On the existence of limiting distributions of some number-theoretic error terms

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Abstract

We prove the existence of the limiting distribution of a class of functions which are bounded and can be approximated by periodic functions in L_1-norm. This had been investigated by Heath-Brown and our work is a generalization. A tool used here is the continuity theorem. By using its quantitative version, we can investigate the rate of convergence of some cases.

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1. INTRODUCTION

In [5], Heath-Brown investigated the distributions (and moments) of some error terms including the error term $\Delta(t)$ in the Dirichlet divisor problem. Actually, he considered a general class of functions which satisfy the following hypothesis.

Hypothesis (H): Let $a_1(t)$, $a_2(t), \ldots$ be continuous real-valued periodic function of period 1. Suppose that there are non-zero constants γ_1, $\gamma_2, \ldots$ such that

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T}\int_1^T \min(1, |F(t) - \sum_{n\le N} a_n(\gamma_n t)|)\, dt = 0.$$

Under the hypothesis and some other conditions, Heath-Brown proved that the limit of $T^{-1}\mu\{t \in [1,T] : F(t) \le u\}$ exists as $T \to \infty$, where μ is the Lebesgue measure. Obviously, $T^{-1}\mu\{t \in [1,T] : F(t) \le u\}$ can be written into the form $T^{-1}\int_1^T \psi_{(-\infty,u]}(F(t))\, dt$ where $\psi_{(-\infty,u]}$ is the characteristic function over the set $(-\infty, u]$. In this paper, we extend his studies by considering a vector-valued function $\underline{F}(t)$ and its 'weighted' distribution, for instance, $\kappa(T)^{-1}\int_1^T \psi_{(-\infty,u]}(F(t))k(t)\, dt$ when F is real-valued. (k satisfies some conditions of regularity.) Such a generalization has practical uses, for example, the case $k(t) = t^{-1}$ is considered in [8]. Here, we are concerned with the existence of the limiting distribution of $\underline{F}(t)$ only. It will be shown that a similar hypothesis (hypothesis (H_k) in Section 2) together with a condition on the L^1-norm of $\|\underline{F}(t)\|$ can yield the existence. This is our main result, Theorem 1. These weak conditions cannot give results as precise and informative as those of Heath-Brown such as [5, Theorem 3]. However, it provides us a unified approach for the existence of limiting distributions of different error terms. This is revealed through Examples 1-3 in Section 5. Results in Examples 1 and 2 are known while Example 3 seems to be new. Moreover, it is interesting to note that the limiting distribution is independent of the weight function k. The ingredients of the proof of Theorem 1 are the Continuity Theorem in Probability Theory and some lemmas analogous to those in [5]. The quantitative version of Continuity Theorem enables us to discuss the rate of convergence in some cases. This is not done in [5]. An illustration (Theorem 2) for $\Delta(t)$ will be given in Example 4 of Section 5.

2. DEFINITIONS AND STATEMENT OF RESULTS

Let $P : \mathbf{R}^n \longrightarrow [0, \infty)$ satisfy the following conditions:

(i) $P(x_1, \ldots, x_n) \to 1$ as $\min\{x_1, \ldots, x_n\} \to \infty$,

(ii) for each i, $P(x_1, \ldots, x_n) \to 0$ as $x_i \to -\infty$,

(iii) $\lim_{\epsilon \to 0+} P(x_1, \ldots, x_i + \epsilon, \ldots, x_n) = P(x_1, \ldots, x_n)$ for each i, (i.e. P is continuous from right in each argument),

(iv) $\sum_{r=0}^{n} (-1)^r \sum_{\underline{\delta} \in \Delta_{r,n}} P(\underline{\delta}) \geq 0$ for any $(\underline{a}, \underline{b}] \subset \mathbf{R}^n$,

where $(\underline{a}, \underline{b}] = \prod_{i=1}^{n} (a_i, b_i]$ and $\Delta_{r,n}$ is a set which contains points of the form $(z_1, \ldots, z_n)$ with $z_i = a_i$ or b_i, and exactly r of z_i's equals a_i (i.e. vertices of $(\underline{a}, \underline{b}]$). Then P is called a (joint) distribution. Define $P_j(x_j) = \lim_{x_i \to \infty, i \neq j} P(x_1, \ldots, x_n)$ $(j = 1, \ldots, n)$. P_j is called the marginal distribution of P. Consider the set

$$C(P) = \{(x_1, \ldots, x_n) : P_j(x_j+) = P_j(x_j-) \text{ for all } 1 \leq j \leq n\},$$

then $C(P)$ is a subset of the set of points of continuity of P. When $n = 1$, these two sets trivially coincide. Suppose $\{P_n\}$ is a sequence of distributions. We say that P_n converges weakly if there is a distribution P such that $\lim_{n \to \infty} P_n(\underline{x}) = P(\underline{x})$ for any $\underline{x} \in C(P)$. P induces a measure, called n-dimensional Lebesgue-Stieltjes measure, defined on the σ-algebra consisting of all borel sets of $\mathbf{R}^n$. Hence we can define the integral $\int_{\mathbf{R}^n} f(\underline{x})\,dP(\underline{x})$ for any borel measurable function f, starting with $\int_{\mathbf{R}^n} \psi_{\underline{b}}(\underline{x})\,dP(\underline{x}) = P(\underline{b})$. Here we denote ψ_S to be the characteristic function over the set S and write $\psi_{\underline{b}}$ for $\psi_{(\underline{a},\underline{b}]}$ when $\underline{a} = (-\infty,\ldots,-\infty)$. In particular, we have

$$\int_{\mathbf{R}^n} \psi_{(\underline{a},\underline{b}]}(\underline{x})\,dP(\underline{x}) = \sum_{r=0}^{n}(-1)^r \sum_{\underline{\delta}\in\Delta_{r,n}} P(\underline{\delta}). \tag{2.1}$$

(A geometrical picture for the case $n = 2$ will be illustrative for its validity.)

