On Optimal Pricing Model for Multiple Dealers in a Competitive Market [1]

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The pricing strategies of dealers have been studied extensively in the micro-structure literature. The two most often addressed sources of risk faced by dealers are: (1) **the inventory risk** arising from uncertainty in the asset's value; and (2) **the asymmetric information risk** arising from informed traders. In this talk, the optimal pricing strategy in Avellande-Stoikov (2008) for a **monopolistic dealer** is extended to a general situation where **multiple dealers** are present in a competitive market. The dealers' trading intensities, their optimal bid and ask prices and therefore their spreads are derived when the dealers are informed the severity of the competition. The effects of various parameters on the bid-ask quotes and profits of the dealers in a competitive market are also discussed.

Keywords: Dynamic Programming (DP); Hamilton-Jacob-Bellman (HJB) Equation; Limit Order Book (LOB); Multiple Dealers.

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1 Background

Technological innovation has completely transformed the fundamentals of the financial market. As a result, automatic and electronic order-driven trading platforms have largely replaced the traditional floor-based trading.

Figure 1: Auction Market (Left)   Order-driven Market (Right)

In an electronic order-driven market, orders arrive at the exchange and wait in the ***Limit Order Book (LOB)*** to be executed. There are **two types** of buy/sell orders for market participants to post:

(1) Market Orders (MO);
(2) Limit Orders (LO).

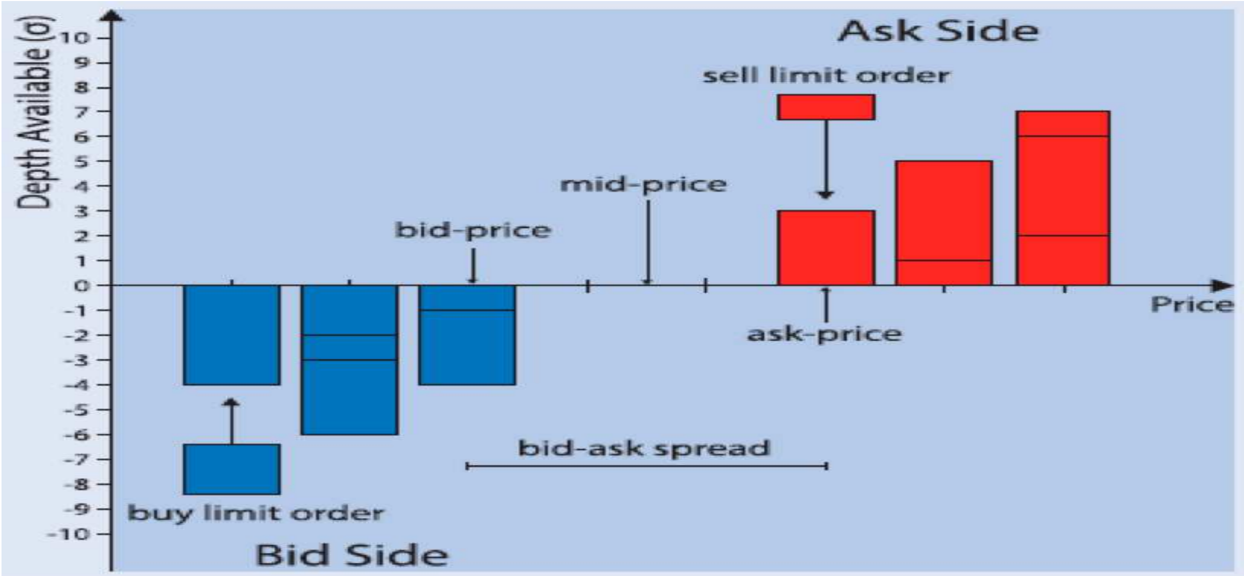


Figure 2: Schematic of an LOB

- **LO**: an order to buy/sell a certain quantity at a ***given*** price
- **Bid Price**: the highest price in limit buy orders
- **Ask Price**: the lowest price in limit sell orders
- **Ask-Bid Spread** = Ask Price - Bid Price
- **Market Price = Mid-Price**= $\frac{\text{Ask Price+Bid Price}}{2}$
- **MO**: an order to trade a certain quantity at ***the best available*** price

The pricing strategies of dealers have been studied extensively in the microstructure literature.

The two most often addressed sources of risk faced by dealers are:

(1) **inventory risk** arising from uncertainty in the asset's value;

-Inventory costs arise from uncertainty about market prices of the securities that the dealer may hold in his portfolio while his limit orders are pending.

(2) **asymmetric information risk** arising from informed trades.

-Information risk arises when some investors have better information than others about a firms prospects.

2 A Review of Two Models

2.1 Ho-Stoll Model

-T. Ho and H. Stoll, *Optimal dealer pricing under transactions and return uncertainty*, Journal of Financial Economics, 9 (1981), 47–73.

Ho and Stoll model (1981) imposes the following assumptions:

(i) Transactions are assumed to evolve over time according to **Poisson jump processes**. Two Poisson processes are used, one for **selling by the dealer** and the other for **buying by the dealer**

$$dq_a = \mathcal{X}_{\{\text{an arrival of a market selling order}\}}\mathbf{Q}\boldsymbol{\lambda_a}dt$$

and

$$dq_b = \mathcal{X}_{\{\text{an arrival of a market buying order}\}}\mathbf{Q}\boldsymbol{\lambda_b}dt$$

where $\mathcal{X}_E$ is the indicator function of the event E, **Q is the MO size**, and λ_a and λ_b are the ***intensities*** of the transactions. Here dq_a and dq_b are the increments of the number of market selling and buying orders, respectively.

(ii) The dealer determines a price of immediacy, b, should a market selling order arrive and a price, a, should a market buying order arrive.

The dealer does not directly quote b and a, instead, he quotes his bid and ask prices, respectively, as follows:

$$p_b = \mathbf{p} - b \quad \text{and} \quad p_a = \mathbf{p} + a.$$

Here p is the dealer's opinion of the true price of the stock at the time he sets the bid-ask quotation and this price is supposed to be a given **constant**.

(iii) The intensities, λ_a and λ_b, depend on the dealer's selling fee and buying fee, respectively.

