Online Portfolio Selection Problems

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Abstract

Online portfolio selection (OLPS) is attracting a lot of attention due to its efficiency and practicability in deriving optimal investment strategies in the investment activities where the market information is constantly renewed in a short period. The first issue in OLPS is the prediction of the future returns of risky assets given historical data. The second issue is to obtain optimal investment strategies based on the predictions. In this talk, we present some models and adaptive algorithms for the captured problems. Numerical experiments are given to validate the effectiveness of the proposed models.

Keywords: Online portfolio selection; Adaptive moving average method; Mean-variance model; Quadratic programming.

Outline

- Introduction to Online Portfolio Selection (OLPS).
- Problem Formulation.
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- Net Profit Maximization Model with Transaction Costs.
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- Adaptive Mean-Variance Model.
- Numerical Experiments (II).
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Introduction to Online Portfolio Selection (OLPS)

• Online portfolio selection attracts both researchers and practitioners. It is different from the traditional portfolio selection theory proposed by Markowitz (1952) in his seminal work.

• In a **traditional portfolio selection problem**, it is usually assumed that the return of a risky asset is subject to a certain **distribution function**.

-Based on the distribution function, the **expected value** and **variance** of the return can be calculated to measure the **expected return** and **risk**, respectively.

-Then investors allocate the capital in different assets to achieve excess **investment return** or avert the **investment risk**.

• In contrast, following Kelly investment theory, **online portfolio selection** concerns more on employing modern techniques to **predict the future returns of risky assets** and **making the optimal investment strategy**.

• In reality, it is difficult to find a distribution function which can well describe the real historical return data of an asset. It is therefore inefficient to estimate all distribution functions of risky assets before making portfolio selection decisions.

• Online portfolio selection focuses on exploring the most **efficient** and **practical computational intelligence techniques** to deal with real online asset trading problems.

• It is a **sequential decision making optimization problem** where the investment strategy is determined at the beginning of each short period.

Introduction to Online Portfolio Selection (OLPS)

Online portfolio strategies can be classified into **five types**.

(I) The first one is called "**Benchmark**".

• One widely adopted Benchmark is the **Uniform Buy-and-Hold strategy**, which is also called the **Market strategy**, Li and Hoi (2014). In this strategy, the available capital is **uniformly distributed** into all the risky assets in each period.

• Another Benchmark is called the **Best stock strategy**, Li and Hoi (2014), where all the capital is invested into **the best asset** in the whole investment process.

• Constant Rebalanced Portfolios (CRP) strategy is a popular Benchmark where the allocation proportions of the risky assets are the **same in all periods**. There are two special CRPs: **Uniform Constant Rebalanced Portfolios** (UCRP) Li and Hoi (2015) and **Best Constant Rebalanced Portfolios** (BCRP) Cover (1991).

(II) The second type of methods focuses on the "**Follow the Winner**" strategy. They are based on the **momentum principle** which assumes that the risky assets performing well currently will continue achieving good performance in the next period.

• Cover (1991) proposed **Universal Portfolio (UP) strategy**, which first distributed the capital to **various portfolio managers** and derived the corresponding returns, then obtain the weighted average of returns of all strategies.

• Helmbold et al. (1998) proposed the **Exponential Gradient (EG) method** in which **exponentiated gradient update** was employed to calculate the investment proportions based on the past return data.

• Agarwal et al. (2006) employed the **Online Newton Step (ONS) method** to tackle online portfolio selection, where the **gradient** and **Hessian matrix** of the **log function of cumulative return** are computed.

(III) There are some “**Follow the Loser**” approaches built on the **mean reversion principle**, which claims that the risky assets performing well in the past may return to normal or perform poorly in the next period. Therefore, it is encouraged to **buy the current under-performing risky assets** and **sell the over-performing assets**.

• Borodin et al. (2004) proposed the **Anti-correlation (Anticor) method** based on the **mean reversion principle**, where the proportions were transferred from the assets performing well to assets performing poorly, and **the explicit amounts of transferred proportions** are determined by the **cross-correlation matrix** of different risky assets.

- Li et al. (2012) proposed the **Passive-aggressive Mean Reversion (PAMR) method** based on a loss function. Current portfolio will be kept if its return is **below a certain return threshold** under the assumption that **under-performing risky assets will perform better** in the next period.

- Similar to PAMR, Li et al. (2011, 2013) proposed the **Confidence Weighted Mean Reversion (CWMR) method** by modeling the portfolio vector with Gaussian distribution and update the distribution constantly following the **mean reversion principle**.

- Huang et al. (2016) proposed the **Robust Median Reversion (RMR) strategy** where the robust L_1-median estimator was adopted to exploit the reversion phenomenon. **The RMR runs in linear time which is easy to implement in real algorithmic trading**.

• The above PAMR and CWMR employed the **single-period mean reversion assumption** where the price of asset in the next period was estimated with **the price of last period**, which may not achieve good performance.

• To improve this, Li et al. (2012, 2015) employed the **Moving Average method** to predict the price of next period based on multiple prices of previous periods and proposed the **Online Moving Average Reversion (OLMAR) method**.

• In this talk, we shall consider **some extensions (AOLMA and AOLPI)**, Guo et al. (2021, 2024) of the **OLMAR method** for predicting prices/returns of risky assets.

(IV) The fourth type of online portfolio selection strategies focuses on "**Pattern Matching Based Approaches**".

• There are usually **two steps** in pattern matching based approaches.

• The **first step** is **sample selection** intended for selecting the historical price patterns which are similar to the latest price pattern. The selected historical price patterns are used to **estimate the relative return vector** of the whole portfolio in the next period.

• The **second step** is to **construct the portfolio optimization model** based on the selected price patterns.

- Györfi et al. (2006) employed the **nonparametric kernel-based sample selection method** to search for similar price patterns by comparing the **Euclidean distance of different patterns**, and constructed a log-optimal portfolio based on the capital growth theory.

- Li et al. (2011) employed the **correlation-driven nonparametric sample selection method** by using the correlation coefficient of different patterns, and proposed the **Correlation-driven Nonparametric (CORN) learning algorithm**.

(V) The fifth type of online portfolio selection strategies is the "**Meta-learning Approach**". In this approach, **multiple base experts** are defined where each expert is equipped with **different strategies and outputs one portfolio**. Then all the output portfolios are combined together into a final portfolio.

Problem Formulation (I)

- In online portfolio selection, an investor makes sequential decisions according to the changing financial market.
- Denote the investment strategy in Period t by $\mathbf{x}_t = (x_{t1}, x_{t2}, \ldots, x_{tm})$, where x_{ti} **is the proportion allocated to risky asset** i, $(t = 1, 2, \ldots, n,\ i = 1, 2, \ldots, m)$.
- Let the **relative return in Period** t be $\mathbf{r}_t = (r_{t1}, r_{t2}, \ldots, r_{tm})$. We note that $\mathbf{x}_t$ **should be determined at the beginning of Period** t and $\mathbf{r}_t$ **is known at the end of Period** t (See Figure 1).

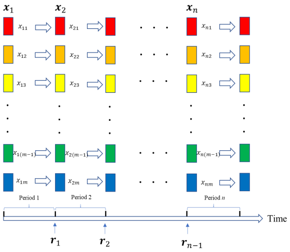


Figure 1: The investment process.

Problem Formulation (I)

- The relative return vector $\mathbf{r}_t = (r_{t1}, r_{t2}, \ldots, r_{tm})$ is calculated as follows:

$$\boxed{\mathbf{r}_t = \mathbf{p}_t / \mathbf{p}_{t-1}}$$

where $\mathbf{p}_t = (p_{t1}, p_{t2}, \ldots, p_{tm})$ is the price at period t and "/" is an element-wise division of two vectors.

- The **Cumulative Return** from the beginning of the investment to **Period** n can be expressed as follows:

$$\boxed{CR_n = \prod_{t=1}^{n} \mathbf{r}_t \mathbf{x}_t^T.} \quad (1)$$

- We note that **transaction cost** is **NOT** considered in Eq. (1). Recall that the decision variables satisfy the following constraints:

$$\boxed{x_{t1} + x_{t2} + \ldots + x_{tm} = 1, \quad t = 1, 2, \ldots, n,} \quad (2)$$

where $0 \le x_{ti} \le 1$, $i = 1, 2, \ldots, m$.

Online Moving Average Method

- Li and Hoi (2012, 2015) proposed **two moving average methods** to predict the single-period return.
- The **first one** is a **Simple Moving Average (SMA) method**: Given the historical stock prices $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_t$ and the truncated **window size** w, the predicted stock price of $\mathbf{p}_{t+1}$ can be calculated as follows:

$$\hat{\mathbf{p}}_{t+1} = \frac{1}{w} \sum_{i=t-w+1}^{t} \mathbf{p}_i.$$

- The **estimated return** for $\mathbf{r}_{t+1}$ can be obtained by

$$\hat{\mathbf{r}}_{t+1} = \frac{\hat{\mathbf{p}}_{t+1}}{\mathbf{p}_t} = \frac{1}{w}\left(\mathbf{1} + \frac{\mathbf{1}}{\mathbf{r}_t} + \frac{\mathbf{1}}{\mathbf{r}_t \cdot \mathbf{r}_{t-1}} + \cdots + \frac{\mathbf{1}}{\prod_{i=0}^{w-2} \mathbf{r}_{t-i}}\right).$$

Here **"1" is the vector of all ones** and the product "·" refers to **the element-wise product of vectors**.

