On Infectious Models for Correlated Default Risk

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Abstract: In this talk, we consider the problem of modeling the temporal dependence of defaults and introduce a novel approach for describing the chain reaction of infectious defaults. The main idea is to extend a Markov chain model for crisis management in epidemiology, namely, Greenwood's model, to describe the chain reaction of infectious defaults of bonds across any pair of industrial sectors. We then employ two important risk measures, namely, Crisis Value-at-Risk (CRVaR) and Crisis Expected Shortfall (CRES), as proxies of risk over a default cycle. Numerical experiments are given to illustrate the practical implementation of the model and identify some main features of the model. We also perform empirical studies of the model and analyze the empirical behaviors of the risk measures arising from the model.

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Introduction

(1) Motivations and Objectives.

(2) The Idea of the Infectious Models.

(3) Crisis Loss, Crisis Value-at-Risk and Crisis Expected Shortfall.

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1. Motivations and Objectives.

• There is a considerable interest in modeling **dependence** of defaults and credit qualities of entities.

• **Why do corporate defaults cluster in time?** [Das et al. (2007) How corporate defaults are correlated? *J. of Finance*]

(i) Firms may be exposed to **some common or correlated risk factors** causing correlated changes in **default probability**;
(ii) The default of a firm can be **contagious** (infectious) and induce the defaults of others, e.g. **Penn Central Railway** in 1970;
(iii) The **lessons** from defaults may generate default correlation. The examples are **Enron** and **WorldCom**. They both hint **accounting irregularities** that might be occurred in other firms and hence affect the **default probability**.

• Das et al. (2007) applied the standard doubly stochastic model of default to practical data (U.S. Corp., 1979-2004) and they find evidence of default clustering **beyond** that predicted by the model.

1.1 A brief review.

• A **comprehensive introduction** to credit risk modeling can be found in the book:
-C. Bluhm, L. Overbeck and C. Wagner, *An introduction to credit risk modeling*. Chapman & Hall/CRC, London, (2003).

• **Bernoulli models (Factor models)**
- **KMV-Model***
- Credit Monitor^{TM}, Portfolio Manager^{TM}
- Based on Merton's asset value model. ($http://www.kmv.com$)
- P. Crosbie, Modeling default risk. KMV Corporation, (1999).

- **CreditMetrics**$^{\text{TM}\dagger}$
M. Gupton, et al., *CreditMetrics-Technical document*.
Morgan Guaranty Trust Co., $http://www.creditmetrics.com$, (1997).

*KMV started as a small company and later it was acquired by Moody's.

†CreditMetric^{TM} is a trademark of RiskMetric Group, a spin-off company from JP Morgan Bank and later it was belonged to the Chase Group.

• **Poisson models (Actuarial model: insurance maths approach)**

-**CreditRisk+**
Credit Suisse Financial Products, *Credit Risk+ a Credit Risk Management Framework*
$http://www.csfb.com/institutional/research/creditrisk.html$ (1997).

• **The use of Copula functions**

-D. Li, On default correlation: a Copula function approach. *Journal of Fixed Income*, 9 (4) 43-54 (2000).

1.2 The Bernoulli Models.

• In a Bernoulli model, the default of a bond B_i in a portfolio of m bonds is assumed to follow the **Bernoulli distribution**

$$B_i = \begin{cases} 1 & \text{with probability } p_i \\ 0 & \text{with probability } 1 - p_i \end{cases}$$

where p_i is called the **default probability**, i.e., $B_i \sim B(1, p_i)$.

• Suppose default probabilities

$$p = (p_1, p_2, \ldots, p_m)$$

follow a **multivariate distribution** F then we have

$$P(B_1 = b_1, \ldots, B_m = b_m) = \int_{[0,1]^m} \prod_{i=1}^{m} p_i^{b_i}(1 - p_i)^{1-b_i} dF(p_1, p_2, \ldots, p_m)$$

where $b_i \in \{0, 1\}$ for $i = 1, 2, \ldots, m$.

1.3 The Poisson Models.

• In a Poisson model, the default probability of Bond i is assumed to follow the **Poisson distribution** with **mean rate (intensity)** λ_i in the way that

$$p_i = P(\tilde{B}_i \geq 1) = 1 - e^{-\lambda_i} = 1 - \sum_{k=0}^{\infty} \frac{(-\lambda_i)^k}{k!} \approx \lambda_i \quad (\lambda_i \text{ is small}).$$

• We note that the default probability is quite close to the intensity. Similarly, we assume the default intensities $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ follow a **multivariate distribution** $\tilde{F}$ then we have

$$P(\tilde{B}_1 = \tilde{b}_1, \ldots, \tilde{B}_m = \tilde{b}_m) = \int_{[0,\infty)^m} e^{-(\lambda_1 + \ldots + \lambda_m)} \prod_{i=1}^{m} \frac{\lambda_i^{\tilde{b}_i}}{\tilde{b}_i!} d\tilde{F}(\lambda_1, \ldots, \lambda_m)$$

where $\tilde{b}_i \in \mathbf{N}$ for $i = 1, 2, \ldots, m$.

• Generally speaking, for a given portfolio, a Bernoulli model yields a **loss distribution** with a **fatter tail** than a Poisson model.

1.4 The use of Copula.

• Li (2000) introduced the use of **Copulas** for credit risk measurement and provides a comprehensive discussion on various copulas functions for modeling **dependent** credit risks.

• A copula is used as a general way of formulating a multivariate distribution in such a way that **various general types of dependence** can be represented.