(iv) In addition to uncertainty about the timing of subsequent transactions, the dealer faces uncertainty about the return on his existing portfolio. Consequently, we have

$$\begin{cases} dF &= rFdt - (p-b)dq_b + (p+a)dq_a \\ dI &= r_I I dt + pdq_b - pdq_a + IdZ_I \\ dY &= r_Y Y dt + Y dZ_Y. \end{cases}$$

- F, I, Y are the balances of the cash account, inventory account, and base wealth, respectively,
- r_I, r_Y represent the mean return of inventory account and base wealth per unit time, respectively,
- r is the constant continuously compounded risk-free rate, and
- Z_I and Z_Y are Wiener processes with mean zero and constant variance rates, σ_I^2 and σ_Y^2, respectively.

The ***objective*** of the dealer is to **maximize the expected utility of his total wealth**, **$E_t[U(W_T)]$, at the terminal time T**, where

$$\boxed{W_T = F_T + I_T + Y_T}.$$

2.2 Avellaneda-Stoikov Model

-M. Avellaneda and S. Stoikov, *High-frequency trading in a limit order book*, Quantitative Finance, 8 (2008), 217–224.

Avellaneda-Stoikov (2008) modified the model of Ho and Stoll (1981) in some aspects.

(i) Assume that the money market pays no interest, and the market price, or mid-price, of the stock evolves over time according to the following zero-drift diffusion process:

$$dS_u = \sigma dW_u \tag{2.1}$$

where the initial value $S_t = s$, $\{W_u\}_{t \le u \le T}$ is a standard Brownian motion and σ is a constant, i.e., a constant volatility model.

(ii) The agent's ***objective*** is to **maximize the expected exponential utility of his portfolio at the terminal time T**. The exponential utility is given by

$$\boxed{u(w) = -\exp(-\gamma w)} \tag{2.2}$$

where γ **is the *risk-aversion* parameter**.

(iii) The ***Poisson intensity*** at which the agent's orders are executed is supposed to be ***exponential***. In the symmetric case, exponential arrival rates are assumed to take the following form:

$$\boxed{\lambda(\delta) = Ae^{-k\delta}} \tag{2.3}$$

where δ is the distance of a LO from the mid-price.

(iv) They introduce the **reservation bid and ask prices** $r^b(s, q, t)$ and $r^a(s, q, t)$, which can be interpreted as the **indifference prices for buying and selling**, respectively. They satisfy

$$\begin{cases} v(x - r^b(s, q, t), s, q + 1, t) = v(x, s, q, t) \\ v(x + r^a(s, q, t), s, q - 1, t) = v(x, s, q, t) \end{cases} \tag{2.4}$$

where $v(x, s, q, t) = E_t[u(W_T)]$, x is the initial wealth at time t and q is the initial inventory level at time t.

-In their model, it is assumed that **there is only one monopolistic dealer in the trading system**. The dealer buys or sells ***one share*** in the market.

-The dealer quotes the bid price p^b and the ask price p^a, and is committed to, respectively, buy and sell one share of stock at these prices.

-The wealth in cash X_t jumps whenever there is a buy order or sell order and it is governed by

$$dX_t = p^a d\mathbf{N_t^a} - p^b d\mathbf{N_t^b}. \tag{2.5}$$

Here N_t^b is the amount of stocks bought by the dealer and N_t^a is the amount of stocks sold. They are assumed to follow Poisson processes with intensities λ^b and λ^a, respectively.

-The number of units of the stock or the inventory level held by the dealer is then governed by

$$dq_t = dN_t^b - dN_t^a. \tag{2.6}$$

-The objective of the dealer who can set limit orders is

$$\boxed{u(s,x,q,t) = \max_{\delta^a,\delta^b} E_t\left[-\exp(-\gamma(X_T + q_T S_T))\right]} \tag{2.7}$$

where $\delta^a = p^a - s$, $\delta^b = s - p^b$ and the dealer holds q stocks at time t.

• For the case of active dealers, who will make decisions to buy or sell before the terminal time T, they derived the following HJB equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 u}{\partial s^2} + \max_{\delta^b} \lambda^b \left[u(s, x - s + \delta^b, q + 1, t) - u(s, x, q, t)\right] \\ + \max_{\delta^a} \lambda^a \left[u(s, x + s + \delta^a, q - 1, t) - u(s, x, q, t)\right] = 0 \\ u(s, x, q, T) = -\exp(-\gamma(x + qs)). \end{cases} \tag{2.8}$$

• To solve the HJB equation, they consider the simplest case by assuming that the Poisson intensities take the following form:

$$\boxed{\lambda^b(\delta) = \lambda^a(\delta) = Ae^{-k\delta}.}$$

Then the following form of solution was adopted:

$$\boxed{u(s, x, q, t) = -\exp(-\gamma x)\exp(-\gamma \theta(s, q, t))}$$

where $\theta(s, q, t)$ is approximated up to the second-order of a Taylor expansion about the inventory variable q:

$$\theta(s, q, t) = \theta^{(0)}(s, t) + \theta^{(1)}(s, t)q + \theta^{(2)}(s, t)q^2 + \ldots + . \tag{2.9}$$

• Substituting θ into Eq. (2.8) yields

$$\begin{cases} \theta_t + \frac{1}{2}\sigma^2\theta_{ss} - \frac{1}{2}\sigma^2\gamma\theta_s^2 + \max_{\delta^b}\left(\frac{\lambda^b(\delta^b)}{\gamma}(1 - e^{\gamma(s-\delta^b-r^b)})\right) \\ + \max_{\delta^a}\left(\frac{\lambda^a(\delta^a)}{\gamma}(1 - e^{-\gamma(s+\delta^a-r^a)})\right) = 0 \\ \theta(s,q,T) = qs. \end{cases} \tag{2.10}$$

• Using the first-order optimality condition, the problem can be transformed into the following one:

$$\begin{cases} \theta_t + \frac{1}{2}\sigma^2\theta_{ss} - \frac{1}{2}\sigma^2\gamma\theta_s^2 + \frac{A}{k+\gamma}\left(e^{-k\delta^a} + e^{-k\delta^b}\right) = 0 \\ \theta(s,q,T) = qs. \end{cases} \tag{2.11}$$

• They consider an asymptotic expansion of θ about q, and higher order terms are assumed to be small enough to be negligible. By considering the coefficients of q and q^2, we obtained the indifference price

$$r(s,q,t) = \frac{r^b(s,q,t) + r^a(s,q,t)}{2} = s - q\gamma\sigma^2(T-t) \tag{2.12}$$

and the bid-ask spread

$$\delta^b + \delta^a = \gamma\sigma^2(T-t) + \frac{2}{\gamma}\ln\left(1 + \frac{\gamma}{k}\right). \tag{2.13}$$

3 Motivations

• The economic setting of the HFT problem is similar. The main differences are the nature of **"true" price** of the underlying asset, the choices of **order types** and the **objective** functions.

