On Some Adaptive Online Portfolio Selection Problems

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Abstract

Abstract In online portfolio selection problems, transaction costs incurred by changes of investment proportions on risky assets have a significant impact on the investment strategy and the return in long-term investment horizon. However, in many online portfolio selection studies, transaction costs are usually neglected in the decision making process. Here we consider an adaptive online portfolio selection problem with transaction costs. We first propose an adaptive online moving average method (AOLMA) to predict the future returns of risky assets by incorporating an adaptive decaying factor into the moving average method, which improves the accuracy of return prediction. The net profit maximization model (NPM) is then constructed where transaction costs are considered in each decision making process. The adaptive online net profit maximization algorithm (AOLNPM) is designed to maximize the cumulative return by integrating AOLMA and NPM together. Numerical experiments show that AOLNPM dominates several state-of-the-art online portfolio selection algorithms in terms of various performance metrics, i.e., cumulative return, mean excess return, Sharpe ratio, Information ratio and Calmar ratio. We then extend our study to the case of constant cash inflow. A novel method to deal with transaction costs and simultaneously calculating the transaction remainder factor and portfolio vector for each period was also proposed.
Keywords: Online portfolio selection; Adaptive moving average method; Transaction cost; Linear programming.

Outline

- Introduction to Online Portfolio Selection (OLPS).
- Problem Formulation (I).
- Online Moving Average Method.
- Adaptive Online Moving Average Method.
- Net Profit Maximization Model with Transaction Costs.
- Numerical Experiments (I).
- OLPS with Constant Cash Inflows.
- Problem Formulation (II).
- Numerical Experiments (II).
- Conclusions.
- References.

Introduction to Online Portfolio Selection (OLPS)

• Online portfolio selection attracts both researchers and practitioners. It is different from the traditional portfolio selection theory proposed by Markowitz (1952) in his seminal work.

• In a **traditional portfolio selection problem**, it is usually assumed that the return of a risky asset is subject to a certain **distribution function**.

-Based on the distribution function, the **expected value** and **variance** of the return can be calculated to measure the **expected return** and **risk**, respectively.

-Then investors allocate the capital in different assets to achieve excess **investment return** or avert the **investment risk**.

Introduction to Online Portfolio Selection (OLPS)

• In contrast, **online portfolio selection** concerns more on employing modern techniques to **predict the future returns of risky assets** and **making the optimal investment strategy**.

• Online portfolio selection focuses on exploring the most **efficient** and **practical computational intelligence techniques** to deal with real online asset trading problems.

• It is a **sequential decision making optimization problem** where the investment strategy is determined at the beginning of each short period.

Online portfolio strategies can be classified into **five types**.

(I) The first one is called "**Benchmark**".

• One widely adopted Benchmark is the **Uniform Buy-and-Hold strategy**, which is also called the **Market strategy**, Li and Hoi (2014). In this strategy, the available capital is uniformly distributed into all the risky assets in each period.

• Another Benchmark is called the **Best stock strategy**, Li and Hoi (2014), where all the capital is invested into the best asset in the whole investment process.

• Constant Rebalanced Portfolios (CRP) strategy is a popular Benchmark where the allocation proportions of the risky assets are the same in all periods. There are two special CRPs: **Uniform Constant Rebalanced Portfolios** (UCRP) Li and Hoi (2015) and **Best Constant Rebalanced Portfolios** (BCRP) Cover (1991).

(II) The second type of methods focuses on the "**Follow the Winner**" strategy. They are based on the **momentum principle** which assumes that the risky assets performing well currently will continue achieving good performance in the next period.

• Cover (1991) proposed the concept of **Universal Portfolio (UP) strategy**, which first distributed the capital to **several base experts** and derived the corresponding returns, then obtain the performance weighted strategy.

• Helmbold et al. (1998) proposed the **Exponential Gradient (EG) method** in which **exponentiated gradient update** was employed to calculate the investment proportions based on the past return data.

• Agarwal et al. (2006) employed the **Online Newton Step (ONS) method** to tackle online portfolio selection, where the **gradient** and **Hessian matrix** of the **log function of cumulative return** are computed.

(III) There are some "**Follow the Loser**" approaches built on the **mean reversion principle**, which claims that the risky assets performing well in the past may return to normal or perform poorly in the next period. Therefore, it is encouraged to **buy the current under-performing risky assets** and **sell the over-performing assets**.

• Borodin et al. (2004) proposed the **Anti-correlation (Anticor) method** based on the **mean reversion principle**, where the proportions were transferred from the assets performing well to assets performing poorly, and the explicit amounts of transferred proportions were determined by the **cross-correlation matrix** of different risky assets.

• Li et al. (2012) proposed the **Passive-aggressive Mean Reversion (PAMR) method** based on a loss function. Current portfolio will be kept if its return is **below a certain return threshold** under the assumption that under-performing risky assets will perform better in the next period.

• Similar to PAMR, Li et al. (2011, 2013) proposed the **Confidence Weighted Mean Reversion (CWMR) method** by modeling the portfolio vector with Gaussian distribution and update the distribution constantly following the **mean reversion principle**.

• Huang et al. (2016) proposed the **Robust Median Reversion (RMR) strategy** where the robust L_1-median estimator was adopted to exploit the reversion phenomenon. The RMR runs in linear time which is easy to implement in real algorithmic trading.

• The above PAMR and CWMR employed the **single-period mean reversion assumption** where the price of asset in the next period was estimated with **the price of last period**, which may not achieve good performance.

• To overcome this, Li et al. (2012, 2015) employed the **Moving Average method** to predict the price of next period based on multiple prices of previous periods and proposed the **Online Moving Average Reversion (OLMAR) method**.

• In this talk, we shall extend the **OLMAR method** for predicting prices/returns of risky assets.

(IV) The fourth type of online portfolio selection strategies focuses on "**Pattern Matching Based Approaches**".

There are usually two steps in pattern matching based approaches.

• The **first step** is **sample selection** intended for selecting the historical price patterns which are similar to the latest price pattern. The selected historical price patterns are used to **estimate the return vector** of the whole portfolio in the next period.

• The **second step** is to **construct the portfolio optimization model** based on the selected price patterns.

Introduction to Online Portfolio Selection (OLPS)

- Györfi et al. (2006) employed the **nonparametric kernel-based sample selection method** to search for similar price patterns by comparing the **Euclidean distance of different patterns**, and constructed a log-optimal portfolio based on the capital growth theory.

- Li et al. (2011) employed the correlation-driven nonparametric sample selection method by using the correlation coefficient of different patterns, and proposed the **Correlation-driven Non-parametric (CORN) learning algorithm**.

(V) The fifth type of online portfolio selection strategies is the "**Meta-learning Approach**". In this approach, multiple base experts are defined where each expert is equipped with different strategies and outputs one portfolio. Then all the output portfolios are combined together into a final portfolio.

Problem Formulation (I)

- In online portfolio selection, an investor makes sequential decisions according to the changing financial market.
- Denote the investment strategy in Period t by $\mathbf{x}_t = (x_{t1}, x_{t2}, \ldots, x_{tm})$, where x_{ti} **is the proportion allocated to risky asset** i, $(t = 1, 2, \ldots, n,\ i = 1, 2, \ldots, m)$.
- Let the **return in Period** t be $\mathbf{r}_t = (r_{t1}, r_{t2}, \ldots, r_{tm})$. We note that $\mathbf{x}_t$ **should be determined at the beginning of Period** t and $\mathbf{r}_t$ **is known at the end of Period** t (See Figure 1).

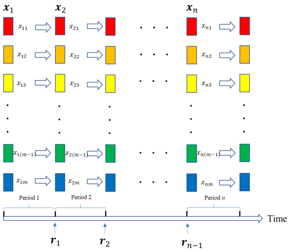


Figure 1: The investment process.

