

A uniform r.v. X that takes values in $(0, t)$ has distribution function

$$\boxed{F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{t} & 0 \le x \le t \\ 1 & x > t \end{cases}}$$

and corresponding density function

$$\boxed{f(x) = \begin{cases} \frac{1}{t} & 0 < x < t \\ 0 & x < 0\ ,\ \ x > t. \end{cases}}$$

We note that

$$\boxed{E(X) = \int_0^t x f(x) dx = \frac{t}{2} \quad \text{and} \quad \mathrm{Var}(X) = \frac{t^2}{12}.}$$

(ii) **(Negative) Exponential r.v.:** A continuous r.v. X is an exponential r.v. with parameter $\lambda > 0$ if its density function is defined by

$$\boxed{f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0. \end{cases}}$$

The distribution function is given by

$$\boxed{F(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0. \end{cases}}$$

We note that

$$\boxed{E(X) = \lambda^{-1} \quad \text{and} \quad \text{Var}(X) = \lambda^{-2}.}$$

• The exponential distribution plays an important role in modeling the inter-arrival and service time in a Markovian queueing system.

(iii) **Erlangian r.v.:** The distribution function of an Erlangian r.v. X is given by

$$\boxed{F(x) = 1 - \sum_{j=0}^{n-1} \frac{(\lambda x)^j}{j!} e^{-\lambda x} \quad (\lambda > 0, x \geq 0 \text{ and } n = 1, 2, \ldots)}$$

and the density function is given by

$$\boxed{f(x) = \frac{(\lambda x)^{n-1}}{(n-1)!} \lambda e^{-\lambda x}}$$

with

$$\boxed{E(X) = n\lambda^{-1} \quad \text{and} \quad \mathrm{Var}(X) = n\lambda^{-2}.}$$

We note that if X_i are independent exponential r.v. having same PDF $\lambda e^{-\lambda x}$ then the PDF of the r.v.

$$\boxed{X = X_1 + X_2 + \ldots + X_n}$$

is given by $f(x)$. This means that the Erlangian distribution is **the sum of n independent exponential random variables** having the same mean.

1.3 Conditional Probability

• What is **conditional probability**?

Consider the following two events:

A: Get three heads in tossing a fair coin three times.

B: Get odd number of heads in tossing a fair coin three times.

All the possible outcomes are listed as follows:

$$\{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}.$$

• We know that

$$\text{Prob}(A) = \frac{1}{8} \quad \text{and} \quad \text{Prob}(B) = \frac{1}{2}$$

What is the probability of getting event A given B? Mathematically the probability is written as $\text{Prob}(A|B)$. Clearly the probability is not $1/8$.

• In fact, in general one has

$$\boxed{\text{Prob}(A|B) = \frac{\text{Prob}(A \text{ and } B)}{\text{Prob}(B)}.}$$

In this case we have $\text{Prob}(A \text{ and } B) = \text{Prob}(A)$ and therefore

$$\text{Prob}(A|B) = \frac{1/8}{1/2} = \frac{1}{4}.$$

1.4 Theorem of Total Probability

If the events $E_1, E_2, \ldots$ form a **partition** of the sample space Ω, that is
(i) $E_i \cap E_j = \phi$ for all $i \neq j$;
(ii) $\bigcup_{i=1}^{\infty} E_i = \Omega$, then, for any event A, we have

$$P\{A\} = \sum_{i=1}^{\infty} P\{A|E_i\}P\{E_i\}.$$

Here $P\{A|E_i\}$ is the conditional probability of A given E_i.

1.5 Discrete Time Markov Chain

Definition: Let $X^{(n)}$ be a r.v. at time n taking values in $M = \{0, 1, 2, \ldots\}$. Suppose there is a fixed probability P_{ij} such that

$$\boxed{P(X^{(n+1)} = j | X^{(n)} = i, X^{(n-1)} = i_{n-1}, \ldots, X^{(0)} = i_0) = P_{ij} \quad n \geq 0}$$

where $i, j, i_0, i_1, \ldots, i_{n-1} \in M$. Then this is called a Markov chain process.
Remark: One can interpret the above probability as follows: the conditional distribution of any future state $X^{(n+1)}$ given the past states $X^{(0)}, X^{(2)}, \ldots, X^{(n-1)}$ and present state $X^{(n)}$, is **independent** of the **past states** and **depends** on the **present state** only.

• The probability P_{ij} represents the probability that the process will make a transition to State j given that currently the process is State i. Clearly one has

$$\boxed{P_{ij} \geq 0, \quad \sum_{j=0}^{\infty} P_{ij} = 1 \quad i = 0, 1, \ldots.}$$

Definition: The matrix containing P_{ij}, the transition probabilities

$$P = \begin{pmatrix} P_{00} & P_{01} & \cdots \\ P_{10} & P_{11} & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}$$

is called the **one-step transition probability matrix** of the process.

1.6 The PageRank Algorithm used by Google

In surfing the Internet, surfers usually use search engines to find the related webpages satisfying their queries. Unfortunately, very often there can be thousands of webpages which are relevant to the queries. Therefore a proper list of the webpages in certain order of importance is necessary.

• The PageRank is defined as follows. Let N be the total number of webpages in the web and we define a matrix Q called the **hyperlink matrix**. Here

$$\boxed{Q_{ij} = \begin{cases} 1/k & \text{if webpage } j \text{ is an outgoing link of webpage } i; \\ 0 & \text{otherwise;} \end{cases}}$$

and k is the total number of outgoing links of webpage j. For simplicity of discussion, here we assume that $Q_{ii} > 0$ for all i. This means for each webpage, there is a link pointing to itself. Hence Q can be regarded as a transition probability matrix of a Markov chain of a random walk.