Suppose P is a distribution. We define $\chi(\underline{u}) = \int_{\mathbf{R}^n} e(\underline{u}\cdot\underline{x})\,dP(\underline{x})$ and call it the characteristic function of P. There is an one-to-one correspondence between characteristic functions and distributions. Besides, the weak convergence is almost equivalent to the convergence of characteristic functions. This is the continuity theorem.

Continuity Theorem Suppose $\{P_n\}$ is a sequence of distributions, and let χ_n be the associated characteristic function of P_n. If χ_n converges to a function χ pointwisely and χ is continuous at $\underline{0}$, then P_n converges weakly and vice versa.

(For more details, one can refer to [7, Section 1.1 and Appendices A and B] or [1, Sections 6.3 and 8.5]. Note that left-continuity is adopted in [1] instead.)

Let $k : [1,\infty) \longrightarrow [0,\infty)$ be a continuous, piecewisely continuously differentiable function which satisfies

(a) $\kappa(T) = \int_1^T k(t)\,dt \to \infty$,

(b) $\int_1^T |k'(t)|\,dt = o(\kappa(T))$,

as $T \to \infty$. We denote this class of functions by $\mathcal{W}$.

Suppose $\underline{F} : [1,\infty)^n \longrightarrow \mathbf{R}$ is (Lebesgue) measurable, and let $\underline{u} = (u_1,\ldots,u_n) \in \mathbf{R}^n$. Then, define

$$D_{\underline{F},T}(\underline{u}) \quad = \quad \frac{1}{\kappa(T)}\int_1^T \psi_{\cap_{i=1}^n F_i^{-1}(-\infty,u_i]}(t)k(t)\,dt$$

$$= \quad \frac{1}{\kappa(T)} \int_1^T \psi_{\underline{u}}(\underline{F}(t)) k(t)\, dt \tag{2.2}$$

where $\underline{F} = (F_1, \ldots, F_n)$. (Recall $\psi_{\underline{u}} = \psi_{(\underline{a}, \underline{u}]}$ with $\underline{a} = (-\infty, \ldots, -\infty)$.) We sometimes write $D_{\underline{F},k,T}(\underline{u})$ for $D_{\underline{F},T}(\underline{u})$ in order to emphasize the weight k. We can verify that $D_{\underline{F},T}$ is a distribution. Conditions (i), (ii) and (iii) can be seen by the dominated convergence theorem. For (iv), we note that

$$\psi_{(\underline{a},\underline{b}]} = \sum_{r=0}^{n} (-1)^r \sum_{\underline{\delta} \in \Delta_{r,n}} \psi_{\underline{\delta}}, \tag{2.3}$$

then

$$\sum_{r=0}^{n} (-1)^r \sum_{\underline{\delta} \in \Delta_{r,n}} D_{\underline{F},T}(\underline{\delta}) = \frac{1}{\kappa(T)} \int_1^T \psi_{(\underline{a},\underline{b}]}(\underline{F}(t)) k(t)\, dt \geq 0.$$

If $D_{\underline{F},T}$ converges weakly to $D_{\underline{F}}$, we call $D_{\underline{F}}$ the limiting distribution of $\underline{F}$.

Besides, we say that $\underline{F}$ satisfies hypothesis (H_k) if it has the following property.

Hypothesis (H_k): Let $a_{rm}(t)$ ($r = 1, \ldots, n$; $m = 1, 2, \ldots$) be (Lebesgue) measurable, real-valued periodic functions of period 1. Suppose that there exist non-zero constants γ_{rm} such that

$$\lim_{N \to \infty} \limsup_{T \to \infty} \frac{1}{\kappa(T)} \int_1^T \min(1, |F_r(t) - \sum_{m \leq N} a_{rm}(\gamma_{rm} t)|) k(t)\, dt = 0$$

for $r = 1, \ldots, n$ and $k \in \mathcal{W}$.

Remark One can observe that for $\alpha > 1$, Hölder's inequality yields

$$\int_1^T \min(1, |\cdot|) k(t)\, dt \leq \kappa(T)^{1-1/\alpha} \left(\int_1^T \min(1, |\cdot|)^\alpha k(t)\, dt \right)^{1/\alpha};$$

while for $0 < \alpha < 1$, we have $\min(1, |\cdot|) \leq \min(1, |\cdot|)^\alpha$. Hence there is no loss of generality in choosing $\min(1, |\cdot|)$ in the hypothesis among all measures $\min(1, |\cdot|)^\alpha$, and $\min(1, |\cdot|^\alpha)$ with $\alpha > 0$. (Note that $\min(1, |\cdot|)^\alpha = \min(1, |\cdot|^\alpha)$.)

Theorem 1 *Suppose $\underline{F}$ satisfies hypothesis (H_k), and $\int_1^T \|\underline{F}(t)\| k(t)\,dt \ll \kappa(T)$ where $\|\cdot\|$ is the usual Euclidean norm. Then, $D_{\underline{F},T}$ converges weakly as $T \to \infty$. Moreover, the limiting distribution is independent of k. (i.e. If $\underline{F}$ can satisfy both (H_{k_1}) and (H_{k_2}) with the same choices of $a_{rm}(t)$ and γ_{rm}, then the two limiting distributions are identical.) If the sequence $\{\gamma_{rm}\}_{\substack{r=1,\dots,n\\ m=1,2,\dots}}$ is linearly independent over $\mathbf{Q}$, then the characteristic function of the limiting distribution is given by*

$$\chi(\alpha_1,\dots,\alpha_n) = \prod_{r=1}^{n}\prod_{m=1}^{\infty}\int_0^1 e(\alpha_r a_{rm}(t))\,dt.$$

Remark It is clear from the proof that the limiting distribution is characterized by $a_{rm}(t)$ and γ_{rm} but not the weight function $k(t)$. More precisely, if F_1 and F_2 satisfy (H_{k_1}) and (H_{k_2}) with the same set of $a_{rm}(t)$ and γ_{rm} and $\int_1^T \|F_i(t)\| k_i(t)\,dt \ll \kappa_i(T)$, then both $D_{F_1,k_1,T}$ and $D_{F_2,k_2,T}$ converge to the same distribution function.