Problem Formulation (I)

• The return vector $\mathbf{r}_t = (r_{t1}, r_{t2}, \ldots, r_{tm})$ is calculated as follows:

$$\boxed{\mathbf{r}_t = \mathbf{p}_t / \mathbf{p}_{t-1}}$$

where $\mathbf{p}_t = (p_{t1}, p_{t2}, \ldots, p_{tm})$ is the price at period t and "$/$" is an element-wise division of two vectors.

• The **Cumulative Return** from the beginning of the investment to **Period** n can be expressed as follows:

$$\boxed{CR_n = \prod_{t=1}^{n} \mathbf{r}_t \mathbf{x}_t^T.} \quad (1)$$

• We note that **transaction cost** is **NOT** considered in Eq. (1). Recall that the decision variables satisfy the following constraints:

$$\boxed{x_{t1} + x_{t2} + \ldots + x_{tm} = 1, \quad t = 1, 2, \ldots, n,} \quad (2)$$

where $0 \leq x_{ti} \leq 1$, $i = 1, 2, \ldots, m$.

Online Moving Average Method

- Li and Hoi (2012, 2015) proposed **two moving average methods** to predict the single-period return.
- The **first one** is a **Simple Moving Average (SMA) method**: Given the historical stock prices $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_t$ and the truncated **window size** w, the predicted stock price of $\mathbf{p}_{t+1}$ can be calculated as follows:

$$\boxed{\hat{\mathbf{p}}_{t+1} = \frac{1}{w} \sum_{i=t-w+1}^{t} \mathbf{p}_i .}$$

- The estimated return for $\mathbf{r}_{t+1}$ can be obtained by

$$\boxed{\hat{\mathbf{r}}_{t+1} = \frac{\hat{\mathbf{p}}_{t+1}}{\mathbf{p}_t} = \frac{1}{w}\left(\mathbf{1} + \frac{\mathbf{1}}{\mathbf{r}_t} + \frac{\mathbf{1}}{\mathbf{r}_t \cdot \mathbf{r}_{t-1}} + \cdots + \frac{\mathbf{1}}{\prod_{i=0}^{w-2} \mathbf{r}_{t-i}}\right) .}$$

Here **"1" is the vector of all ones** and the product "·" refers to **the element-wise product of vectors**.

Online Moving Average Method

- The **second one** is the **Exponential Moving Average (EMA) method** which uses all the historical stock prices by assigning each stock price an **exponential weight**.
- The predicted stock price can be calculated as follows:

$$\boxed{\hat{\mathbf{p}}_{t+1} = \alpha \mathbf{p}_t + (1-\alpha)\hat{\mathbf{p}}_t = \alpha \mathbf{p}_t + \alpha(1-\alpha)\mathbf{p}_{t-1} + \cdots + (1-\alpha)^{t-1}\mathbf{p}_1}$$

and the estimated return is

$$\boxed{\hat{\mathbf{r}}_{t+1} = \alpha \mathbf{1} + (1-\alpha)\frac{\hat{\mathbf{r}}_t}{\mathbf{r}_t}}$$

where α is the **decaying factor**.

Adaptive Online Moving Average Method

- We propose the **Adaptive Online Moving Average (AOLMA) method** where the decaying factor can be **adjusted automatically** according to the performances of risky assets.
- Define the **decaying vector** of the whole portfolio at Period t by $\boldsymbol{\alpha}_t = (\alpha_{t1}, \alpha_{t2}, \ldots, \alpha_{tm})$, where α_{ti} is the decaying factor of risky asset i $(i = 1, 2, \ldots, m)$.
- Then the **predicted price at Period $(t+1)$** can be expressed as follows:

$$\boxed{\hat{\mathbf{p}}_{t+1} = \boldsymbol{\alpha}_{t+1} \cdot \mathbf{p}_t + (\mathbf{1} - \boldsymbol{\alpha}_{t+1}) \cdot \hat{\mathbf{p}}_t} \tag{3}$$

and the **predicted return** is

$$\boxed{\hat{\mathbf{r}}_{t+1} = \boldsymbol{\alpha}_{t+1} \cdot \mathbf{1} + (\mathbf{1} - \boldsymbol{\alpha}_{t+1}) \cdot \frac{\hat{\mathbf{r}}_t}{\mathbf{r}_t}.} \tag{4}$$

Adaptive Online Moving Average Method

- The key of the adaptive moving average method lies in the **decaying factor** α_t.

- Consider risky asset i at Period t: the predicted return following from Eq. (4) is

$$\boxed{\hat{r}_{ti} = \alpha_{ti} + (1 - \alpha_{ti})\frac{\hat{r}_{(t-1)i}}{r_{(t-1)i}}}$$

and the **corresponding error** is

$$\boxed{r_{ti} - \hat{r}_{ti} = r_{ti} - \frac{\hat{r}_{(t-1)i}}{r_{(t-1)i}} - \left(1 - \frac{\hat{r}_{(t-1)i}}{r_{(t-1)i}}\right)\alpha_{ti}.} \tag{5}$$

Adaptive Online Moving Average Method

- Our aim is to improve (reduce) the **prediction error** in the online portfolio selection process. Once r_{ti} is known, the error can be obtained and can be applied to determine the decaying factor for the next period.
- There are **four cases** that one needs to consider:

Case 1: $r_{ti} > \hat{r}_{ti}$ and $r_{(t-1)i} > \hat{r}_{(t-1)i}$.

Case 2: $r_{ti} > \hat{r}_{ti}$ and $r_{(t-1)i} \leq \hat{r}_{(t-1)i}$.

Case 3: $r_{ti} \leq \hat{r}_{ti}$ and $r_{(t-1)i} > \hat{r}_{(t-1)i}$.

Case 4: $r_{ti} \leq \hat{r}_{ti}$ and $r_{(t-1)i} \leq \hat{r}_{(t-1)i}$.

Adaptive Online Moving Average Method

- For **Case 1**, it can be derived from Eq. (5) that the coefficient of α_{ti} is $-\left(1-\frac{\hat{r}_{(t-1)i}}{r_{(t-1)i}}\right) < 0$.
- Then the decaying factor for the next period $\alpha_{(t+1)i}$ can be **increased** by one step size to reduce the prediction error:

$$\alpha_{(t+1)i} = \alpha_{ti} + \tau \tag{6}$$

where τ is the given **step size** of the decaying factor.

- For **Case 2**, the coefficient is $-\left(1-\frac{\hat{r}_{(t-1)i}}{r_{(t-1)i}}\right) \geq 0$, then the decaying factor $\alpha_{(t+1)i}$ can be **decreased** to reduce the prediction error as follows:

$$\alpha_{(t+1)i} = \alpha_{ti} - \tau. \tag{7}$$

Adaptive Online Moving Average Method

- Similarly, for **Case 3** and **Case 4**, the decaying factor can be updated by Eqs. (6) and (7), respectively.

Remark

-It is reasonable to employ the above decaying factor updating mechanism in online portfolio selection.

-For example, in **Case 1**, both $\hat{r}_{(t-1)i}$ and $\hat{r}_{ti}$ are underestimated. Then in the next period, it is necessary to increase the value of $\hat{r}_{(t+1)i}$ by using a larger $\alpha_{(t+1)i}$ following from Eq. (5) (As $\frac{\hat{r}_{ti}}{r_{ti}} < 1$).

-The **initial value** of the decaying factor is set to be $\alpha_{1i} = 0.5$. If the iterated decaying factor α_{ti} is **outside the interval** $[0, 1]$, then it is **reset to** 0.5.

Adaptive Online Moving Average Method

Example

-To verify the effectiveness of our proposed AOLMA method, we employ the classical benchmark data set **MSCI which contains the historical daily returns of 24 stocks from April 1, 2006 to March 31, 2010**, Li and Ho (2015).