Online Moving Average Method

- The **second one** is the **Exponential Moving Average (EMA) method** which uses all the historical stock prices by assigning each stock price an **exponential weight**.
- The predicted stock price can be calculated as follows:

$$\boxed{\hat{\mathbf{p}}_{t+1} = \alpha \mathbf{p}_t + (1-\alpha)\hat{\mathbf{p}}_t = \alpha \mathbf{p}_t + \alpha(1-\alpha)\mathbf{p}_{t-1} + \cdots + (1-\alpha)^{t-1}\mathbf{p}_1}$$

and the **estimated return** is

$$\boxed{\hat{\mathbf{r}}_{t+1} = \alpha \mathbf{1} + (1-\alpha)\frac{\hat{\mathbf{r}}_t}{\mathbf{r}_t}}$$

where α is the **decaying factor**.

Adaptive Online Moving Average Method

- We then propose the **Adaptive Online Moving Average (AOLMA) method** where the decaying factor can be **adjusted automatically** according to the performances of risky assets.
- Define the **decaying vector** of the whole portfolio at Period t by $\boldsymbol{\alpha}_t = (\alpha_{t1}, \alpha_{t2}, \ldots, \alpha_{tm})$, where α_{ti} is the decaying factor of risky asset i ($i = 1, 2, \ldots, m$).
- Then the **predicted price at Period $(t+1)$** can be expressed as follows:

$$\boxed{\hat{\mathbf{p}}_{t+1} = \boldsymbol{\alpha}_{t+1} \cdot \mathbf{p}_t + (\mathbf{1} - \boldsymbol{\alpha}_{t+1}) \cdot \hat{\mathbf{p}}_t} \qquad (3)$$

and the **predicted return** is

$$\boxed{\hat{\mathbf{r}}_{t+1} = \boldsymbol{\alpha}_{t+1} \cdot \mathbf{1} + (\mathbf{1} - \boldsymbol{\alpha}_{t+1}) \cdot \frac{\hat{\mathbf{r}}_t}{\mathbf{r}_t}.} \qquad (4)$$

Adaptive Online Moving Average Method

- The key of the adaptive moving average method lies in the **decaying factor** α_t. We believe that a time-dependent decaying factor will further improve the prediction accuracy.
- Consider **risky asset** i at **Period** t: the predicted return following from Eq. (4) is given by

$$\boxed{\hat{r}_{ti} = \alpha_{ti} + (1 - \alpha_{ti}) \frac{\hat{r}_{(t-1)i}}{r_{(t-1)i}}}$$

and the **corresponding error** is

$$\boxed{r_{ti} - \hat{r}_{ti} = r_{ti} - \frac{\hat{r}_{(t-1)i}}{r_{(t-1)i}} - \left(1 - \frac{\hat{r}_{(t-1)i}}{r_{(t-1)i}}\right) \alpha_{ti}.} \tag{5}$$

Adaptive Online Moving Average Method

• Our aim is to **improve** (reduce) the **prediction error** in the online portfolio selection process. Once r_{ti} is known, the error can be obtained and can be applied to determine the decaying factor for the next period.

• There are **four cases** that one needs to consider:

Case 1: $r_{ti} > \hat{r}_{ti}$ and $r_{(t-1)i} > \hat{r}_{(t-1)i}$.

Case 2: $r_{ti} > \hat{r}_{ti}$ and $r_{(t-1)i} \leq \hat{r}_{(t-1)i}$.

Case 3: $r_{ti} \leq \hat{r}_{ti}$ and $r_{(t-1)i} > \hat{r}_{(t-1)i}$.

Case 4: $r_{ti} \leq \hat{r}_{ti}$ and $r_{(t-1)i} \leq \hat{r}_{(t-1)i}$.

Adaptive Online Moving Average Method

• For **Case 1**, it can be derived from Eq. (5) that the coefficient of α_{ti} is $-\left(1 - \frac{\hat{r}_{(t-1)i}}{r_{(t-1)i}}\right) < 0$.

• Then, the decaying factor for the next period $\alpha_{(t+1)i}$ can be **increased** by one step size to reduce the prediction error:

$$\alpha_{(t+1)i} = \alpha_{ti} + \tau \tag{6}$$

where τ is the given **step size** of the decaying factor.

• For **Case 2**, the coefficient is $-\left(1 - \frac{\hat{r}_{(t-1)i}}{r_{(t-1)i}}\right) \geq 0$, then the decaying factor $\alpha_{(t+1)i}$ can be **decreased** to reduce the prediction error as follows:

$$\alpha_{(t+1)i} = \alpha_{ti} - \tau. \tag{7}$$

- Similarly, for **Case 3** and **Case 4**, the decaying factor can be updated by Eqs. (6) and (7), respectively.

Remark

-It is reasonable to employ the above decaying factor updating mechanism in online portfolio selection.
-For example, in **Case 1**, both $\hat{r}_{(t-1)i}$ and $\hat{r}_{ti}$ are underestimated. Then in the next period, it is necessary to increase the value of $\hat{r}_{(t+1)i}$ by using a larger $\alpha_{(t+1)i}$ following from Eq. (5) (As $\frac{\hat{r}_{ti}}{r_{ti}} < 1$).
-The **initial value** of the decaying factor is set to be $\alpha_{1i} = 0.5$. If the iterated decaying factor α_{ti} is **outside the interval** $[0, 1]$, then it is **reset to** 0.5.

Adaptive Online Moving Average Method

Example

-To verify the effectiveness of our proposed AOLMA method, we employ the classical benchmark data set **MSCI which contains the historical daily returns of 24 stocks from April 1, 2006 to March 31, 2010**, Li and Hoi (2015).

-For each stock i, its **prediction relative error** at the j-th trading day is given by

$$\boxed{Er(j) = \frac{|\hat{r}_{ji} - r_{ji}|}{r_{ji}} \times 100\%}$$

and the **average relative error** is

$$\boxed{\bar{Er} = \frac{1}{n}\sum_{j=1}^{n} \frac{|\hat{r}_{ji} - r_{ji}|}{r_{ji}} \times 100\%.}$$

Example

-We apply **SMA ($w = 6$), EMA ($\alpha = 0.5$), AOLMA ($\tau = 0.0006$)** to estimate the daily returns and make comparisons with the real returns.
-The average relative errors are shown in Table 1.
-It is clear that AOLMA achieves the **lowest relative error** in each stock, meaning that AOLMA performs better than both SMA and EMA.

Table 1: Average relative errors of SMA, EMA and AOLMA.

Stock	SMA(%)	EMA(%)	AOLMA(%)	Stock	SMA(%)	EMA(%)	AOLMA(%)
1	2.06	1.16	1.14	13	1.88	1.06	1.04
2	3.08	1.75	1.69	14	3.69	2.07	2.05
3	2.57	1.44	1.42	15	2.53	1.43	1.39
4	2.11	1.19	1.16	16	3.48	1.96	1.92
5	3.39	1.90	1.87	17	2.72	1.53	1.48
6	2.80	1.58	1.53	18	2.68	1.51	1.48
7	2.62	1.48	1.43	19	3.18	1.79	1.77
8	2.26	1.28	1.25	20	2.72	1.53	1.48
9	4.00	2.25	2.21	21	2.87	1.62	1.57
10	2.62	1.48	1.46	22	2.83	1.59	1.56
11	2.60	1.47	1.45	23	3.52	1.98	1.93
12	2.72	1.53	1.50	24	2.30	1.29	1.29

Example

-To test the **robustness** of AOLMA, we conduct multiple experiments with **step size** τ **ranging from** 0.0001 **to** 0.0010.
-The final average relative errors are shown in Fig. 2.
-For all the stocks, the maximum difference of average relative errors with different τ **does not exceed** 0.09%.
-To show the advantages of AOLMA, for all step sizes τ, we select **the worst case** of average relative error for each stock, and compare it with SMA and EMA (See Fig. 3).

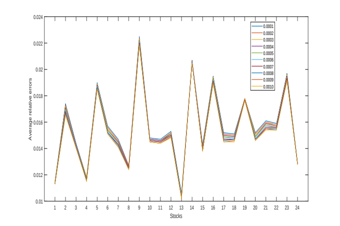


Figure 2: Average relative errors for different τ

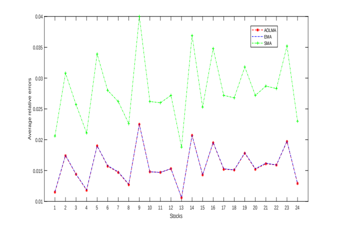


Figure 3: A comparison with SMA and EMA

• We propose the **Net Profit Maximization Model with Transaction Costs**. It is worth noting that several general assumptions are made in the model.

– Firstly, we assume **proportional transaction costs** on risky assets **purchases and sales**.

– Secondly, we assume that each asset share is **arbitrarily divisible**, and that any required quantities of shares, even fractional, can be **bought and sold** at the last closing price in any trading period.

– Thirdly, we assume that **market behavior** and **stock prices** are **NOT affected** by any trading strategy / market impact.

– Fourthly, **NO capital** is added or removed from the portfolio.