An immediate consequence is the following Corollary.

Corollary 1 *Suppose $a_{rm}(t)$ is periodic of period 1 and integrable on $[0,1]$, and*

$$\lim_{N\to\infty}\limsup_{T\to\infty}\frac{1}{\kappa(T)}\int_1^T |F_r(t) - \sum_{m\le N} a_{rm}(\gamma_{rm}t)| k(t)\,dt = 0$$

for $r = 1,\dots,N$. Then, $D_{\underline{F},T}$ converges weakly as $T \to \infty$.

Under some circumstances, $\underline{F}(t)$ can satisfy the hypothesis only after a transformation.

Corollary 2 *Let $a \in \mathbf{R}$ and $\alpha : [a,\infty) \longrightarrow [1,\infty)$ be surjective, strictly increasing and continuously differentiable. Suppose $F \circ \alpha$ satisfies hypothesis (H_k) and conditions in Theorem 1 or Corollary 1. Then, $D_{\underline{F},h,T}$ converges weakly as $T \to \infty$ where $h = (\kappa \circ \alpha^{-1})'$.*

Sometimes, we are interested in the limiting distribution which counts on integers only. In particular, we can have the following result, which is a case of $n = 1$ and $k(t) = 1$.

Corollary 3 *Let $F(t)$ satisfy the conditions in Theorem 1 or Corollary 1. Suppose that for $t \in [n, n+1)$,*

$$F(t) = F(n) + C(\{t\} - \frac{1}{2}) - \lambda + o(1) \qquad \text{as } n \to \infty$$

where C and λ are absolute constants. Define

$$\mathsf{D}_{F,X}(u) = \frac{1}{X} Card\{1 \le n \le X : F(n) \le u\}.$$

Then, $\mathsf{D}_{F,X}$ converges weakly as $X \to \infty$.

Remark: We can make use of $\mathsf{D}_{F,X}$ with some other properties to investigate the sign-changes (including zeros perhaps) of $F(t)$ on integers, see [6] for example.

3. SOME PREPARATIONS

Lemma 3.1 *Let $h : \mathbf{R} \longrightarrow \mathbf{C}$ be an integrable periodic function of period 1. Then*

$$\frac{1}{\kappa(T)} \int_1^T |h(\gamma t)| k(t)\, dt \le 2 \left(1 + (\gamma \kappa(T))^{-1} \int_1^T |k'(u)|\, du\right) \int_0^1 |h(u)|\, du$$

if $T \ge 1 + |\gamma|^{-1}$, where $\gamma \ne 0$ is real.

Proof We may assume $\gamma > 0$; for otherwise, we consider $h^-(|\gamma| t)$ where $h^-(t) = h(-t)$. Choose an integer m_0 such that $\gamma(T-1) - 2 < m_0 \le \gamma(T-1) - 1$, then after a change of variable, we have

$$\int_1^T |h(\gamma t)| k(t)\, dt$$

$$
\begin{aligned}
&= \gamma^{-1}\Big(\int_{\gamma}^{\gamma T-m_0-1} + \sum_{m=0}^{m_0}\int_{\gamma T-m-1}^{\gamma T-m}\Big)|h(u)|k(\gamma^{-1}u)\,du \\
&\leq \gamma^{-1}\Big(\sup_{\gamma\leq u\leq \gamma T-m_0-1} k(\gamma^{-1}u)\int_{\gamma}^{\gamma T-m_0-1} \\
&\qquad + \sum_{m=0}^{m_0}\sup_{\gamma T-m-1\leq u\leq \gamma T-m} k(\gamma^{-1}u)\int_{\gamma T-m-1}^{\gamma T-m}\Big)|h(u)|\,du \\
&\leq \gamma^{-1}\Big(\sup_{1\leq u\leq T-(m_0+1)/\gamma} k(u) + \sum_{m=0}^{m_0}\sup_{T-(m+1)/\gamma\leq u\leq T-m/\gamma} k(u)\Big)\int_0^1 |h(u)|\,du.
\end{aligned}
$$

Since

$$
\begin{aligned}
\sup_{a\leq u\leq b} k(u) &= \Big(\sup_{a\leq u\leq b} - \inf_{a\leq u\leq b}\Big)k(u) + \inf_{a\leq u\leq b} k(u) \\
&\leq \sup_{a\leq v_1\leq v_2\leq b}\left|\int_{v_1}^{v_2} k'(u)\,du\right| + \frac{1}{b-a}\int_a^b k(u)\,du \\
&\leq \int_a^b (|k'(u)| + (b-a)^{-1}k(u))\,du,
\end{aligned}
$$

we have, as $T \geq 1+\gamma^{-1}$,

$$
\begin{aligned}
&\int_1^T |h(\gamma t)|k(t)\,dt \\
&\leq \gamma^{-1}\Big(\int_1^{1+1/\gamma} + \sum_{m=0}^{m_0}\int_{T-(m+1)/\gamma}^{T-m/\gamma}\Big)(|k'(u)| + \gamma k(u))\,du \int_0^1 |h(u)|\,du \\
&\leq 2(\kappa(T) + \gamma^{-1}\int_1^T |k'(u)|\,du)\int_0^1 |h(u)|\,du.
\end{aligned}
$$

Lemma 3.2 *Let $p : \mathbf{R}^n \longrightarrow \mathbf{R}$ be uniformly continuous and bounded. Then,*

$$
\int_{\mathbf{R}^n} p(\underline{u})\,dD_{\underline{F},T}(\underline{u}) = \frac{1}{\kappa(T)}\int_1^T p(\underline{F}(t))k(t)\,dt.
$$

Proof As every uniformly continuous function can be approximated by step functions of the form $\sum_i c_i\psi_{(\underline{a}_i,\underline{b}_i]}$ in supremum norm, it suffices to consider the discontinuous case $p = \psi_{(\underline{a},\underline{b}]}$. Now, by (2.1) and (2.3),

$$
\begin{aligned}
\int_{\mathbf{R}^n} \psi_{(\underline{a},\underline{b}]}(\underline{u})\,dD_{\underline{F},T}(\underline{u}) &= \sum_{r=0}^{n}(-1)^r\sum_{\underline{\delta}\in\Delta_{r,n}} D_{\underline{F},T}(\underline{\delta}) \\
&= \frac{1}{\kappa(T)}\int_1^T \psi_{(\underline{a},\underline{b}]}(\underline{F}(t))k(t)\,dt.
\end{aligned}
$$