• One may regard a surfer as a random walker and the webpages as the states of the Markov chain. Assuming that this underlying Markov chain is irreducible, then the steady-state probability distribution $(p_1, p_2, \ldots, p_N)^T$ of the states (webpages) exists.

• Here p_i is the **proportion of time** that the random walker (surfer) is visiting state (webpage) i. The higher the value of p_i is, the more important webpage i will be. Thus the PageRank of webpage i is then defined as p_i.

Example: Let us consider a web of three webpages: $0, 1, 2$. Suppose that the links are given as follow: $0 \to 1, \quad 0 \to 2, \quad 1 \to 0 \quad$ and $\quad 2 \to 1$. The out-degrees of States $0, 1, 2$ are $3, 2, 2$ respectively.

• The transition probability of this Markov chain is given by

$$P = \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}.$$

The steady state probability distribution

$$\mathbf{p} = (p_0, p_1, p_2)$$

satisfies

$$\mathbf{p} = \mathbf{p}P$$

and

$$p_0 + p_1 + p_2 = 1.$$

Solving the linear system of equations, we get

$$(p_0, p_1, p_2) = \left(\frac{3}{9}, \frac{4}{9}, \frac{2}{9} \right).$$

Thus the ranking of the webpages is Webpage 1 > Wepbage 0 > Webpage 2.

• It is clear that both Webpages 0 and 2 point to Webpage 1 and therefore it must be the most important.

• Since the most important Webpage 1 points to Webpage 0 only, Webpage 0 is more important than Webpage 2.

• We remark that the steady state probability distribution may not exist as the Markov chain may not be irreducible.

• But one can always consider the following transition probability matrix:

$$\tilde{P} = (1-\alpha)P + \frac{\alpha}{N}(1,1,\ldots,1)^t(1,1,\ldots,1)$$

for very small positive α. Then $\tilde{P}$ is irreducible.

2 Poisson Distribution and Exponential Distribution

We introduce some more properties of the Poisson distribution and the exponential distribution.

2.1 Probability Generating Function

Let K be a non-negative integer-valued random variable with probability function $\{p_j\}$ where $p_j = P\{K = j\}$ $(j = 0, 1, 2, \ldots)$. The power series

$$\boxed{g(z) = p_0 + p_1 z + p_2 z^2 + \cdots}$$

is called the **probability generating function** for the r.v. K. It is different form the moment generating function. The followings are two examples

(i) The probability generating function of a Bernoulli r.v. is simply

$$g(z) = q + pz \quad \text{where} \quad q = 1 - p.$$

(ii) For the Poisson r.v. with distribution

$$p_j = \frac{(\lambda t)^j}{j!} e^{-\lambda t} \quad (j = 0, 1, \ldots)$$

The probability generating function is

$$g(z) = \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} e^{-\lambda t} z^j = e^{-\lambda t} \sum_{j=0}^{\infty} \frac{(\lambda t z)^j}{j!} = e^{-\lambda t(1-z)} .$$

Here are some properties of probability generating function.

(i) $E(X) = \sum_{j=1}^{\infty} jp_j = g'(1)$.

(ii) Variance of

$$
\begin{aligned}
X &= \text{Var}(X) = E(X^2) - E(X)^2 \\
&= \sum_{j=1}^{\infty} j^2 p_j - [g'(1)]^2 \\
&= g''(1) + g'(1) - [g'(1)]^2.
\end{aligned}
$$

For the Bernoulli r.v. $\text{Var}(X) = pq$, and for the Poisson r.v. $\text{Var}(X) = \lambda t$.

(iii) **Convolution:** Suppose $K = K_1 + K_2$ where K_1 and K_2 are independent, non-negative, integer-valued random variables. Then

$$P\{K = k\} = \sum_{j=0}^{k} P\{K_1 = j\} P\{K_2 = k - j\} \ .$$

If $g_1(z)$ and $g_2(z)$ are the probability generating function of K_1 and K_2, respectively, i.e.

$$g_i(z) = \sum_{j=0}^{\infty} P\{K_i = j\} z^j \quad (i = 1, 2) \ ,$$

then term-by-term multiplication shows that the product $g_1(z)g_2(z)$ is given by

$$g_1(z)g_2(z) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} P\{K_1 = j\}P\{K_2 = k - j\} \right) z^k \ .$$

If K has the generating function

$$g(z) = \sum_{k=0}^{\infty} P\{K = k\}z^k.$$

Hence we have

$$g(z) = g_1(z)g_2(z).$$

Thus we have the important result: The probability generating function of a sum of mutually independent r.v. is equal to the product of their respective probability generating functions.

2.2 Sum of Two Poisson r.v.

Let $N = N_1 + N_2$ where N_j is a Poisson r.v. with mean λ_j. Then N has the probability generating function

$$g(z) = e^{-\lambda_1(1-z)}e^{-\lambda_2(1-z)} = e^{-(\lambda_1+\lambda_2)(1-z)} \ .$$

This shows that the sum of two independent Poisson r.v. with means λ_1 and λ_2 is itself a Poisson r.v. with mean $\lambda_1 + \lambda_2$. Hence the sum of any independent Poisson r.v.s is still a Poisson r.v.

2.3 The Exponential Distribution and Markov Property

Definition 5.1 A probability distribution (let say having non-negative r.v. X) is said to have the **Markov property** if for any two non-negative values t and x we have

$$\boxed{P\{X > t + x | X > t\} = P\{X > x\}.}$$

Proposition 1 The negative exponential distribution has the Markov property.

Proof: This follows from

$$\boxed{P\{X > t + x | X > t\} = \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} = e^{-\lambda x} = P\{X > x\}.} \tag{1}$$

□

• In quite a few important applications, observation has shown that the negative exponential distribution provides a very good description of service time distribution (which is therefore called **exponential service times or negative exponential distribution**).