• It seems that the existing literature paid more attention to trading strategies in a single agent's framework while relatively little attention has been paid to a multi-agent case. Part of the reason for this is that the accounting for multiple agents leads to
(i) **high dimensional** problems and
(ii) **complicated interactions**, and
(iii) typically **renders the analysis intractable** using standard techniques in stochastic optimal control.

• Intuitively speaking, other than in economic field, large population dynamical multiple-agent competitive phenomena occur in various other field including **communication networks**, **biological systems** and **social sciences**.

• We extend Avellaneda-Stoikov (2008)'s quantitative model for a single agent to the case of multiple agents, and investigate the optimal High Frequency Trading (HFT) strategy in a **competitive** trading environment.

• Our research work contributes to the HFT literature in various aspects:

$\boxed{(i)}$ We derive the optimal bid and ask prices for each agent when the agent is informed the severity of the competition (for example, how many active agents are in the market)

$\boxed{(ii)}$ We compare quoting prices with those obtained in Avellaneda and Stoikov (2008) to shed some lights on trading competition in the market

$\boxed{(iii)}$ We also conduct comparison of the profit generated by agents in competitive markets with that in single agent markets. This comparison may hopefully enhance our understanding on how HFT agents gain profit by providing stock liquidity.

4 The Limit Order Book Rates

• ***(Single-agent).*** In Avellaneda-Stoikov's model, the arrival rates of market buy/sell orders that will reach a **monopolistic agent** is given by

$$\boxed{\lambda^a(\delta) = \lambda^b(\delta) = Ae^{-k\delta}}$$

where δ represents the distance of a LO from the mid-price.

• ***(Multi-agent).*** In the case of multi-agent market under competition (e.g., N agents), suppose that agents' impacts on the overall frequency of orders

$$\lambda^a(\delta_1^a, \ldots, \delta_N^a) \quad \text{and} \quad \lambda^b(\delta_1^b, \ldots, \delta_N^b)$$

are **"separable"** and have an **"identical functional form"**, i.e.,

$$\lambda^a(\delta_1^a, \cdots, \delta_N^a) = f_1(\delta_1^a) f_2(\delta_2^a) \cdots f_N(\delta_N^a) \quad \text{and} \quad f_i(\delta_i^a) = f(\delta_i^a)^{\beta_i}.$$

• Here β_i describes Agent i's impact on orders' overall frequency.

• The assumption that the intensities are "separable" and of "identical form" means that in our framework, the market is assumed to be perfect (no asymmetric information risk) and the sensitivity of the market demand with respect to individual supply changes remain the same.

• Then we have

$$\boxed{\lambda^a(\delta_1^a, \ldots, \delta_N^a) = Ae^{-k(\beta_1\delta_1^a + \cdots + \beta_N\delta_N^a)}.}$$

• Furthermore, if the distribution of the size of orders Q obeys a "power law"

$$f^Q(x) \propto x^{-1-\alpha}$$

and the market impact follows a "log law", i.e.,

$$\Delta p \propto \ln(Q)$$

then it can be shown that

$$\boxed{\lambda_i^a = Ae^{-k(\beta_1\delta_1^a + \cdots + \beta_N\delta_N^a)}e^{-(1-\frac{1}{N})\beta_i\delta_i^a}}$$

5 The Multiple-Agent Problem

5.1 The Basic Model

• Assume that the mid-price evolves according to

$$dS_t = \sigma dW_t$$

• Each agent (e.g., Agent i) has controls over his ask/bid prices $p_i^a(t)$ and $p_i^b(t)$ during the trading horizon $[0, T]$

• The wealth (in cash) of Agent i jumps whenever there is a buy or a sell order executed

$$\boxed{dX_i(t) = p_i^a(t)dN_i^a(t) - p_i^b(t)dN_i^b(t)}$$

where

$$N_i^a(t) \sim Poi(\lambda_i^a(t)) \quad \text{and} \quad N_i^b(t) \sim Poi(\lambda_i^b(t))$$

with intensities given by

$$\boxed{\begin{cases} \lambda_i^a(t) = Ae^{-k(\beta_1\delta_1^a + \cdots + \beta_N\delta_N^a)}e^{-\beta_i(1-\frac{1}{N})\delta_i^a} \\ \lambda_i^b(t) = Ae^{-k(\beta_1\delta_1^b + \cdots + \beta_N\delta_N^b)}e^{-\beta_i(1-\frac{1}{N})\delta_i^b} \end{cases}} \tag{5.1}$$

• The number of shares held by Agent i, $q_i(t)$, satisfies

$$\boxed{dq_i(t) = dN_i^b(t) - dN_i^a(t)}.$$

• Let

$$\delta_i^a(t) = p_i^a(t) - S_t \quad \text{and} \quad \delta_i^b(t) = S_t - p_i^b(t)$$

Agent i's mark-to-market wealth

$$Y_i = X_i + q_i S$$

evolves according the following dynamics:

$$\boxed{dY_i(t) = \delta_i^a(t) dN_i^a(t) + \delta_i^b(t) dN_i^b(t) + \sigma q_i(t) dW_t} \tag{5.2}$$

• For any agent (e.g., Agent i), suppose his objective is to find the optimal control for

$$\boxed{u_i(s, x_i, q_i, t) = \max_{\delta_i^a, \delta_i^b} E_t \left[-\exp(-\gamma_i Y_i(T)) \right]} \tag{5.3}$$

5.2 An Inactive Agent

• Consider an inactive agent i, who does not have any LOs and simply holds an inventory of q_i stocks until the terminal time T. It is a special case of the feedback control problem in which $(\delta_i^a, \delta_i^b) = (\infty, \infty)$. Under this setting,

$$\boxed{u_i(s, x_i, q_i, t) = -\exp(-\gamma_i x_i)\exp(-\gamma_i q_i s)\exp\left(\frac{\gamma_i^2 q_i^2 \sigma^2 (T-t)}{2}\right)} \tag{5.4}$$

which is the same as the one calculated in the monopolistic market.

