A NEW BOUND $k^{2/3+\varepsilon}$ FOR RANKIN-SELBERG L-FUNCTIONS FOR HECKE CONGRUENCE SUBGROUPS

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Abstract

Let f be a holomorphic Hecke eigenform for $\Gamma_0(\mathcal{N})$ of weight k, or a Maass eigenform for $\Gamma_0(\mathcal{N})$ with Laplace eigenvalue $1/4+k^2$. Let g be a fixed holomorphic or Maass cusp form for $\Gamma_0(\mathcal{N})$. A subconvexity bound for central values of the Rankin-Selberg L-function $L(s, f\otimes g)$ is proved in the k-aspect: $L(1/2+it, f\otimes g) \ll_{\mathcal{N},g,t,\varepsilon} k^{2/3+\varepsilon}$, while a convexity bound is only $\ll k^{1+\varepsilon}$. This new bound improves earlier subconvexity bounds for these Rankin-Selberg L-functions by Sarnak, the authors, and Blomer. Techniques used include a result of Good, spectral large sieve, meromorphic continuation of a shifted convolution sum to $\Re\mathrm{e}\, s > -1/2$ passing through all Laplace eigenvalues, and a weighted stationary phase argument.

Content

1. Introduction

A central problem in the theory of L-functions is to investigate their sizes on the critical line. Following from the Phragmén-Lindelöf principle with the functional equation, one can

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obtain an estimate referred as a convexity bound. For example, the convexity bound of the Riemann zeta-function is

$$\zeta(1/2+it) \ll |t|^{1/4+\varepsilon}.$$

Any improvement of the exponent 1/4, referred as a subconvexity estimate, is interesting and significant. The method of Weyl on exponential sums improves 1/4 to 1/6, which is apparently a great progress. Weyl's method has been sharpened many times to-date, with important ideas and machineries introduced. Numerically, though, the improvement seems rather small. The Riemann zeta-function is only a classic example of GL_1 functions. The subconvexity estimate of general L-functions in various aspects is of great interest and importance, with many applications, for instance, to equidistribution problems. For a Dirichlet L-function $L(s,\chi)$, Burgess established the subconvexity estimate $L(1/2+it,\chi) \ll_t q^{3/16+\varepsilon}$ on the conductor aspect, where q denotes the modulus of the character χ. (The exponent of the convexity bound is 1/4.) A bound analogous to Weyl's (i.e. replacing 3/16 by 1/6) for real characters χ was only achieved recently by Conrey and Iwaniec [5].

In the same paper [5], Conrey and Iwaniec actually derived subconvexity estimates for the central values of some GL_2 L-functions attached to Maass or holomorphic Hecke eigenforms on the level aspect. They proved $L_f(1/2,\chi) \ll q^{1/3+\varepsilon}$ for real characters χ and for self-dual forms f, which is of the same quality as Weyl's in view of the convexity bound $O(q^{1/2+\varepsilon})$.[3] An analogous bound $k^{1/3+\varepsilon}$ was done in Ivić [16] for a Maass Hecke eigenform f with eigenvalue $1/4+k^2$, and in Peng's dissertation [27] for holomorphic forms of weight k respectively. Recently, Jutila and Motohashi [18] obtained a beautiful uniform bound in both the weight (or eigenvalue) and t aspects with the same exponent 1/3 for the full modular group. The subconvexity problem for higher rank cases is plausibly difficult; still some results are currently known for Rankin-Selberg L-functions $L(s,f\otimes g)$ which are in the GL_4 case. A challenge is to establish the desirable subconvexity estimate whose quality is comparable to the generic results in GL_1 and GL_2 cases.

Let f be a holomorphic Hecke eigenform for $\Gamma_0(\mathcal{N})$ of weight k, and g a fixed holomorphic or Maass cusp form for $\Gamma_0(\mathcal{N})$. Or, we may let g be a cusp form for $\Gamma_0(\mathcal{N}')$ with $(\mathcal{N},\mathcal{N}')=1$. The Rankin-Selberg L-function $L(s,f\otimes g)$ is an Euler product of degree 4 and satisfies a functional equation. The convexity bound is $L(1/2+it,f\otimes g) \ll k^{1+\varepsilon}$ and the subconvexity estimate

$$L(1/2+it,f\otimes g) \ll_{\mathcal{N},g,t,\varepsilon} k^{576/601+\varepsilon}. \tag{1.1}$$

