Pigeonhole Principle (鴿巢原理)

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Abstract: In this talk, we present a well-known mathematical result, the ***Pigeonhole Principle*** with a simple proof (proof by contradiction). Some interesting applications of the result and a two-player game ***SIM*** will also be introduced.

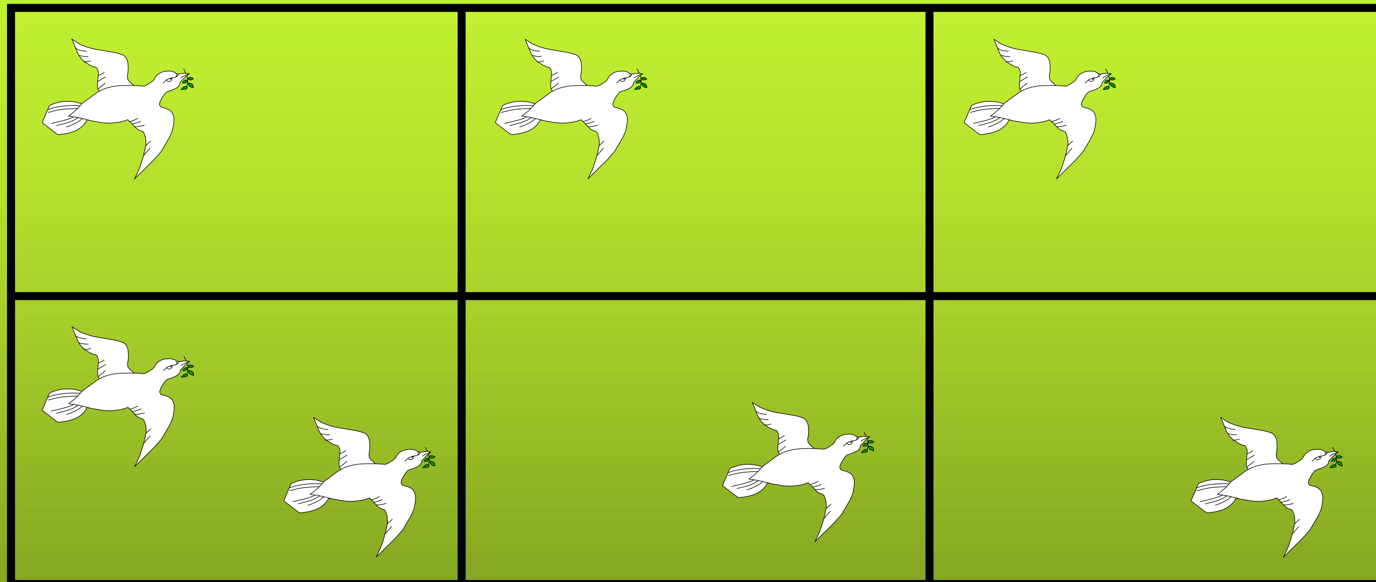
Mathematics in Education

- Mathematics is about ``**Proof**'' (**證明**)

言登日月

- Mathematics in **STEAM** education is more about **Mathematical Modeling** & **Computational Thinking**.
- Strike a good balance.

Pigeonhole Principle (鴿巢原理)



If **7** pigeons are to live in **6** boxes (holes), then there is at least one box containing **two or more** pigeons.

Dirichlet (狄里克利)

The first statement of the Pigeonhole Principle is believed to have been made by the German Mathematician Dirichlet in 1834 under the name *Schubfachprinzip* (drawer principle).

Dirichlet (狄里克利)

- Dirichlet（ 1805 ~ 1859 ）德國數學家， 生於現德國 Duren（ 當時屬法國 ）。 他是**解析數論 (Analytic Number Theory)**的奠基者,也是現代**函數 (Functions)** 觀念的定義者。
- Dirichlet 人緣不錯, 勤奮寡言, 不修邊幅又健忘, 卻是一個觀念清晰的好老師 。(He is a nice and hard working teacher)

Continue

- Dirichlet 於1825年證明(proved)費瑪最後定理(Fermat's Last Theorem) (when) n=5 情況而得盛名, 後因心臟病發(heart attack)於瑞士, 最後病逝於哥廷根。
- 費瑪最後定理: $X^n + Y^n = Z^n$, $n=3,4,5,\ldots,$
- n=2: $3^2 + 4^2 = 5^2$.

Taken from: http://episte.math.ntu.edu.tw/people/p_dirichlet/

Pigeonhole Principle (鴿巢原理)

If we put n+1 balls into n boxes, then at least one box must contain two or more balls.

將 n+1 個球放入 n 個盒子內, 最小有一個盒子藏有2個或以上的球。

Proof (證明)

- We want to prove that '**at least one box must contain two or more balls**' by '**Proof by Contradiction**'.
- Assume that the statement above is wrong then all the n boxes contains 1 or 0 ball. Therefore the total number of balls is less than or equal to n. This is a **contradiction** (矛盾). The contradiction is due to our assumption that the 'statement is wrong'.
- We conclude that the statement must be true.

Examples

- For any **368** people, **at least two** of them must have the **same birthday (366)**.
- There are at least **least two** people in the world having **same number of hair**.
- At **least two** of you in this class (assuming the class size >12) born in the **same month**.

Exercise

- There are 10 married couples. How many of the 20 people must be selected in order to guarantee that one has selected a married couple ?
- Answer: 11

Mark Six (六合彩)

Mark Six is a popular lottery game in Hong Kong. Similar lottery game can be found all over the world. There are 49 balls (number 1 to 49) in the urn. Six balls are first drawn without replacement. The 7th ball is then drawn as the special number.