Exponential times have the nice feature that by the Markov property in Eq. (1), the distribution of the remaining holding time after a customer has been served for any length of time $t > 0$ is the same as that initially at $t = 0$, i.e. the remaining holding time does not depend on how long a customer has already been served.

Here are some properties of this probability distribution.

(i) If X_1 and X_2 are independent non-negative random variables with density functions $f_1(t)$ and $f_2(t)$. Then the probability that X_2 exceeds X_1 is

$$P\{X_2 > X_1\} = \int_0^\infty \int_s^\infty f_1(s) f_2(t) dt\, ds$$

(i.e. we integrate the joint density function over the region

$$\{(s,t) \in \mathbf{R}^2 | t > s\}).$$

If X_1 and X_2 are exponential with means λ_1^{-1} and λ_2^{-1} then the above integral becomes

$$\int_0^\infty \int_s^\infty \lambda_1 e^{-\lambda_1 s} \lambda_2 e^{-\lambda_2 t} dt ds = \int_0^\infty \lambda_1 e^{-\lambda_1 s} e^{-\lambda_2 s} ds = \frac{\lambda_1}{\lambda_1 + \lambda_2} . \tag{2}$$

(ii) Suppose that $X_1, X_2, \ldots, X_n$ are independent, identical, exponential random variables with mean λ^{-1}, and consider the corresponding order statistics

$$X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}.$$

Observe that $X_{(1)} = \min(X_1, X_2, \ldots, X_n)$. $X_{(1)} > x$ if and only if all $X_i > x$ $(i = 1, 2, \ldots, n)$. Hence

$$P\{X_{(1)} > x\} = P\{X_1 > x\} P\{X_2 > x\} \cdots P\{X_n > x\} = (e^{-\lambda x})^n = e^{-n\lambda x} .$$

Note that $X_{(1)}$ is again exponentially distributed with mean $\frac{1}{n}$ times the mean of the original random variables.

Proposition 2 A random variable X has a negative exponential distribution if and only if

$$P\{X < t+h | X > t\} = \lambda h + o(h) \quad \text{as } h \to 0. \quad \text{Here} \quad \lim_{h\to 0} \frac{o(h)}{h} = 0.$$

Proof: Suppose X follows the exponential distribution with parameter λ, then we have

$$\begin{aligned} P\{X < t+h | X > t\} &= 1 - e^{-\lambda h} \qquad (\text{by the Markov property}) \\ &= 1 - (1 - \lambda h + o(h)) \quad (\text{by Taylor's series}) \\ &= \lambda h + o(h) \quad \text{as } h \to 0\ . \end{aligned} \tag{3}$$

Conversely, suppose that

$$P\{X < t+h | X > t\} = \lambda h + o(h) \quad \text{as} \quad h \ \to 0$$

then

$$P\{X > t+h | X > t\} = 1 - \lambda h + o(h).$$

Using

$$P\{X > t+h | X > t\} = \frac{P\{X > t+h\}}{P\{X > t\}},$$

re-arranging and letting $h \to 0$, we obtain the differential equation

$$\frac{d}{dt} P\{X > t\} = -\lambda P\{X > t\}, \qquad (\text{O.D.E of the form: } \frac{dy}{dt} = -\lambda y(t), \quad y(t) = P(X > t))$$

which has the unique solution

$$P\{X > t\} = e^{-\lambda t}$$

satisfying the initial condition $P\{X > 0\} = 1$, i.e. X has an exponential distribution. $\square$

3 Poisson Process

Definition: Poisson process occurs frequently in later sections.

It is more convenient if the postulates of such a process is defined simply as:

At any epoch t,

$$P\{\text{one occurrence during}(t, t+h)\} = \lambda h + o(h) \quad \text{as } h \to 0$$

and

$$P\{\text{two or more occurrence during}(t, t+h)\} = o(h) \quad \text{as } h \to 0$$

(instead of being always referred to as a pure birth process with constant coefficients).

Poisson process, Poisson distributions and negative exponential distributions can be related in the following manner.

Proposition 3 Suppose in a certain process, we let T_i $(i = 1, 2, 3, \cdots)$ be the epoch of the i^{th} occurrence.
Let $A_i = T_i - T_{i-1}$ $(i = 1, 2, 3, \cdots)$; $T_0 =$ epoch that we start to count the number of occurrences.
Let $X(t) =$ number of occurrences in a time interval of length t.
Then the following statements are equivalent.

(a) The process is Poisson (with coefficient λ).

(b) $X(t)$ is a Poisson random variable with parameter λt, i.e.

$$\boxed{P\{X(t) = j\} = \frac{(\lambda t)^j}{j!} e^{-\lambda t}\ , \quad j = 0, 1, 2, \cdots\ .}$$

(c) A_i's are mutually independent identically distributed exponential random variables with mean λ^{-1}, i.e.

$$\boxed{P\{A_i \leq t\} = 1 - e^{-\lambda t}\ , \quad i = 1, 2, \cdots\ .}$$

Proof: **(a) $\Rightarrow$ (b):** Let $P_j(t)$ be the probability that there are j occurrences in the time interval $[0,t]$. It follows from the above postulates that

$$\boxed{\begin{aligned} P_j(t+h) &= \underbrace{[\lambda h + o(h)]}_{1 \text{ occurrence in } (t,t+h)} \times \underbrace{P_{j-1}(t)}_{j-1 \text{ in } [0,t]} + \underbrace{[1-(\lambda h + o(h))]}_{0 \text{ occurrence in } (t,t+h)} \times \underbrace{P_j(t)}_{j \text{ in } [0,t]} \\ &\quad + \underbrace{[o(h)]}_{\text{more than 1 occurrence in } (t,t+h)} \times \underbrace{(P_{j-2}(t) + P_{j-3}(t) + \ldots +)}_{\leq j-2 \text{ in } [0,t]} \\ &= \lambda h P_{j-1}(t) + [1-\lambda h]P_j(t) + o(h) \end{aligned}}$$

where $h \to 0$ and $j = 0, 1, \ldots$.