Net Profit Maximization Model with Transaction Costs

- The **net profit maximization model (NPM)** considering **transaction cost** in each trading period:

$$\begin{cases} \max & \sum_{i=1}^{m} \hat{r}_{ti} x_{ti} - \gamma \sum_{i=1}^{m} |x_{ti} - \tilde{x}_{(t-1)i}|. \\ \text{s.t.} & x_{t1} + x_{t2} + \ldots + x_{tm} = 1, 0 \le x_{ti} \le 1, i = 1, 2, \ldots, m. \end{cases} \tag{8}$$

- Here γ is the **unit transaction cost rate** for buying/selling assets, $\tilde{\mathbf{x}}_{t-1}$ is the actually investment strategy in period $(t-1)$.
- The model can be transformed into the following **LP problem**:

$$\begin{cases} \max & \sum_{i=1}^{m} \hat{r}_{ti} \tilde{x}_{(t-1)i} + \sum_{i=1}^{m} (\tilde{x}_{(t-1)i} - \gamma) u_{ti} - \sum_{i=1}^{m} (\tilde{x}_{(t-1)i} + \gamma) v_{ti}. \\ \text{s.t.} & \sum_{i=1}^{m} (u_{ti} - v_{ti}) = 0, \\ & 0 \le \tilde{x}_{(t-1)i} + u_{ti} - v_{ti} \le 1, i = 1, 2, \ldots, m, u_{ti} \ge 0, v_{ti} \ge 0, i = 1, \end{cases} \tag{9}$$

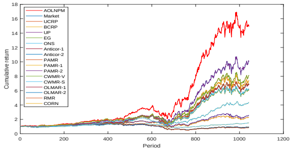
- Integrating AOLMA and NPM together, we have **Adaptive On-Line Net Profit Maximization (AOLNPM) Algorithm**.

Numerical Experiments (I)

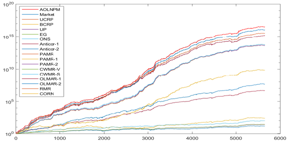
- **MSCI**, **NYSE-O**, **NYSE-N** and **TSE** are employed as benchmark data sets for testing the performances of different online portfolio selection algorithms.
- **MSCI** contains **24 stocks** which has been employed for verifying the effectiveness of AOLMA method.
- **NYSE-O** and **NYSE-N** contain historical return data of stocks selected from **American stock market**, where **NYSE-O** contains the data of **36 stocks ranging from June 3, 1962 to December 31, 1984**, and **NYSE-N** contains the data of **23 stocks ranging from January 1, 1985 to June 30, 2010**.
- **TSE** contains **88 stocks** selected from **Canadian stock market** ranging from January 4, 1994 to December 31, 1998.
- The total numbers of the trading days for MSCI, NYSE-O, NYSE-N and TSE are **1043**, **5651**, **6431** and **1259**, respectively.

Numerical Experiments (I)

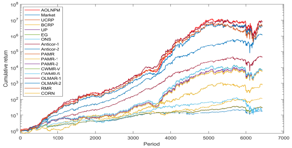
• Numerical results for demonstrating the effectiveness of AOL-NPM algorithm over other algorithms on benchmark data sets: MSCI (Li and Hoi (2015)), NYSE-O (Konno and Yamazaki (1991)), NYSE-N (Cover (1991)) and TSE (Borodin et al. (2004)).



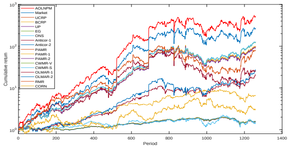
MSCI



NYSE-O



NYSE-N



TSE

Figure 4: **Cumulative returns** on different data sets.

Table 2: **Mean excess return** on MSCI, NYSE-O, NYSE-N and TSE

Method	MSCI[1]	NYSE-O[1]	NYSE-N[2]	TSE[1]
AOLNPM	**0.5781**	**0.9910**	**0.8651**	**0.6791**
Anticor-1	0.5271	0.8816	0.8011	0.5982
Anticor-2	0.5305	0.9089	0.8606	0.6066
BCRP	0.5133	0.6707	0.6500	0.5618
CWMR-V	0.5624	0.9841	0.7596	0.6527
CWMR-S	0.5624	0.9841	0.7591	0.6530
CORN	0.5477	0.9509	0.7044	0.5372
EG	0.4973	0.6048	0.6089	0.5149
Market	0.4968	0.5862	0.5931	0.5156
ONS	0.4938	0.6438	0.5947	0.5137
OLMAR-1	0.5649	0.9897	0.8872	0.6049
OLMAR-2	0.5745	0.9920	0.8786	0.6752
PAMR	0.5583	0.9835	0.7550	0.6459
PAMR-1	0.5589	0.9835	0.7550	0.6459
PAMR-2	0.5621	0.9835	0.7588	0.6443
TCO-1	0.5546	0.9806	0.8602	0.6325
TCO-2	0.5471	0.9753	**0.8723**	0.6421
UCRP	0.4974	0.6048	0.6093	0.5149
UP	0.4970	0.6032	0.6081	0.5143

$$MER = \frac{1}{n}\sum_{t=1}^{n}(R_t - R_t^{\star}) = \bar{R} - \bar{R}^{\star}.$$

$R_t^{\star}$ is the return of the portfolio in period t by using Market strategy, and R_t is the return of the portfolio in period t.

Numerical Experiments (I)

Table 3: **Sharpe ratios** on MSCI, NYSE-O, NYSE-N and TSE.

Method	MSCI[1]	NYSE-O[2]	NYSE-N[7]	TSE[1]
AOLNPM	**0.1034**	**0.1907**	**0.0799**	**0.1046**
Anticor-1	0.0513	0.1583	0.0862	0.0982
Anticor-2	0.0538	0.1502	**0.0929**	0.0882
BCRP	0.0381	0.0597	0.0166	0.0725
CWMR-V	0.0920	0.1907	0.0594	0.1020
CWMR-S	0.0921	0.1907	0.0591	0.1023
CORN	0.0821	0.1383	0.0573	0.0428
EG	0.0030	0.0722	0.0501	0.0485
Market	0.0017	0.0552	0.0457	0.0505
ONS	0.0002	0.0767	0.0305	0.0264
OLMAR-1	0.0897	0.1913	0.0863	0.0714
OLMAR-2	0.1003	0.2014	0.0840	0.1027
PAMR	0.0866	0.1886	0.0589	0.1016
PAMR-1	0.0874	0.1886	0.0589	0.1016
PAMR-2	0.0922	0.1901	0.0600	0.1008
TCO-1	0.0893	**0.2119**	0.0902	0.0899
TCO-2	0.0768	0.1945	0.0887	0.0929
UCRP	0.0031	0.0725	0.0501	0.0485
UP	0.0023	0.0715	0.0496	0.0467

$$SR = \frac{1}{\sigma}(\bar{R} - r_f).$$

Here r_f is the risk-free return in financial market, $\bar{R}$ is the average return of the portfolio and σ is the corresponding standard deviation of daily returns.

Numerical Experiments (I)

Table 4: **Information ratios** on MSCI, NYSE-O, NYSE-N and TSE.

Method	MSCI[2]	NYSE-O[4]	NYSE-N[7]	TSE[1]
AOLNPM	**0.1522**	**0.1871**	**0.0701**	**0.0998**
Anticor-1	0.1235	0.1576	0.0765	0.0903
Anticor-2	0.1057	0.1447	**0.0837**	0.0802
BCRP	0.0359	0.0386	−0.0057	0.0617
CWMR-V	0.1375	0.1863	0.0469	0.0963
CWMR-S	0.1375	0.1863	0.0466	0.0965
CORN	0.1161	0.1302	0.0399	0.0331
EG	0.0281	0.0345	0.0242	−0.0082
ONS	−0.0027	0.0394	0.0121	0.0069
OLMAR-1	0.1297	0.1870	0.0771	0.0659
OLMAR-2	0.1466	0.1982	0.0745	0.0976
PAMR	0.1291	0.1839	0.0462	0.0956
PAMR-1	0.1305	0.1839	0.0462	0.0956
PAMR-2	0.1400	0.1856	0.0473	0.0948
TCO-1	**0.1665**	**0.2123**	0.0797	0.0838
TCO-2	0.1410	0.1940	0.0790	0.0872
UCRP	0.0277	0.0337	0.0238	−0.0075
UP	0.0128	0.0306	0.0221	−0.0139

$$IR = (\bar{R} - \bar{R}^{\star})/\sigma(R - R^{\star}).$$

Here $\sigma(R - R^{\star})$ is the standard deviation of the excess return over Market strategy.

Numerical Experiments (I)

Table 5: **Calmar ratios** on MSCI, NYSE-O, NYSE-N and TSE.

Method	MSCI[1]	NYSE-O[9]	NYSE-N[7]	TSE[6]
AOLNPM	**0.1609**	**0.3724**	**0.1277**	**0.1810**
Anticor-1	0.0751	0.2862	0.1368	0.1635
Anticor-2	0.0797	0.2726	**0.1541**	0.1452
BCRP	0.0520	0.0941	0.0244	0.1199
CWMR-V	0.1377	0.3853	0.0960	0.1905
CWMR-S	0.1378	0.3853	0.0959	**0.1910**
CORN	0.1289	0.2607	0.0916	0.0696
EG	0.0041	0.1106	0.0704	0.0649
Market	0.0023	0.0835	0.0637	0.0675
ONS	0.0002	0.1252	0.0457	0.0406
OLMAR-1	0.1365	0.3737	0.1420	0.1233
OLMAR-2	0.1549	0.4001	0.1380	0.1788
PAMR	0.1281	0.3798	0.0946	0.1828
PAMR-1	0.1294	0.3798	0.0946	0.1828
PAMR-2	0.1370	0.3842	0.0965	0.1814
TCO-1	0.1359	**0.4443**	0.1484	0.1646
TCO-2	0.1165	0.3850	0.1478	0.1788
UCRP	0.0042	0.1113	0.0704	0.0650
UP	0.0032	0.1096	0.0697	0.0626

$$CR = \bar{R}_{net}/MDD, \quad MDD = \sqrt{\frac{1}{n}\sum_{t=1}^{n}\min\{R_t - 1, 0\}^2}.$$

Here $\bar{R}_{net}$ is the average daily net profit return rate, and MDD (the maximum drawdown of return) only covers the return which is less than 1.