-For each stock i, its **prediction relative error** at the j-th trading day is given by

$$Er(j) = \frac{|\hat{r}_{ji} - r_{ji}|}{r_{ji}} \times 100\%$$

and the **average relative error** is

$$\bar{Er} = \frac{1}{n} \sum_{j=1}^{n} \frac{|\hat{r}_{ji} - r_{ji}|}{r_{ji}} \times 100\%.$$

Example

-We apply **SMA ($w = 5$), EMA ($\alpha = 0.5$), AOLMA ($\tau = 0.0006$)** to estimate the daily returns and make comparisons with the real returns. The average relative errors are shown in Table 1.
-It is clear that AOLMA achieves the **lowest relative error** in each stock, meaning that AOLMA performs better than both SMA and EMA.

Table 1: Average relative errors of SMA, EMA and AOLMA.

Stock	SMA(%)	EMA(%)	AOLMA(%)	Stock	SMA(%)	EMA(%)	AOLMA(%)
1	2.06	1.16	1.14	13	1.88	1.06	1.04
2	3.08	1.75	1.69	14	3.69	2.07	2.05
3	2.57	1.44	1.42	15	2.53	1.43	1.39
4	2.11	1.19	1.16	16	3.48	1.96	1.92
5	3.39	1.90	1.87	17	2.72	1.53	1.48
6	2.80	1.58	1.53	18	2.68	1.51	1.48
7	2.62	1.48	1.43	19	3.18	1.79	1.77
8	2.26	1.28	1.25	20	2.72	1.53	1.48
9	4.00	2.25	2.21	21	2.87	1.62	1.57
10	2.62	1.48	1.46	22	2.83	1.59	1.56
11	2.60	1.47	1.45	23	3.52	1.98	1.93
12	2.72	1.53	1.50	24	2.30	1.29	1.29

Adaptive Online Moving Average Method

Example

-To test the **robustness** of AOLMA, we conduct multiple experiments with **step size τ ranging from 0.0001 to 0.0010**. The final average relative errors are shown in Fig. 2.
-For all the stocks, the maximum difference of average relative errors with different τ **does not exceed** 0.09%.
-To show the advantages of AOLMA, for all step sizes τ, we select **the worst case** of average relative error for each stock, and compare it with SMA and EMA (See Fig. 3).

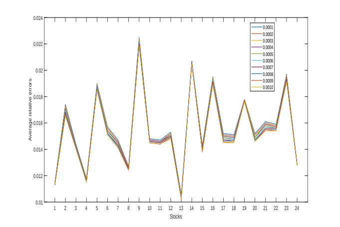


Figure 2: Average relative errors for different τ

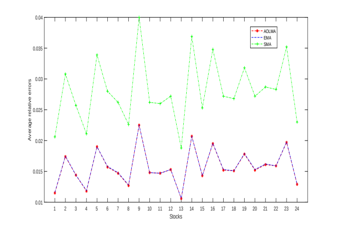


Figure 3: A comparison with SMA and EMA

Net Profit Maximization Model with Transaction Costs

- We propose the **Net Profit Maximization Model with Transaction Costs**. It is worth noting that several general assumptions are made in the model.

– Firstly, we assume **proportional transaction costs** on risky assets purchases and sales.

– Secondly, we assume that each asset share is **arbitrarily divisible**, and that any required quantities of shares, even fractional, can be bought and sold at the last closing price in any trading period.

– Thirdly, we assume that market behavior and stock prices are not affected by any trading strategy / market impact.

– Fourthly, no additional capital is introduced or withdrawn in the investment period.

Net Profit Maximization Model with Transaction Costs

- The **net profit maximization model (NPM)** considering **transaction cost** in each trading period:

$$\begin{cases} \max & \sum_{i=1}^{m} \hat{r}_{ti} x_{ti} - \gamma \sum_{i=1}^{m} |x_{ti} - \tilde{x}_{(t-1)i}|. \\ \text{s.t.} & x_{t1} + x_{t2} + \ldots + x_{tm} = 1, \\ & 0 \le x_{ti} \le 1, i = 1, 2, \ldots, m. \end{cases} \quad (8)$$

- Here γ is the **unit transaction cost rate** for buying/selling assets and $\tilde{\mathbf{x}}_{t-1}$ is the actually investment strategy in period $(t-1)$.
- The model can be transformed into the following **LP problem**:

$$\begin{cases} \max & \sum_{i=1}^{m} \hat{r}_{ti} \tilde{x}_{(t-1)i} + \sum_{i=1}^{m} (\tilde{x}_{(t-1)i} - \gamma) u_{ti} - \sum_{i=1}^{m} (\tilde{x}_{(t-1)i} + \gamma) v_{ti}. \\ \text{s.t.} & \sum_{i=1}^{m} (u_{ti} - v_{ti}) = 0, \\ & 0 \le \tilde{x}_{(t-1)i} + u_{ti} - v_{ti} \le 1, i = 1, 2, \ldots, m, \\ & u_{ti} \ge 0, v_{ti} \ge 0, i = 1, 2, \ldots, m. \end{cases} \quad (9)$$

Net Profit Maximization Model with Transaction Costs

- Based on the above NPM model, we can also incorporate the **risk constraint** into online portfolio selection decision making. Similar to the estimation of the return $\mathbf{r}_t$, we use the **absolute deviation**, Konno and Yamazaki (1991), of the most recent w prices to measure the risk of the whole portfolio at period t:

$$\frac{1}{w}\sum_{j=1}^{w}\left|\sum_{i=1}^{m} x_{ti}(r_{(t-j)i} - \hat{r}_{ti})\right|.$$

- Then the NPM model with risk constraint can be expressed as

$$\begin{cases} \max & \sum_{i=1}^{m} \hat{r}_{ti} x_{ti} - \gamma \sum_{i=1}^{m} |x_{ti} - \tilde{x}_{(t-1)i}| \\ \text{s.t.} & \frac{1}{w}\sum_{j=1}^{w}\left|\sum_{i=1}^{m} x_{ti}(r_{(t-j)i} - \hat{r}_{ti})\right| \le \theta \\ & x_{t1} + \ldots + x_{tm} = 1,\ 0 \le x_{ti} \le 1, i = 1, \ldots, m, \end{cases} \quad (10)$$

where θ **is the acceptable risk level of the investor.**

Net Profit Maximization Model with Transaction Costs

- To solve Model (10), we transform the first nonlinear constraint into a linear one. For each $j = 1, 2, \ldots, w$,

$$\left| \sum_{i=1}^{m} x_{ti}(r_{(t-j)i} - \hat{r}_{ti}) \right|$$

can be expressed as

$$\max\left\{ \sum_{i=1}^{m} x_{ti}(r_{(t-j)i} - \hat{r}_{ti}), 0 \right\} + \max\left\{ \sum_{i=1}^{m} x_{ti}(\hat{r}_{ti} - r_{(t-j)i}), 0 \right\}.$$

Set

$$\max\left\{ \sum_{i=1}^{m} x_{ti}(r_{(t-j)i} - \hat{r}_{ti}), 0 \right\} = d_j$$

and

$$\max\left\{ \sum_{i=1}^{m} x_{ti}(\hat{r}_{ti} - r_{(t-j)i}), 0 \right\} = g_j.$$

- Then Model (10) can be transformed into

$$\begin{cases} \max & \sum_{i=1}^{m} \hat{r}_{ti} x_{ti} - \gamma \sum_{i=1}^{m} |x_{ti} - \tilde{x}_{(t-1)i}|. \\ \text{s.t.} & \frac{1}{w} \sum_{j=1}^{w} (d_j + g_j) \leq \theta, \\ & \sum_{i=1}^{m} x_{ti} (r_{(t-j)i} - \hat{r}_{ti}) \leq d_j, \\ & \sum_{i=1}^{m} x_{ti} (\hat{r}_{ti} - r_{(t-j)i}) \leq g_j, \\ & x_{t1} + x_{t2} + \ldots + x_{tm} = 1, \\ & 0 \leq x_{ti} \leq 1, i = 1, 2, \ldots, m, \\ & d_j \geq 0, g_j \geq 0, j = 1, 2, \ldots, w. \end{cases} \tag{11}$$