• A Copula is a **multivariate** (m-variable) distribution (e.g. m assets) such that its marginal distributions follows the **standard uniform distribution** and denote as follows:

$$\boxed{C(u_1, u_2, \ldots, u_m) : [0, 1]^m \to [0, 1].} \tag{1}$$

• The success of Copula in modeling loss distribution depends on the following two results by Sklar (1959).

Let F be a joint distribution function with marginal distributions $F_1, \ldots, F_m$. Then we have (i) and (ii).

(i) There **exists** a Copula C such that

$$\boxed{F(x_1, x_2, \ldots, x_m) = C(F_1(x_1), F_2(x_2), \ldots, F_m(x_m)).} \qquad (2)$$

If further that all the marginal distributions are **continuous** then C is **unique**.

(ii) Conversely if C is a Copula and $F_1, F_2, \ldots, F_m$ are **univariate distributions** then the function defined in (2) is a joint distribution with marginal distributions $F_1, F_2, \ldots, F_m$.

- A popular Copula is the **normal Copula** defined as follows:

$$\boxed{C(u_1, u_2, \ldots, u_m) = N_m\left(N^{-1}(u_1), N^{-1}(u_2), \ldots, N^{-1}(u_m); \Sigma\right)}$$

-$N_m(.;\Sigma)$ is the cumulative multivariate centered Gaussian distribution with correlation matrix Σ;

-$N(.)$ is the cumulative standard normal distribution.

- It was employed in both **CreditMetrics**$^{\text{TM}}$ and **KMV-Model**.

1.5 Binomial expansion techniques (BET).

• Moody's **Binomial Expansion Techniques** (BET) is another technique for modeling credit default.

-Moody's Investment Services, *The binomial expansion method applied to CBO/CLO analysis*, (1999).

• It actually assumes **independence** between firms and therefore the number of defaults in a portfolio can be assumed to follow a **binomial distribution**.

2. The idea of infectious models

• Davis (Imperial College) and Lo (Deutsche Bank) (2001) proposed an infectious default model.
-M. Davis and V. Lo, Infectious defaults, *Quantitative Finance*, **1** 382-387 (2001).

• In their models, they consider n **identical bonds** and assume that the default process Z_i (0 = **survive** and 1 = **default**) of a bond i consists of **two components** such that

$$\boxed{Z_i = X_i + (1 - X_i)\left(1 - \prod_{j \neq i}(1 - X_j Y_{ji})\right).}$$

Here for $i, j = 1, 2, \ldots, n$, X_i and $Y_{ij} (i \neq j)$, are **independent Bernoulli random variables**, $X_i, Y_{ij} \in \{0, 1\}$.

• The bond i may **default directly** ($X_i = 1$) or it may be **infected** by another defaulted bond j when $X_i = 0$ and $X_j = Y_{ji} = 1$.

• We note that if $X_i = 0$, then the second term is 1 when at least there is one $j (j \neq i)$, such that $X_j = Y_{ji} = 1$.

• The Bernoulli random variable Y_{ji} is used to model the **infectious effect** from a **defaulted bond** j to a **survival bond** i.

• They then extended this idea to a portfolio of bonds in **multiple industrial sectors** but the sectors are assumed to be **independent**.

• It results in a **combinatorial problem of huge size** though an algorithm was developed to increase the computational efficiency. The computational cost is huge for moderate number of bonds in a portfolio.

2.1. Greenwood's model (one-sector model)

• We first **ignore the default dependence within a sector** but **focus on cross-sector dependence**. We then handle the case of both intra- and inter-sector dependence.

• In the **Greenwood's model**, (Daley and Gani (1999)) we assume that the **default probability** of each surviving bond is **the same and independent** in the sector.

• Let α denote the probability that the default of a surviving bond is triggered by the defaulted bonds in the sector, where $\alpha \in (0, 1)$.

• For each $t \in \mathcal{T}$, let X_t **be the number of survival bonds at time** t and Y_t **be the number of bonds defaulted at time** t.

• The sum of the numbers of the defaulted bonds and the surviving bonds at time $t+1$ equals to the number of surviving bonds at time t, i.e,

$$\boxed{X_{t+1} + Y_{t+1} = X_t.}$$

• The one-sector infectious model can be described by a **bivariate Markov chain model**.

• Similar to Binomial Expansion Techniques (BET), under which the **joint conditional probability distribution** for X_{t+1} and Y_{t+1} given $X_t = x_t$ and $Y_t = y_t$ is given by

$$\begin{aligned} p_{x_t,y_t}(x_{t+1}, y_{t+1}) &= P\{(X_{t+1}, Y_{t+1}) = (x_{t+1}, y_{t+1}) \mid (X_t, Y_t) = (x_t, y_t)\} \\ &= \begin{pmatrix} x_t \\ y_{t+1} \end{pmatrix} \alpha^{y_{t+1}}(1-\alpha)^{x_{t+1}} \\ &= \begin{pmatrix} x_t \\ x_t - x_{t+1} \end{pmatrix} \alpha^{x_t - x_{t+1}}(1-\alpha)^{x_{t+1}}. \end{aligned} \tag{3}$$

• Therefore $\{X_t,\ t = 0, 1, 2, \ldots,\}$ is a **Markov chain process**.