[3] For general f and χ, the best known exponent is $3/8+\theta/4$ with $\theta=7/64$ by Blomer, Harcos, and Michel [4].

on weight aspect was firstly obtained in Sarnak [29]. The proof of this subconvexity bound made use of a bound toward the Ramanujan conjecture with $\theta = 7/64$ (Kim and Sarnak [19]):

$$\begin{aligned} |\alpha_\pi^{(j)}(p)| \leq p^\theta \quad & \text{for } p \text{ at which } \pi \text{ is unramified,} \\ |\Re\mathrm{e}\, \mu_\pi^{(j)}(\infty)| \leq \theta \quad & \text{if } \pi \text{ is unramified at } \infty, \end{aligned} \tag{1.2}$$

where π is an automorphic cuspidal representation of $GL_2(\mathbb{Q}_\mathbb{A})$ with unitary central character and local Hecke eigenvalues $\alpha_\pi^{(j)}(p)$ for $p < \infty$ and $\mu_\pi^{(j)}(\infty)$ for $p = \infty$, $j = 1, 2$. In terms of (1.2), the exponent in Sarnak's bound (1.1) is $18/(19 - 2\theta) + \varepsilon$.

If f is a Maass Hecke eigenform for $\Gamma_0(\mathcal{N})$ with Laplace eigenvalue $1/4 + k^2$, Liu and Ye [23] proved a similar subconvexity bound (cf. Liu and Ye [24])

$$L(1/2 + it, f \otimes g) \ll_{\mathcal{N},g,t,\varepsilon} k^{(15+2\theta)/16+\varepsilon}, \tag{1.3}$$

When $\theta = 7/64$, this is $\ll k^{487/512+\varepsilon}$.

Here both results in (1.1) and (1.3) depend crucially on the value of θ. In fact, if we take the trivial estimate $\theta = 1/2$, no subconvexity bound will be concluded in either case. This phenomenon is somewhat unexpected, as was explained in Sarnak [30]. Recently we showed in [22] that such a heavy dependence on nontrivial θ can be avoided, and our result gives the exponent $1 - 1/(8 + 4\theta) + \varepsilon$ for both holomorphic or Maass f, which still yields a subconvexity bound $k^{9/10+\varepsilon}$ even for $\theta = 1/2$. Using $\theta = 7/64$, it is sharper than (1.1) and (1.3). On the other hand, a recent work of Blomer [3] gave a new bound $k^{(6-2\theta)/(7-4\theta)+\varepsilon}$. This superseded our result in [22], although it does require a nontrivial θ to produce a subconvexity bound. The goal of this paper is to produce a better bound which corresponds to the important exponent $1/6$ in GL_1 case.

Theorem 1 *Let f be a holomorphic Hecke eigenform for $\Gamma_0(\mathcal{N})$ of weight k, or a Maass Hecke eigenform for $\Gamma_0(\mathcal{N})$ with Laplace eigenvalue $1/4 + k^2$, and let g be a fixed holomorphic or Maass cusp form for $\Gamma_0(\mathcal{N})$, or for $\Gamma_0(\mathcal{N}')$ with $(\mathcal{N}, \mathcal{N}') = 1$. Then for any small $\varepsilon > 0$, we have*

$$\sum_{K-L \leq k \leq K+L} |L(1/2 + it, f \otimes g)|^2 \ll_{\mathcal{N},g,t,\varepsilon} (KL)^{1+\varepsilon} \tag{1.4}$$

whenever $K^{1/3+\varepsilon} \leq L \leq K^{1-\varepsilon}$. The implied constant in $\ll_{\mathcal{N},g,t,\varepsilon}$ can be made explicity into $\mathcal{N}^{A(\varepsilon)}(|t| + 1)^{3/2} C(g, \varepsilon)$ where $A(\varepsilon) > 0$ depends only on ε and $C(g, \varepsilon) > 0$ is a constant depending on g and ε

Corollary 2 *Let f and g be defined as in Theorem 1. Then*

$$L(1/2+it, f\otimes g) \ll_{\mathcal{N},g,t,\varepsilon} k^{2/3+\varepsilon}, \tag{1.5}$$

where the implied constant takes an explicit form as in Theorem 1.

Remark 0. The polynomial growth of the level $\mathcal{N}$ of f and $|t|$ for fixed g is useful in applications. We are unable to keep track on the dependence of the implied constants on g due to the lack of such information in Good's estimate.