- Rearranging terms, we have

$$\frac{P_j(t+h) - P_j(t)}{h} = \lambda P_{j-1}(t) - \lambda P_j(t) + \frac{o(h)}{h}.$$

Letting $h \to 0$, we get the following set of differential-difference equations

$$\boxed{\frac{d}{dt}P_j(t) = \lambda P_{j-1}(t) - \lambda P_j(t).}$$

If at time $t = 0$ there is no occurrence, the initial conditions are

$$P_{-1}(t) \equiv 0, \quad P_0(0) = 1 \quad \text{and} \quad P_j(0) = 0 \text{ for } j = 1, 2, \ldots.$$

- For $j = 0$, $\quad P_0'(t) = -\lambda P_0(t)$, hence $P_0(t) = a_0 e^{-\lambda t}$. From the initial conditions, we get $a_0 = 1$ and we have $P_0(t) = e^{-\lambda t}$.

• Inductively, we can prove that if

$$P_{j-1}(t) = \frac{(\lambda t)^{j-1}}{(j-1)!}e^{-\lambda t}$$

then the equation

$$P_j'(t) = \lambda \left(\frac{(\lambda t)^{j-1}}{(j-1)!}e^{-\lambda t} \right) - \lambda P_j(t)$$

gives (exercise) the solution

$$\boxed{P_j(t) = \frac{(\lambda t)^j}{j!}e^{-\lambda t}.} \tag{4}$$

(b) ⇒ (c):

$$P\{A_1 \le t\} = P\{T_1 \le t + T_0\} = \underbrace{1 - P\{X(t) = 0\}}_{\text{At least 1 occurrence in [0,t]}} = 1 - e^{-\lambda t}.$$

Suppose that $T_1 = \tau$, then

$$P\{A_2 \le t\} = P\{T_2 \text{ occurs in } (\tau, \tau + t)\} = 1 - P\{X(t) = 0\} = 1 - e^{-\lambda t}.$$

Arguing similarly for the other A_i's, (c) then follows.

(c) $\Rightarrow$ (a): Let $T_{j-1} \leq t < T_j$ for some $j \in \mathbb{N}$. There is **at least one occurrence** in $(t, t+h)$ if $B_j < h$ where $B_j = T_j - t$. Now

$$P\{B_j < h\} = P\{A_j < h + t - T_{j-1} | A_j \geq t - T_{j-1}\} = P\{A_j < h\}.$$

The last equality follows from the Markov property of the exponential variable A_j.

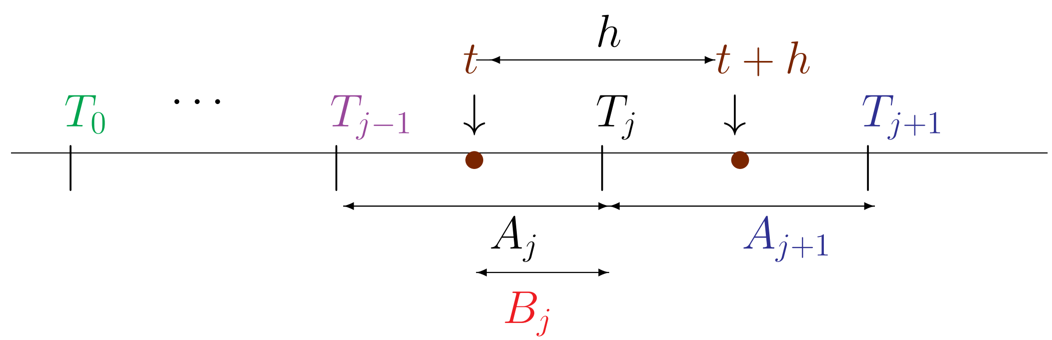


Figure 1.

- Thus we have

$$\begin{aligned} P\{\text{there is } \textbf{at least one} \text{ occurrence in } (t, t+h)\} &= P\{A_j < h\} \\ &= 1 - e^{-\lambda h} \\ &= \lambda h + o(h) \quad \text{(by Taylor's series)} . \end{aligned} \tag{5}$$

• Next consider the event that there is **exactly one occurrence** in $(t, t+h)$. This event occurs if $B_j < h$ and $A_{j+1} > h - B_j$. The probability is therefore

$$\boxed{\int_0^h \int_{h-x}^{\infty} f_{A_{j+1}}(y) f_{B_j}(x) dy dx}$$

where $f_{A_{j+1}}(y)$ and $f_{B_j}(x)$ are the density functions of A_{j+1} and B_j, respectively.

• Since A_{j+1} and B_j are both exponentially distributed, the above integral becomes

$$\begin{aligned}
\int_0^h \boxed{\int_{h-x}^{\infty} \lambda e^{-\lambda y} \lambda e^{-\lambda x} dy} dx &= \int_0^h e^{-\lambda(h-x)} \lambda e^{-\lambda x} dx \\
&= \lambda h e^{-\lambda h} = \lambda h (1 - \lambda h / 1! + \ldots) \\
&= \lambda h + o(h).
\end{aligned}$$