Numerical Experiments (I)

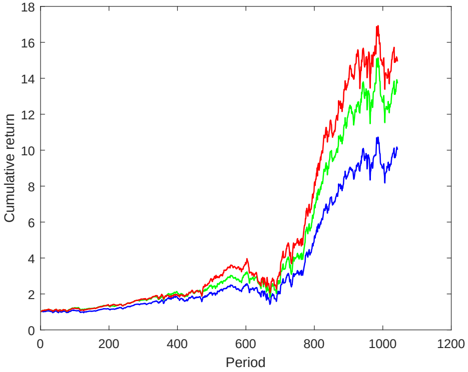
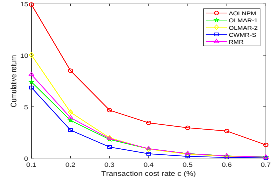
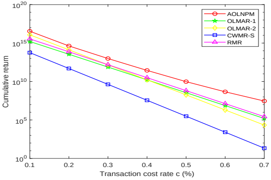


Figure 5: Impact of AOLMA and NPM: AOLNPM, NPM with EMA and OLMAR

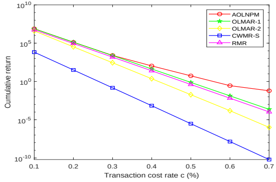
The **blue curve** is the **cumulative return** derived by the **OLMAR method**. The **green curve** refers to the return by using **EMA and our NPM model**. The **red curve** is obtained by using **AOLMA and NPM (AOLNPM) simultaneously**.



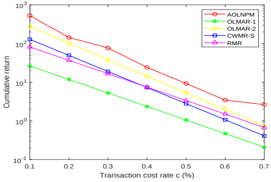
MSCI



NYSE-O



NYSE-N



TSE

Figure 6: Cumulative returns with different transaction cost rates.

To study the relationship between the transaction cost rate γ and the cumulative return, we set different rates ranging from 0.1% to 0.7%. It is found that AOLNPM dominates other algorithms with high or low transaction cost rate.

Online Moving Average with Peer Impact Method (Motivation)

- In a financial market, the performance of one asset may be affected by that of **other assets** and the financial market. Therefore, it is useful to consider the **impact of peer assets**.
- The price fluctuation of different assets may differ a lot and it is therefore better to **employ different decaying factors** for different assets.
- In real applications, risk-neural and risk-averse investors treat controlling investment risk as one important objective. It is necessary and important to incorporate **investment risk** into the objective of OLPS.

Adaptive Online Moving Average with Peer Impact Method

- Recall that the investor invests into m risky assets within the investment horizon of n periods. At the beginning of period t, the investment strategy $\mathbf{x}_t = (x_{t,1}, x_{t,2}, \ldots, x_{t,m})$ should be determined. Suppose that the **price relative vector** at period t is $\mathbf{r}_t = (r_{t,1}, r_{t,2}, \ldots, r_{t,m})$. The **final wealth** at the end of period n is as follows:

$$S_n = S_0 \prod_{t=1}^{n} \left(\mathbf{x}_t \mathbf{r}_t^\top - \gamma \parallel \mathbf{x}_t - \tilde{\mathbf{x}}_{t-1} \parallel_1 \right) . \tag{10}$$

- The prediction method EMA only depends on the past historical data sequence $\mathbf{D}_{t,i} = (r_{1,i}, r_{2,i}, \ldots, r_{(t-1),i})$. In this talk, we introduce the **peer impact**. For Asset i, we denote the average return data of all the other assets as $\bar{\mathbf{D}}_{t,i}$, where

$$\bar{\mathbf{D}}_{t,i} = \left(R_{1,i}, R_{2,i}, \ldots, R_{(t-1),i}\right) = \left(\frac{\sum_{k \neq i} r_{1,k}}{m-1}, \frac{\sum_{k \neq i} r_{2,k}}{m-1}, \ldots, \frac{\sum_{k \neq i} r_{(t-1),k}}{m-1} \right).$$

Adaptive Online Moving Average with Peer Impact Method

- From the data sequence $\mathbf{D}_{t,i}$, we derive

$$\boxed{u_{t,i} = \bar{\theta}^{(1)}_{t,i} + (1 - \bar{\theta}^{(1)}_{t,i}) \cdot \frac{\hat{r}_{(t-1),i}}{r_{(t-1),i}}.}$$

  Similarly, for data sequence $\bar{\mathbf{D}}_{t,i}$, we derive

$$\boxed{v_{t,i} = \bar{\theta}^{(2)}_{t,i} + (1 - \bar{\theta}^{(2)}_{t,i}) \cdot \frac{\hat{R}_{(t-1),i}}{R_{(t-1),i}},}$$

  where $\bar{\theta}^{(1)}_{t,i}$ and $\bar{\theta}^{(2)}_{t,i}$ are the **decaying factors** of asset i at period t for $\mathbf{D}_{t,i}$ and $\bar{\mathbf{D}}_{t,i}$, respectively.
- Then, we estimate the future return $r_{t,i}$ based on $u_{t,i}$ and $v_{t,i}$, which can be expressed as follows:

$$\boxed{\hat{r}_{t,i} = \alpha_i u_{t,i} + \beta_i v_{t,i}}$$

  where α_i and β_i are the **weighting factors**, meaning that $\hat{r}_{t,i}$ is both affected by the historical data $\mathbf{D}_{t,i}$ and $\bar{\mathbf{D}}_{t,i}$.

Adaptive Online Moving Average with Peer Impact Method

- At the beginning of period t, all the historical data $\mathbf{D}_{t,i}$ and $\bar{\mathbf{D}}_{t,i}$ are available and the values of $u_{t,i}$ and $v_{t,i}$ can be derived. Then, the **sum of mean squared errors** over the first $(t-1)$ periods is

$$\boxed{MSE_i = \sum_{k=1}^{t-1}(\alpha_i u_{k,i} + \beta_i v_{k,i} - r_{k,i})^2.}$$

  To minimize the sum of **mean squared errors** (MSE), we set the derivatives with respect to α_i and β_i.
- We derive the **estimations** for α_i and β_i:

$$\hat{\alpha}_i = \frac{(\sum_{k=1}^{t-1} r_{k,i}u_{k,i})(\sum_{k=1}^{t-1} v_{k,i}^2) - (\sum_{k=1}^{t-1} u_{k,i}v_{k,i})(\sum_{k=1}^{t-1} r_{k,i}v_{k,i})}{(\sum_{k=1}^{t-1} u_{k,i}^2)(\sum_{k=1}^{t-1} v_{k,i}^2) - (\sum_{k=1}^{t-1} u_{k,i}v_{k,i})^2}$$

$$\hat{\beta}_i = \frac{(\sum_{k=1}^{t-1} r_{k,i}v_{k,i})(\sum_{k=1}^{t-1} u_{k,i}^2) - (\sum_{k=1}^{t-1} u_{k,i}v_{k,i})(\sum_{k=1}^{t-1} u_{k,i}r_{k,i})}{(\sum_{k=1}^{t-1} u_{k,i}^2)(\sum_{k=1}^{t-1} v_{k,i}^2) - (\sum_{k=1}^{t-1} u_{k,i}v_{k,i})^2}.$$

Adaptive Online Moving Average with Peer Impact Method

- Then, we have

$$\hat{r}_{t,i} = \hat{\alpha}_i \left[\bar{\theta}^{(1)}_{t,i} + (1 - \bar{\theta}^{(1)}_{t,i}) \cdot \frac{\hat{r}_{(t-1),i}}{r_{(t-1),i}} \right] + \hat{\beta}_i \left[\bar{\theta}^{(2)}_{t,i} + (1 - \bar{\theta}^{(2)}_{t,i}) \cdot \frac{\hat{R}_{(t-1),i}}{R_{(t-1),i}} \right],$$

and the **corresponding estimation error** is

$$r_{t,i} - \hat{r}_{t,i} = r_{t,i} - \hat{\alpha}_i \left(\frac{\bar{\theta}^{(1)}_{t,i} r_{(t-1),i} + (1 - \bar{\theta}^{(1)}_{t,i}) \hat{r}_{(t-1),i}}{r_{(t-1),i}} \right) - \hat{\beta}_i \left(\frac{\bar{\theta}^{(2)}_{t,i} R_{(t-1),i} + (1 - \bar{\theta}^{(2)}_{t,i}) \hat{R}_{(t-1),i}}{R_{(t-1),i}} \right).$$

- Here $\bar{\theta}^{(1)}_{t,i}$ can be updated to potentially **reduce the estimation error** of the next period, see Table 6.

Table 6: Decaying factor iteration table.

	$\hat{\alpha}_i(r_{(t-1),i} - \hat{r}_{(t-1),i}) \geq 0$	$\hat{\alpha}_i(r_{(t-1),i} - \hat{r}_{(t-1),i}) < 0$
$r_{t,i} \geq \hat{r}_{t,i}$	$\bar{\theta}^{(1)}_{t,i} = \bar{\theta}^{(1)}_{t,i} + \delta$	$\bar{\theta}^{(1)}_{t,i} = \bar{\theta}^{(1)}_{t,i} - \delta$
$r_{t,i} < \hat{r}_{t,i}$	$\bar{\theta}^{(1)}_{t,i} = \bar{\theta}^{(1)}_{t,i} - \delta$	$\bar{\theta}^{(1)}_{t,i} = \bar{\theta}^{(1)}_{t,i} + \delta$

For the factor $\bar{\theta}^{(2)}_{t,i}$, we use similar iteration mechanism. The initial value of the **decaying factor** is set to be 0.5.