The proof starts with the approximate functional equations, an averaging process with Kuznetsov formula (or Petersson trace formula), and an application of a Voronoi summation formula. This will be sketched in Section 4, and full details can be found in [29] and [23]. The core part is the estimation of a shifted convolution sum. We follow Sarnak [29] to apply the spectral theory to transform the shifted sum into a series, whose summands are products of gamma functions and the Fourier coefficients of the form g.

A key feature in the present paper is to provide the analytic continuation of the shifted convolution sum to a bigger region. In [29] and [23], the shifted convolution sum was holomorphically continued to the half plane $\Re\mathrm{e}\, s > 1/2+\theta$. In [22], this same sum was meromorphically extended to the half plane $\Re\mathrm{e}\, s > 1/2$, with possible poles in the interval $[1/2, 1/2+\theta]$ associated with possible exceptional Laplace eigenvalues. We found that their contribution is controllable and indeed negligible in this problem. In the present paper, we will further extend this shifted sum meromorphically to the left of $\Re\mathrm{e}\, s = 1/2$.

The poles due to the non-exceptional eigenvalues come into play. To handle them we apply a substantial estimate of Good, which is also applied in our work [22]. However, we control the Fourier coefficients of $g(z)$ and some exponential integrals with the trivial pointwise bound there, and hence, waste their oscillatory properties and cause great losses. The spectral large sieve inequality due to Deshouillers and Iwaniec [6] is an effective tool to exploit the cancellation of the Fourier coefficients. Blomer [2] made use of this large sieve to derive cancellation among the shifted convolution sums, which were, unlike our approach, treated by the δ-method there.

In order to fit the spectral large sieve into our context, we need to bypass a technicality of separating the two intertwined parameters h and t_j appropriately. To this end, we provide an asymptotic formula with an admissible error term for the exponential integrals, which will be done in Section 5. This process also enables us to utilize the cancellation in the sum over h, so that we exploit the cancellations in various parts to yield our result. In short, here we extend our method in [22] and utilize, inspired by [3], the spectral large sieve but the way of using it is different from Blomer. Among other things, we need some results on exponential integrals in Huxley [14] (or see [13]) to complete the delicate analysis.

Remark 1. In [18], Jutila and Motohashi pointed out that their method might be generalized to get a subconvexity bound for our Rankin-Selberg L-functions for the full modular group in the k aspect with the merit of being uniform in small $|t|$.

Remark 2. We have not discussed the vital and novel development of the subconvexity estimate for Rankin-Selberg L-functions on level aspects. Interested readers are referred to the important works of Kowalski, Michel and VanderKam [20], Michel [26], and Harcos and Michel [11].

Arguments in Ye [32] are applicable to the present paper, and yield the following improved result on the fourth power moment of a GL_2 L-function over a short interval.

Theorem 3 *Let g be a fixed self-contragredient holomorphic or Maass Hecke eigenform for $\Gamma_0(\mathcal{N})$, and χ a real, primitive character* mod $\mathcal{Q}$ *with $\mathcal{N}|\mathcal{Q}$. Then*

$$\int_K^{K+L} \Big|L\Big(\frac{1}{2}+it, g\otimes\chi\Big)\Big|^4 \, dt \ll_{\mathcal{N},g,\mathcal{Q},\varepsilon} (KL)^{1+\varepsilon}$$

for $L = K^{1/3+\varepsilon}$.

By a standard argument (cf. Ivic [15, p.197]), our Theorem 3 further implies

Corollary 4 *Let g and χ be defined as in Theorem 3. Then for any real t,*

$$L\Big(\frac{1}{2}+it,\ g\otimes\chi\Big) \ll_{\mathcal{N},g,\mathcal{Q},\varepsilon} (1+|t|)^{1/3+\varepsilon}.$$

This is of the same strength as the classical results of Good for holomorphic cusp form g in [8], [9], and [10], and of Meurman for Maass g in [25].

Notation. The Vinogradov symbol $\ll$ is, as usual, defined as $|A| \le CB$ for some constant C when we write $A \ll B$. Sometimes, we write $\ll_*$ to emphasize that the implied constant C depends on $*$. The symbol $A \asymp B$ represents both $A \ll B$ and $B \ll A$. Finally $a \sim A$ means the set of integers a for which $A \le a < 2A$. In the sequel, $\varepsilon > 0$ is arbitrarily small but fixed. We write $\epsilon = C\varepsilon$ where C is an unspecified absolute positive constant. The implicit constant C may differ at each occurrence of ϵ.