• The $(x_0+1)\times(x_0+1)$ transition probability matrix $P(\alpha)$ of the **Markov chain model** can be obtained as follows:

$$P(\alpha) = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \alpha & 1-\alpha & 0 & \cdots & 0 \\ \alpha^2 & \binom{2}{1}\alpha(1-\alpha) & (1-\alpha)^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \alpha^{x_0} & \binom{x_0}{1}\alpha(1-\alpha)^{x_0-1} & \binom{x_0}{2}\alpha^{x_0-2}(1-\alpha)^2 & \cdots & (1-\alpha)^{x_0} \end{pmatrix}. \tag{4}$$

• It is **a lower triangular matrix** and the elements of the i^{th} row of $P(\alpha)$ are given by the i^{th} term in the expansion of $\{\alpha+(1-\alpha)\}^{i-1}$, for $i = 1, 2, \ldots, x_0+1$.

2.2. Two-sector model

• We extend the version of Greenwood's model (one-sector model) to describe the chain reaction of infectious defaults across any two related sectors (two-sector model), say **sector A** and **sector B**.

• The key ideas of the two-sector model are the followings:

(i) to model the chain reaction of infectious defaults in a sector, say **sector A**;
(ii) to describe the impact of the default state of another sector, say **sector B**, on the likelihood of the default of a bond in **sector A**.

• For the purpose of illustration, we assume that there are **two states** in **sector** B. We say that the default state of sector B in **state** 0 if there is **no default** observed and is **in state** 1 if there is **at least one default observed**.

• Suppose $\{H_t\}_{t \in \mathcal{T}}$ denotes a stochastic process with state space $\{0, 1\}$, where H_t **represents the default state of sector B at time** t.

• We assume that $\{H_t\}_{t \in \mathcal{T}}$ follows a **two-state, observable** Markov chain with the following 2×2 transition probability matrix:

$$\begin{pmatrix} p^{(0,0)} & p^{(0,1)} \\ p^{(1,0)} & p^{(1,1)} \end{pmatrix}$$

where

$$\begin{cases} p^{(0,0)} + p^{(0,1)} = 1 \\ p^{(1,0)} + p^{(1,1)} = 1 \\ 0 \leq p^{(i,j)}. \end{cases} \quad (5)$$

- Each time Sector B may have **one or more default bonds (1)** or **no default bond (0)**. It is a 2-state Markov chain process having transition probability matrix:

$$\begin{array}{c} \mathbf{0} \\ \mathbf{1} \end{array} \begin{pmatrix} p^{(0,0)} & p^{(0,1)} \\ p^{(1,0)} & p^{(1,1)} \end{pmatrix}$$

- The following is a possible realization of the process:

$$\begin{array}{lcccccccccc}
\textbf{Sector B} & \underbrace{\boxed{1}}_{Default} & \rightarrow & \underbrace{\boxed{1}}_{Default} & \rightarrow & \underbrace{\boxed{0}}_{No\ default} & \rightarrow & \underbrace{\boxed{0}}_{No\ default} & \rightarrow & \underbrace{\boxed{1}}_{Default} & \rightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \cdots \\
\textbf{Sector A} & \underbrace{\alpha_1}_{Default} & & \underbrace{\alpha_1}_{Default} & & \underbrace{\alpha_0}_{Default} & & \underbrace{\alpha_0}_{Default} & & \underbrace{\alpha_1}_{Default} & \cdots \\
 & prob. & & prob. & & prob. & & prob. & & prob. & \cdots
\end{array}$$

• Here we assume the initial conditions are given as follow:

$$\boxed{X_0 = x_0, \quad Y_0 = y_0 \quad \text{and} \quad x_0 + y_0 = N.}$$

For each $s = 0, 1$, let α_s denote the probability that the default of a surviving bond is infected by the defaulted bonds in sector A if the default state of **sector B is** s, where $\alpha_s \in (0, 1)$, for $s = 1, 2$.

• The chain reaction of the infectious defaults of bonds in sector A **depends on** the state of sector B.

• Thus both the chain reaction of infectious defaults in sector A and the impact of default state in sector B on the likelihood of defaults of sector A can be modeled.

• Then, under the two-sector model, the joint probability distribution of $(X_{t+1}, Y_{t+1}, H_{t+1})$ given (X_t, Y_t, H_t) is:

$$
\begin{aligned}
& p_{x_t,y_t,h_t}(x_{t+1}, y_{t+1}, h_{t+1}) \\
= \; & P\left\{(X_{t+1}, Y_{t+1}, H_{t+1}) = (x_{t+1}, y_{t+1}, h_{t+1}) | (X_t, Y_t, H_t) = (x_t, y_t, h_t)\right\} \\
= \; & \begin{pmatrix} x_t \\ x_t - x_{t+1} \end{pmatrix} \alpha_{h_t}^{x_t - x_{t+1}} (1 - \alpha_{h_t})^{x_{t+1}} p^{(h_t, h_{t+1})}, \quad h_t, h_{t+1} = 0, 1,
\end{aligned} \tag{6}
$$

where $p^{(h_t,h_{t+1})}$ is given by (5).

• The two-sector model can be considered as a **Markovian regime-switching Markov chain modulated by another Markov chain with two regimes**. The transition probability matrix of the Markov chain model is a $2(x_0 + 1) \times 2(x_0 + 1)$ matrix given as follows:

$$
\begin{pmatrix} p^{(0,0)} P(\alpha_0) & p^{(0,1)} P(\alpha_0) \\ p^{(1,0)} P(\alpha_1) & p^{(1,1)} P(\alpha_1) \end{pmatrix}
$$

where x_0 is the number of survival bonds in sector A at time $t = 0$, and, for $s = 0, 1$, $P(\alpha_s)$ is defined in (4).

3. Crisis Loss, Crisis Value-at-Risk and Crisis Expected Shortfall

• We define the duration of the default crisis T as the **default cycle**, (from defaults begin until no more default).