The following is some ten draws of the Mark Six Lottery in reverse order.

Date	Draw Number	Draw Results							
20/12/2002	02/110	13	18	23	24	26	33	+	15
17/12/2002	02/109	6	18	39	40	41	42	+	9
12/12/2002	02/108	7	15	16	23	31	35	+	8
10/12/2002	02/107	5	36	37	38	46	49	+	17
05/12/2002	02/106	11	21	27	31	37	44	+	1
03/12/2002	02/105	9	11	14	17	24	28	+	46
28/11/2002	02/104	17	19	26	31	37	43	+	38
26/11/2002	02/103	19	21	40	42	46	47	+	33
21/11/2002	02/102	4	16	18	25	29	41	+	21
19/11/2002	02/101	3	15	22	23	42	47	+	18

Something to Think About

- One observes that at least 2 drawn numbers having the same first digit in all the cases (we assume that 1=01, 2=02, 3=03, 4=04,…).
- Do you think there is something wrong with the machine?

A Solution

- There are 6 numbers to be drawn.
- For each number, the first digit can be {0,1,2,3,4}. There are only 5 choices.
- This is equivalent to put 6 balls into 5 boxes with labels: 0,1,2,3,4. Therefore at least one of the boxes contains 2 or more balls.

Pigeonhole Principle : Strong Form
(鴿巢原理:加強版)

If we put (k*n+1) balls in n boxes, then at least one box must contain k+1 or more balls.

將 (k*n+1) 個球放入 n 個盒子內, 最小有一個盒子藏有k+1個或以上的球。

Proof (證明)

- We want to prove that at least one box must contain (k+1) or more balls.
- Assume that the statement above is wrong then all the n boxes contains k or less balls. Therefore the total number of balls is less than or equal to n*k. This is a contradiction. The contradiction is due to the assumption that ‘the statement is wrong’.
- We conclude that the statement must be true.

Exercise

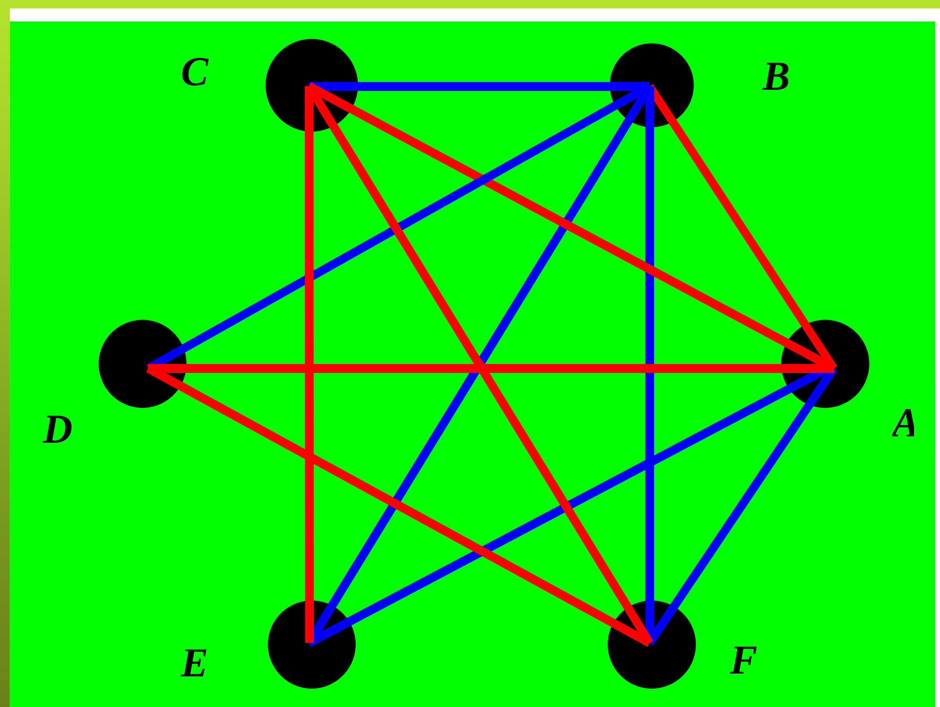
- There are **90** people in a hall. Some of them know each other, some are not.

  Prove that there are at **least two** persons in the hall who know the **same number of people** in the hall.
- **What should be the holes and pigeons? How many holes are there**?

Solution

- If there is a person in the hall who ***does not know*** any other people, then each of the other persons in the hall may know either 0, 1, 2, 3, ... , or 88 people. Therefore we have **89 holes**: 0, 1, ... , 88, and have to distribute among **90 people**.
- Next, assume that every person in the hall ***know at least one*** other person. Again, we have **89 holes**: 1, 2, ... , 89 and **90 people**.
- Therefore in either case, we can apply the Pigeonhole Principle to get the conclusion.

SIM



How to Play SIM (遊戲規則) ?

- The game board consists of **6 points** in the form of a hexagon. The points are labelled as A, B, C, D, E, F.
- **Two different colored strings** are distributed to two players. The players take turns to join any two points with their own strings.
- The aim of the game is to **force your opponent to complete a triangle of his color**. The points of the triangle must be three points of the hexagon A,B,C,D,E,F.