Net Profit Maximization Model with Transaction Costs

- Then Model (11) can be transformed into

$$\begin{cases} \max & \sum_{i=1}^{m} \hat{r}_{ti}\tilde{x}_{(t-1)i} + \sum_{i=1}^{m} \left(\tilde{x}_{(t-1)i} - \gamma\right) u_{ti} - \sum_{i=1}^{m} \left(\tilde{x}_{(t-1)i} + \gamma\right) v_{ti} \\ \text{s.t.} & \sum_{j=1}^{w}(d_j + g_j) \leq w\theta, \\ & \sum_{i=1}^{m}(r_{(t-j)i} - \hat{r}_{ti})u_{ti} - \sum_{i=1}^{m}(r_{(t-j)i} - \hat{r}_{ti})v_{ti} - d_j \leq \sum_{i=1}^{m}(\hat{r}_{ti} - r_{(t-j)i})\tilde{x}_{(t-1)i}, \\ & \sum_{i=1}^{m}(\hat{r}_{ti} - r_{(t-j)i})u_{ti} - \sum_{i=1}^{m}(\hat{r}_{ti} - r_{(t-j)i})v_{ti} - g_j \leq \sum_{i=1}^{m}(r_{(t-j)i} - \hat{r}_{ti})\tilde{x}_{(t-1)i}, \\ & \sum_{i=1}^{m}(u_{ti} - v_{ti}) = 0, \\ & 0 \leq \tilde{x}_{(t-1)i} + u_{ti} - v_{ti} \leq 1, \\ & u_{ti} \geq 0, v_{ti} \geq 0, i = 1, 2, \ldots, m, \\ & d_j \geq 0, g_j \geq 0, j = 1, 2, \ldots, w. \end{cases} \tag{12}$$

It is a **LP problem** of $(2m+2w)$ variables, including u_{ti}, v_{ti}, d_j and g_j, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, w$.

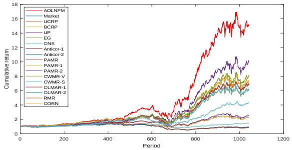
- By integrating the AOLMA and NPM together, we have the **Adaptive OnLine Net Profit Maximization (AOLNPM) Algorithm**.

Numerical Experiments (I)

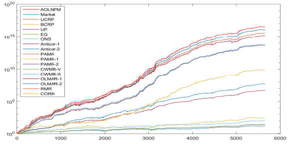
- **MSCI**, **NYSE-O**, **NYSE-N** and **TSE** are employed as benchmark data sets for testing the performances of different online portfolio selection algorithms.
- **MSCI** contains **24 stocks** which has been employed for verifying the effectiveness of AOLMA method.
- **NYSE-O** and **NYSE-N** contain historical return data of stocks selected from **American stock market**, where **NYSE-O** contains the data of **36 stocks ranging from June 3, 1962 to December 31, 1984**, and **NYSE-N** contains the data of **23 stocks ranging from January 1, 1985 to June 30, 2010**.
- **TSE** contains **88 stocks** selected from **Canadian stock market** ranging from January 4, 1994 to December 31, 1998.
- The total numbers of the trading days for MSCI, NYSE-O, NYSE-N and TSE are **1043**, **5651**, **6431** and **1259**, respectively.

Numerical Experiments (I)

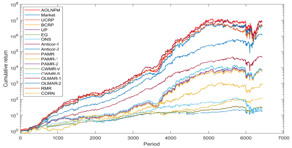
- Numerical results for demonstrating the effectiveness of AOL-NPM algorithm over other algorithms on benchmark data sets: MSCI (Li and Ho (2015)), NYSE-O (Konno and Yamazaki (1991)), NYSE-N (Cover (1991)) and TSE (Borodin et al. (2004)).



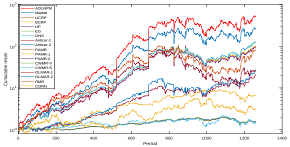
MSCI



NYSE-O



NYSE-N



TSE

Figure 4: **Cumulative returns** on different data sets.

Numerical Experiments (I)

Table 2: **Mean excess returns** on MSCI, NYSE-O, NYSE-N and TSE.

Method	MSCI	NYSE-O	NYSE-N	TSE
AOLNPM	$\mathbf{2.8819 \times 10^{-3}}$	$\mathbf{6.8728 \times 10^{-3}}$	$\mathbf{2.5673 \times 10^{-3}}$	$\mathbf{6.2786 \times 10^{-3}}$
UCRP	2.2166×10^{-5}	9.6691×10^{-5}	9.0060×10^{-5}	-1.6165×10^{-5}
BCRP	4.4008×10^{-4}	6.3441×10^{-4}	3.5015×10^{-4}	1.4139×10^{-3}
UP	1.0274×10^{-5}	8.7757×10^{-5}	8.3683×10^{-5}	-3.0125×10^{-5}
EG	2.1204×10^{-5}	9.7442×10^{-5}	8.6917×10^{-5}	-1.6750×10^{-5}
ONS	-2.3200×10^{-5}	3.5072×10^{-4}	2.9386×10^{-4}	1.3161×10^{-4}
Anticor-1	9.5500×10^{-4}	2.3782×10^{-3}	1.3924×10^{-3}	2.4246×10^{-3}
Anticor-2	1.0876×10^{-3}	2.8591×10^{-3}	2.0296×10^{-3}	2.9419×10^{-3}
PAMR	1.9613×10^{-3}	5.5542×10^{-3}	1.2918×10^{-3}	4.2222×10^{-3}
PAMR-1	1.9799×10^{-3}	5.5542×10^{-3}	1.2921×10^{-3}	4.2222×10^{-3}
PAMR-2	2.0692×10^{-3}	5.5443×10^{-3}	1.3098×10^{-3}	4.1742×10^{-3}
CWMR-V	2.0837×10^{-3}	5.6009×10^{-3}	1.3325×10^{-3}	4.4938×10^{-3}
CWMR-S	2.0845×10^{-3}	5.6001×10^{-3}	1.3328×10^{-3}	4.5034×10^{-3}
OLMAR-1	2.2026×10^{-3}	6.3133×10^{-3}	$\mathbf{2.5806 \times 10^{-3}}$	3.8425×10^{-3}
OLMAR-2	2.5019×10^{-3}	6.6669×10^{-3}	2.4707×10^{-3}	5.7325×10^{-3}
RMR	2.2857×10^{-3}	6.4443×10^{-3}	2.4826×10^{-3}	4.6607×10^{-3}
CORN	1.5714×10^{-3}	4.0408×10^{-3}	7.6987×10^{-4}	1.1347×10^{-3}

$$MER = \frac{1}{n}\sum_{t=1}^{n}(R_t - R_t^{\star}) = \bar{R} - \bar{R}^{\star}.$$

$R_t^{\star}$ is the return of the portfolio in period t by using Market strategy, and R_t is the return of the portfolio in period t.

Table 3: **Sharpe ratios** on MSCI, NYSE-O, NYSE-N and TSE.