• The reservation bid and ask prices are given implicitly by the relations

$$\begin{cases} u_i(s, x_i - r_i^b(s, q_i, t), q_i + 1, t) = u_i(s, x_i, q_i, t) \\ u_i(s, x_i + r_i^a(s, q_i, t), q_i - 1, t) = u_i(s, x_i, q_i, t) \end{cases} \tag{5.5}$$

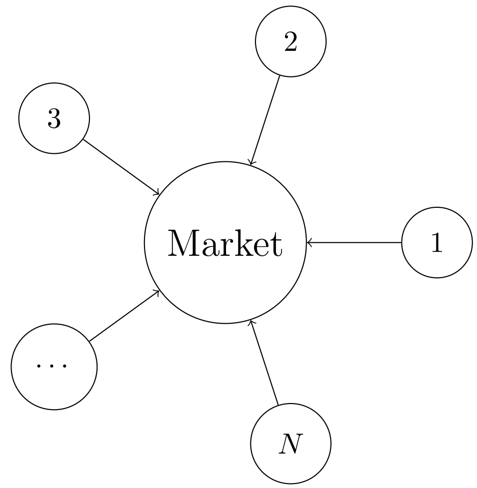
which means that the agent is indifferent between keeping inactive and buying one stock at the reservation bid price r_i^b (or, selling one stock at the reservation ask price r_i^a). It is straightforward to calculate that

$$\boxed{\begin{cases} r_i^b(s, q_i, t) = s - (1 + 2q_i)\frac{\gamma_i \sigma^2 (T-t)}{2} \\ r_i^a(s, q_i, t) = s + (1 - 2q_i)\frac{\gamma_i \sigma^2 (T-t)}{2} \end{cases}} \tag{5.6}$$

and hence the reservation (or indifference) price is given by

$$\boxed{r_i(s, q_i, t) = \frac{r_i^a(s, q_i, t) + r_i^b(s, q_i, t)}{2} = s - q_i \gamma_i \sigma^2 (T-t)} \tag{5.7}$$

5.3 An Active Agent



• In general, it is difficult determine the optimal quoting strategies for dealers in a competitive market.

• In the market, **each active agent's action depends not only on his own but also his competitor's characteristics**.

• They all need to solve a relatively complex Dynamic Programming (DP) problem than the one encountered in the single dealer case.

5.3.1 The One-period Model

• Assume that trades occur immediately after time t_{n-1}. Agents choose their bid/ask quotes at the beginning of the trading session, t_{n-1}, defined through the controls $(\delta^{i,b}_{n-1}, \delta^{i,a}_{n-1})^N_{i=1}$.

• These quotes influence the arrival rates of market orders over the time interval $(t_{n-1,}, t_n)$. By Eq. (5.1), the arrival rates take the following forms:

$$\begin{cases} \lambda^{i,a}_{n-1} &= Ae^{-k(\beta_1\delta^{1,a}_{n-1}+\cdots+\beta_N\delta^{N,a}_{n-1})}e^{-(1-\frac{1}{N})\beta_i\delta^{i,a}_{n-1}} \\ \lambda^{i,b}_{n-1} &= Ae^{-k(\beta_1\delta^{1,b}_{n-1}+\cdots+\beta_N\delta^{N,b}_{n-1})}e^{-(1-\frac{1}{N})\beta_i\delta^{i,b}_{n-1}}. \end{cases} \tag{5.8}$$

• For any agent in this competitive market, the objective is to determine the optimal bid/ask quotes to maximize his own expected utility function:

$$\boxed{\begin{aligned} & V^i(s_{n-1}, x^i_{n-1}, \gamma_1, \cdots, \gamma_N, q^1_{n-1}, \cdots, q^N_{n-1}, t_{n-1}) \\ = & \max_{\delta^{i,a}_{n-1}, \delta^{i,b}_{n-1}} \left\{E\left[-\exp(-\gamma_i\left(X^i_T + q^i_T S_T\right))|\mathcal{F}_{n-1}\right]\right\} \end{aligned}} \tag{5.9}$$

• For any agent, he can only determines his own bid/ask quotes $\delta_{n-1}^{i,b}$ and $\delta_{n-1}^{i,a}$. However, the stochastic feedback problem is also related to other agents' bid/ask quotes $\delta_{n-1}^{1,b}, \cdots, \delta_{n-1}^{N,b}$ and $\delta_{n-1}^{1,a}, \cdots, \delta_{n-1}^{N,a}$.

• Suppose all agents achieve their Nash equilibrium in this game problem, then the optimal quoting policy in the one-period case is

$$\boxed{\begin{cases} \delta_{n-1}^{i,a} = \dfrac{1}{\gamma_i}\ln\left(1+\dfrac{\gamma_i}{(k+1-\frac{1}{N})\beta_i}\right)+\dfrac{\gamma_i\sigma^2(T-t_{n-1})}{2}(-2q_{n-1}^i+1) \\ \delta_{n-1}^{i,b} = \dfrac{1}{\gamma_i}\ln\left(1+\dfrac{\gamma_i}{(k+1-\frac{1}{N})\beta_i}\right)+\dfrac{\gamma_i\sigma^2(T-t_{n-1})}{2}(2q_{n-1}^i+1) \end{cases}} \tag{5.10}$$

and Agent i's utility is given by

$$\begin{aligned} & V^i(s_{n-1}, x_{n-1}^i, \gamma_1, \cdots, \gamma_N, q_{n-1}^1, \cdots, q_{n-1}^N, t_{n-1}) \\ = \; & -\exp\left(-\gamma_i(x_{n-1}^i+q_{n-1}^i s_{n-1})\right)\exp\left(\frac{\gamma_i^2\sigma^2(q_{n-1}^i)^2(T-t_{n-1})}{2}\right) \\ & \left[1-\tfrac{\gamma_i\Delta t_{n-1}}{(k+1-\frac{1}{N})\beta_i+\gamma_i}\left(\lambda_{n-1}^{i,a}+\lambda_{n-1}^{i,b}\right)\right] \end{aligned} \tag{5.11}$$

where $\Delta t_{n-1} = t_n - t_{n-1}$.

We remark that

(i) Only in the one-period case, agents' bid/ask quotes are independent of their competitors. However, even in the one-period case, their value functions are not independent of the inventory position and other parameters, such as risk aversion of their competitors.