• Hence

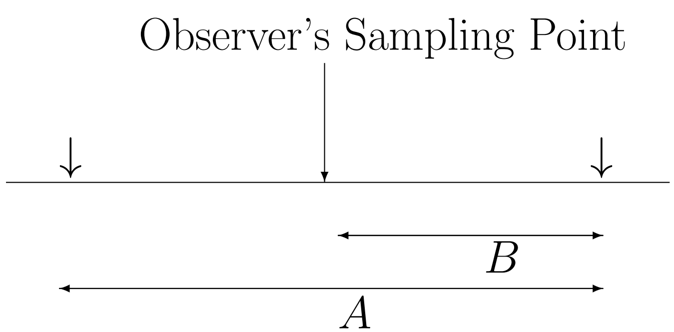
$$\begin{aligned}
P\{\text{two or more occurrences in } (t, t+h)\} &= P\{\text{at least one occurrence in } (t, t+h)\} \\
&\quad -P\{\text{exactly one occurrence in } (t, t+h)\} \\
&= o(h) \text{ as } h \to 0 .
\end{aligned}$$

and (a) follows $\square$

3.1 Random Property of a Poisson Process

• In many queueing situations, the Poisson process provides rather a good approximation of the input process (such input process is called the **Poisson input**). By proposition above, an input process is Poisson (with coefficient λ) if and only if the **inter-arrival times** (i.e. the lengths of time between successive customer arrivals) are mutually independent exponentially distributed random variables with mean λ^{-1}.

Note that if the mean arrival rate is λ then the mean inter-arrival time is λ^{-1}. By the Markov property of exponential random variables, the distribution of lengths of time from an arbitrarily chosen epoch to the next arrival (called the **next-arrival times**) is the same as the distribution of inter-arrival times.



Distribution of A = Distribution of B

Figure 3.1.

• This nice property much simplifies the mathematical analysis of queueing systems with Poisson input. We have shown that for Poisson process,

$$P\{\text{exactly one occurrence in}(t, t+h)\} = \lambda h e^{-\lambda h}.$$

Now for a fixed t and any x in $(0, t)$,

$$\begin{aligned} & P\{\text{the epoch of occurrence is in } (0, x)|\text{ exactly one occurrence in } (0, t)\} \\ = & \frac{P\{\text{exactly one occurrence in } (0, x)\text{, and no occurrence in } (x, t)\}}{P\{\text{exactly one occurrence in } (0, t)\}} \\ = & \frac{\lambda x e^{-\lambda x} \times e^{-\lambda(t-x)}}{\lambda t e^{-\lambda t}} \\ = & \frac{x}{t}, \end{aligned}$$

which is a **uniform distribution**.

• This means that if we know there is exactly one occurrence in $(0, t)$ then the epoch of that occurrence is equally likely throughout $(0, t)$. In this sense we say that a **Poisson process is random**.

3.2 Erlangian (Gamma) Distribution

Consider points on a line such that the distances (denoted by X_j, $j = 1, 2, \cdots$) between successive points are independently, identically, exponentially distributed random variables each with mean λ^{-1} [therefore the number of points occurring in $(0, t)$ has the Poisson distribution].

• We wish to find the distribution of the distance spanning n consecutive points, i.e. the distribution of

$$S_n = X_1 + \cdots + X_n.$$

Now

$$\begin{aligned} P\{S_n \leq x\} &= P\{\text{number of occurrences in } (0, x) \text{ is at least } n\} \\ &= 1 - \sum_{j=0}^{n-1} \frac{(\lambda x)^j}{j!} e^{-\lambda x} \ . \end{aligned}$$

Thus S_n has an **Erlangian distribution**.

Remarks:

(i) The component random variables $\{X_n\}$ are sometimes considered to be **phases**, and consequently S_n is called the n-phase Erlangian r.v.

(ii) Suppose S_n is composed of n independent, identically distributed exponential phases, each with mean λ^{-1}. If we let $n \to \infty$ and $\lambda^{-1} \to 0$ such that

$$E(S_n) = n\lambda^{-1} = c, (\text{ a constant})$$

then

$$Var(S_n) = n\lambda^{-2} \to 0.$$

Thus in the limit $S_\infty = c$ (a constant). Hence the Erlangian distribution provides a model for a range of input processes (or service times) characterized by complete randomness when $n = 1$ and no randomness when $n = \infty$.

4 Birth-and-Death Processes

In this section, we shall discuss the theory of birth-and-death processes, the analysis of which is relatively simple and has important applications in the context of queueing theory.

Let us consider a system that can be represented by a family of random variables $\{N(t)\}$ parameterized by the time variable t. This is called a **stochastic process** or simply a **process**.

In particular, let us assume that for each t, $N(t)$ is a non-negative integral-valued random variable. Examples are the followings.

(i) a telephone switchboard, where $N(t)$ is the number of calls occurring in an interval of length t.

(ii) a queue, where $N(t)$ is the number of customers waiting or in service at time t.

We say that the system is in **state** E_j at time t if $N(t) = j$. Our aim is then to compute the **state probabilities** $P\{N(t) = j\}$, $j = 0, 1, 2, \cdots$.

Definition 4.1: A process obeying the following postulates is called a **birth-and-death process**:

(1) At any time t, $P\{E_j \to E_{j+1}$ during $(t, t+h) | E_j$ at $t\} = \lambda_j h + o(h)$ as $h \to 0$ $(j = 0, 1, 2, \cdots)$. λ_j is a constant depending on j.

(2) At any time t, $P\{E_j \to E_{j-1}$ during $(t, t+h) | E_j$ at $t\} = \mu_j h + o(h)$ as $h \to 0$ $(j = 1, 2, \cdots)$. μ_j is a constant depending on j.

(3) At any time t, $P\{E_j \to E_{j\pm k}$ during $(t, t+h) | E_j$ at $t\} = o(h)$ as $h \to 0$ if $k \geq 2$ $(j = 0, 1, \cdots)$.

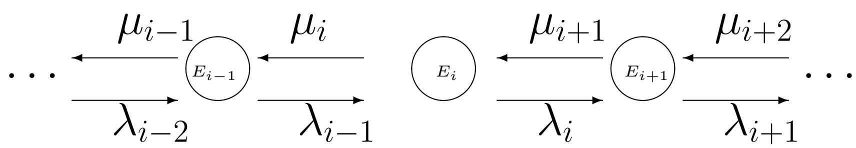


Figure 2.1: The Birth and Death Process.