Method	MSCI	NYSE-O	NYSE-N	TSE
AOLNPM	**0.1160**	**0.2047**	**0.0872**	**0.1111**
Market	0.0017	0.0552	0.0457	0.0505
UCRP	0.0031	0.0725	0.0501	0.0485
BCRP	0.0381	0.0597	0.0546	0.0725
UP	0.0023	0.0715	0.0496	0.0467
EG	0.0030	0.0722	0.0501	0.0485
ONS	0.0002	0.0767	0.0305	0.0264
Anticor-1	0.0513	0.1583	0.0862	0.0982
Anticor-2	0.0538	0.1502	**0.0929**	0.0882
PAMR	0.0866	0.1886	0.0589	0.1016
PAMR-1	0.0874	0.1886	0.0589	0.1016
PAMR-2	0.0922	0.1901	0.6000	0.1008
CWMR-V	0.0920	0.1907	0.0594	0.1020
CWMR-S	0.0921	0.1907	0.0591	0.1023
OLMAR-1	0.0897	0.1913	0.0863	0.0714
OLMAR-2	0.1003	0.2014	0.0840	0.1027
RMR	0.0939	0.1960	0.0840	0.0873
CORN	0.0821	0.1383	0.0573	0.0428

$$SR = \frac{1}{\sigma}(\bar{R} - r_f).$$

Here r_f is the risk-free return in financial market, $\bar{R}$ is the average return of the portfolio and σ is the corresponding standard deviation of daily returns.

Table 4: **Information ratios** on MSCI, NYSE-O, NYSE-N and TSE.

Method	MSCI	NYSE-O	NYSE-N	TSE
AOLNPM	**0.1643**	**0.2016**	**0.0778**	**0.1063**
UCRP	0.0277	0.0337	0.0238	−0.0075
BCRP	0.0359	0.0386	0.0280	0.0617
UP	0.0128	0.0306	0.0221	−0.0139
EG	0.0281	0.0345	0.0242	−0.0082
ONS	−0.0027	0.0394	0.0121	0.0069
Anticor-1	0.1235	0.1576	0.0765	0.0903
Anticor-2	0.1057	0.1447	**0.0837**	0.0802
PAMR	0.1291	0.1839	0.0462	0.0956
PAMR-1	0.1305	0.1839	0.0462	0.0956
PAMR-2	0.1400	0.1856	0.0473	0.0948
CWMR-V	0.1375	0.1863	0.0469	0.0963
CWMR-S	0.1375	0.1863	0.0466	0.0965
OLMAR-1	0.1297	0.1870	0.0771	0.0659
OLMAR-2	0.1466	0.1982	0.0745	0.0976
RMR	0.1373	0.1918	0.0746	0.0820
CORN	0.1161	0.1302	0.0399	0.0331

$$IR = (\bar{R} - \bar{R}^\star)/\sigma(R - R^\star).$$

Here $\sigma(R - R^\star)$ is the standard deviation of the excess return over Market strategy.

Numerical Experiments (I)

Table 5: **Calmar ratios** on MSCI, NYSE-O, NYSE-N and TSE.

Method	MSCI	NYSE-O	NYSE-N	TSE
AOLNPM	**0.1802**	**0.4126**	**0.1438**	**0.1937**
Market	0.0023	0.0835	0.0637	0.0675
UCRP	0.0042	0.1113	0.0704	0.0650
BCRP	0.0520	0.0941	0.0804	0.1199
UP	0.0032	0.1096	0.0697	0.0626
EG	0.0041	0.1106	0.0704	0.0649
ONS	0.0002	0.1252	0.0457	0.0406
Anticor-1	0.0751	0.2862	0.1368	0.1635
Anticor-2	0.0797	0.2726	**0.1541**	0.1452
PAMR	0.1281	0.3798	0.0946	0.1828
PAMR-1	0.1294	0.3798	0.0946	0.1828
PAMR-2	0.1370	0.3842	0.0965	0.1814
CWMR-V	0.1377	0.3853	0.0960	0.1905
CWMR-S	0.1378	0.3853	0.0959	0.1910
OLMAR-1	0.1365	0.3737	0.1420	0.1233
OLMAR-2	0.1549	0.4001	0.1380	0.1788
RMR	0.1430	0.3907	0.1389	0.1539
CORN	0.1289	0.2607	0.0916	0.0696

$$CR = \bar{R}_{net}/MDD, \quad MDD = \sqrt{\frac{1}{n}\sum_{t=1}^{n}\min\{R_t - 1, 0\}^2}.$$

Here $\bar{R}_{net}$ is the average daily net profit return rate, and MDD (the maximum drawdown of return) only covers the return which is less than 1.

Numerical Experiments (I)

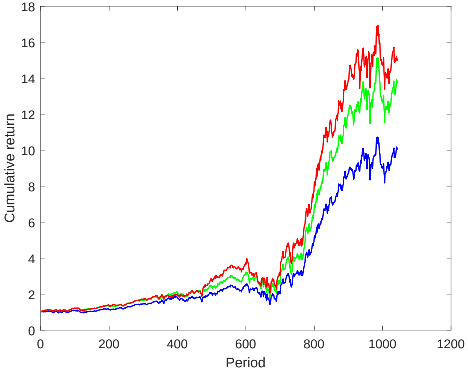
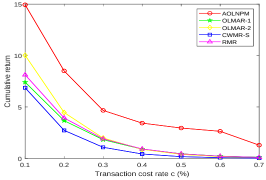
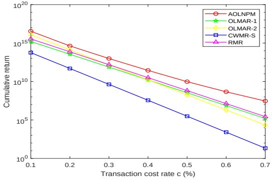


Figure 5: Impact of AOLMA and NPM: AOLNPM, NPM with EMA and OLMAR

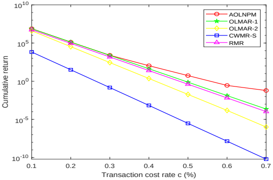
The **blue curve** is the **cumulative return** derived by the **OLMAR method**. The **green curve** refers to the return by using **EMA and our NPM model**. The **red curve** is obtained by using **AOLMA and NPM (AOLNPM) simultaneously**.



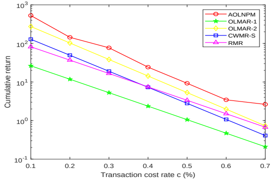
MSCI



NYSE-O



NYSE-N



TSE

Figure 6: Cumulative returns with different transaction cost rates.

To study the relationship between the transaction cost rate γ and the cumulative return, we set different rates ranging from 0.1% to 0.7%. It is found that AOLNPM dominates other algorithms with high or low transaction cost rate.

Numerical Experiments (I)

To study the **impact of the risk constraint** on online portfolio selection, we solve Model (12) where the risk levels θ are set as constants ranging from 0.070 to 0.100. For our NPM model without considering risk constraint, the corresponding value of θ is set as infinity. Table 6 presents the cumulative returns of our NPM model in all the four data sets.

Table 6: Cumulative returns under different risk levels.

θ	MSCI	NYSE-O	NYSE-N	TSE
0.070	8.4688	1.0144×10^{15}	2.5632×10^{6}	49.7796
0.075	9.1502	1.6913×10^{15}	3.3654×10^{6}	59.9744
0.080	10.0596	2.9023×10^{15}	4.0816×10^{6}	68.4651
0.085	11.1557	4.4478×10^{15}	4.6710×10^{6}	74.7941
0.090	11.9467	6.3118×10^{15}	5.2353×10^{6}	82.0197
0.095	12.5697	8.9063×10^{15}	5.9611×10^{6}	93.4080
0.100	13.3096	1.1688×10^{16}	6.3234×10^{6}	106.2034
Infinity	**14.9357**	$\mathbf{3.4535 \times 10^{16}}$	$\mathbf{7.7277 \times 10^{6}}$	**529.6150**

- We consider a portfolio selection task in a financial market over n periods with $(m+1)$ assets, including **one cash asset** and m **risky assets**.

- In the t^{th} period, the prices of all assets are represented by the **closing price vector** $\mathbf{p}_t \in \mathbb{R}_+^{m+1}$ with each element p_{ti} representing the closing price of asset i.