• The **loss in crisis** is a function of **the default cycle** T and **the severity of defaults measured by the number of default bonds** W_T over the **duration** T.

• Let $L(\cdot,\cdot)(\omega) : \mathcal{T} \times \mathcal{R} \times \Omega \to \mathcal{R}$ denote a real-valued function $L(T, W_T)(\omega)$ of T and W_T. We then suppose that for a fixed $\omega \in \Omega$,

$$T(\omega) = t, \quad W_t(\omega) = w, \quad \text{and} \quad L(t,w)(\omega) = l(t,w) \in \mathcal{R}.$$

The loss from the default crisis is $l(t,w)$ when the **duration of default crisis** $T = t$ and **the number of defaulted bonds in the crisis** $W_t = w$.

• Here we assume some hypothetical values for the **loss** $L(T, W_T)$, for each $T = 0, 1, \ldots, X_0$ and $W_T = 0, 1, \ldots, X_0$, as below:

$$\begin{cases} L(i, 0) = 0, \text{ for each } i = 0, 1, \ldots, X_0; \\ L(0, j) = j - 1 + 0.1, \text{ for each } j = 1, 2, \ldots, X_0; \\ L(i, j) = L(0, j) + i - 1, \text{ for each } j = 1, 2, \ldots, X_0 \text{ and } i = 1, 2, \ldots, j; \\ L(i, j) = 0, \text{ otherwise.} \end{cases}$$

We remark that $T \leq X_0$.

• The hypothetical values for the loss variable is used for illustration. We note that $l(i, j)$ is **monotonic increasing** in i and j.

• The values are consistent with the economic intuition that the loss from a portfolio **increases** if either the number of defaulted bonds W_T **increases** or the duration of the default crisis becomes **longer**.

3.2. CRisis Values-at-Risk (CRVaR)

• The **CRVaR with probability level** β is defined as a functional $V_\beta(\cdot) : \mathcal{L}(T, W_T) \to \mathcal{R}$ such that for each $L(T, W_T) \in \mathcal{L}(T, W_T)$,

$$\boxed{V_\beta(L(T, W_T)) := \inf\{l \in \mathcal{R} | \mathcal{P}(L(T, W_T) > l) \leq \beta\}.}$$

• In the language of statistics, $V_\beta(L(T, W_T))$ is the generalized β-**quantile** of the distribution of the **loss variable** $L(T, W_T)$.

• The loss from the default crisis $L(T, W_T)$ is completely determined when T and W_T are given, $\mathcal{P}(L(T, W_T) > l)$ is completely determined by the joint pdf of W_T and T.

3.3. CRisis Expected Shortfall (CRES)

• The **CRES with probability level** β is defined as a functional $E_{\beta}(\cdot) : \mathcal{L}(T, W_T) \to \mathcal{R}$ such that for each $L(T, W_T) \in \mathcal{L}(T, W_T)$,

$$\boxed{E_{\beta}(L(T, W_T)) := E_{\mathcal{P}}[L(T, W_T) | L(T, W_T) \geq V_{\beta}(L(T, W_T))].}$$

• In other words, $E_{\beta}(L(T, W_T))$ is the **average of the loss from the default crisis** when the loss **exceeds the CRVaR** of the default crisis with **probability level** β.

4. Parameters Estimation
(One-sector and two-sector model)

- To estimate the **default probability** in the **one-sector model**.

- To compute the joint **probability distribution** of (T, W_T) in the **one-sector model**.

- To estimate the **default probability** and the **transition probabilities** in the **two-sector model**.

- To compute the joint **probability distribution** of (T, W_T) in the **two-sector model**.

4.1. The default probability in one-sector model

- The **likelihood function** is given by

$$L(\alpha|x_0, x_1, \ldots, x_N) = K(1-\alpha)^{x_1+\ldots+x_N}\alpha^{x_0-x_N}$$

where $K := \binom{x_0}{x_1}\binom{x_1}{x_2} \times \cdots \times \binom{x_{N-1}}{x_N}$ is independent of α.
It is more convenient to work with the **log-likelihood function**

$$\ln L(\alpha|x_0, x_1, \ldots, x_N) = \ln(K) + \sum_{i=1}^{N} x_i \ln(1-\alpha) + (x_0 - x_N)\ln(\alpha) \ .$$

- By solving

$$\frac{\partial \ln L(\alpha|x_0, x_1, \ldots, x_N)}{\partial \alpha} = 0,$$

we obtain the **maximum likelihood estimate** of α:

$$\boxed{\hat{\alpha} = \frac{x_0 - x_N}{x_0 + \sum_{i=1}^{N-1} x_i}.}$$

4.2. The joint P.D.F. of (T, W_T) in one-sector model

- Let

$$\boxed{p_j(n) = P\{X_n = j, Y_n > 0\} = \sum_{i=j+1}^{x_0-(n-1)} p_i(n-1) r_{i,j}} \tag{7}$$

where

$$\boxed{r_{i,j} = P\{X_{n+1} = j \mid X_n = i, Y_n > 0\} = \binom{i}{j} \alpha^{i-j}(1-\alpha)^j.}$$

Since

$$\boxed{p_i(0) = \begin{cases} 1 & \text{if } i = x_0 \\ 0 & \text{otherwise} \end{cases}}$$

all $p_j(n)$ can be computed recursively using (7). We have

$$\boxed{P\{(T, W_T) = (n, k) \mid X_0 = i, Y_0 > 0\} = p_{i-k}(n-1) r_{i-k,i-k}.}$$