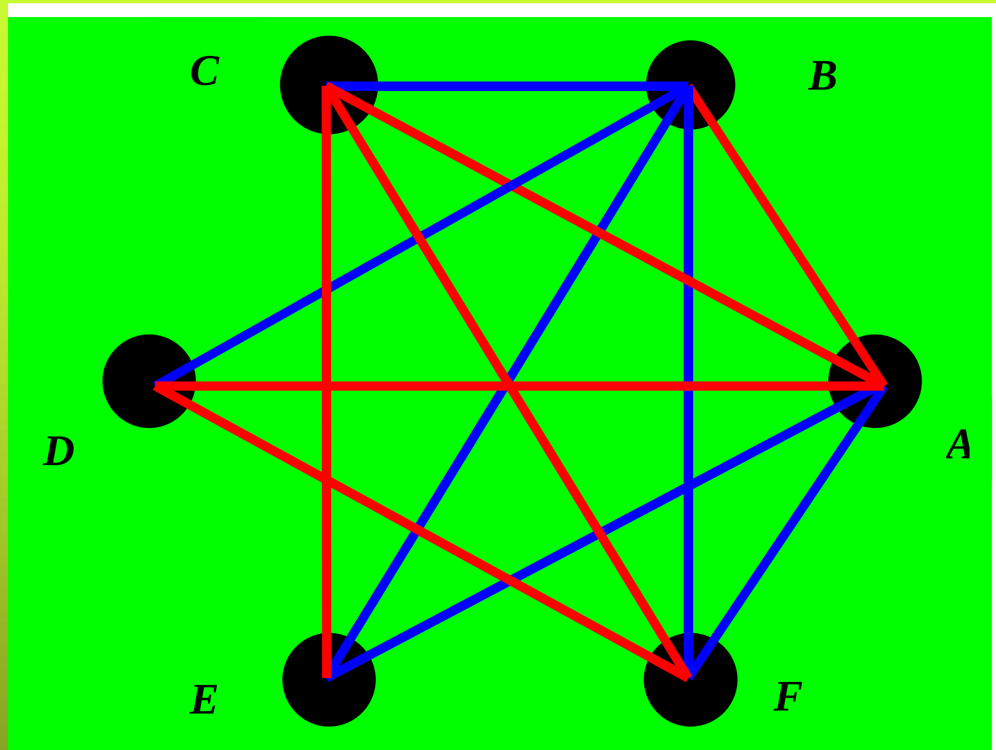
Gustavus J. Simmons

SIM is invented by well-known graph theorists and cryptographer **Gustavus J. Simmons**.

Simmons described the game in his 1969 paper. He is a "Senior Fellow for National Security Studies" at the Sandia National Laboratories, Albuquerque (NM), USA.

http://hkumath.hku.hk/~wkc/MathModel/

Game in Progress



In this game, both Peter (red lines) and Paul (blue lines) have made six moves. If Peter now chooses DE, then Paul is forced to choose either EF (and Paul loses by completing triangle BEF and triangle AEF) or CD (and Paul loses by completing triangle BCD).

A Question You Should Ask

After playing this game several times, you may ask the following question.

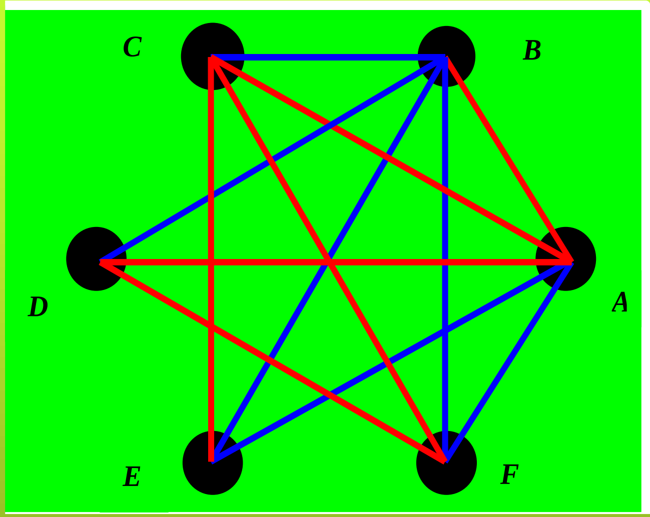
Is it possible to have a draw ?

The answer to this question is NO. But why?

We will give a **proof** by using Pigeonhole Principle.

Proof (證明)

- We **assume a draw is possible** (all lines are drawn) and hope to arrive at a **contradiction**.
- We choose a point, say A. Then, there are five lines joining A to the other points. They are AB, AC, AD, AE and AF.
- Since we have two different colours, by the Pigeonhole Principle, at least 3 lines must be in the same color, say AB, AC and AD.
- We may assume these three lines are red.