- Their price changes in period t are represented by a **price relative (return) vector**, i.e., $\mathbf{r}_t \in \mathbb{R}_+^{m+1}$. The element

$$\boxed{r_{ti} = \frac{p_{ti}}{p_{(t-1)i}}}$$

represents the **ratio of price change** of Asset i in Period t.

Problem Formulation (II)

- The allocation over the portfolio at period t is specified by the **portfolio vector**

$$\mathbf{x}_t = (x_{t0}, x_{t1}, \ldots, x_{tm})$$

where x_{ti} is the proportion of capital invested in the ith risky asset at period t, $i = 1, 2, \ldots, m$, and x_{t0} represents the proportion of capital invested in cash asset. Typically, we assume **no short selling is allowed**, and then $\mathbf{x}_t \in \Delta_m$, where

$$\boxed{\Delta_m = \left\{ \mathbf{x}_t : \mathbf{x}_t \in \mathbb{R}_+^{m+1}, \quad \sum_{i=0}^{m} x_{ti} = 1 \right\}.}$$

We assume that the portfolio generally starts with **uniform portfolio**, i.e., $\mathbf{x}_1 = (\frac{1}{m+1}, \ldots, \frac{1}{m+1})$.

Problem Formulation (II)

- At the beginning of period t, the manager decides a new portfolio $\mathbf{x}_t$ and rebalances from the allocation $\bar{\mathbf{x}}_{t-1}$ to $\mathbf{x}_t$. We note that at the end of period $(t-1)$, the allocation $\bar{\mathbf{x}}_{t-1}$ differs from $\mathbf{x}_{t-1}$ due to price changes during period $(t-1)$ where

$$\boxed{\bar{x}_{(t-1)i} = \frac{x_{(t-1)i} r_{(t-1)i}}{\mathbf{x}_{t-1} \mathbf{r}_{t-1}^{\top}}, \quad i = 0, 1, \ldots, m.}$$

- After investing the **cash inflow** K, the allocation changes from $\bar{\mathbf{x}}_{t-1}$ to

$$\boxed{\hat{\mathbf{x}}_{t-1} = \frac{S_{t-1} \bar{\mathbf{x}}_{t-1} + K\mathbf{c}}{S_{t-1} + K}}. \quad (13)$$

Here S_{t-1} is the capital at the end of period $(t-1)$ and $\mathbf{c} = (1, 0, \ldots, 0)$ is the $(m+1)$-dimensional vector with $c_0 = 1, c_i = 0, i = 1, \ldots, m$. The portfolio's capital becomes $(S_{t-1} + K)$.

Problem Formulation (II)

- Recall $\gamma \in [0, 1]$ is **transaction cost rate** for selling and buying assets. And w_{t-1} is the **transaction remainder factor**, which is the net proportion after transaction costs are deducted.
- It is important to note that there is **no transaction cost when trading the cash asset**.
- We have

$$\boxed{1 = w_{t-1} + \gamma \sum_{i=1}^{m} |x_{ti} w_{t-1} - \hat{x}_{(t-1)i}|} \quad (14)$$

which means that the summation of the transaction remainder factor and the transaction costs always equals to 1. As a result of rebalancing, the remaining capital becomes $(S_{t-1} + K) \times w_{t-1}$.

Problem Formulation (II)

- During the period t, the allocation $\mathbf{x}_t$ changes the wealth by a factor of

$$\mathbf{x}_t\mathbf{r}_t^\top = \sum_{i=0}^{m} x_{ti} \cdot r_{ti}.$$

To summarize, in period t, the portfolio wealth changes from $(S_{t-1} + K)$ to

$$(S_{t-1} + K) \times w_{t-1} \times (\mathbf{x}_t\mathbf{r}_t^\top).$$

- Since we re-invest and use relative prices, the wealth grows in a **multiplicative manner**. The **cumulative wealth of the portfolio** at the end of period n is given by

$$\boxed{S_n = S_0 \prod_{t=1}^{n} w_{t-1} \times (\mathbf{x}_t\mathbf{r}_t^\top) + K \sum_{k=2}^{n} \prod_{t=k}^{n} w_{t-1} \times (\mathbf{x}_t\mathbf{r}_t^\top)} \quad (15)$$

where the **initial wealth S_0 is set to 1** for convenience.

- Eq. (14) reveals the relationship between transaction remainder factor and transaction costs. It is obvious that $|x_{ti}w_{t-1}-\hat{x}_{(t-1)i}|$ is a nonlinear function.
- We employ the method of **change of variables** to transform Eq. (14) into a linear function that is much easier to solve. We suppose that there are u_{ti} and v_{ti} satisfying

$$\begin{cases} |x_{ti}w_{t-1} - \hat{x}_{(t-1)i}| = u_{ti} + v_{ti}. \\ x_{ti}w_{t-1} - \hat{x}_{(t-1)i} = u_{ti} - v_{ti} \qquad i = 0, 1, \ldots, m. \\ u_{ti} \geq 0, v_{ti} \geq 0. \end{cases} \tag{16}$$

- In case that $x_{ti}w_{t-1} - \hat{x}_{(t-1)i} \geq 0$, we have $u_{ti} \geq 0, v_{ti} = 0$; otherwise, $u_{ti} = 0, v_{ti} > 0$.
- Therefore, we can always find corresponding solutions of $u_{t,i}$ and $v_{t,i}$ that fall into:

$$(1) u_{ti} > 0,\ v_{ti} = 0;\ \ (2) u_{ti} = 0,\ v_{ti} = 0;\ \ (3) u_{ti} = 0,\ v_{ti} > 0.$$

Problem Formulation (II) [Model Optimization]

- Then we can transform Eq. (14) into the following equation:

$$w_{t-1} = 1 - \gamma \sum_{i=1}^{m} (u_{ti} + v_{ti}) = 1 - \gamma(\mathbf{u}_t + \mathbf{v}_t)\mathbf{1} + \gamma(u_{t0} + v_{t0}). \qquad (17)$$

- The portfolio vector $\mathbf{x}_t$ can be expressed as follows:

$$\boxed{\mathbf{x}_t = \frac{\hat{\mathbf{x}}_{t-1} + \mathbf{u}_t - \mathbf{v}_t}{w_{t-1}} = \frac{\hat{\mathbf{x}}_{t-1} + \mathbf{u}_t - \mathbf{v}_t}{1 - \gamma(\mathbf{u}_t + \mathbf{v}_t)\mathbf{1} + \gamma(u_{t0} + v_{t0})}.} \qquad (18)$$

Here $\mathbf{1}$ is the $(m+1)$-dimensional column vector of all ones.

Problem Formulation (II) [Price Relative Vector Prediction]

- In order to implement the portfolio selection strategies, we make prediction on the price relative vector $\tilde{\mathbf{r}}_t$ for the period t.
- Various methods have been proposed to predict $\tilde{\mathbf{r}}_t$. We employ two popular methods: **Exponential Moving Average (EMA)** and **Robust Median Reversion (RMR)**.
- **RMR:** To predict the next price relative $\tilde{\mathbf{r}}_t$, we calculate the multivariate L_1-median of historical prices. The L_1-median of a k-**historical price window** is the solution of following optimization:

$$\boldsymbol{\mu} = \underset{\boldsymbol{\mu}}{\operatorname{argmin}} \sum_{i=1}^{k} \|\mathbf{p}_{t-i} - \boldsymbol{\mu}\|_2$$

where $\|\cdot\|_2$ is the Euclidean norm. RMR predicts the next price using a robust L_1-median estimator at the end of $(t-1)$th period. The predicted value of price relative vector is

$$\tilde{\mathbf{r}}_t(k) = \boldsymbol{\mu}/\mathbf{p}_{t-1}.$$

Formulation (II)

- We then formulate the framework for online portfolio selection with transaction costs and constant cash inflows.