Adaptive Online Moving Average with Peer Impact Method

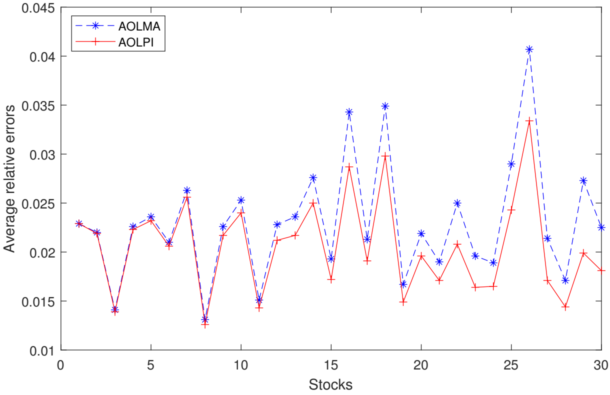


Figure 7: A comparison of the average relative errors of AOLMA and AOLPI.

Adaptive Mean-Variance Model

- Then, we propose **Adaptive Mean-Variance** (AMV) model to solve the practical OLPS problem. The variance of the risky asset is employed as a measure of the investment risk. The AMV model is given as follows:

$$\left\{\begin{array}{ll} \max & \mathbf{x}_t \mathbf{r}_t^\top - \eta \mathbf{x}_t \boldsymbol{\Sigma}_t \mathbf{x}_t^\top - \gamma \parallel \mathbf{x}_t - \tilde{\mathbf{x}}_{t-1} \parallel_1 \\ \text{s.t.} & \mathbf{x}_t \mathbf{1}^\top = 1, \\ & \mathbf{0} \leq \mathbf{x}_t \leq \mathbf{1} \end{array}\right. \quad (11)$$

  Here $\boldsymbol{\Sigma}_t$ is the **covariance matrix** of all assets estimated at the beginning of period t, η is the **weighting factor** ($\eta > 0$).
- The updating process of $\boldsymbol{\Sigma}_t$ can be expressed as follows:

$$\boldsymbol{\Sigma}_t = \begin{pmatrix} \sigma_{1,1}^{(t)} & \sigma_{1,2}^{(t)} & \sigma_{1,3}^{(t)} & \cdots & \sigma_{1,m}^{(t)} \\ \sigma_{2,1}^{(t)} & \sigma_{2,2}^{(t)} & \sigma_{2,3}^{(t)} & \cdots & \sigma_{2,m}^{(t)} \\ \sigma_{3,1}^{(t)} & \sigma_{3,2}^{(t)} & \sigma_{3,3}^{(t)} & \cdots & \sigma_{3,m}^{(t)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{m,1}^{(t)} & \sigma_{m,2}^{(t)} & \sigma_{m,3}^{(t)} & \cdots & \sigma_{m,m}^{(t)} \end{pmatrix},$$

  where each $\sigma_{i,j}^{(t)}$ is the **covariance of Assets i and j** at the beginning of period t.

Adaptive Mean-Variance Model

- For Asset i, the **average value** of the past returns $\mu_{t,i}$ is defined as follows:

$$\mu_{t,i} = \frac{\sum_{k=1}^{t-1} r_{k,i}}{t-1}.$$

The iteration formula relating $\mu_{t,i}$ and $\mu_{(t+1),i}$ can be derived as follows:

$$\mu_{(t+1),i} = \frac{\sum_{k=1}^{t} r_{k,i}}{t} = \frac{(t-1)\mu_{t,i} + r_{t,i}}{t} = \left(\frac{t-1}{t}\right)\mu_{t,i} + \frac{1}{t} r_{t,i}.$$

- The **variance of Asset i** can be calculated by

$$\sigma_{i,i}^{(t)} = \frac{1}{t-2} \sum_{k=1}^{t-1} (r_{k,i} - \mu_{t,i})^2,$$

and the iteration formula relating $\sigma_{i,i}^{(t)}$ and $\sigma_{i,i}^{(t+1)}$ is

$$\boxed{\sigma_{i,i}^{(t+1)} = \frac{t-2}{t-1}\sigma_{i,i}^{(t)} + \frac{1}{t}(r_{t,i} - \mu_{t,i})^2.}$$

Adaptive Mean-Variance Model

- For Assets i and j, we have

$$\sigma_{i,j}^{(t)} = \frac{1}{t-2}\sum_{k=1}^{t-1}(r_{k,i}-\mu_{t,i})(r_{k,j}-\mu_{t,j}).$$

  Then, the iteration formula relating $\sigma_{i,j}^{(t)}$ and $\sigma_{i,j}^{(t+1)}$ can be derived as follows:

$$\boxed{\sigma_{i,j}^{(t+1)} = \frac{t-2}{t-1}\sigma_{i,j}^{(t)} + \frac{1}{t}(r_{t,i}-\mu_{t,i})(r_{t,j}-\mu_{t,j}).}$$

- The **covariance matrix** can be updated when new return data is obtained at the end of period t, which is given by

$$\boxed{\boldsymbol{\Sigma}_{t+1} = \left(\frac{t-2}{t-1}\right)\boldsymbol{\Sigma}_t + \frac{1}{t}\boldsymbol{M}_t}$$

  where $\boldsymbol{M}_t$ is an $n \times n$ matrix with its (i,j)-th entry being given by $(r_{t,i}-\mu_{t,i})(r_{t,j}-\mu_{t,j})$.

Adaptive Mean-Variance Model

- We then combine the **AOLPI method** with the **AMV model** together in solving the following optimization problem:

$$\left\{ \begin{array}{ll} \max & \mathbf{x}_t \hat{\mathbf{r}}_t^\top - \eta \mathbf{x}_t \Sigma_t \mathbf{x}_t^\top - \gamma \parallel \mathbf{x}_t - \tilde{\mathbf{b}}_{t-1} \parallel_1 \\ \text{s.t.} & \mathbf{x}_t \mathbf{1}^\top = 1, \\ & \mathbf{0} \le \mathbf{x}_t \le \mathbf{1}, \end{array} \right. \quad (12)$$

  where the first two terms $\mathbf{x}_t \hat{\mathbf{r}}_t, \eta \mathbf{x}_t \Sigma_t \mathbf{x}_t^\top$ in the objective function is standard **quadratic programming**.

- We employ the **method of change of variables** to transform $\gamma ||\mathbf{x}_t - \tilde{\mathbf{x}}_{t-1}||_1$ into a linear one. Suppose that there are non-negative variables $u_{t,i}$ and $v_{t,i}$ such that

$$\left(\begin{array}{c} |x_{t,i} - \tilde{x}_{(t-1),i}| \\ x_{t,i} - \tilde{x}_{(t-1),i} \end{array} \right) = \left(\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) \left(\begin{array}{c} u_{t,i} \\ v_{t,i} \end{array} \right), \quad i = 1, 2, \ldots, m.$$

Adaptive Mean-Variance Model

- It can be derived that $x_{t,i} = \tilde{x}_{(t-1),i} + u_{t,i} - v_{t,i}$. Then,

$$\mathbf{x}_t = (\mathbf{u}_t, \mathbf{v}_t)N + \tilde{\mathbf{x}}_{t-1},$$

where $N = (\mathbf{I}_m, -\mathbf{I}_m)^\top$ and $\mathbf{I}_m$ is an identity matrix of size $m \times m$. The transaction cost term can be transformed into

$$\gamma \parallel \mathbf{x}_t - \tilde{\mathbf{b}}_{t-1} \parallel_1 = \gamma \sum_{i=1}^{m} (u_{t,i} + v_{t,i}) = \gamma(\mathbf{u}_t, \mathbf{v}_t)(\mathbf{1}, \mathbf{1})^\top,$$

where $(\mathbf{1}, \mathbf{1})$ is the $1 \times 2m$ row vector of all ones.

- Model (12) can be reformulated as follows:

$$\begin{cases} \max & (\mathbf{u}_t, \mathbf{v}_t)\boldsymbol{F}_t^\top - \eta(\mathbf{u}_t, \mathbf{v}_t)\boldsymbol{H}_t(\mathbf{u}_t, \mathbf{v}_t)^\top + \boldsymbol{C}_t \\ \text{s.t.} & (\mathbf{u}_t - \mathbf{v}_t)\mathbf{1}^\top = 0, \\ & -\tilde{\mathbf{x}}_{t-1} \leq (\mathbf{u}_t, \mathbf{v}_t)N \leq \mathbf{1} - \tilde{\mathbf{x}}_{t-1}, \\ & \mathbf{0} \leq \mathbf{u}_t, \mathbf{0} \leq \mathbf{v}_t, \end{cases} \tag{13}$$

where $\boldsymbol{F}_t = \hat{\mathbf{r}}_t N^\top - \gamma(\mathbf{1}, \mathbf{1}) - 2\eta N \Sigma_t \tilde{\mathbf{x}}_{t-1}^\top$, $\boldsymbol{H}_t = N\Sigma_t N^\top$ and $\boldsymbol{C}_t = \tilde{\mathbf{x}}_{t-1}\hat{\mathbf{r}}_t^\top - \eta\tilde{\mathbf{x}}_{t-1}\Sigma_t\tilde{\mathbf{x}}_{t-1}^\top$.

Theorem

The matrix $\boldsymbol{H}_t$ in Model (13) is semi-positive definite for $t = 3, 4, \ldots, n$.

Theorem

There is at least one optimal solution for Model (13) if the feasible region is not empty.

In addition, we can also derive the following theorem.