Remark: $o(h)$ is a function of h such that

$$\boxed{\lim_{h \to 0} \frac{o(h)}{h} = 0.}$$

Possible examples of $o(h)$ are $o(h) = h^2$ and $o(h) = h \sin(h)$. However, $o(h)$ cannot take the form $\sqrt{h}$ or $h \log(h)$.

Notation: Let $P_j(t) = P\{N(t) = j\}$ and let $\lambda_{-1} = \mu_0 = P_{-1}(t) = 0$.

It follows from the above postulates that (where $h \to 0; j = 0, 1, \ldots.$)

$$P_j(t+h) = \underbrace{(\lambda_{j-1}h + o(h))}_{an\ arrival} P_{j-1}(t) + \underbrace{(\mu_{j+1}h + o(h))}_{a\ departure} P_{j+1}(t) + \underbrace{[1 - ((\lambda_j + \mu_j)h + o(h))]}_{no\ arrival\ or\ departure} P_j(t)$$

$$\boxed{P_j(t+h) = (\lambda_{j-1}h)P_{j-1}(t) + (\mu_{j+1})hP_{j+1}(t) + [1 - (\lambda_j + \mu_j)h]P_j(t) + o(h)}.$$

Rearranging terms, we have

$$\boxed{\frac{P_j(t+h) - P_j(t)}{h} = \lambda_{j-1}P_{j-1}(t) + \mu_{j+1}P_{j+1}(t) - (\lambda_j + \mu_j)P_j(t) + \frac{o(h)}{h}.}$$

Letting $h \to 0$, we get the following set of **differential-difference equations**

$$\frac{d}{dt}P_j(t) = \lambda_{j-1}P_{j-1}(t) + \mu_{j+1}P_{j+1}(t) - (\lambda_j + \mu_j)P_j(t). \tag{6}$$

If at time $t = 0$ the system is in state E_i, the initial conditions are then $P_j(0) = \delta_{ij}$ where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Definition 4.2: The coefficients $\{\lambda_j\}$ and $\{\mu_j\}$ are called the **birth** and **death rates** respectively. When $\mu_j = 0$ for all j, the process is called a **pure birth process**; and when $\lambda_j = 0$ for all j, the process is called a **pure death process**.

In the case of either a pure birth process or a pure death process, Eq. (6) can be solved by recurrence.

4.1 Pure Birth Process with Constant Rates

In this section, we consider a **Pure birth process ($\mu_i = 0$) with constant $\lambda_j = \lambda$ and initial state E_0.**

Then Eq. (6) become

$$\boxed{\frac{d}{dt}P_j(t) = \lambda P_{j-1}(t) - \lambda P_j(t) \quad (j = 0, 1, \cdots)}$$

where $P_{-1}(t) = 0$ and $P_j(0) = \delta_{0j}$.

Here

$$j = 0, \quad P_0'(t) = -\lambda P_0(t),$$

hence

$$\boxed{P_0(t) = a_0 e^{-\lambda t}.}$$

From the initial conditions, we get $a_0 = 1$.

Inductively, we can prove that if

$$\boxed{P_{j-1}(t) = \frac{(\lambda t)^{j-1}}{(j-1)!}e^{-\lambda t}}$$

then the equation

$$\boxed{P_j'(t) = \lambda \left(\frac{(\lambda t)^{j-1}}{(j-1)!} e^{-\lambda t} \right) - \lambda P_j(t)}$$

gives (exercise) the solution

$$\boxed{P_j(t) = \frac{(\lambda t)^j}{j!} e^{-\lambda t}.} \tag{7}$$

Note:

(i) The probabilities (7) satisfy the normalization condition

$$\sum_{j=0}^{\infty} P_j(t) = 1 \quad (t \geq 0).$$

(ii) For each t, $N(t)$ has the Poisson distribution, given by $\{P_j(t)\}$. We say that $N(t)$ describes a **Poisson process**. We will discuss it more later.

Remark: Since the assumption $\lambda_j = \lambda$ is often a realistic one, the simple formula (7) plays a central role in queueing theory.

Another Approach

- **Generating function** approach for solving the pure birth problem.

- Let $\{p_i\}$ be a discrete probability density distribution for a random variable X, i.e.,

$$P(X = i) = p_i \quad i = 0, 1, \ldots,$$

- Recall that the probability generating function is defined as

$$g(z) = \sum_{n=0}^{\infty} p_n z^n.$$

Let the probability generating function for $p_n(t)$ be

$$\boxed{g(z,t) = \sum_{n=0}^{\infty} p_n(t) z^n.}$$

The idea is that if we can find $g(z,t)$ and obtain its coefficients when expressed in a power series of z then one can solve $p_n(t)$.

From the differential-difference equations, we have

$$\boxed{\sum_{n=0}^{\infty} \frac{dP_n(t)}{dt} z^n = z \sum_{n=0}^{\infty} \lambda P_n(t) z^n - \sum_{n=0}^{\infty} \lambda P_n(t) z^n.}$$

Assuming one can inter-change the operation between the summation and the differentiation, then we have

$$\frac{dg(z,t)}{dt} = \lambda(z-1)g(z,t)$$

when z is regard as a constant.

Then we have

$$\boxed{g(z,t) = Ke^{\lambda(z-1)t}.}$$

Since

$$g(z,0) = \sum_{n=0}^{\infty} P_n(0) z^n = 1$$

we have $K = 1$. Hence we have

$$g(z,t) = e^{-\lambda t}\left(1 + \lambda z + \frac{(\lambda t z)^2}{2!} + \ldots +\right) = \left(e^{-\lambda t} + e^{-\lambda t}\lambda z + \frac{e^{-\lambda t}(\lambda t)^2}{2!} z^2 + \ldots +\right).$$

The result follows.