Lemma 3.3 *Let $b_i : \mathbf{R} \longrightarrow \mathbf{C}$ $(1 \le i \le l)$ be measurable functions of period 1. Suppose that $|b_i(t)| \le 1$, then the limit*

$$\mathcal{L}_k = \lim_{T\to\infty} \frac{1}{\kappa(T)} \int_1^T e(\gamma t) b_1(\gamma_1 t) \cdots b_l(\gamma_l t) k(t)\, dt$$

exists for any real γ, $\gamma_1, \ldots, \gamma_l$. The limit is independent of k, i.e. $\mathcal{L}_{k_1} = \mathcal{L}_{k_2}$.

Proof When $l = 0$, we have $\int_1^T e(\gamma t)k(t)\, dt = \kappa(T)$ if $\gamma = 0$. Otherwise,

$$\begin{aligned} \int_1^T e(\gamma t)k(t)\, dt &= \frac{1}{2\pi i\gamma}\{e(\gamma t)k(t)|_1^T - \int_1^T k'(t)e(\gamma t)\, dt\} \\ &\ll \gamma^{-1} \int_1^T |k'(t)|\, dt = o(\kappa(T)). \end{aligned}$$

Thus, the lemma holds for this case. Suppose it holds for some $l \ge 0$. Write $f(t) = e(\gamma t)b_1(\gamma_1 t) \ldots b_l(\gamma_l t)$ and let $\gamma_{l+1} \ne 0$ (otherwise it goes back to the case l), then following Heath-Brown[5, Lemma 1], we pick a Fourier series $S_N(t) = \sum_{|n|\le N} c_n e(nt)$ for $b_{l+1}(t)$ which converges to it in the mean. This can be done as b_{l+1} is square-integrable. Thus, we have $\lim_{N\to\infty} \int_0^1 |b_{l+1}(t) - S_N(t)|\, dt = 0$. Applying Lemma 3.1, we get

$$\frac{1}{\kappa(T)} \int_1^T |b_{l+1}(\gamma_{l+1}t) - S_N(\gamma_{l+1}t)| k(t)\, dt \le 3 \int_0^1 |b_{l+1}(t) - S_N(t)|\, dt$$

for all $T \ge T_0(\gamma_{l+1}, k)$. Hence, as $|f(t)| \le 1$,

$$\begin{aligned} &\left| \kappa(T)^{-1} \int_1^T f(t) b_{l+1}(\gamma_{l+1}t)\, dt - \kappa(T)^{-1} \int_1^T f(t) S_N(t)\, dt \right| \\ \le\ & 3 \int_0^1 |b_{l+1}(t) - S_N(t)|\, dt \\ <\ & \epsilon \end{aligned}$$

for any $N \ge N_0(\epsilon)$ and for all $T \ge T_0(\gamma_{l+1}, k)$. Induction assumption yields that $L_N = \lim_{T\to\infty} \kappa(T)^{-1} \int_1^T f(t)S_N(t)\, dt$ exists and its value is independent of k. Parallel to the argument in [5, Lemma1], Cauchy criterion shows the convergence for the case $l + 1$.

Suppose $\mathcal{L}_{k_1}$ and $\mathcal{L}_{k_2}$ are the limits corresponding to two different weight functions k_1 and k_2 respectively. Then for any $\epsilon > 0$, we have, by taking sufficiently large N, that

$$|\mathcal{L}_{k_1} - \mathcal{L}_{k_2}| \leq |\mathcal{L}_{k_1} - L_N| + |\mathcal{L}_{k_2} - L_N| < \epsilon.$$

Our assertion follows.

Lemma 3.4 *Let $b_i : \mathbf{R} \longrightarrow \mathbf{C}$ be measureable periodic functions of period 1, and $|b_i(t)| \leq 1$. Then $\lim_{T\to\infty} \kappa(T)^{-1} \int_1^T b_1(\gamma_1 t) \cdots b_l(\gamma_l t) k(t)\, dt$ exists and the limit is independent of k. Moreover, if $\{\gamma_1, \dots, \gamma_l\}$ is linearly independent over $\mathbf{Q}$, then the limit is equal to*

$$\prod_{i=1}^{l} \int_0^1 b_i(t)\, dt.$$

The proof follows closely the argument in Heath-Brown[5, Lemma 2], with Lemma 3.3.

4. PROOFS OF RESULTS

We begin to prove Theorem 1. From Lemma 3.2, we see that the characteristic function of $D_{\underline{F},T}$ is

$$\chi_T(\underline{\alpha}) = \int_{\mathbf{R}^n} e(\underline{\alpha} \cdot \underline{u})\, dD_{\underline{F},T}(\underline{u}) = \frac{1}{\kappa(T)} \int_1^T e(\underline{\alpha} \cdot \underline{F}(t)) k(t)\, dt.$$

Define

$$\chi_{N,T}(\underline{\alpha}) = \frac{1}{\kappa(T)} \int_1^T e(\sum_{r=1}^{n} \alpha_r \sum_{m\leq N} a_{rm}(\gamma_{rm} t)) k(t)\, dt$$

We divide our proof into the following steps:

Step 1. $\chi_N(\underline{\alpha}) = \lim_{T\to\infty} \chi_{N,T}(\underline{\alpha})$ exists.

The existence follows from Lemma 3.4.