- The objective of this task is to maximize the portfolio's cumulative wealth S_n, which is equivalent to maximizing the portfolio's expected return in each period.

- **Objective:** As for period t:

$$\mathbf{x}_t = \arg\max_{\mathbf{x}\in\Delta_m} w_{t-1} \times (\mathbf{x}\tilde{\mathbf{r}}_t^\top) - \lambda||\mathbf{x}w_{t-1} - \hat{\mathbf{x}}_{t-1}||_1. \quad (19)$$

- The **first term** of the objective function is the **expected return** and the **second term** is the L_1**-penalty term** for period t.
- Here $\lambda \geq 0$ is the **trade-off parameter** to balance the expected return and L_1-penalty term.

Formulation (II)

- By using Eqs. (16-18), we can transform Problem (19) into the following problem, **Transaction cost with Cash Inflow** (TCI):

$$\begin{aligned} (\mathbf{u}_t, \mathbf{v}_t) = \arg\max_{\mathbf{u},\mathbf{v}\in\mathbb{R}^{m+1}} \ & (\hat{\mathbf{x}}_{t-1} + \mathbf{u} - \mathbf{v})\tilde{\mathbf{r}}_t^\top - \lambda(\mathbf{u}+\mathbf{v})\mathbf{1} \\ \text{s.t.} \quad & (\mathbf{u}-\mathbf{v})\mathbf{1} + \gamma(\mathbf{u}+\mathbf{v})\mathbf{1} - \gamma(u_0 + v_0) = 0, \\ & \hat{\mathbf{x}}_{t-1} + \mathbf{u} - \mathbf{v} \geq \mathbf{0}, \mathbf{u} \geq \mathbf{0}, \mathbf{v} \geq \mathbf{0} \end{aligned} \tag{20}$$

where the constraint condition

$$(\mathbf{u}-\mathbf{v})\mathbf{1} + \gamma(\mathbf{u}+\mathbf{v})\mathbf{1} - \gamma(u_{t0} + v_{t0}) = 0$$

is a transformation of $\sum_{i=0}^{m} x_i = 1$.

Theorem

Problem (19) achieves the optimal solution $\boldsymbol{x}_t^\star$ if and only if Problem (20) achieves the optimal solution $(\boldsymbol{u}_t^\star, \boldsymbol{v}_t^\star)$, where $\boldsymbol{x}_t^\star$ and $(\boldsymbol{u}_t^\star, \boldsymbol{v}_t^\star)$ satisfies equation (16).

Numerical Experiments (II) [Transaction Costs Analysis]

• Numerical experiments are given to demonstrate the effectiveness of our algorithm over other online portfolio selection algorithms.

• In the numerical experiments, we use four public data sets: NYSE-O, and its following data set, NYSE-N, TSE and MSCI.

• We implement the optimization Model (20) with two different predictors RMR and EMA. We compare the performance of TCIR (RMR) and TCIE (EMA) with 13 other algorithms.

• For the parameters in TCIR and TCIE, we set the trade off parameter to $\lambda = 5 * \gamma$, the window size in RMR prediction to $k = 5$, and the decaying factor in EMA prediction $\alpha = 0.5$.

Numerical Experiments (II) [Cumulative Wealth $K = 0$]

Table 1: Cumulative wealth achieved by various strategies on the data sets with common transaction cost rates (0%, 0.25% and 0.5%) and zero constant cash inflow.

	NYSE-O			NYSE-N			TSE			MSCI		
Algorithms	0	0.25%	0.5%	0	0.25%	0.5%	0	0.25%	0.5%	0	0.25%	0.5%
BAH	14.21	14.17	14.14	18.23	18.18	18.14	1.60	1.59	1.58	0.90	0.89	0.89
Best	0.39	0.14	0.05	9.91	4.27	1.96	0.30	0.22	0.16	0.42	0.24	0.13
BCRP	248.50	180.25	130.74	120.76	99.54	82.04	6.59	6.32	6.06	1.49	1.48	**1.47**
UCRP	26.67	22.66	19.26	31.82	26.34	21.80	1.57	1.49	1.43	0.92	0.89	0.87
UP	26.75	22.66	19.41	31.35	26.13	21.82	1.56	1.49	1.42	0.92	0.90	0.88
EG	26.70	22.81	19.50	31.26	26.14	21.86	1.57	1.50	1.43	0.92	0.90	0.88
ONS	25.15	23.76	22.44	27.86	25.68	23.67	0.38	0.36	0.35	0.88	0.84	0.80
Anticor	9.98E+07	1295.00	0.02	1.72E+06	29.99	0.00	22.55	1.81	0.14	4.21	0.80	0.15
CWMR	5.72E+15	2.86E+05	0.00	9.18E+05	0.00	0.00	186.55	1.79	0.02	**15.56**	0.19	0.00
PAMR	5.08E+15	2.60E+05	0.00	1.30E+06	0.00	0.00	257.86	2.09	0.02	14.99	0.18	0.00
OLMAR	7.66E+16	4.89E+08	2.85	4.21E+08	1.70	0.00	59.00	1.10	0.02	14.56	0.53	0.02
RMR	1.77E+17	6.13E+08	1.92	6.24E+08	1.30	0.00	245.17	3.65	0.05	15.26	0.49	0.02
TCO	6.76E+12	6.70E+08	2.79E+04	2.43E+07	4.21E+03	56.83	153.05	**9.57**	0.91	5.33	1.54	0.75
TCIR	7.10E+16	**2.68E+11**	**1.47E+07**	**1.27E+09**	6.21E+04	**865.64**	299.13	4.05	1.20	10.37	1.08	1.20
TCIE	**8.41E+18**	1.87E+11	9.62E+04	1.13E+09	**2.50E+05**	488.36	**932.39**	5.45	**10.85**	14.91	**1.88**	0.94

Numerical Experiments (II) [Cumulative Wealth $K = 0.1$]

Table 2: Cumulative wealth achieved by various strategies on the data sets with transaction cost rates (0%, 0.25% and 0.5%) and constant cash inflow ($K = 0.1$).

	NYSE-O			NYSE-N			TSE			MSCI		
Algorithms	0	0.25%	0.5%	0	0.25%	0.5%	0	0.25%	0.5%	0	0.25%	0.5%
BAH	2393.30	2387.32	2381.34	2554.50	2548.10	2541.72	155.21	154.82	154.43	99.97	99.72	99.47
Best	641.02	482.33	374.46	2040.29	1452.53	1076.31	110.57	103.67	97.61	78.88	65.30	54.97
BCRP	2.07E+04	1.61E+04	1.26E+04	1.35E+04	1.15E+04	9.79E+03	345.46	335.44	325.75	141.55	140.83	**140.11**
UCRP	3893.12	3460.65	3081.39	4191.71	3604.29	3104.19	150.03	145.79	141.69	100.93	99.63	98.35
UP	3883.47	3462.28	3097.40	4128.42	3587.50	3103.84	150.06	145.97	141.77	100.98	99.69	98.39
EG	3904.08	3484.09	3114.12	4131.99	3583.06	3111.56	150.06	146.02	142.11	100.89	99.65	98.43
ONS	6441.42	6416.18	6391.22	4299.60	4197.89	4099.64	60.75	60.51	60.28	94.66	93.76	92.86
Anticor	2.42E+09	6.89E+04	77.83	7.50E+07	5470.29	88.77	1016.38	184.63	58.45	238.17	98.59	51.31
CWMR	8.05E+16	8.73E+06	16.43	4.34E+07	48.25	18.46	3529.13	118.17	26.61	**701.73**	64.08	18.53
PAMR	7.18E+16	7.92E+06	15.63	6.13E+07	49.01	18.63	4759.22	130.66	26.82	679.49	63.49	18.65
OLMAR	9.09E+17	8.32E+09	220.60	1.62E+10	649.63	49.77	996.29	69.48	23.85	640.36	91.85	28.20
RMR	1.99E+18	9.64E+09	162.06	2.43E+10	539.51	46.75	3491.68	153.53	38.42	699.36	95.27	29.10
TCO	1.01E+14	1.29E+10	3.16E+06	9.88E+08	2.90E+05	467.69	2666.12	**443.84**	112.73	323.79	152.40	88.16
TCIR	7.09E+17	3.37E+12	**6.17E+08**	**5.21E+10**	3.23E+06	**6.65E+04**	5174.64	225.38	85.74	487.15	123.56	123.52
TCIE	**9.21E+19**	**5.01E+12**	3.53E+06	4.76E+10	**1.07E+07**	4.56E+04	**1.46E+04**	287.85	**586.15**	680.91	**173.68**	94.58

Numerical Experiments (II) [Sharpe Ratio $K = 0.1$]

Table 3: Sharpe ratio achieved by various strategies on the data sets with transaction cost rates (0%, 0.25% and 0.5%) and constant cash inflow ($K = 0.1$).