(ii) Agent i's bid-ask spread

$$\delta_{n-1}^{i,b} + \delta_{n-1}^{i,a} = \frac{2}{\gamma_i} \ln\left(1 + \frac{\gamma_i}{(k+1-\frac{1}{N})\beta_i}\right) + \gamma_i \sigma^2 (T - t_{n-1})$$

is independent of the inventory. After taking a first-order approximation of the order arrival terms, we have

$$\lambda_{n-1}^{i,a} + \lambda_{n-1}^{i,b} = A\left[2 - (k+1-\frac{1}{N})\beta_i(\delta_{n-1}^{i,a} + \delta_{n-1}^{i,b}) - k\sum_{j\neq i}\beta_j(\delta_{n-1}^{j,a} + \delta_{n-1}^{j,b}) + \cdots + \right]$$

where the linear term does not depend on the inventory variables. Therefore, if we substitute the approximation into Eq. (5.11), we arrive at the conclusion that Agents' utilities depend only on their own inventories. We define this approximation as

$$f^i(s_{n-1}, x_{n-1}^i, q_{n-1}^i, \gamma_1, \cdots, \gamma_N, t_{n-1})$$

which equals

$$-\exp\left(-\gamma_i(x_{n-1}^i+q_{n-1}^i s_{n-1})\right)\exp\left(\frac{\gamma_i^2\sigma^2(q_{n-1}^i)^2(T-t_{n-1})}{2}\right)h_{n-1}^i$$

where

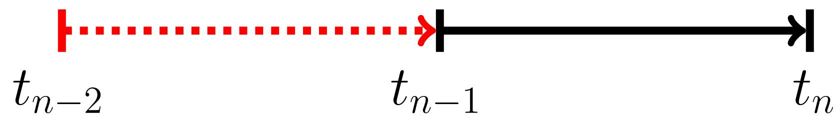
$$\begin{aligned} h_{n-1}^i \;&=\; 1-\frac{A\gamma_i\Delta t_{n-1}}{(k+1-\frac{1}{N})\beta_i+\gamma_i}\Big\{2-(k+1-\frac{1}{N})\beta_i\Big[\frac{2}{\gamma_i}\ln\left(1+\frac{\gamma_i}{(k+1-\frac{1}{N})\beta_i}\right) \\ &+\;\gamma_i\sigma^2(T-t_{n-1})\Big]-k\sum_{j\neq i}\beta_j\Big[\frac{2}{\gamma_i}\ln\left(1+\frac{\gamma_j}{(k+1-\frac{1}{N})\beta_j}\right)+\gamma_j\sigma^2(T-t_{n-1})\Big]\Big\}. \end{aligned}$$

(iii) Notice that

$$\begin{aligned} & V^i(s_{n-1},x_{n-1}^i,\gamma_1,\cdots,\gamma_N,q_{n-1}^1,\cdots,q_{n-1}^N,t_{n-1}) \\ =\; & -\exp\left(-\gamma_i(x_{n-1}^i+q_{n-1}^i s_{n-1})\right)\exp\left(\frac{\gamma_i^2\sigma^2(q_{n-1}^i)^2(T-t_{n-1})}{2}\right) \\ & \left[1-\tfrac{\gamma_i\Delta t_{n-1}}{(k+1-\frac{1}{N})\beta_i+\gamma_i}\left(\lambda_{n-1}^{i,a}+\lambda_{n-1}^{i,b}\right)\right] \\ >\; & -\exp\left(-\gamma_i(x_{n-1}^i+q_{n-1}^i s_{n-1})\right)\exp\left(\frac{\gamma_i^2\sigma^2(q_{n-1}^i)^2(T-t_{n-1})}{2}\right) \end{aligned}$$

which means that active agents will always have advantage over the inactive agents.

5.3.2 The Two-period Model



• Assume that agents may only trade in the intervals (t_{n-2}, t_{n-1}) and (t_{n-1}, t_n). They can only choose their bid/ask quotes at time t_{n-2} and t_{n-1}. Suppose trades occur immediately after time t_{n-2} and t_{n-1}. Adopting the above linear approximation, we have

$$\boxed{\begin{cases} \delta^{i,b}_{n-2} = \dfrac{1}{\gamma_i} \ln \left(1 + \dfrac{\gamma_i}{(k+1-\frac{1}{N})\beta_i} \right) + \dfrac{\gamma_i \sigma^2 (T - t_{n-2})}{2} (2q^i_{n-2} + 1) \\ \delta^{i,a}_{n-2} = \dfrac{1}{\gamma_i} \ln \left(1 + \dfrac{\gamma_i}{(k+1-\frac{1}{N})\beta_i} \right) + \dfrac{\gamma_i \sigma^2 (T - t_{n-2})}{2} (-2q^i_{n-2} + 1) \end{cases}} \tag{5.12}$$

and the utility is given by

$$\begin{aligned} & V^i \left(s_{n-2}, x^i_{n-2}, \gamma_1, \cdots, \gamma_N, q^1_{n-2}, \cdots, q^N_{n-2}, t_{n-2} \right) \\ = \; & -\exp(-\gamma_i (x^i_{n-2} + q^i_{n-2} s_{n-2})) \exp \left(\frac{\gamma_i^2 \sigma^2 (q^i_{n-2})^2 (T - t_{n-2})}{2} \right) \\ & \left[1 - \tfrac{\gamma_i \Delta t_{n-2}}{(k+1-\frac{1}{N})\beta_i} \left(\lambda^{i,a}_{n-2} + \lambda^{i,b}_{n-2} \right) \right] h^i_{n-1} \end{aligned} \tag{5.13}$$

where $\Delta t_{n-2} = t_{n-1} - t_{n-2}$.