Step 2. $\chi(\underline{\alpha}) = \lim_{N\to\infty} \chi_N(\underline{\alpha})$ exists. Using $|\prod w_i - \prod z_i| \le \sum |w_i - z_i|$ for $|w_i|, |z_i| \le 1$, $|e(u) - 1| \le 2\pi \min(1, |u|)$ and $\min(1, |a||b|) \le (|a| + 1)\min(1, |b|)$, we have for any N and N',

$$
\begin{aligned}
& |\chi_N(\underline{\alpha}) - \chi_{N'}(\underline{\alpha})| \\
\le\ & \sum_{M=N,N'} \limsup_{T\to\infty} \left| \kappa(T)^{-1} \int_1^T (e(\sum_{r=1}^n \alpha_r \sum_{m\le M} a_{rm}(\gamma_{rm}t)) - e(\underline{\alpha}\cdot \underline{F}(t)))k(t)\,dt \right| \\
\le\ & 2\pi \sum_{M=N,N'} \sum_{r=1}^n \limsup_{T\to\infty} \kappa(T)^{-1} \int_1^T \min(1, |\alpha_r||F_r(t) - \sum_{m\le M} a_{rm}(\gamma_{rm}t)|)k(t)\,dt \\
\le\ & 2\pi \sum_{M\le N,N'} \sum_{r=1}^n (|\alpha_r| + 1) \limsup_{T\to\infty} \kappa(T)^{-1} \int_1^T \min(1, |F_r(t) - \sum_{m\le M} a_{rm}(\gamma_{rm}t)|)k(t)\,dt.
\end{aligned}
$$

This tends to zero as N, $N' \longrightarrow \infty$ by hypothesis (H_k). By Cauchy criterion, $\chi_N(\underline{\alpha}) \longrightarrow \chi(\underline{\alpha})$ pointwisely for some function χ.

Step 3. $\lim_{T\to\infty} \chi_T(\underline{\alpha}) = \chi(\underline{\alpha})$.

For each fixed $\underline{\alpha}$ and for any $\epsilon > 0$, we have

$$
\begin{aligned}
& |\chi_T(\underline{\alpha}) - \chi(\underline{\alpha})| \\
\le\ & |\chi_T(\underline{\alpha}) - \chi_{N,T}(\underline{\alpha})| + |\chi_{N,T}(\underline{\alpha}) - \chi_N(\underline{\alpha})| + |\chi_N(\underline{\alpha}) - \chi(\underline{\alpha})| \\
\le\ & 2\pi \sum_{r=1}^n (|\alpha_r| + 1)\kappa(T)^{-1} \int_1^T \min(1, |F_r(t) - \sum_{m\le N} a_{rm}(\gamma_{rm}t)|)k(t)\,dt \\
& + |\chi_{N,T}(\underline{\alpha}) - \chi_N(\underline{\alpha})| + |\chi_N(\underline{\alpha}) - \chi(\underline{\alpha})| \\
\le\ & \sum_{r=1}^n (|\alpha_r| + 1)\epsilon
\end{aligned}
$$

whenever $T \ge T(N, \epsilon, \underline{\alpha})$ and $N \ge N(\epsilon, \underline{\alpha})$.

If $\{\gamma_{rm}\}_{\substack{r=1,\ldots,n \\ m=1,2,\ldots}}$ is linearly independent over $\mathbf{Q}$, then we have from Lemma 3.4 that

$$
\chi(\alpha_1, \ldots, \alpha_n) = \prod_{r=1}^n \prod_{m=1}^\infty \int_0^1 e(\alpha_r a_{rm}(t))\,dt.
$$

Step 4. $\chi(\underline{\alpha})$ is continuous at $\underline{\alpha} = \underline{0}$.

Here we use the condition $\int_1^T \|\underline{F}(t)\|k(t)\,dt \ll \kappa(T)$. The continuity at $\underline{\alpha} = 0$ follows from

$$\begin{aligned} |\chi_T(\underline{\alpha}) - \chi_T(\underline{0})| &= \left|\kappa(T)^{-1}\int_1^T (e(\underline{\alpha}\cdot\underline{F}(t)) - 1)k(t)\,dt\right| \\ &\le 2\pi \sum_{r=1}^n \frac{|\alpha_r|}{\kappa(T)} \int_1^T |F_r(t)|k(t)\,dt \\ &\ll \|\underline{\alpha}\|\kappa(T)^{-1}\int_1^T \|\underline{F}(t)\|k(t)\,dt \ll \|\underline{\alpha}\| \end{aligned}$$

where the implied constants are independent of $\underline{\alpha}$.

This completes the proof of Theorem 1 by Continuity Theorem.

To prove Corollary 1, it suffices to show $\kappa(T)^{-1}\int_1^T \|\underline{F}(t)\|k(t)\,dt \ll 1$ in view of Theorem 1. This follows from

$$\begin{aligned} &\kappa(T)^{-1}\int_1^T \|\underline{F}(t)\|k(t)\,dt \\ \le\ & \sum_{r=1}^n \kappa(T)^{-1}\int_1^T |F_r(t)|k(t)\,dt \\ \le\ & \sum_{r=1}^n \left\{ \sum_{m\le N} \kappa(T)^{-1}\int_1^T |a_{rm}(\gamma_{rm}t)|k(t)\,dt + \kappa(T)^{-1}\int_1^T |F_r(t) - \sum_{m\le N} a_{rm}(\gamma_{rm}t)|k(t)\,dt \right\} \\ \ll\ & \sum_{r=1}^n \left\{ \sum_{m\le N} \int_0^1 |a_{rm}(t)|\,dt + \kappa(T)^{-1}\int_1^T |F_r(t) - \sum_{m\le N} a_{rm}(\gamma_{rm}t)|k(t)\,dt \right\} \ll 1 \end{aligned}$$

for all sufficiently large T, by using the conditions in Corollary 1 and Lemma 3.1.

Now we prove Corollary 2. Write $H(T) = \int_1^T h(t)\,dt$, then

$$\frac{1}{H(T)}\int_1^T \psi_{\underline{u}}(\underline{F}(t))h(t)\,dt = \frac{1}{\kappa(\alpha^{-1}(T))}\int_a^{\alpha^{-1}(T)} \psi_{\underline{u}}(\underline{F}(\alpha(v)))k(v)\,dv$$

after a change of variable. Since $\alpha^{-1}(T) \to \infty$ as $T \to \infty$, the result follows from Theorem 1 or Corollary 1 accordingly.