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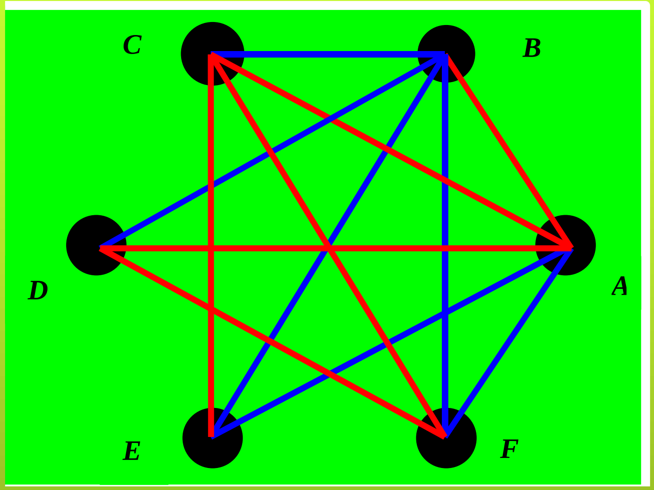
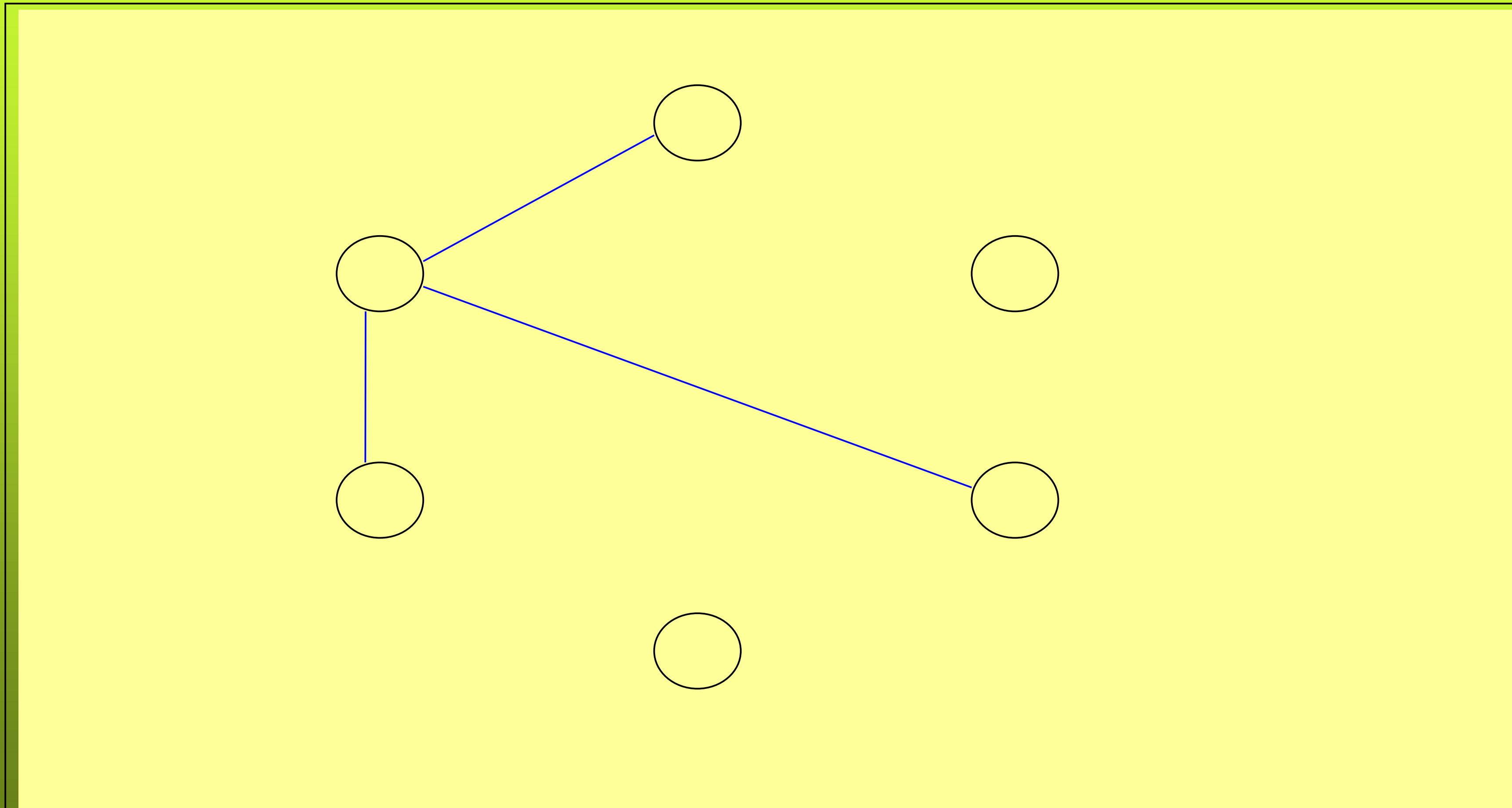
- Now, we consider the triangle BCD.
- If **one** side of triangle **BCD** is red, then we have a red triangle, otherwise BCD will be a **blue triangle**.
- Therefore, either a red or a blue triangle must exist and it is impossible to have a draw.