- We remark that the spread

$$\delta^{i,b}_{n-2} + \delta^{i,a}_{n-2} = \frac{2}{\gamma_i} \ln\left(1 + \frac{\gamma_i}{(k+1-\frac{1}{N})\beta_i}\right) + \gamma_i \sigma^2 (T - t_{n-2})$$

is independent of the inventory. By taking a first-order approximation of the order arrival terms, we have

$$\lambda^{i,b}_{n-2} + \lambda^{i,a}_{n-2} = A\left[2 - (k+1-\frac{1}{N})\beta_i(\delta^{i,a}_{n-2} + \delta^{i,b}_{n-2}) - k\sum_{j\neq i}\beta_j(\delta^{j,a}_{n-2} + \delta^{j,b}_{n-2}) + \cdots +\right].$$

The linear term does not depend on the inventory. Similar to the one-period case, substituting the linear approximation into Eq. (5.12), one can get an approximation of the utility V^i, i.e.,

$$f^i(s_{n-2}, q^i_{n-2}, x^i_{n-2}, \gamma_1, \cdots, \gamma_N, t_{n-2})$$

which equals to

$$-\exp(-\gamma_i(x^i_{n-2} + q^i_{n-2}s_{n-2}))\exp\left(\frac{\gamma_i^2\sigma^2(q^i_{n-2})^2(T - t_{n-2})}{2}\right)h^i_{n-2}h^i_{n-1}$$

and only depends on the inventory q^i_{n-2}.

5.3.3 The Multi-period Model

By repeating the argument of this analysis, one can get the following result for the multi-period model:

- In the n-period model, agents' optimal bid/ask quotes are given by

$$\boxed{\begin{cases} \delta_l^{i,b} & = \dfrac{1}{\gamma_i}\ln\left(1+\dfrac{\gamma_i}{(k+1-\frac{1}{N})\beta_i}\right)+\dfrac{\gamma_i\sigma^2(T-t_l)}{2}(2q_l^i+1) \\ \delta_l^{i,a} & = \dfrac{1}{\gamma_i}\ln\left(1+\dfrac{\gamma_i}{(k+1-\frac{1}{N})\beta_i}\right)+\dfrac{\gamma_i\sigma^2(T-t_l)}{2}(-2q_l^i+1) \end{cases}} \tag{5.14}$$

and their utilities are

$$\begin{aligned} & V^i\left(s_l, x_l^i, \gamma_1, \ldots, \gamma_N, q_l^1, \ldots, q_l^N, t_{n-2}\right) \\ = \; & -\exp(-\gamma_i(x_l^i+q_l^i s_l))\exp\left(\frac{\gamma_i^2\sigma^2(q_l^i)^2(T-t_l)}{2}\right) \\ & \left[1-\tfrac{\gamma_i\Delta t_l}{(k+1-\frac{1}{N})\beta_i}\left(\lambda_l^{i,a}+\lambda_l^{i,b}\right)\right]\textstyle\prod_{m=l+1}^{n-1} h_m^i \end{aligned} \tag{5.15}$$

where $\Delta t_l = t_{l+1} - t_l$.

5.3.4 The Continuous Model

In every step of the back-forward model, we adopt the first-order approximation of the arrival terms appearing in the utility function. Then we find that an approximate agent's utility functions depend *only* on their own inventories. We then consider the case of continuous model. Define the approximate utility as $u_i(s, x_i, q_i, t)$.

The following theorem results from applying the principle of Dynamic Programming (DP).

- The optimal bid/ask quotes in dealer markets under competition are given by

$$\boxed{\begin{cases} \delta_t^{i,b} &= \dfrac{1}{\gamma_i} \ln\left(1 + \dfrac{\gamma_i}{(k+1-\frac{1}{N})\beta_i}\right) + \dfrac{\gamma_i \sigma^2 (T-t)}{2}(2q_i + 1) \\ \delta_t^{i,a} &= \dfrac{1}{\gamma_i} \ln\left(1 + \dfrac{\gamma_i}{(k+1-\frac{1}{N})\beta_i}\right) + \dfrac{\gamma_i \sigma^2 (T-t)}{2}(-2q_i + 1) \end{cases}} \tag{5.16}$$

and agents' approximate utility functions under the quoting strategy are greater than those for the inactive case, i.e.,

$$\boxed{u_i(s, x_i, q_i, t) > -\exp(-\gamma_i(x_i + q_i s)) \exp\left(\frac{\gamma_i^2 q_i^2 \sigma^2 (T-t)}{2}\right)} \tag{5.17}$$

We remark that

(i) For the "frozen inventory" problem,

$$\begin{cases} E_t[x_i + q_i S_T] = x_i + q_i s \\ u_i(s, x_i, q_i, t) = -\exp(-\gamma_i x_i)\exp(-\gamma_i q_i s)\exp\left(\frac{\gamma_i^2 q_i^2 \sigma^2 (T-t)}{2}\right). \end{cases}$$

For the active dealer,

$$\begin{cases} E_t[X_i(T) + q_i(T) S_T] = x_i + q_i s + E_t\left[\int_t^T \delta_i^a dN_i^a + \int_t^T \delta_i^b dN_i^b\right] > x_i + q_i s \\ u_i(s, x_i, q_i, t) > -\exp(-\gamma_i x_i)\exp(-\gamma_i q_i s)\exp\left(\frac{\gamma_i^2 q_i^2 \sigma^2 (T-t)}{2}\right). \end{cases}$$

This means that active agents using our strategy to quoting always have an advantage over inactive ones.

(ii) When $N = 1$, then $\beta_i = 1$,

$$\delta_i^b + \delta_i^a = \gamma_i \sigma^2 (T - t) + \frac{2}{\gamma_i} \ln\left(1 + \frac{\gamma_i}{k}\right)$$

which coincides with the results in Avellanede and Stoikov (2008).

6 Numerical Results

• We adopt the algorithm below to solve the problem:

Step 1: Given the state variables at time t, compute δ_i^b and δ_i^a for each agent i, $(i = 1, 2, \ldots, N)$

Step 2: At time $t + dt$, the state variables are updated:

with probability $\lambda_i^a(\delta_i^a)dt$, Agent i's inventory decreases by 1, wealth increases by $s + \delta_i^a$;

with probability $\lambda_i^b(\delta_i^b)dt$, Agent i's inventory increases by 1, wealth decreases by $s - \delta_i^b$.

The mid-price is updated by a random increment $\pm\sigma\sqrt{dt}$.

• Suppose that there are N dealers in a market. In the numerical experiments, we assume β_i's to be identical, i.e., $\beta_i = 1/N$ $(i = 1, 2, \ldots, N)$. As far as our simulation is concerned, we chose the following parameters:

$$s = 100, t = 0, T = 1, \sigma = 2, dt = 0.005, q_i = 0, \gamma_i = 0.1, k = 1.5, A = 140$$

The values of the parameters are chosen to be the same as those in Avellaneda & Stoikov (2008).