Finally we prove Corollary 3 and suppose that $F(t)$ satisfies the conditions in Theorem 1. Define $F^*(t) = F(n) - \lambda$ if $t \in [n, n+1)$. By taking $a_0(t) = -C(\{t\} - 1/2)$

and $\gamma_0 = 1$, we see that

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_1^T \min(1, |F^*(t) - \sum_{0\le n\le N} a_n(\gamma_n t)|)\, dt = 0$$

and

$$\int_1^T |F^*(t)|\, dt \ll T.$$

Our assertion follows from Theorem 1. The case that $F(t)$ satisfies conditions in Corollary 1 can be proved similarly.

5. APPLICATIONS

Example 1. Let q be a natural number and $(a, q) = 1$. We denote $\pi(x, q, a)$ to be the number of primes $p \le x$ with $p \equiv a \pmod q$. Write $E(x, q, a) = (\phi(q)\pi(x, q, a) - \pi(x))x^{-1/2}\log x$, and

$$E_{q;a_1,\dots,a_n}(x) = (E(x, q, a_1), \dots, E(x, q, a_n)),$$

we have by [8, (2.5) and Lemma 2.2] and assuming G.R.H.,

$$E(x, q, a) = -c(q, a) - \sum_{\chi\ne\chi_0} \overline{\chi}(a) \sum_{|\gamma_\chi|\le N} \frac{x^{i\gamma_\chi}}{1/2 + i\gamma_\chi} + \epsilon_a(x, N, X)$$

and $\int_{\log 2}^Y |\epsilon_a(e^y, N, e^Y)|^2\, dy \ll_q YN^{-1}\log^2 N + N^{-1}\log^3 N$ where $c(q, a)$ is a constant, $\sum_{\chi\ne\chi_0}$ and $\sum_{\gamma_\chi}$ sum over the non-principal Dirichlet characters modulo q and zeros of the corresponding L-functions respectively.

We apply Corollary 2 by taking $F_r(x) = E(x, q, a_r) + c(q, a)$, $\alpha(t) = e^t$, $k(t) = 1$ (so $h(t) = 1/t$) and $-\Re e\, \overline{\chi}(a)e^{it}/(1/2 + i\gamma)$ to be $a_{rm}(t)$. Then $D_{\underline{F},h,T}$ converges weakly. This gives back the result of [8, Theorem 1.1] after a translation of $(c(q, a_1), \dots, c(q, a_n))$.

Let $E_{q;N,R}(x) = (\pi_N(x, q) - \pi_R(x, q))x^{-1/2}\log x$ where $\pi_R(x, q)$ (and $\pi_N(x, q)$) is the number of prime quadratic residues (and nonresidues respectively) not exceeding

x. Applying the same argument and assuming G.R.H., we can show the existence of the limiting distribution of $E_{q;N,R}$ too.

Example 2. Let $\phi(n)$ be the Euler function (i.e. $\phi(n)$ denotes the number of integers less than n which are relatively prime to n). Define

$$E(x) = \sum_{n \le x} \phi(n) - \frac{3}{\pi^2}x^2 \text{ and } H(x) = \sum_{n \le x} \frac{\phi(n)}{n} - \frac{6}{\pi^2}x.$$

From Chowla[1, Lemma 2], we have

$$H(u) = - \sum_{n \le u/\log^5 u} \frac{\mu(n)}{n}\psi(\frac{u}{n}) + O(\frac{1}{\log^{20} u})$$

where $\mu(n)$ is the Möbius function and $\psi(x) = \{x\} - 1/2$ ($\{x\}$ is the fractional part of x). This yields (see [6, Main Lemma] for more details) that for $1 \le N \le T/\log^5 T$,

$$\int_1^T (H(x) + \sum_{n \le N} \frac{\mu(n)}{n}\psi(\frac{x}{n}))^2 \, dx \ll TN^{-1} + T\log^{-4} T.$$

Besides, Chowla[2, Lemma 13] gives us $x^{-1}E(x) = H(x) + O((\log x)^{-4})$. By Corollaries 2 and 3 with $a_n(t) = \mu(n)n^{-1}\psi(t)$ and $\gamma_n = n^{-1}$, we see that all $D_H(u)$, $\mathsf{D}_H(u)$, $D_R(u)$ and $\mathsf{D}_R(u)$ exist where $R(x) = E(x)/x$. It should be remarked that Erdös and Shapiro[4] had proved the existence of $\mathsf{D}_H(u)$ by a different argument, and their argument can show that $\mathsf{D}_H(u)$ is continuous.

Example 3. Let $\sigma_a(n) = \sum_{d|n} d^a$ and define

$$\Delta_a(t) = \sum_{n \le t} \sigma_a(n) - \zeta(1-a)t - \frac{\zeta(1+a)}{1+a}t^{1+a} + \frac{1}{2}\zeta(-a).$$

We shall consider the case $-1 \le a < -1/2$. (The case $a = -1$ is defined by taking $a \longrightarrow -1^+$.) It is known (see [2, Lemma 15]) that

$$\Delta_a(t) = - \sum_{n \le \sqrt{t}} n^a \psi(\frac{t}{n}) - t^a \sum_{n \le \sqrt{t}} n^{|a|}\psi(\frac{t}{n}) + O(t^{a/2})$$

where $\psi(x)$ is defined as in Example 2. Using this formula, one can show (with the argument in [6, Main Lemma] again) that for $N \leq \sqrt{T}$,

$$\int_1^T |\Delta_a(t) + \sum_{n\leq N} n^a \psi(\frac{t}{n})|^2 \, dt \ll TN^{1+2a} + T^{3/2+a} \log T.$$

Hence, we can conclude the existence of the limiting distribution of $\Delta_a(t)$ by Corollary 1.

Example 4. Let $d(n) = \sum_{d|n} 1$ and define

$$\Delta(t) = \sum_{n\leq t} d(n) - t(\log t + 2\gamma - 1)$$

where γ is the Euler constant. Taking $G(t) = \Delta(t)/t^{1/4}$, then Heath-Brown[5] showed that $D_G(u)$ exists and possesses a (probability) density function $f(\alpha)$. Here, we focus on the rate of convergence and obtain the following result.