4.3. The default probability in two-sector model

• We employ the maximum likelihood method to estimate the parameters α_0 and α_1 based on the observed numbers of defaults in both sectors A and B. Given these observations $x_0, x_1, \ldots, x_N$ and $h_0, h_1, \ldots, h_N$, the likelihood function:

$$L(\alpha | x_0, x_1, \ldots, x_N, h_0, h_1, \ldots, h_N)$$
$$= \binom{x_0}{x_1} (1-\alpha_{h_0})^{x_1} \alpha_{h_0}^{x_0 - x_1} \cdots \binom{x_{N-1}}{x_N} (1-\alpha_{h_{N-1}})^{x_N} \alpha_{h_{N-1}}^{x_{N-1}-x_N} .$$

• Using the same techniques, the estimates of α_0 and α_1 are:

$$\boxed{\widehat{\alpha}_0 = \frac{\sum_{i=0}^{N-1} (1-h_i) y_{i+1}}{\sum_{i=0}^{N-1} (1-h_i) x_i} \quad \text{and} \quad \widehat{\alpha}_1 = \frac{\sum_{i=0}^{N-1} h_i y_{i+1}}{\sum_{i=0}^{N-1} h_i x_i}.} \tag{8}$$

• Given the default data of sector B, the transition probabilities $p^{(i,j)}$ $(i, j = 0, 1)$ are estimated by the proportion of observed frequencies of transitions from state i to state i and state j.

4.4. The Joint P.D.F. of (T, W_T) in two-sector model

- The joint p.d.f. of (T, W_T) is as follows. For $s = 0, 1$ and $0 < t \le T$,

$$\begin{aligned} p_j^s(t) &= P\{X_t = j, H_t = s, Y_t > 0\} \\ &= \sum_{i=j+1}^{x_0-(t-1)} (p_i^0(t-1) r_{i,j}^{(0,s)} + p_i^1(t-1) r_{i,j}^{(1,s)}) \\ &= \sum_{k=0}^{1} \sum_{i=j+1}^{x_0-(t-1)} p_i^k(t-1) r_{i,j}^{(k,s)}, \end{aligned} \tag{9}$$

$$\begin{aligned} r_{i,j}^{(k,s)} &= P\{(X_{t+1}, H_{t+1}) = (j, s) | (X_t, H_t) = (i, k)\} \\ &= \binom{i}{j} \alpha_k^{i-j} (1-\alpha_k)^j p^{(k,s)}, \end{aligned} \tag{10}$$

and

$$p_i^s(0) = \begin{cases} 1 & \text{if } i = x_0, s = H_0 \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

• Again, the probability $p_j^s(t)$ can be obtained recursively using the above initial conditions and Equations (5) and (9) as follows:

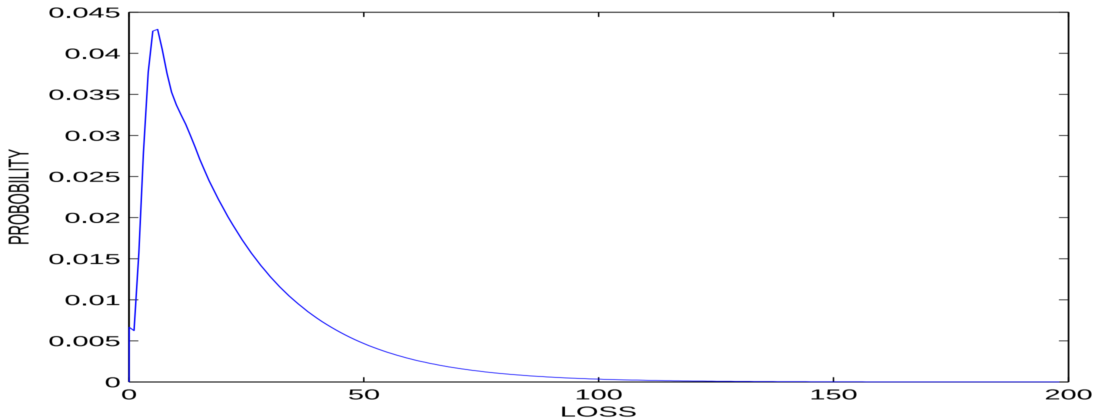
$$\begin{array}{l} P\{(T, W_T) = (t, k) | X_0 = i, Y_0 > 0\} \\ = p_{i-k}^1(t-1) r_{i-k,i-k}^{(1,0)} + p_{i-k}^0(t-1) r_{i-k,i-k}^{(0,0)} \\ + p_{i-k}^1(t-1) r_{i-k,i-k}^{(1,1)} + p_{i-k}^0(t-1) r_{i-k,i-k}^{(0,1)} \\ = p_{i-k}^1(t-1)(1-\alpha_1)^{i-k} + p_{i-k}^0(t-1)(1-\alpha_0)^{i-k}. \end{array} \quad (12)$$

• Here we remark that even when the size of the Markov chain is huge, the closed-form recursive formula also provide an efficient and convenient way to evaluate the joint probability distribution T and W_T.

• This makes our model practically useful even when we have large number of bonds in a portfolio.

4.5. An example.

$\bullet X_0 = 10^3$, $\alpha_0 = 5\times 10^{-4}$, $\alpha_1 = 5\times 10^{-3}$, $\begin{pmatrix} p^{(0,0)} & p^{(0,1)} \\ p^{(1,0)} & p^{(1,1)} \end{pmatrix} = \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$.



The loss distribution.

Table 1: The CRVaR and CRES.