• We show the case $N = 2$ and $\beta_i = 0.5$ for $i = 1, 2$.

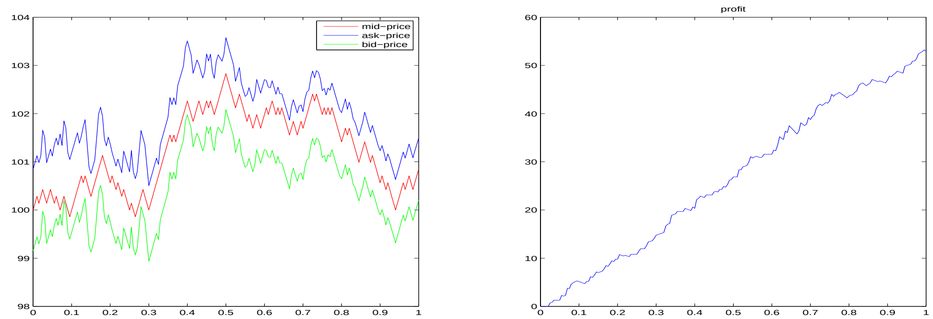
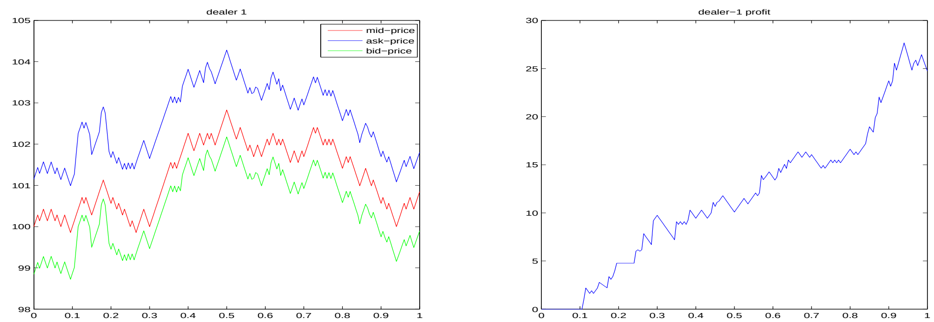
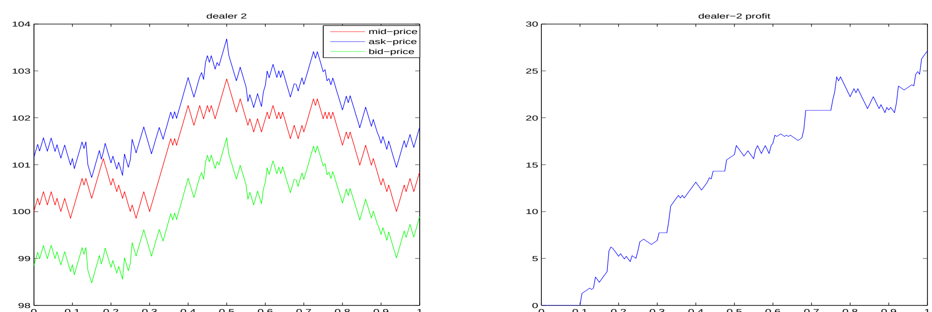


Figure 3: The mid-price and the optimal bid-ask quotes of one monopolistic dealer.



(a) Dealer 1



(b) Dealer 2

Figure 4: The mid-price and the optimal bid-ask quotes of two dealers.

6.1 The Effect of N

Table 6.1.1: 1000 simulations of one dealer with $\gamma = 0.1$ and $\beta_1 = 1$.

Agent	Average Spread	Profit	Std (Profit)	q_T	Std (q_T)
Dealer	1.49	64.26	5.68	0.20	3.40

Table 6.1.2: 1000 simulations of two dealers with $\gamma_i = 0.1$ and $\beta_i = 0.5$, $N = 2$.

Agent	Average Spread	Profit	Std (Profit)	q_T	Std (q_T)
Dealer 1	2.11	29.15	6.09	-0.02	2.88
Dealer 2	2.11	29.40	6.22	0.08	2.79

Table 6.1.3: 1000 simulations of three dealers with $\gamma_i = 0.1$ and $\beta_i = 1/3$, $N = 3$.

Agent	Average Spread	Profit	Std (Profit)	q_T	Std (q_T)
Dealer 1	2.79	15.69	5.65	0.14	2.51
Dealer 2	2.79	15.85	5.69	-0.11	2.58
Dealer 3	2.79	15.88	5.53	-0.01	2.44

Table 6.1.4: 1000 simulations of seven dealers with $\gamma_i = 0.1$ and $\beta_i = 1/7$, $N = 7$.

Agent	Average Spread	Profit	Std (Profit)	q_T	Std (q_T)
Dealer 1	5.40	1.76	2.34	0.028	0.79
Dealer 2	5.40	1.93	2.52	-0.045	0.84
Dealer 3	5.40	1.95	2.57	0.003	0.82
Dealer 4	5.40	1.79	2.46	-0.03	0.80
Dealer 5	5.40	1.83	2.49	0.003	0.79
Dealer 6	5.40	1.86	2.50	-0.001	0.83
Dealer 7	5.40	1.86	2.60	-0.019	0.86

6.2 Sensitivity Study of γ

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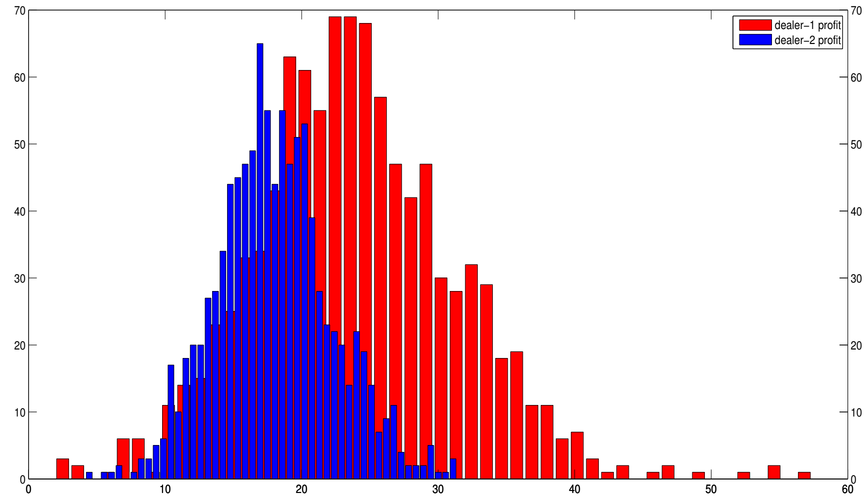

Figure 5: $\gamma_1 = 0.01$, $\gamma_2 = 1$ and $\beta_1 = \beta_2 = 0.5$

Table 6.2.1: 1000 simulations of two dealers with $\gamma_1 = 0.01$, $\gamma_2 = 1$ and $\beta_i = 0.5$.