Theorem 2 *Let $D_{G,T} = T^{-1}\mu\{t \in [1,T] : t^{-1/4}\Delta(t) \leq u\}$, and $D_G(u)$ be its limit. Then, for all $u \in \mathbf{R}$,*

$$D_G(u) = D_{G,T}(u) + O((\log\log T)^{-1/8}(\log\log\log T)^{3/4})$$

as $T \to \infty$.

We denote $F(t) = t^{-1/2}\Delta(t^2)$ and $\psi_u(t) = \psi_{F^{-1}(-\infty,u]}(t)$, the characteristic function over the set $F^{-1}(-\infty,u]$. Then,

$$D_{G,T}(u) = \frac{1}{T}\mu\{t \in [1,T] : G(t) \leq u\} = \frac{1}{T}\int_1^T \psi_u(\sqrt{t})\, dt.$$

Integration by parts yields $D_{G,T}(u) = 2(D_{F,\sqrt{T}}(u) - T^{-1}\int_1^{\sqrt{T}} vD_{F,v}(u)\, dv)$ as $D_{F,v}(u) = v^{-1}\int_1^v \psi_u(w)\, dw$. We have for any $r > 2$,

$$\begin{aligned} & D_{G,T}(u) - D_G(u) \\ \ll & \sup_{T^{1/r}\leq v\leq T^{1/2}} |D_{F,v}(u) - D_G(u)| + T^{2/r-1}. \end{aligned} \tag{5.1}$$

Hence it suffices to consider $D_G(u) - D_{F,T}(u)$. By Berry-Esseen Theorem (see [3, Lemma 1.47]) and $\sup_{\alpha\in\mathbf{R}} |f(\alpha)| \ll 1$ (see [5]),

$$D_G(u) - D_{F,T}(u) \ll \frac{1}{R} + \int_{-R}^{R} \left|\frac{\chi_{F,T}(\alpha) - \chi(\alpha)}{\alpha}\right| d\alpha \tag{5.2}$$

where $\chi_{F,T}(\alpha)$ and $\chi(\alpha)$ are characteristic functions of $D_{F,T}$ and D_G respectively.

We define $\chi_{N,T}$ and χ_N to be those characteristic functions in the proof of Theorem 1, and take

$$a_n(t) = \frac{1}{\pi\sqrt{2}} \frac{\mu(n)^2}{n^{3/4}} \sum_{r=1}^{\infty} \frac{d(nr^2)}{r^{3/2}} \cos(2\pi rt - \frac{\pi}{4}),$$

and $\gamma_n = 2\sqrt{n}$ if n is squarefree, and any suitable value otherwise. Then one can see that

$$\chi_{N,T}(\alpha) = \frac{1}{T}\int_1^T \prod_{n=1}^{N} e(\alpha a_n(\gamma_n t))\, dt \text{ and } \chi_N(\alpha) = \prod_{n=1}^{N} \int_0^1 e(\alpha a_n(t))\, dt$$

as $\{\gamma_n\}$ is linearly independent over $\mathbf{Q}$ (see [5, Lemma 2 and (3.4)]). We consider

$$\begin{aligned} & \chi_{F,T}(\alpha) - \chi(\alpha) \\ = \ & \chi_{F,T}(\alpha) - \chi_{N,T}(\alpha) + \chi_{N,T}(\alpha) - \chi_N(\alpha) + \chi_N(\alpha) - \chi(\alpha). \end{aligned} \tag{5.3}$$

Recalling that $\chi_{F,T}(\alpha) = T^{-1}\int_1^T e(\alpha F(t))\, dt$, we have

$$\chi_{F,T}(\alpha) - \chi_{N,T}(\alpha) \ll |\alpha| \frac{1}{T}\int_1^T |F(t) - \sum_{n\le N} a_n(\gamma_n t)|\, dt$$

by using $e(u) - 1 \ll |u|$. Suppose $N \le \log T$. Using [5, (5.2)] with the estimate $\sum_{n\ge N} d(n)^2 n^{-3/2} \ll N^{-1/2}\log^3 N$ (instead of $N^{\epsilon-1/2}$), we obtain

$$\int_T^{2T} |F(t) - \sum_{n\le N} a_n(\gamma_n t)|^2\, dt \ll TN^{-1/2}\log^3 N,$$

and from (5.2) and (5.3),

$$D_G(u) - D_{F,T}(u)$$

$$\begin{aligned}
&\ll \frac{1}{R} + \int_{-R}^{R} \frac{1}{T}\int_1^T |F(t) - \sum_{n\le N} a_n(\gamma_n t)|\,dt\,d\alpha \\
&\quad + \int_{-R}^{R} \limsup_{T\to\infty} \frac{1}{T}\int_1^T |F(t) - \sum_{n\le N} a_n(\gamma_n t)|\,dt\,d\alpha \\
&\quad + \int_{-R}^{R} |\frac{\chi_{N,T}(\alpha) - \chi_N(\alpha)}{\alpha}|\,d\alpha \\
&\ll \frac{1}{R} + RN^{-1/4}(\log N)^{3/2} + \int_{-R}^{R} |\frac{\chi_{N,T}(\alpha) - \chi_N(\alpha)}{\alpha}|\,d\alpha. \qquad (5.4)
\end{aligned}$$

We shall take $N = 2[(\log\log T)/4]$, $R = N^{1/8}(\log N)^{-3/4}$. Let

$$K_M(x) = \sum_{k=-M}^{M} (1 - \frac{|k|}{M}) e(kx) = \frac{1}{M}(\frac{\sin \pi M x}{\sin \pi x})^2.$$