An Example



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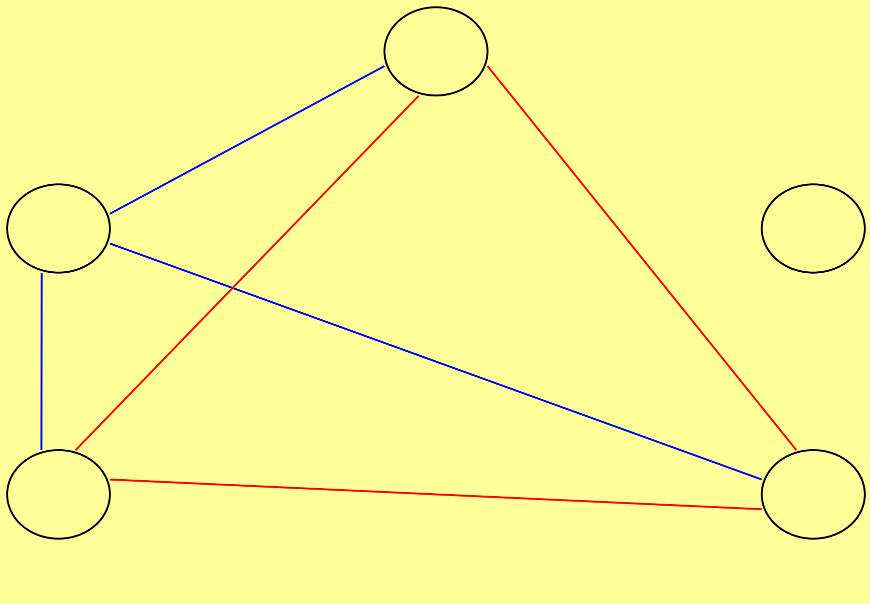
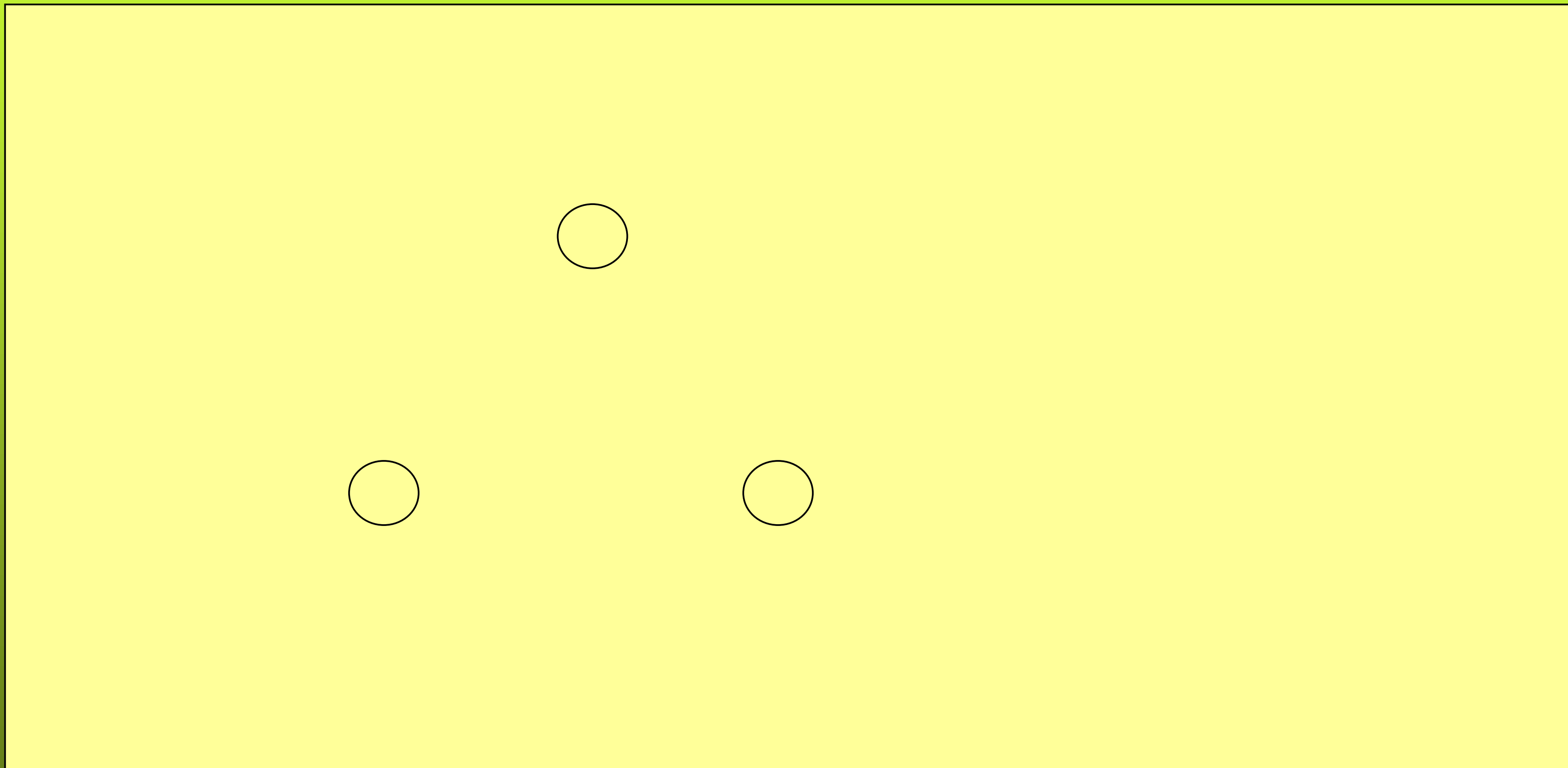


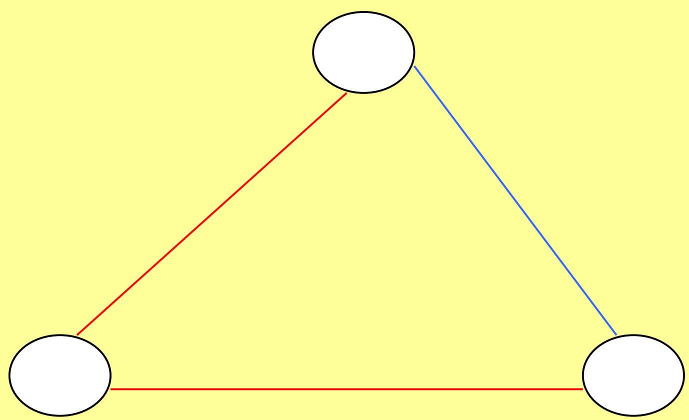
Excise 1: Extension of the Proof

- You can play SIM on a polygon (n vertices) and there will be no draw if the polygon is NOT a **Triangle, a Square or a Pentagon**.
- This means that for $n > 5$, playing SIM will result in no draw and the proof is the same as before. Follow the previous proof (try and see).