	CRVaR	CRES
β=0.01	91.1	109.5
β=0.05	61.1	79.8

5. Application to credit default data.

• We adopted the quarterly bond defaults data of **four sectors** taken from Giampieri et al. (2005):
(i) **Consumer**;
(ii) **Energy**;
(iii) **Media**;
(iv) **Transportation**;
in the United States taken from Standard & Poors' ProCredit6.2 database.

• The data set covers the period from **January 1981 to December 2002**. A total of 88 quarters.

• At the beginning there are $1024, 420, 650$ and 281 non-default bonds in the above sectors respectively. At the end of observation period, there are $251, 71, 133$ and 59 default bonds in the above sectors respectively.

5.1 How to choose a sector B?

• To choose a sector B, we consider the correlations of the default sequences of the sectors.

Table 2: Correlations of the sectors

	Consumer	Energy	Media	Transport
Consumer	-	0.0224	0.6013*	0.3487
Energy	0.0224	-	0.1258*	0.1045
Media	0.6013*	0.1258	-	0.3708
Transport	0.3487	0.1045	0.3708*	-

• We note that all the correlations are positive.

• **Likelihood Ratio Test** (LRT) is employed to test the one-sector model against the two-sector model. The test statistic of the LRT is the log-likelihood ratio, which follows approximately $\chi^2(1)$ **distribution**. The critical values are 3.843 and 6.637 at 95% and 99% significant levels, respectively.

• Those log-likelihood ratios greater than the critical value are signified with a "$*$" in Table 3. Table 3 presents the log-likelihood ratios of the one-sector model to the two-sector model.

Table 3: The log-likelihood ratio: one-sector to two-sector

Sector A	Consumer	Energy	Media	Transport
Sector B	Media	Media	Consumer	Media
Log-likelihood Ratio	68.99*	1.21	43.44*	12.84*

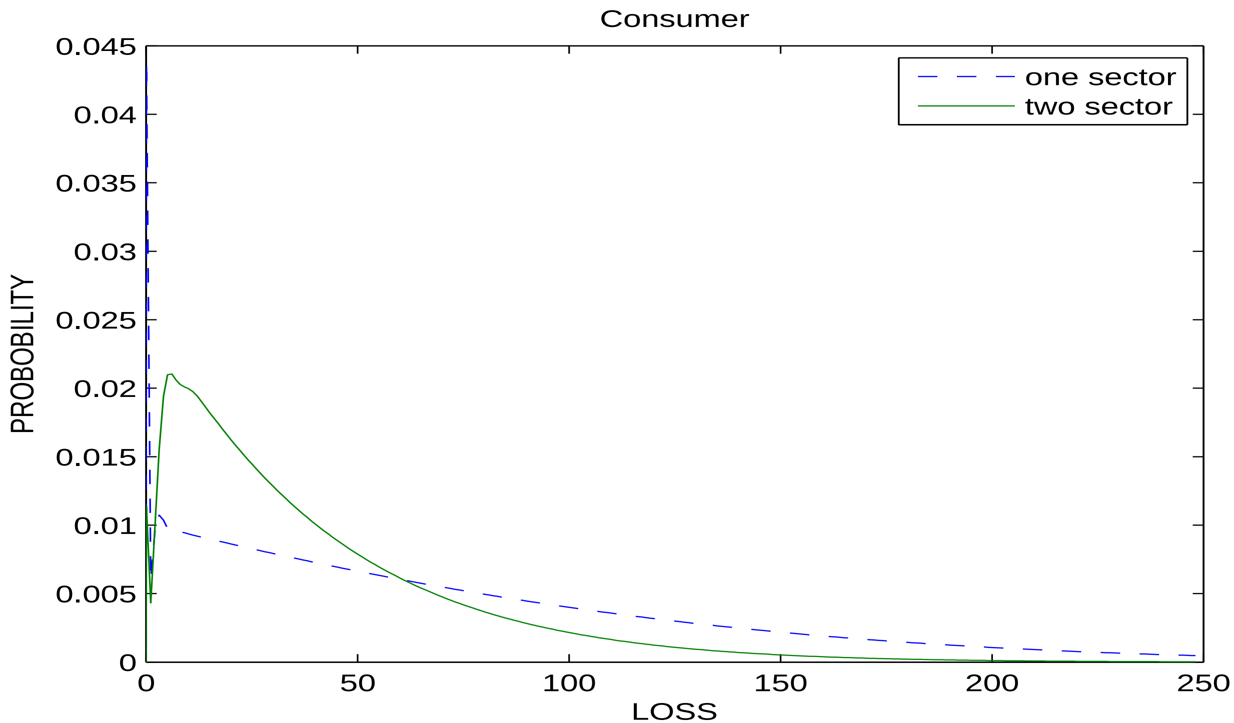
Table 4: Results on choosing Sector B.