Then,

$$\begin{aligned}
&|\chi_{N,T}(\alpha) - \chi_N(\alpha)| \\
\le\ & \frac{1}{T}\int_1^T \left| \prod_{n=1}^{N} e(\alpha a_n(\gamma_n t)) - \prod_{n=1}^{N} \int_0^1 e(\alpha a_n(u)) K_M(\gamma_n t - u)\,du \right| dt \\
& + \left| \frac{1}{T}\int_1^T \prod_{n=1}^{N} \int_0^1 e(\alpha a_n(u)) K_M(\gamma_n t - u)\,du\,dt - \prod_{n=1}^{N}\int_0^1 e(\alpha a_n(t))\,dt \right| \\
=\ & I_T + |J_T|, \text{ say.} \qquad (5.5)
\end{aligned}$$

Noting that $K_M(u)$ is periodic of period 1, $K_M(u) > 0$ and $\int_0^1 K_M(u)\,du = 1$, we have

$$\begin{aligned}
I_T &= \frac{1}{T}\int_1^T \left| \prod_{n=1}^{N} e(\alpha a_n(\gamma_n t)) - \prod_{n=1}^{N} \int_0^1 e(\alpha a_n(u)) K_M(\gamma_n t - u)\,du \right| dt \\
&\le \sum_{n\le N} \frac{1}{T}\int_1^T \left| e(\alpha a_n(\gamma_n t)) - \int_0^1 e(\alpha a_n(u)) K_M(\gamma_n t - u)\,du \right| dt \\
&\ll \sum_{n\le N} \int_0^1 \left| \int_0^1 (e(\alpha a_n(t)) - e(\alpha a_n(t-u)))\, K_M(u)\,du \right| dt \\
&\ll \sum_{n\le N} |\alpha| \int_0^1 \int_0^\delta |a_n(t) - a_n(t-u)| K_M(u)\,du\,dt + \sum_{n\le N} \int_{\delta<u\le 1} K_M(u)\,du \\
&\ll |\alpha| \sum_{n\le N} \int_0^1 \int_0^\delta \frac{1}{n^{3/4}} \sum_{r=1}^{\infty} \frac{d(nr^2)}{r^{3/2}} |\sin(2\pi r(t - \frac{u}{2}) - \frac{\pi}{4}) \sin(\pi r u)| K_M(u)\,du\,dt \\
&\quad + \frac{1}{M} \sum_{n\le N} \int_{\delta<u\le 1} \frac{du}{u^2}
\end{aligned}$$

$$
\begin{aligned}
&\ll |\alpha| \sum_{n\le N} \frac{1}{n^{3/4-\epsilon}}\Big(\sum_{r\le 1/\delta} \frac{r\delta}{r^{3/2-\epsilon}} + \sum_{r>1/\delta} \frac{1}{r^{3/2-\epsilon}}\Big) + \frac{N}{M\delta} \\
&\ll |\alpha|\delta^{1/2-\epsilon}N^{1/4+\epsilon} + \frac{N}{M\delta}.
\end{aligned}
$$

Taking $M = [(\log T)^{3/4}]$ and $\delta = (\log T)^{-1/2}$, we get

$$I_T \ll (|\alpha|+1)(\log T)^{\epsilon-1/4}. \tag{5.6}$$

Now,

$$
\begin{aligned}
J_T &= \frac{1}{T}\int_1^T \prod_{n=1}^N \int_0^1 e(\alpha a_n(u))K_M(\gamma_n t-u)\,du\,dt - \prod_{n=1}^N \int_0^1 e(\alpha a_n(t))\,dt \\
&= \sum_{|k_1|\le M} \cdots \sum_{\substack{|k_N|\le M \\ |k_1|+\cdots+|k_N|\ne 0}} \prod_{n=1}^N \Big((1-\frac{|k_n|}{M})\int_0^1 e(\alpha a_n(u_n))e(-k_n u_n)\,du_n\Big) \\
&\qquad \times \frac{1}{T}\int_1^T e((k_1\gamma_1+\cdots+k_N\gamma_N)t)\,dt \\
&\le 2T^{-1} \sum_{|k_1|\le M} \cdots \sum_{\substack{|k_N|\le M \\ |k_1|+\cdots+|k_N|\ne 0}} |k_1\gamma_1+\cdots+k_N\gamma_N|^{-1} \\
&\ll \frac{M^N}{T}(M\sqrt{N})^{2^N} \ll (\log T)^{\epsilon-1/4} \qquad (5.7)
\end{aligned}
$$

by [5, Lemma 5]. (Note that (5.7) determines our choice of the order of magnitude of N.) Hence, $|\chi_{N,T}(\alpha)-\chi_N(\alpha)| \ll (|\alpha|+1)(\log T)^{\epsilon-1/4}$ by putting (5.6) and (5.7) into (5.5).

On the other hand, suppose $|\alpha| \le (\log T)^{-1}$,

$$
\begin{aligned}
&|\chi_{N,T}(\alpha)-\chi_N(\alpha)| \\
&\ll |\frac{1}{T}\int_1^T (e(\alpha\sum_{n\le N} a_n(\gamma_n t)) - 1)\,dt| + |\int_0^1 \cdots \int_0^1 (e(\alpha\sum_{n\le N} a_n(u_n)) - 1)\,du_1\cdots du_N| \\
&\ll \frac{|\alpha|}{T}\int_1^T |\sum_{n\le N} a_n(\gamma_n t)|\,dt + |\alpha|\sum_{n\le N}\int_0^1 |a_n(u)|\,du \\
&\ll |\alpha|N^{1/4+\epsilon}
\end{aligned}
$$

since $a_n(u) \ll n^{-3/4+\epsilon}$. Therefore,

$$\int_{-R}^R |\frac{\chi_{N,T}(\alpha)-\chi_a(\alpha)}{\alpha}|\,d\alpha$$

$$\begin{aligned} &\ll N^{1/4+\epsilon}\int_{|\alpha|\le(\log T)^{-1}} d\alpha + \int_{(\log T)^{-1}\le|\alpha|\le R}(|\alpha|+1)(\log T)^{\epsilon-1/4}\frac{d\alpha}{|\alpha|} \\ &\ll (\log T)^{\epsilon-1/4} \end{aligned}$$

and this yields our result with (5.4) and (5.1).

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