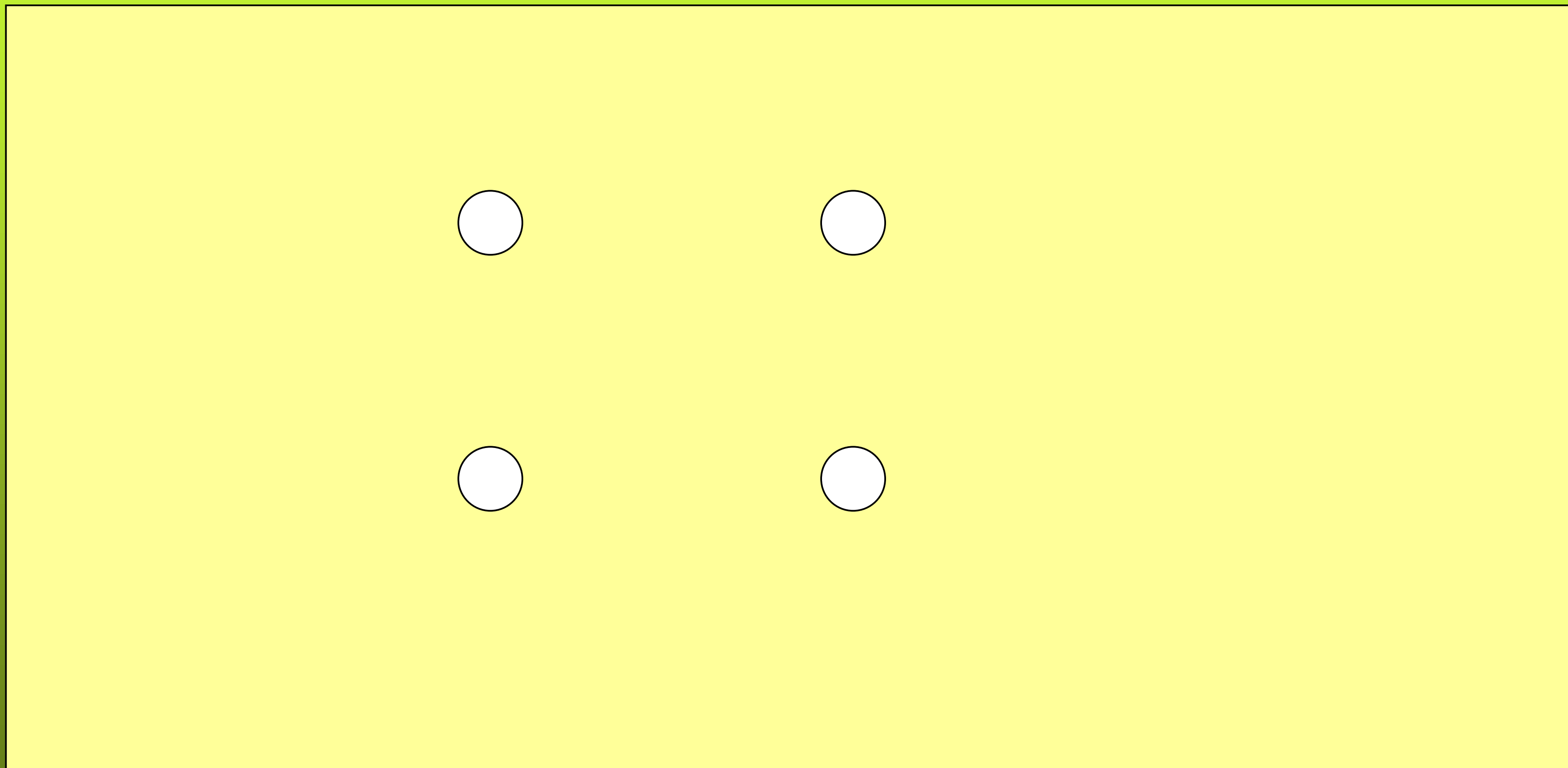
A Triangle



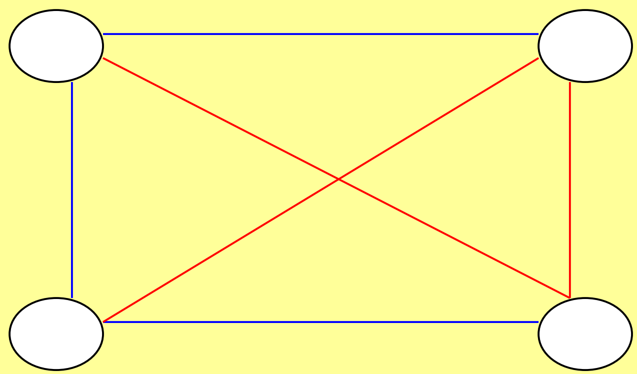
A Draw is Possible