Sector A	Sector B (LRT)	Sector B (Correlation)
Consumer	**Media**	**Media**
Energy	**Transport**	**Media**
Media	**Consumer**	**Consumer**
Transport	**Media**	**Media**

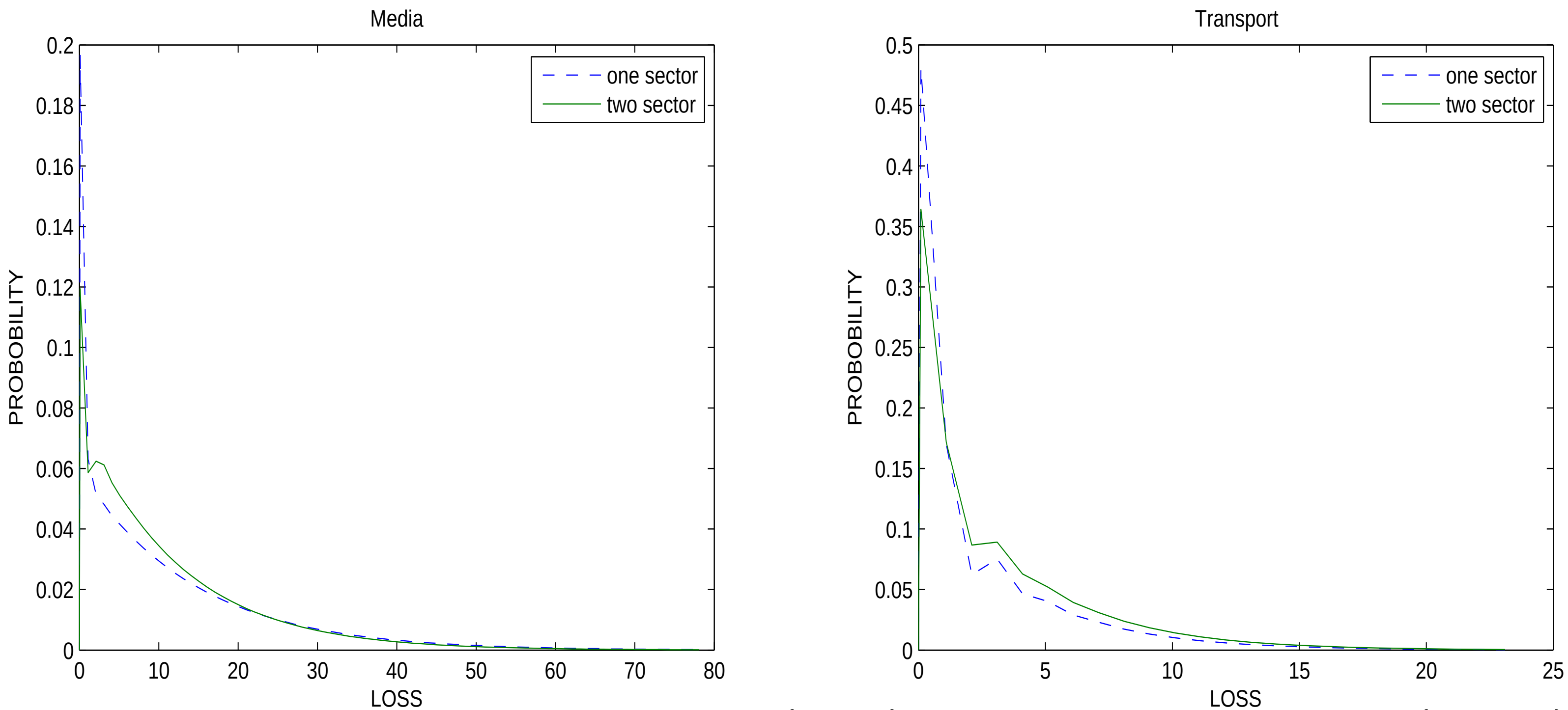
5.2 CRVaR and CRES: One-sector model against two-sector model.