Theorem

Model (13) achieves the optimal solution $\mathbf{u}_t^* = (u_{t,1}^*, u_{t,2}^*, \ldots, u_{t,m}^*)$ and $\mathbf{v}_t^* = (v_{t,1}^*, v_{t,2}^*, \ldots, v_{t,m}^*)$ if and only if Model (12) achieves the optimal solution $\mathbf{x}_t^* = (x_{t,1}^*, x_{t,2}^*, \ldots, x_{t,m}^*)$.

- We conduct numerical experiments to validate the effectiveness of our proposed AOLPIMV algorithm over some other OLPS algorithms. Some real data sets are employed, including **MSCI, NYSE-O, NYSE-N and TSE**.
- In addition, we collect the historical return data of another 20 stocks in the American stock market ranging from January 3, 2006, to October 7, 2010, which are contained in the data set **NASTDA**.
- AOLPIMV employs the adaptive decaying factors, and the corresponding **iteration step size** δ is set as 0.00040, 0.00045, 0.00010, 0.00065, and 0.00010 for MSCI, NYSE-O, NYSE-N, TSE, and NASTDA, respectively. The **risk weighting factor** η is set as 0.6, 0.05, 0.1, 0.7 and 0.005, respectively. The **window size** w is set to 6. The **transaction cost** γ is set to be 0.0005.

Numerical Experiments

Table 7: **Mean excess return** on MSCI, NYSE-O, NYSE-N, TSE, and NASTDA.

Method	MSCI[1]	NYSE-O[1]	NYSE-N[1]	TSE[1]	NASTDA[2]
AOLPIMV	**0.5819**	**0.9922**	**0.8902**	**0.6817**	**0.5528**
AOLNPM	0.5781	0.9910	0.8651	0.6791	0.5288
Anticor-1	0.5271	0.8816	0.8011	0.5982	0.5342
Anticor-2	0.5305	0.9089	0.8606	0.6066	0.5410
BCRP	0.5133	0.6707	0.6500	0.5618	**0.5563**
CWMR-V	0.5624	0.9841	0.7596	0.6527	0.5234
CWMR-S	0.5624	0.9841	0.7591	0.6530	0.5234
CORN	0.5477	0.9509	0.7044	0.5372	0.5000
EG	0.4973	0.6048	0.6089	0.5149	0.5161
Market	0.4968	0.5862	0.5931	0.5156	0.5155
ONS	0.4938	0.6438	0.5947	0.5137	0.5350
OLMAR-1	0.5649	0.9897	0.8872	0.6049	0.5467
OLMAR-2	0.5745	0.9920	0.8786	0.6752	0.5279
PAMR	0.5583	0.9835	0.7550	0.6459	0.5204
PAMR-1	0.5589	0.9835	0.7550	0.6459	0.5204
PAMR-2	0.5621	0.9835	0.7588	0.6443	0.5222
TCO-1	0.5546	0.9806	0.8602	0.6325	0.5478
TCO-2	0.5471	0.9753	0.8723	0.6421	0.5489
UCRP	0.4974	0.6048	0.6093	0.5149	0.5187
UP	0.4970	0.6032	0.6081	0.5143	0.5187

It can be seen that AOLPIMV gains the largest Mean excess return on MSCI, NYSE-O, NYSE-N and TSE, and the second largest return on NASTDA.

Numerical Experiments

Table 8: **Sharpe ratios** on MSCI, NYSE-O, TSE, and NASTDA.

Method	MSCI[1]	NYSE-O[2]	NYSE-N[3]	TSE[1]	NASTDA[3]
AOLPIMV	**0.1115**	**0.2032**	**0.0870**	**0.1072**	**0.0552**
AOLNPM	0.1034	0.1907	0.0799	0.1046	0.0382
Anticor-1	0.0513	0.1583	0.0862	0.0982	0.0472
Anticor-2	0.0538	0.1502	**0.0929**	0.0882	0.0498
BCRP	0.0381	0.0597	0.0166	0.0725	**0.0627**
CWMR-V	0.0920	0.1907	0.0594	0.1020	0.0344
CWMR-S	0.0921	0.1907	0.0591	0.1023	0.0344
CORN	0.0821	0.1383	0.0573	0.0428	0.0155
EG	0.0030	0.0722	0.0501	0.0485	0.0321
Market	0.0017	0.0552	0.0457	0.0505	0.0317
ONS	0.0002	0.0767	0.0305	0.0264	0.0492
OLMAR-1	0.0897	0.1913	0.0863	0.0714	0.0503
OLMAR-2	0.1003	0.2014	0.0840	0.1027	0.0376
PAMR	0.0866	0.1886	0.0589	0.1016	0.0322
PAMR-1	0.0874	0.1886	0.0589	0.1016	0.0322
PAMR-2	0.0922	0.1901	0.0600	0.1008	0.0335
TCO-1	0.0893	**0.2119**	0.0902	0.0899	0.0549
TCO-2	0.0768	0.1945	0.0887	0.0929	0.0556
UCRP	0.0031	0.0725	0.0501	0.0485	0.0359
UP	0.0023	0.0715	0.0496	0.0467	0.0358

It is clear that AOLPIMV performs the best in MSCI and TSE, achieving the second largest ratio on NYSE-O, and the third largest ratio on NYSE-N and NASTDA. This shows that AOLPIMV achieves relatively good and steady performance.

Numerical Experiments

Table 9: **Information ratios** on MSCI, NYSE-O, TSE, and NASTDA.

Method	MSCI[2]	NYSE-O[2]	NYSE-N[5]	TSE[1]	NASTDA[6]
AOLPIMV	**0.1584**	**0.2001**	**0.0770**	**0.1022**	**0.0495**
AOLNPM	0.1522	0.1871	0.0701	0.0998	0.0291
Anticor-1	0.1235	0.1576	0.0765	0.0903	0.0459
Anticor-2	0.1057	0.1447	**0.0837**	0.0802	0.0477
BCRP	0.0359	0.0386	−0.0057	0.0617	**0.0562**
CWMR-V	0.1375	0.1863	0.0469	0.0963	0.0241
CWMR-S	0.1375	0.1863	0.0466	0.0965	0.0241
CORN	0.1161	0.1302	0.0399	0.0331	−0.0029
EG	0.0281	0.0345	0.0242	−0.0082	0.0096
ONS	−0.0027	0.0394	0.0121	0.0069	0.0515
OLMAR-1	0.1297	0.1870	0.0771	0.0659	0.0445
OLMAR-2	0.1466	0.1982	0.0745	0.0976	0.0284
PAMR	0.1291	0.1839	0.0462	0.0956	0.0214
PAMR-1	0.1305	0.1839	0.0462	0.0956	0.0213
PAMR-2	0.1400	0.1856	0.0473	0.0948	0.0229
TCO-1	**0.1665**	**0.2123**	0.0797	0.0838	0.0535
TCO-2	0.1410	0.1940	0.0790	0.0872	0.0542
UCRP	0.0277	0.0337	0.0238	−0.0075	0.0503
UP	0.0128	0.0306	0.0221	−0.0139	0.0378

The benchmark is set as the Market strategy. It is clear that AOLPIMV achieves the largest Information ratios on TSE and the second largest ratio on MSCI and NYSE-O.

Table 10: **Calmar ratios** on MSCI, NYSE-O, TSE, and NASTDA.

Method	MSCI[1]	NYSE-O[2]	NYSE-N[4]	TSE[3]	NASTDA[2]
AOLPIMV	**0.1721**	**0.4062**	**0.1427**	**0.1898**	**0.0851**
AOLNPM	0.1609	0.3724	0.1277	0.1810	0.0578
Anticor-1	0.0751	0.2862	0.1368	0.1635	0.0731
Anticor-2	0.0797	0.2726	**0.1541**	0.1452	0.0790
BCRP	0.0520	0.0941	0.0244	0.1199	**0.1001**
CWMR-V	0.1377	0.3853	0.0960	0.1905	0.0523
CWMR-S	0.1378	0.3853	0.0959	**0.1910**	0.0524
CORN	0.1289	0.2607	0.0916	0.0696	0.0226
EG	0.0041	0.1106	0.0704	0.0649	0.0457
Market	0.0023	0.0835	0.0637	0.0675	0.0448
ONS	0.0002	0.1252	0.0457	0.0406	0.0744
OLMAR-1	0.1365	0.3737	0.1420	0.1233	0.0786
OLMAR-2	0.1549	0.4001	0.1380	0.1788	0.0572
PAMR	0.1281	0.3798	0.0946	0.1828	0.0491
PAMR-1	0.1294	0.3798	0.0946	0.1828	0.0490
PAMR-2	0.1370	0.3842	0.0965	0.1814	0.0511
TCO-1	0.1359	**0.4443**	0.1484	0.1646	0.0878
TCO-2	0.1165	0.3850	0.1478	0.1788	0.0880
UCRP	0.0042	0.1113	0.0704	0.0650	0.0513
UP	0.0032	0.1096	0.0697	0.0626	0.0512

The benchmark is set as the Market strategy. It is clear that AOLPIMV achieves the largest Information ratios on MSCI, the second on NYSE-O and NASTDA and third on TSE.

To further study the influence of introducing peer impact, we use the adaptive moving average mean-variance (AOLMAMV) algorithm, where the AOLMA is used to predict the future returns of risky assets without considering the peer impact.

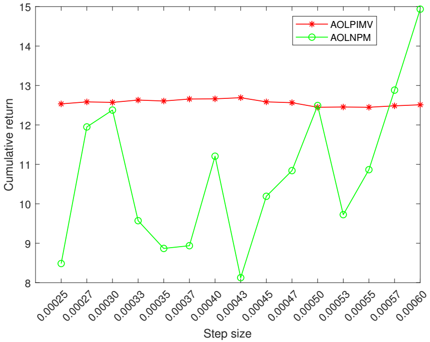


Figure 8: Cumulative returns of AOLPIMV and AOLMAMV (Different Step Size δ).