Agent	Average Spread	Profit	Std (Profit)	q_T	Std (q_T)
Dealer 1	2.01	23.97	7.44	0.07	4.30
Dealer 2	3.39	17.98	4.25	-0.074	1.61

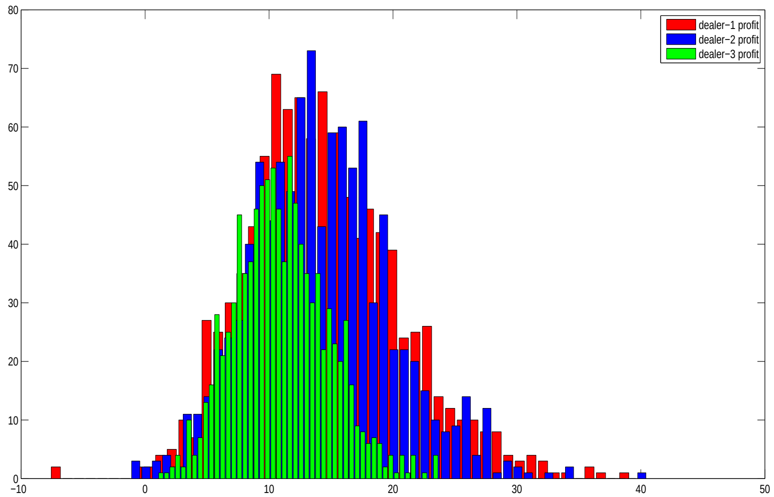

Figure 6: $\gamma_1 = 0.01$, $\gamma_2 = 0.1$, $\gamma_3 = 1$ and $\beta_1 = \beta_2 = \beta_3 = 1/3$

Table 6.2.2: 1000 simulations of three dealers with $\gamma_1 = 0.01$, $\gamma_2 = 0.1$, $\gamma_3 = 1$ and $\beta_i = 1/3$.

Agent	Average Spread	Profit	Std (Profit)	q_T	Std (q_T).
Dealer 1	2.77	14.36	6.26	-0.09	3.43
Dealer 2	2.79	14.21	5.68	-0.01	2.70
Dealer 3	3.74	10.99	3.74	0.08	1.47

6.3 Sensitivity Study of Initial Inventory Positions

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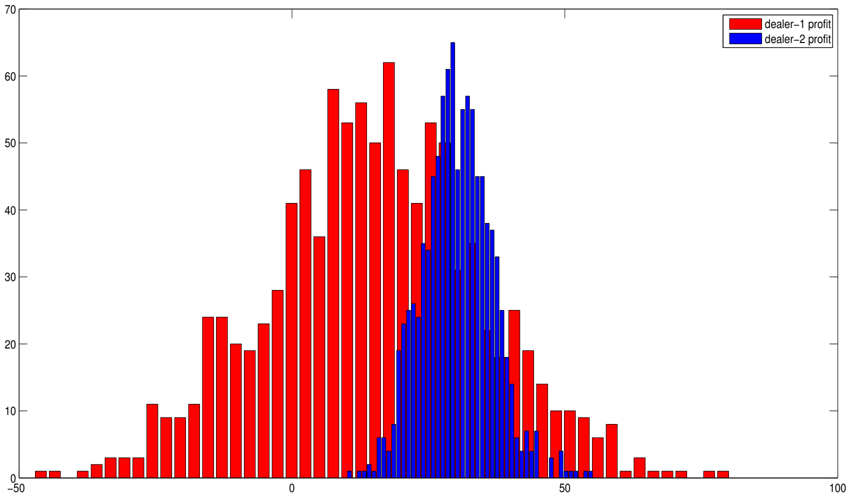

Figure 7: $\gamma_1 = \gamma_2 = 0.1$, $\beta_1 = \beta_2 = 0.5$, $q_1 = 10$ and $q_2 = 1$.

Table6.3.1: 1000 simulations of two dealers with $\gamma_1 = \gamma_2 = 0.1$, $\beta_1 = \beta_2 = 0.5$, $q_1 = 10$ and $q_2 = 1$.

Agent	Average Spread	Profit	Std (Profit)	q_T	Std (q_T)
Dealer 1	2.11	14.85	19.59	0.19	2.94
Dealer 2	2.11	30.24	6.34	-0.13	3.00

Table 6.3.2: 1000 simulations of two dealers with $\gamma_1 = \gamma_2 = 0.1$, $\beta_1 = \beta_2 = 0.5$, $q_1 = 50$ and $q_2 = 0$.

Agent	Average Spread	Profit	Std (Profit)	q_T	Std (q_T)
Dealer 1	2.11	-391.98	84.14	2.09	2.99
Dealer 2	2.11	37.55	12.45	-1.98	2.91

Table 6.3.3:1000 simulations of two dealers with $\gamma_1 = \gamma_2 = 0.01$, $\beta_1 = \beta_2 = 0.5$, $q_1 = 50$ and $q_2 = 0$.

Agent	Average Spread	Profit	Std (Profit)	q_T	Std (q_T)
Dealer 1	2.01	13.11	40.42	25.31	4.74
Dealer 2	2.01	32.69	16.18	-7.12	4.57

7 Summary

• A framework for modeling and investigating the optimal HFT strategies in a competitive market is proposed. The modeling framework is built based on Avellande-Stoikov's quantitative model for a monopolistic agent.

• We recursively derived an approximate optimal bid/ask prices for each agent in the financial market when informed the severity of the competition.

• We also analyze the effect of various parameters in our model on the bid-ask quotes and profits of trading.

• Further works:
(i) Take in to account the presence of additional market factors, such as order handling costs, asymmetric information and inter-dealer trading, in our model;
(ii) Consider a more general model for the mid-price;
(iii) Combine the use of limit orders and market orders.

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