Table 5: One-sector model against two-sector model

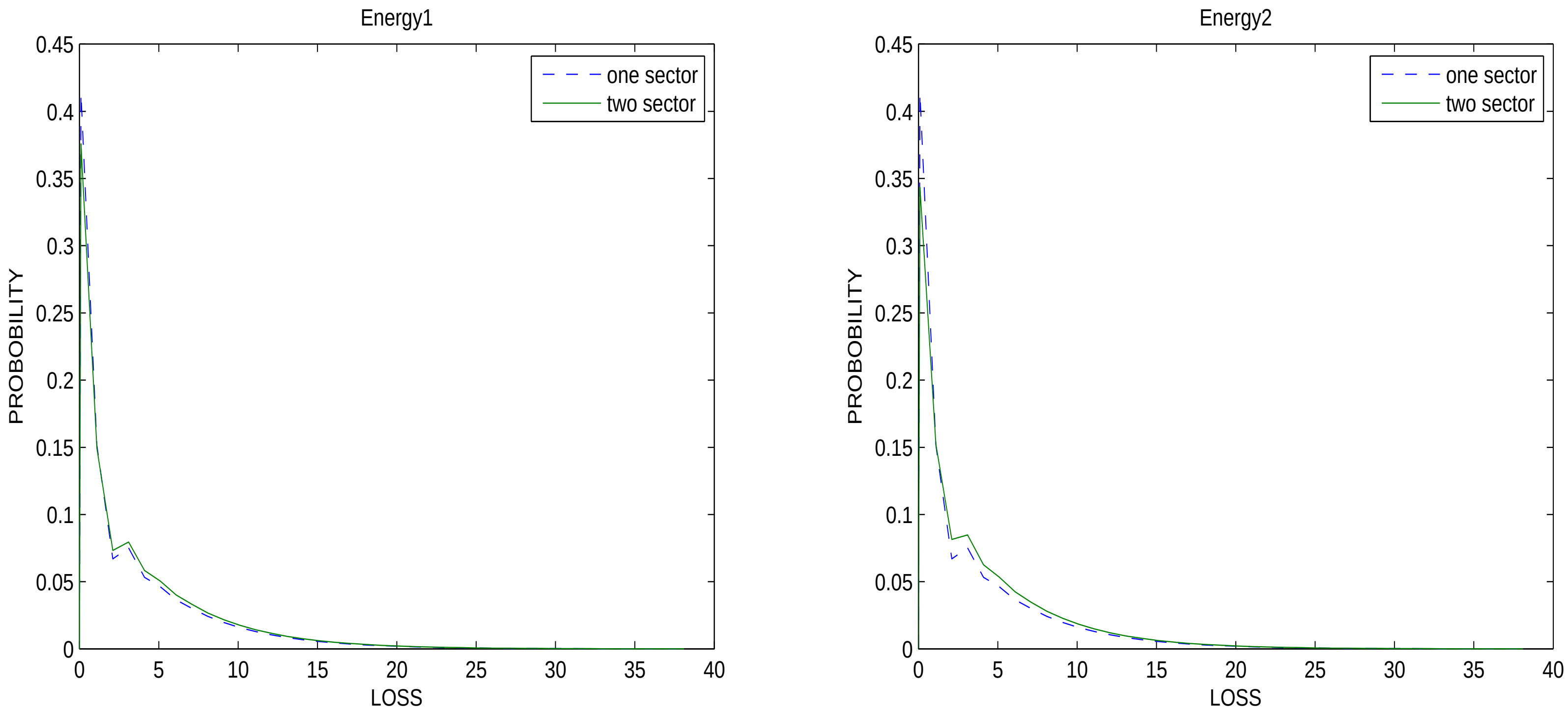
Sector A	Consumer*	Energy	Media*	Transport*
Sector B	Media	Media	Consumer	Media
α_0 (two-sector model)	0.0013	0.0018	0.0005	0.0013
α_1 (two-sector model)	0.0043	0.0023	0.0033	0.0036
CRVaR ($\beta = 0.01$)	166.1	20.1	52.1	16.1
CRES ($\beta = 0.01$)	195.6	24.5	63.3	20.2
CRVaR ($\beta = 0.05$)	114.1	12.1	34.1	10.1
CRES ($\beta = 0.05$)	146.1	17.1	45.7	14.1
α (one-sector model)	0.0030	0.0021	0.0025	0.0026
CRVaR($\beta = 0.01$)	284.1	19.1	58.1	15.1
CRES ($\beta = 0.01$)	322.0	24.1	69.8	18.5
CRVaR($\beta = 0.05$)	208.1	12.1	38.1	9.1
CRES ($\beta = 0.05$)	254.4	16.6	52.2	12.6



Loss distribution of consumer sector



Loss distribution of Media sector (Left) and Transport sector (Right)



Loss distribution of energy sector:
Sector B = Media (Left), Sector B = Transport (Right).

6. Concluding Remarks.

1. We note that
(i) **quarantining policy** (to identify and isolate the infectives); and
(ii) **vaccination policy** (as a precaution)
are effective ways to reduce the effect of the spread of a disease.

• In our model, we did not consider
(i) **intervention policy** (to identify and provide special help to firms in crisis); and
(i) **impose regulations** (stress test)
in (or before) the default crisis.
We will extend our model by investigating the possibility of including such intervention policy.

2. We will extend the two-sector model to the case of a **network of sectors**, i.e., replace state of Sector B by the **state of the whole network of sectors**.

3. Default correlation within a sector has been ignored in all the models. We introduce this important feature by assuming the default probability p_i of Bond i in the sector follows a **common probability distribution** $f(p)$.

• This results in a model of **uniform default probability and uniform default correlation**. The probability of having k defaults given m survival bonds is given by

$$\boxed{p_{m,k} = \binom{m}{k} \int_0^1 p^k (1-p)^{m-k} f(p)dp.} \tag{13}$$

• This idea can be easily implemented into all the discussed models. Take for example, if the assumption of the uniform default correlation is introduced to the one-sector model, then the **transition probability matrix** has to be modified as follows:

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \int_0^1 pf(p)dp & \int_0^1 (1-p)f(p)dp & 0 & \cdots & 0 \\ \int_0^1 p^2 f(p)dp & \begin{pmatrix} 2 \\ 1 \end{pmatrix} \int_0^1 p(1-p)f(p)dp & \int_0^1 p^2 f(p)dp & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \int_0^1 p^{x_0} f(p)dp & \begin{pmatrix} x_0 \\ 1 \end{pmatrix} \int_0^1 p(1-p)^{x_0-1} f(p)dp & \cdots & \cdots & \int_0^1 (1-p)^{x_0} f(p)dp \end{pmatrix}. \tag{14}$$

To construct (14) we have to determine the p.d.f of the default probability $f(p)$.

4. We have mentioned that it might be more realistic to consider the case that the **default probability** depends on the **current number of defaulted bonds** in the sector. It will be interesting to extend our work in this direction. One possible way is to consider **Reed-Frost model**:

$$\begin{aligned} p_{x_t,y_t}(x_{t+1},y_{t+1}) &= P\{(X_{t+1},Y_{t+1})=(x_{t+1},y_{t+1}) \mid (X_t,Y_t)=(x_t,y_t)\} \\ &= \binom{x_t}{x_{t+1}} (\alpha^{Y_t})^{x_{t+1}}(1-\alpha^{Y_t})^{x_t-x_{t+1}}. \end{aligned}$$

• In this case, the common default probability is $(1-\alpha^{Y_t})$ and we have a two-dimensional Markov chain process in the state space $S=\{(x,y): 0 \le x, y \le X_0\}$.

• The estimation method for α can be done by a method similarly The CRVaR and CRES can be extended easily to this case. In general, one can consider a general probability function of Y_t for the common default probability.

5. Finally it is interesting to apply the develop models and concepts to other credit default data.

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