Numerical Experiments

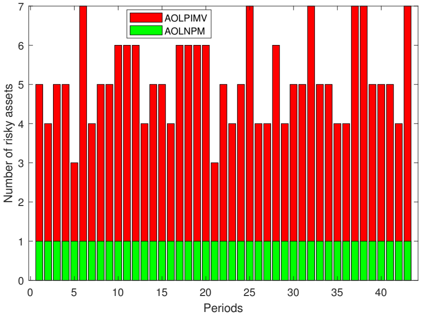


Figure 9: Number of assets in the last 43 periods.

Conclusions

- To accurately predict the future returns of assets, we propose the **AOLPI method**, which considers the historical returns of assets and the **peer impact of other assets**.
- Meanwhile, we construct the **AMV model** where the investment return and risk are considered simultaneously in the decision making process.
- We **integrate AOLPI and AMV** and propose the **AOLPIMV algorithm** to solve practical online portfolio selection issues. Numerical experiments are provided to verify the effectiveness of the AOLPIMV algorithm.

Conclusions

- We shall study other **time series model** such as **ARIMA** for return predictions.
- Consider other **risk measure** such as Conditional Risk-at-Risk (**CVaR**).
- Apply machine learning and deep learning methods, such as Long Short-Term Memory Network (LSTM) to return predictions.

ARIMA Models

• ARIMA models can be employed to predict stock prices based on historical price data and are defined by three primary parameters: p, d, and q:

- p denotes the order of the autoregressive part, indicating the number of lagged observations included in the model.
- d represents the degree of differencing required to render the time series stationary.
- q signifies the order of the moving average component, reflecting the number of lagged forecast errors in the prediction formula.

The ARIMA model is mathematically expressed as:

$$(1-\sum_{i=1}^{p}\vartheta_i L^i)(1-L)^d \mathbf{p}_t = (1+\sum_{i=1}^{q}\theta_i L^i)\epsilon_t$$

ARIMA Models

- L is the lag operator,
- ϑ_i are the coefficients of the autoregressive component,
- θ_i are the coefficients of the moving average section,
- $\mathbf{p}_t$ is the stock price time series being analyzed,
- ϵ_t represents white noise,
- d is the degree of differencing.

• Parameters p, d, and q are determined through statistical criteria such as the **Akaike Information Criterion (AIC)**, which balance model accuracy and complexity to avoid over-fitting.
• The order p of the AR part indicates that the model considers data from the previous p time points.
• The order d is utilized to achieve stationarity in the series. The determination of d is crucial to whether the series is stationary, and this does not heavily depend on the state of the market.
• The Moving Average (MA) component: the order q reflects the moving average of the random error terms.

Conditional Value-at-Risk

• Let $\ell(\mathbf{b}_t, \mathbf{r}_t)$ be the loss associated with the decision portfolio vector $\mathbf{b}_t$, to be chosen from a certain set Δ_m, and the random vector $\mathbf{r}_t$ in $\mathbb{R}^m$.

• The probability of $\ell(\mathbf{b}_t, \mathbf{r}_t)$ not exceeding a threshold β is given by

$$\Psi(\mathbf{b}_t, \beta) = \int_{\ell(\mathbf{b}_t, \mathbf{r}_t) \leq \beta} p(\mathbf{r}_t) d\mathbf{r}_t.$$

As a function of β for fixed $\mathbf{b}_t$, $\Psi(\mathbf{b}_t, \beta)$ is the cumulative distribution function for the loss associated with $\mathbf{b}_t$. It completely determines the behavior of this random variable and is fundamental in defining VaR and CVaR.

Conditional Value-at-Risk

• The α-VaR and α-CVaR values denoted by $\beta_\alpha(\mathbf{b}_t)$ and $\phi_\alpha(\mathbf{b}_t)$ are the loss random variable associated with $\mathbf{b}_t$ and any specified probability level α in $(0, 1)$, which are given by

$$\alpha\text{-VaR} = \beta_\alpha(\mathbf{b}_t) = \min\{\beta \in \mathbb{R} : \Psi(\mathbf{b}_t, \beta) \geq \alpha\}$$

and

$$\alpha\text{-CVaR} = \phi_\alpha(\mathbf{b}_t) = (1-\alpha)^{-1} \int_{\ell(\mathbf{b}_t, \mathbf{r}_t) \geq \beta_\alpha(\mathbf{b}_t)} \ell(\mathbf{b}_t, \mathbf{r}_t) p(\mathbf{r}_t) d\mathbf{r}_t.$$

• In the first formula, $\beta_\alpha(\mathbf{b}_t)$ comes out as the left endpoint of the nonempty interval consisting of the values β such that $\Psi(\mathbf{b}_t, \beta) = \alpha$ indeed.

• In the second formula, the probability that $\ell(\mathbf{b}_t, \mathbf{r}_t) \geq \beta_\alpha(\mathbf{b}_t)$ equals to $1 - \alpha$, consequently. Thus, $\phi_\alpha(\mathbf{b}_t)$ is the expectation of the loss associated with $\mathbf{b}_t$ conditional on the loss no less than $\beta_\alpha(\mathbf{b}_t)$.

Conditional Value-at-Risk

The key to our approach is a characterization of $\phi_\alpha(\mathbf{b}_t)$ and $\beta_\alpha(\mathbf{b}_t)$ in terms of the function F_α on $\Delta_m \times \mathbb{R}$ that is defined by

$$F_\alpha(\mathbf{b}_t, \beta) = \beta + (1-\alpha)^{-1} \int_{\mathbf{r}_t \in \mathbb{R}^m} [\ell(\mathbf{b}_t, \mathbf{r}_t) - \beta]^+ p(\mathbf{r}_t) d\mathbf{r}_t.$$

Lemma

As a function of $\beta, F_\alpha(\mathbf{b}_t, \beta)$ is convex and continuously differentiable. The $\alpha - CVaR$ of the loss associated with any $\mathbf{b}_t$ can be determined from the formula: $\phi_\alpha(\mathbf{b}_t) = \min_{\beta \in \boldsymbol{R}} F_\alpha(\mathbf{b}_t, \beta)$. In this formula, the set consisting of the values of β where the minimum value is achieved, namely $A_\alpha(\mathbf{b}_t) = \underset{\beta \in \boldsymbol{R}}{\operatorname{argmin}} F_\alpha(\mathbf{b}_t, \beta)$. is a nonempty, closed, bounded interval, and the α-VaR of the loss is given by $\beta_\alpha(\mathbf{b}_t) =$ left endpoint of $A_\alpha(\mathbf{b}_t)$. In particular, one always has $\beta_\alpha(\mathbf{b}_t) \in \underset{\beta \in \boldsymbol{R}}{\operatorname{argmin}} F_\alpha(\mathbf{b}_t, \beta)$ and

$\phi_\alpha(\mathbf{b}_t) = F_\alpha(\mathbf{b}_t, \beta_\alpha(\mathbf{b}_t))$.

Conditional Value-at-Risk

Lemma

Minimizing the α-CVaR of the loss associated with $\mathbf{b}_t$ over $\mathbf{b}_t \in \Delta_m$ is equivalent to minimizing $F_\alpha(\mathbf{b}_t, \beta)$ over $(\mathbf{b}_t, \beta) \in \Delta_m \times \mathbb{R}$, in the sense that $\min_{\mathbf{b}_t \in \Delta_m} \phi_\alpha(\mathbf{b}_t) = \min_{(\mathbf{b}_t,\beta) \in \Delta_m \times \mathbb{R}} F_\alpha(\mathbf{b}_t, \beta)$. Furthermore, $F_\alpha(\mathbf{b}_t, \beta)$ is convex with respect to $(\mathbf{b}_t, \beta)$, and $\phi_\alpha(\mathbf{b}_t)$ is convex with respect to $\mathbf{b}_t$ when $\ell(\mathbf{b}_t, \mathbf{r}_t)$ is convex with respect to $\mathbf{b}_t$, in which case, if the feasible set Δ_m is a convex set, the joint minimization is an instance of convex programming.

According to the above lemma, it is not necessary to determine an $\mathbf{b}_t$ that yields minimized α-CVaR by working with the function $\phi_\alpha(\mathbf{b}_t)$ directly, which may be hard since its definition is based on the α-VaR value $\beta_\alpha(\mathbf{b}_t)$ whose mathematical properties are often tricky.