4.2 Pure Death Process

In this section, we consider a **pure death process** with $\mu_j = j\mu$ and initial state E_n. The equations (6) become

$$\boxed{\frac{d}{dt}P_j(t) = (j+1)\mu P_{j+1}(t) - j\mu P_j(t) \quad j = n, n-1, \cdots, 0} \tag{8}$$

where

$$P_{n+1}(t) = 0 \quad \text{and} \quad P_j(0) = \delta_{nj}.$$

• We solve these equations by recursively starting from the case $j = n$.

$$\boxed{\frac{d}{dt}P_n(t) = -n\mu P_n(t)\ , \quad P_n(0) = 1}$$

implies that

$$\boxed{P_n(t) = e^{-n\mu t}.}$$

• The equation with $j = n - 1$ is

$$\boxed{\frac{d}{dt}P_{n-1}(t) = n\mu P_n(t) - (n-1)\mu P_{n-1}(t) = n\mu e^{-n\mu t} - (n-1)\mu P_{n-1}(t).}$$

• Solving this differential equation and we get

$$\boxed{P_{n-1}(t) = n(e^{-\mu t})^{n-1}(1 - e^{-\mu t}).}$$

Recursively, we get

$$\boxed{P_j(t) = \binom{n}{j}(e^{-\mu t})^j(1 - e^{-\mu t})^{n-j} \quad (j = 0, 1, \cdots, n).} \tag{9}$$

Note:

(i) For each t, the probabilities (9) comprise a binomial distribution.

(ii) The number of equations in (8) is finite in number. For pure birth process, the number of equations is infinite.

5 More on Birth-and-Death Process

A simple queueing example is given as follows (An illustration of birth-and-death process in queueing theory context). We consider a queueing system with one server and no waiting position, with

$$P\{\text{one customer arriving during } (t, t+h)\} = \lambda h + o(h)$$

and

$$P\{\text{service ends in } (t, t+h)|\text{ server busy at } t\} = \mu h + o(h) \text{ as } h \to 0.$$

• This corresponds to a two state birth-and-death process with $j = 0, 1$. The arrival rates are $\lambda_0 = \lambda$ and $\lambda_j = 0$ for $j \neq 0$ (an arrival that occurs when the server is busy has no effect on the system since the customer leaves immediately); and the departure rates are $\mu_j = 0$ when $j \neq 1$ and $\mu_1 = \mu$ (no customers can complete service when no customers are in the system).

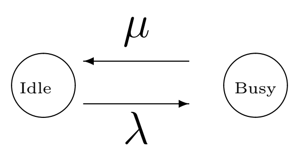


Figure 3.1: The Two-state Birth and Death Process.

The equations for the birth-and-death process are given by

$$\boxed{\frac{d}{dt}P_0(t) = -\lambda P_0(t) + \mu P_1(t) \quad \text{and} \quad \frac{d}{dt}P_1(t) = \lambda P_0(t) - \mu P_1(t).} \tag{10}$$

One convenient way of solving this set of simultaneous linear differential equations (not a standard method!) is as follows:

- Adding the equations in Eq. (10), we get

$$\frac{d}{dt}[P_0(t) + P_1(t)] = 0,$$

hence we have $P_0(t) + P_1(t) =$ constant.

- Initial conditions are $P_0(0) + P_1(0) = 1$; thus $P_0(t) + P_1(t) = 1$. Hence we get

$$\boxed{\frac{d}{dt}P_0(t) + (\lambda + \mu)P_0(t) = \mu.}$$

The solution (exercise) is given by

$$P_0(t) = \frac{\mu}{\lambda + \mu} + (P_0(0) - \frac{\mu}{\lambda + \mu})e^{-(\lambda+\mu)t} \ .$$

Since $P_1(t) = 1 - P_0(t)$,

$$P_1(t) = \frac{\lambda}{\lambda + \mu} + (P_1(0) - \frac{\lambda}{\lambda + \mu})e^{-(\lambda+\mu)t} \ . \tag{11}$$

• For the three examples (including two examples in the previous chapter) of birth-and-death processes that we have considered, the system of differential-difference equations are much simplified and can therefore be solved very easily. In general, the solution of differential-difference equations is no easy matter. Here we merely state the properties of its solution without proof.

Proposition 4 For arbitrarily prescribed coefficients $\lambda_n \geq 0$, $\mu_n \geq 0$ there always exists a positive solution $\{P_n(t)\}$ of differential-difference equations (6) such that $\sum P_n(t) \leq 1$. If the coefficients are bounded, this solution is unique and satisfies the regularity condition $\sum P_n(t) = 1$.

Remark: Fortunately in all cases of practical significance, the regularity condition $\sum P_n(t) = 1$ and uniqueness of solution are satisfied.

5.1 Statistical Equilibrium (Steady State Probability Distribution)

Consider the state probabilities of the above example when $t \to \infty$, from (11) we have

$$\begin{cases} P_0 &= \lim_{t\to\infty} P_0(t) = \dfrac{\mu}{\lambda+\mu} \\ P_1 &= \lim_{t\to\infty} P_1(t) = \dfrac{\lambda}{\lambda+\mu}. \end{cases} \tag{12}$$

We note that $P_0 + P_1 = 1$ and they are called the **steady state probabilities** of the system.

Note:

(i) P_0 and P_1 are independent of the initial values $P_0(0)$ and $P_1(0)$.

(ii) If at time $t = 0$,

$$P_0(0) = \frac{\mu}{\lambda + \mu} = P_0$$

and

$$P_1(0) = \frac{\lambda}{\lambda + \mu} = P_1,$$

(come from (11)) clearly show that these initial values will persist for ever.