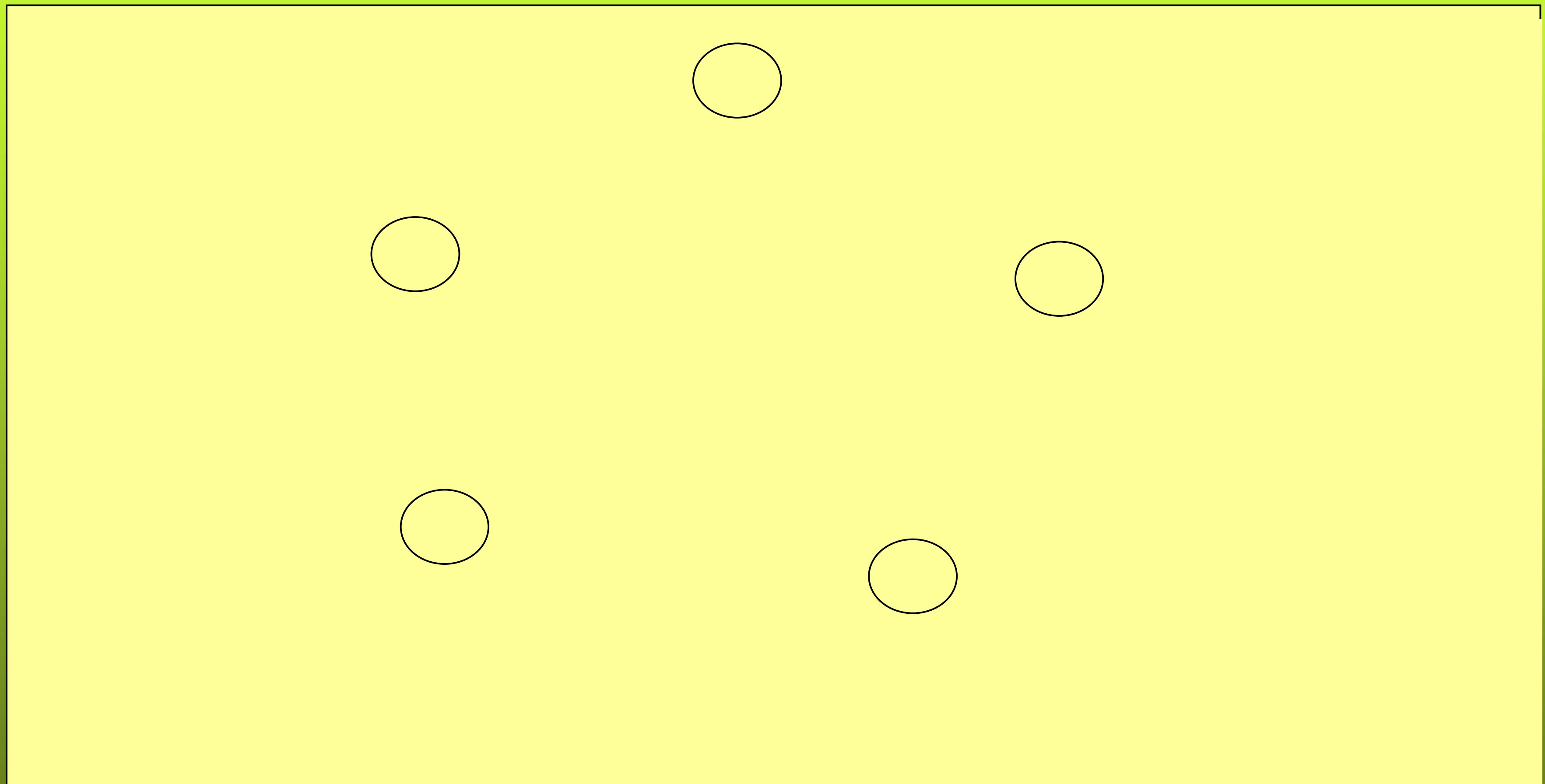
A Square



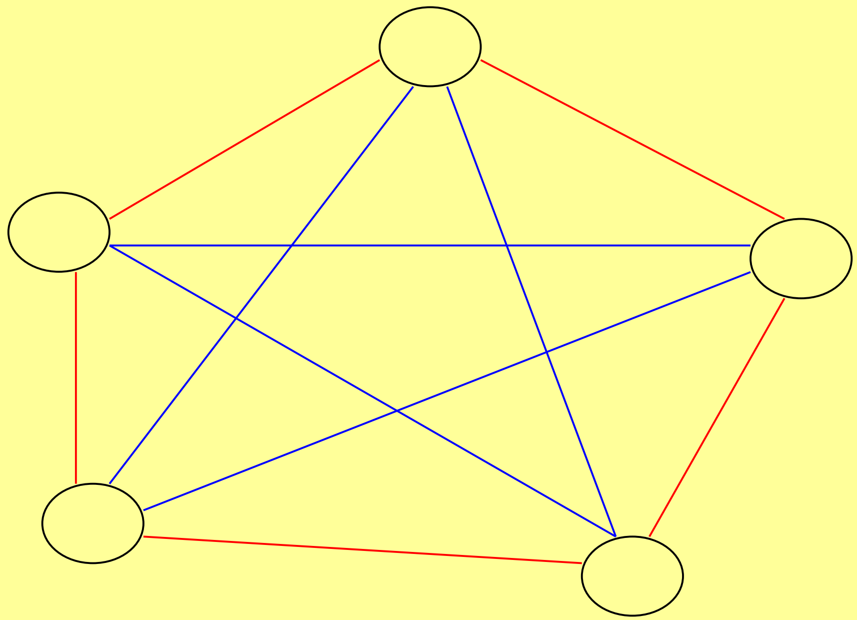
A Draw is Possible !



A Pentagon



A Draw is Possible!