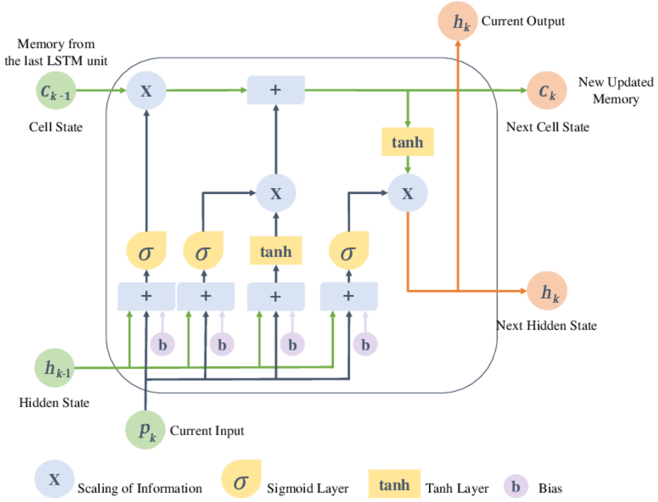


Figure 10: One LSTM block Framework

References

- Agarwal, A., Hazan, E., Kale, S., and Schapire, R. (2006). Algorithms for portfolio management based on the newton method. In Proceedings of International Conference on Machine Learning, Pittsburgh, PA, 9-16.
- Aboussalah, A., and Lee, C. (2020). Continuous control with stacked deep dynamic recurrent reinforcement learning for portfolio optimization. *Expert Systems with Applications,* 140, 112891.
- Akcoglu, K., Drineas, P., and Kao, M. (2002). Fast universalization of investment strategies with provably good relative returns. International Colloquium on Automata, Languages, and Programming, 888-900.
- Akcoglu, K., Drineas, P., and Kao, M. Y. (2004). Fast universalization of investment strategies. *SIAM Journal on Computing*, *34*(1), 1-22.
- Borodin, A., El-Yaniv, R., and Gogan, V. (2004). Can we learn to beat the best stock? *Journal of Artificial Intelligence Research*, *21*(1), 579-594.
- Brandtner, Y., Kürsten, W., and Rischau, R. (2020). Beyond expected utility: Subjective risk aversion and optimal portfolio choice under convex shortfall risk measures. *European Journal of Operational Research*, *285*(3), 1114-1126.
- Brown, D., and Jennings, R. (1989). On technical analysis. *The Review of Financial Studies*, *2*(4), 527-551.
- Cover, T. (1991). Universal portfolios. *Mathematical Finance*, *1*(1), 1-29.

- Cui, X., Gao, J., Shi, Y., and Zhu, S. (2019). Time-consistent and self-coordination strategies for multi-period mean-conditional Value-at-Risk portfolio selection. *European Journal of Operational Research*, *276*(2), 781-789.
- Das, P., and Banerjee, A. (2011). Meta optimization and its application to portfolio selection. In Proceedings of International Conference on Knowledge Discovery and Data Mining, 1163-1171.
- Duchi, J., Shalev-Shwartz, S., Singer, Y., and Chandra, T. (2008). Efficient projections onto the ℓ_1-ball for learning in high dimensions. In Proceedings of International Conference on Machine Learning, 272-279.
- Gaivoronski, A., and Stella, F. (2000). Stochastic nonstationary optimization for finding universal portfolios. *Annals of Operations Research*, *100*, 165-188.
- Gaivoronski, A., and Stella, F. (2003). Online portfolio selection using stochastic programming. *Journal of Economic Dynamics and Control*, *27*(6), 1013-1043.
- Gatzert, N., Martin, A., Schmidt, M., Seith, B., and Vogl, N. (2021). Portfolio optimization with irreversible long-term investments in renewable energy under policy risk: A mixed-integer multistage stochastic model and a moving-horizon approach. *European Journal of Operational Research*, *290*(2), 734-748.
- Guan, H., and An, Z. (2019). A local adaptive learning system for online portfolio selection. *Knowledge-Based Systems*, *186*, 104958.
- Guo, S., Yu, L. A., Li, X., and Kar, S. (2016). Fuzzy multi-period portfolio selection with different investment horizons. *European Journal of Operational Research*, *254*(3), 1026-1035.

- Guo, S., Ching, W., Li, W., Siu, T., and Zhang, Z. (2020). Fuzzy hidden Markov-switching portfolio selection with capital gain tax. *Expert Systems with Applications*, *149*, 113304.
- Guo, S., and Ching, W. (2021). High-order Markov-switching portfolio selection with capital gain tax. *Expert Systems with Applications*, *165*, 113915.
- Guo, S., Gu, J. and Ching, W. (2021). Adaptive online portfolio selection with transaction costs, *European Journal of Operational Research*, *295*(3), 1074-1086.
- Guo, S., Gu, J., Fok, C. and Ching, W. (2023). Online portfolio selection with state-dependent price estimators and transaction costs, *European Journal of Operational Research*, *311*, 333-353.
- Guo, S., Gu, J., Ching, W. and Lyu, B. (2024). Adaptive online mean-Variance portfolio selection with transaction costs, *Quantitative Finance*, *24*(1), 59-82.
- Györfi, L., Lugosi, G., and Udina, F. (2006). Nonparametric kernel-based sequential investment strate. *Mathematical Finance*, *16*(2), 337-357.
- Györfi, L., Udina, F., and Walk, H. (2008). Nonparametric nearest neighbor based empirical portfolio selection strategies. *Statistics and Decisions*, *26*(2), 145-157.
- Ha, Y., and Zhang, H. (2020). Algorithmic trading for online portfolio selection under limited market liquidity. *European Journal of Operational Research*, *286*(3), 1033-1051.
- Hazan, E., and Seshadhri, C. (2009). Efficient learning algorithms for changing environments. In Proceedings of the International Conference on Machine Learning, 393-400.
- Helmbold, D., Schapire, R., Singer, Y., and Warmuth, M. (1998). On-line portfolio selection using multiplicative updates. *Mathematical Finance*, *8*(4), 325-347.

- Huang, D., Zhou, J., Li, B., Hoi, S., and Zhou, S. (2016). Robust median reversion strategy for online portfolio selection. *IEEE Transactions on Knowledge and Data Engineering*, *28*(9), 2480-2493.
- Kar, M. B., Kar, S., Guo, S., Li, X., and Majumder, S. (2019). A new bi-objective fuzzy portfolio selection model and its solution through evolutionary algorithms. *Soft Computing*, *23*(12), 4367-4381.
- Kelly, J. (1956). A new interpretation of information rate. *Bell Systems Technical Journal*, *35*, 917-926.
- Konno, H., and Yamazaki, H. (1991). Mean-absolute deviation portfolio optimization model and its applications to Tokyo stock market. *Management Science*, *37*(5), 519-531.
- Li, X., Qin, Z. F., and Kar, S. (2010). Mean-variance-skewness model for portfolio selection with fuzzy returns. *European Journal of Operational Research*, *202*(1), 239-247.
- Li, X., Jiang, H., Guo, S., Ching, W., and Yu, L. (2020). On product of positive L-R fuzzy numbers and its application to multi-period portfolio selection problems. *Fuzzy Optimization and Decision Making*, 19, 53-79.
- Li, X., Guo, S., and Yu, L. (2015). Skewness of fuzzy numbers and its applications in portfolio selection. *IEEE Transactions on Fuzzy Systems*, *23*(6), 2135-2143.
- Li, B., and Hoi, S. (2012). On-line portfolio selection with moving average reversion. In Proceedings of International Conference on Machine Learning, 273-280.

- Li, B., and Hoi, S. (2014). Online portfolio selection: A survey. *ACM Computing Surveys*, *46*(3), Article 35, https://doi.org/10.1145/2512962.
- Li, B., and Hoi, S. (2015). Online Portfolio Selection: Principles and Algorithms, CRC Press.
- Li, B., Hoi, S. C., Zhao, P., and Gopalkrishnan, V. (2011). Confidence weighted mean reversion strategy for on-line portfolio selection. In Proceedings of the International Conference on Artificial Intelligence and Statistics, 434-442.
- Li, B., Zhao, P., Hoi, S., and Gopalkrishnan, V. (2012). PAMR: Passive-aggressive mean reversion strategy for portfolio selection. *Machine Learning*, *87*(2), 221-258.
- Li, B., Hoi, S., Zhao, P., Gopalkrishnan, V. (2013). Confidence weighted mean reversion strategy for on-line portfolio selection. ACM Transactions on Knowledge Discovery from Data, Article 4.
- Li, B., Hoi, S., Gopalkrishnan, V. (2011). CORN: Correlation-driven nonparametric learning approach for portfolio selection. ACM Transactions on Intelligent Systems and Technology, *2*(3).
- Ling, A., Sun, J., Wang, M. (2020). Robust multi-period portfolio selection based on downside risk with asymmetrically distributed uncertainty set. *European Journal of Operational Research*, *285*(1), 81-95.
- Leal, M., Ponce, D., and Puerto, J. (2020). Portfolio problems with two levels decision-makers: Optimal portfolio selection with pricing decisions on transaction costs. *European Journal of Operational Research*, *284*(2), 712-727.
- Lyu, B., Wu, B., Guo, S., Gu, J. and Ching, W. (2024). Robust online portfolio optimization with cash flows, *Omega*, 129: 103169.

- Markowitz, H. (1952). Portfolio selection. *Journal of Finance, 3*(1), 77-91.
- Markowitz, H. (1959). Portfolio selection: efficient diversification of investments, Wiley, New York.
- Sharpe, W. (1966). Mutual fund performance. *The Journal of Business, 39*(1), 119-138.
- Staino, A., Russo, E. (2020). Nested Conditional Value-at-Risk portfolio selection: A model with temporal dependence driven by market-index volatility. *European Journal of Operational Research*, *280*(2), 741-753.
- Vovk, V. (1990). Aggregating strategies. In Proceedings of the Annual Conference on Learning Theory, 371-386.
- Vovk, V., Watkins, C. (1998). Universal portfolio selection. In Proceedings of the Annual Conference on Learning Theory, 12-23.
- Young, T. (1991). Calmar ratio: A smoother tool. *Futures*, *20*(1), 40.
- Zhang, W., Liu, Y., Xu, W. (2012). A possibilistic mean-semivariance-entropy model for multi-period portfolio selection with transaction costs. *European Journal of Operational Research*, *222*(2), 341-349.
- Zhang, Y., Li, X., Guo, S. (2018). Portfolio selection problems with Markowitz's mean-variance framework: a review of literature. *Fuzzy Optimization and Decision Making*, *17*(2), 125-158.