	NYSE-O			NYSE-N			TSE			MSCI		
Algorithms	0	0.25%	0.5%	0	0.25%	0.5%	0	0.25%	0.5%	0	0.25%	0.5%
BAH	0.0547	0.0547	0.0546	0.0458	0.0458	0.0457	0.0498	0.0495	0.0493	0.0012	0.001	0.0008
Best	0.0085	0.0022	-0.0040	0.0305	0.0223	0.0144	-0.0087	-0.0150	-0.0226	-0.0455	-0.0773	-0.1077
BCRP	0.0599	0.0570	0.0541	0.0555	0.0535	**0.0516**	0.0740	0.0726	**0.0713**	0.0378	**0.0372**	**0.0366**
UCRP	0.0729	0.0694	0.066	0.0507	0.0482	0.0457	0.0479	0.0434	0.0388	0.0029	0.0014	0.0000
UP	0.0727	0.0693	0.0659	0.0507	0.0483	0.0458	0.0478	0.0432	0.0388	0.0028	0.0014	0.0000
EG	0.0726	0.0692	0.0659	0.0507	0.0483	0.0459	0.0479	0.0436	0.0392	0.0028	0.0014	0.0000
ONS	0.0334	0.0332	0.0329	0.0325	0.0322	0.0319	-0.0089	-0.0097	-0.0105	0.0023	-0.0004	-0.0030
Anticor	0.1384	0.0607	-0.0201	0.0919	0.0302	-0.0330	0.0818	0.0313	-0.0205	0.0731	-0.0044	-0.0833
CWMR	0.2152	0.0846	-0.0525	0.0852	-0.0554	-0.2036	0.1132	0.0311	-0.0535	**0.1259**	-0.0700	-0.2770
PAMR	0.2149	0.0841	-0.0531	0.0865	-0.0529	-0.1996	0.1179	0.0338	-0.0531	0.1246	-0.0716	-0.2789
OLMAR	0.2106	0.1181	0.0197	0.1038	0.0162	-0.076	0.0823	0.0285	-0.0271	0.1157	-0.0205	-0.1654
RMR	0.2141	0.1188	0.0176	0.1056	0.0149	-0.0805	0.1018	0.0446	-0.0145	0.1179	-0.0231	-0.1730
TCO	0.1979	0.1405	0.0850	0.0988	0.0565	0.0148	0.0945	**0.0673**	0.0218	0.0859	0.0324	-0.0174
TCIR	0.2089	**0.1475**	**0.1015**	**0.1086**	0.0652	0.0466	0.1048	0.0468	0.0294	0.1042	0.0168	0.0117
TCIE	**0.2334**	0.1464	0.0768	0.1078	**0.0705**	0.0427	**0.1186**	0.0503	0.0586	0.1172	0.0343	-0.0009

Numerical Experiments (II) [Information Ratio $K = 0.1$]

Table 4: Information ratio achieved by various strategies on the data sets with transaction cost rates (0%, 0.25% and 0.5%) and constant cash inflow ($K = 0.1$).

	NYSE-O			NYSE-N			TSE			MSCI		
Algorithms	0	0.25%	0.5%	0	0.25%	0.5%	0	0.25%	0.5%	0	0.25%	0.5%
BAH	-0.0354	-0.0356	-0.0357	-0.0253	-0.0254	-0.0255	0.006	0.005	0.0041	-0.0343	-0.0371	-0.0396
Best	-0.0141	-0.0208	-0.0273	-0.0092	-0.0184	-0.0267	-0.0208	-0.0273	-0.0351	-0.0584	-0.0959	-0.1307
BCRP	0.0338	0.0303	0.0268	0.021	0.0186	0.0161	0.0621	0.0606	**0.0591**	0.0323	0.0317	**0.0312**
UP	-0.0038	-0.2009	-0.441	-0.0108	-0.1388	-0.2575	-0.0274	-0.2545	-0.5139	-0.0058	-0.254	-0.2811
EG	0.0022	-0.1055	-0.2087	-0.0121	-0.1106	-0.2047	-0.0053	-0.0227	-0.3813	-0.0258	-0.003	-0.2773
ONS	0.0176	0.0173	0.017	0.0188	0.0185	0.0181	-0.0219	-0.0227	-0.0236	-0.0002	-0.0107	-0.0057
Anticor	0.1254	0.04	-0.0493	0.0818	0.01	-0.0637	0.0759	0.0225	-0.0323	0.1146	-0.1131	-0.1414
CWMR	0.2105	0.07	-0.0784	0.0744	-0.0876	-0.2608	0.1094	0.0234	-0.0655	**0.1898**	-0.1157	-0.4489
PAMR	0.2102	0.0695	-0.0791	0.0761	-0.0845	-0.256	0.1144	0.0263	-0.065	0.1877	-0.1157	-0.4519
OLMAR	0.2059	0.1072	0.0015	0.0965	-0.0014	-0.1057	0.0779	0.0225	-0.035	0.169	-0.0344	-0.2649
RMR	0.2097	0.1081	-0.0006	0.0987	-0.003	-0.1111	0.0980	0.0391	-0.0221	0.1726	-0.0386	-0.2776
TCO	0.1954	0.1317	0.0701	0.0915	0.0431	-0.0053	0.0900	**0.0691**	0.0147	0.1549	0.0522	-0.0363
TCIR	0.2042	**0.1390**	**0.0895**	**0.1017**	0.0536	**0.0330**	0.101	0.0413	0.0234	0.1507	0.022	0.0145
TCIE	**0.2303**	0.1374	0.0617	0.1009	**0.0595**	0.0282	**0.1151**	0.0449	0.0531	0.1713	**0.0500**	-0.005

Conclusions:

- A new **adaptive decaying factor method** is introduced, by which we gain predicted returns with better relative errors.
  - **Transaction costs** are introduced in maximizing the net profit of the whole portfolio in each period.
  - The nonlinear **Net Profit Model** is transformed into an equivalent linear programming problem, which is easy to solve and implement.
  - The **AOLNPM algorithm** outperforms traditional online portfolio selection algorithms in multiple numerical experiments with different benchmark data sets.

Conclusions

- We propose a framework for online portfolio selection with **transaction costs and constant cash inflows**.
  - We propose a new method to calculate **transaction remainder factor** and **portfolio vector** simultaneously.
  - Two algorithms, TCIR and TCIE, are developed to solve the proposed online portfolio selection model, and a series of numerical experiments are conducted to verify the effectiveness of our algorithms.
- **Future Works:**
  - Employ **regime-switching model** in the prediction step.
  - To consider **mean-variance, VaR, CVaR**, etc. in the optimization step.
  - To consider the **market impact** on the price.
  - Using **machine learning method** to combine the two steps: **prediction and optimization**.

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