Exercise 2: Extension of the Game

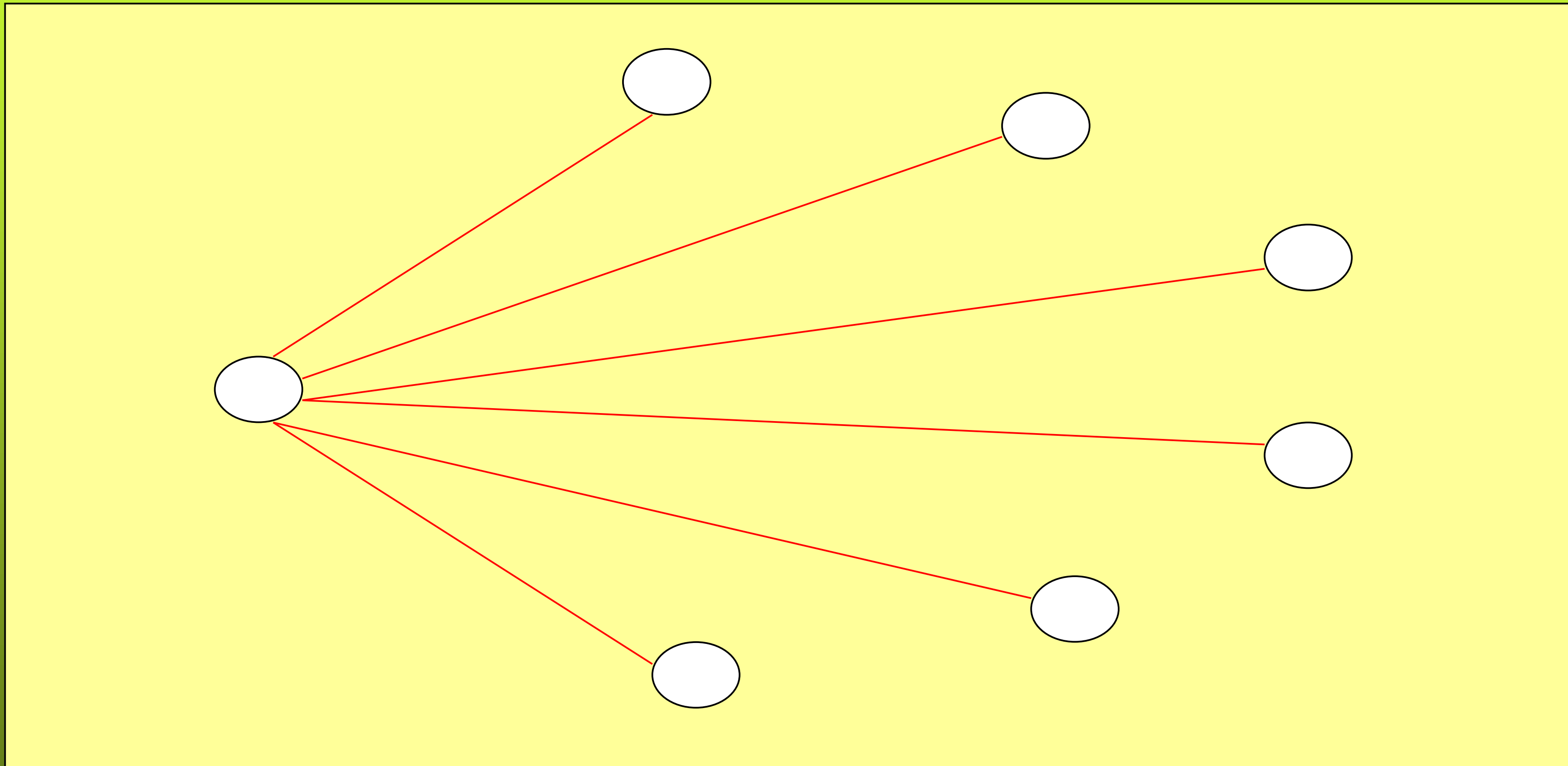
- Suppose the game is now played by 3 players with 3 different colors (say Red, Blue and Green). The rules are the same.
- Can you show that if the game is played on a regular polygon of **17 vertices** then it is **Not** possible to have a **draw**?

Proof (證明)

- We **assume a draw is possible** (all lines are drawn and there is no triangle of same color) and we hope to arrive at a **contradiction**.

- Pick up a vertex, it must has 16 edges. By pigeonhole principle of strong form, we can find 6 edges of the same color (say Red). The corresponding 6 vertices form a hexagon. For this hexagon, its edges must be of the other two colors (Blue and Green). But this means that there must be at least one triangle of same color in the hexagon (we have proved before).

An Example



Exercise 3: Further Extension

- How about 4 players with 4 colors?
- It is impossible to color (edges) of a polygon of 17 vertices with 3 colors and find no triangle of same color.
- Suppose the edges connected to a vertex are to be colored by 4 colors. What is the minimum number of edges such that one can find at least 17 edges are colored with the same color?
- The answer is 4(17-1)+1 = 65.
- Therefore if the game is played by 4 players on a polygon of 65+1= 66 vertices then there will not be a draw at the end.

Exercise 4: Further Generalization

- Suppose n players with n different colors are playing the game on a polygon of f(n) vertices.

  Give a value of f(n) such that the game has no draw.

- $f(2) = 6$,
- $f(3) = (2+1)(f(2)-1)+1+1 = 17$,
- $f(4) = (3+1)(f(3)-1)+1+1 = 66$,
- $f(n+1) = (n+1)(f(n)-1)+1+1 = (n+1)f(n)-n+1$

References

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