• This leads us to the important notion of statistical equilibrium. We say that a system is in **statistical equilibrium** (or the state distribution is **stationary**) if its state probabilities are constant in time.

• Note that the system still fluctuate from state to state, but there is no net trend in such fluctuations.

• In the above queueing example, we have shown that the system attains statistical equilibrium as $t \to \infty$. Practically speaking, this means the system is in statistical equilibrium after sufficiently long time (so that initial conditions have no more effect on the system). For the general birth-and-death processes, the following holds.

Proposition 5 (a) Let $P_j(t)$ be the state probabilities of a birth-and-death process. Then

$$\lim_{t\to\infty} P_j(t) = P_j$$

exist and are independent of the initial conditions; they satisfy the system of linear difference equations obtained from the difference-differential equations in previous chapter by replacing the derivative on the left by zero.
(b) If all $\mu_j > 0$ and the series

$$\boxed{S = 1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0\lambda_1}{\mu_1\mu_2} + \cdots + \frac{\lambda_0\lambda_1\cdots\lambda_{j-1}}{\mu_1\mu_2\cdots\mu_j} + \ldots +} \tag{13}$$

converges, then

$$\boxed{P_0 = S^{-1} \quad \text{and} \quad P_j = \frac{\lambda_0\lambda_1\cdots\lambda_{j-1}}{\mu_1\mu_2\cdots\mu_j} S^{-1} \quad (j = 1, 2, \cdots).}$$

If the series (13) diverges, then

$$P_j = 0 \quad (j = 0, 1, \cdots) \ .$$

Proof: We shall not attempt to prove part (a) of the above proposition, but rather we assume the truth of (a) and use it to prove part (b).

By using part (a) of the proposition, we obtain the following linear difference equations.

$$\begin{aligned}&(\lambda_j+\mu_j)P_j=\lambda_{j-1}P_{j-1}+\mu_{j+1}P_{j+1}\\&(\lambda_{-1}=\mu_0=0\ ;\quad j=0,1,\cdots)\quad P_{-1}=0\ .\end{aligned}\tag{14}$$

Rearranging terms, we have

$$\lambda_j P_j-\mu_{j+1}P_{j+1}=\lambda_{j-1}P_{j-1}-\mu_j P_j\ .\tag{15}$$

If we let

$$f(j)=\lambda_j P_j-\mu_{j+1}P_{j+1},$$

then (15) is simply

$$f(j)=f(j-1)\quad \text{for } j=0,1,\cdots$$

as $f(-1)=0$. Hence

$$f(j)=0\ (j=0,1,\cdots).$$

This implies

$$\lambda_j P_j=\mu_{j+1}P_{j+1}.$$

By recurrence, we get (if $\mu_1,\cdots,\mu_j>0$)

$$\boxed{P_j=\frac{\lambda_0\lambda_1\cdots\lambda_{j-1}}{\mu_1\cdots\mu_j}P_0\quad (j=1,2,\cdots).}\tag{16}$$

Finally, by using the normalization condition $\sum P_j=1$ we have the result in part (b). □

Remarks:

(i) Part (a) of the above proposition suggests that to find the statistical equilibrium distribution

$$\boxed{\lim_{t\to\infty} P_j(t) = P_j.}$$

We set the derivatives on the left side of difference-differential equations to be zero and replace $P_j(t)$ by P_j and then solve the linear difference equations for P_j.

In most cases, the latter method is much **easier** and **shorter**.

(ii) If $\mu_j = 0$ for some $j = k$ ($\lambda_j > 0$ for all j), then, as equation (16) shows,

$$P_j = 0 \quad \text{for } j = 0, 1, \cdots, k-1.$$

In particular, for pure birth process, $P_j = 0$ for all j.

Example 1: Suppose that for all i we have

$$\lambda_i = \lambda \quad \text{and} \quad \mu_j = j\mu$$

then

$$S = 1 + \frac{\lambda}{\mu} + \frac{1}{2!}\left(\frac{\lambda}{\mu}\right)^2 + \frac{1}{3!}\left(\frac{\lambda}{\mu}\right)^3 + \ldots + = e^{\frac{\lambda}{\mu}}.$$

Therefore we have the Poisson distribution

$$P_j = \frac{1}{j!}\left(\frac{\lambda}{\mu}\right)^j e^{\frac{-\lambda}{\mu}}.$$

Example 2: Suppose that for all i we have

$$\lambda_i = \lambda \quad \text{and} \quad \mu_j = \mu$$

such that $\lambda < \mu$ then

$$S = 1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3 + \ldots + = \frac{1}{1 - \frac{\lambda}{\mu}}.$$

Therefore we have the Geometric distribution

$$P_j = \left(\frac{\lambda}{\mu}\right)^j \left(1 - \frac{\lambda}{\mu}\right).$$

A Summary on Stochastic Process and Probability Theory

• The definition of a birth-and-death process.
- the meaning of $o(h)$: $\lim_{h\to 0} o(h)/h = 0$.
- the general solutions for the pure birth process, the pure death process and the two-state birth-and-death process.
- what is the meaning of the steady state solution of a birth-and-death process? Under what condition do we have the steady state probability of a birth-and-death process ? How to get this steady state probability ?

• The Poisson distribution and the Exponential distribution: the functional form, the mean and the variance.
- Proof for the sum of two Poisson r.v. is again a Poisson r.v.
- Proof for the Markov property of the Exponential distribution.

• The definition for a Poisson process.
- the relationship between the Poisson and the Exponential distribution in a Poisson Process.
- if the arrival process of customers is a Poisson process with mean λ then (i) the inter-arrival time of the customers is independent and follows the exponential distribution $\lambda e^{-\lambda t}$ and (ii) the number of arrived customers in the time interval $[a, a+t]$ follows the Poisson distribution $(\lambda t)^k/k!e^{-\lambda t}$.