2. Spectral theory on $L^2(\Gamma \setminus \mathbb{H})$ and large sieve inequalities

Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ and $\mathbb{H}$ the upper half-plane. Then $\Gamma \setminus \mathbb{H}$ is a hyperbolic Riemannian manifold of curvature -1, equipped with a measure $dxdy/y^2$.

Automorphic functions (defined on $\mathbb{H}$) of weight 0 for Γ are regarded as functions on $\Gamma \setminus \mathbb{H}$. Introducing the Petersson inner product

$$\langle f_1, f_2 \rangle = \int_{\Gamma\setminus\mathbb{H}} f_1(z)\overline{f_2(z)}\,\frac{dxdy}{y^2},$$

the space $L^2(\Gamma \setminus \mathbb{H})$ of all square integrable functions forms a Hilbert space. We denote the non-Euclidean Laplacian (the Laplacian on this manifold) by

$$\Delta = -y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big).$$

By Maass-Selberg theory (see Deshouillers and Iwaniec [6, p.227]), $L^2(\Gamma\setminus\mathbb{H})$ admits a spectral decomposition with respect to Δ. The spectrum of Δ consists of two components: the discrete spectrum $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$, and the continuous spectrum covering the segment $[1/4, \infty)$. Each eigenvalue in the discrete spectrum has finite order, and $\lambda_j \to \infty$ as $j \to \infty$. The value of the first non-zero eigenvalue remains an unsettled problem to-date. Selberg conjectured that $\lambda_1 \geq 1/4$ for congruence groups, however, the currently best known result is $\lambda_1 \geq 1/4 - \theta^2$, where θ is the value in (1.2), due to Kim and Sarnak [19]. Hence the eigenvalues are divided into two types. Writing $\lambda_j = s_j(1 - s_j)$ and $s_j = 1/2 + it_j$, we may assume

$$0 < it_j \leq \theta \text{ if } \tfrac{1}{4} - \theta^2 \leq \lambda_j < \tfrac{1}{4}, \text{ or } t_j \in [0, \infty) \text{ otherwise.}$$

We call the corresponding eigenvalues exceptional and non-exceptional respectively. Concerning the number of eigenvalues, we have Weyl's law

$$\#\{j : t_j \leq T\} = cT^2 + O(T \log T)$$

for some constant $c > 0$.

Let $\{\phi_0, \phi_1, \cdots\}$ be an orthonormal basis of the eigenfunctions for the discrete spectrum, and simultaneously the eigenfunctions of the reflection operators, i.e. $\overline{\phi_j(-\overline{z})} = \epsilon_j \phi_j(z)$ where $\epsilon_j = \pm 1$. Besides, for a cusp $\mathfrak{a}$ we denote by $\{E_{\mathfrak{a}}(z, 1/2 + i\tau) : \tau \in \mathbb{R}\}$ the corresponding Eisenstein series which composes the eigenpacket for the continuous part. Then for any $f \in L^2(\Gamma \setminus \mathbb{H})$, we can expand it into an infinite series

$$f(z) = \sum_{j\geq 0} \langle f, \phi_j\rangle \phi_j(z) + \frac{1}{4\pi} \sum_{\mathfrak{a}} \int_{-\infty}^{\infty} \langle f, E_{\mathfrak{a}}(\cdot, 1/2 + i\tau)\rangle E_{\mathfrak{a}}(z, 1/2 + i\tau)\, d\tau$$

convergent in L^2 sense. Here the summation $\sum_{\mathfrak{a}}$ runs over all cusps of Γ, and there are $O_\Gamma(1)$ cusps. Both $\phi_j(z)$ and $E_{\mathfrak{a}}(z, s)$ have Fourier series expansions: for $z = x + iy$,

$$\phi_j(z) = \sqrt{y} \sum_{m\neq 0} \rho_j(m) K_{it_j}(2\pi|m|y) e(mx),$$

and

$$\begin{aligned} E_{\mathfrak{a}}(z,s) &= \delta_{\mathfrak{a}\infty}y^s + \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\rho_{\mathfrak{a}}(s,0)y^{1-s} \\ &\quad + \frac{2\pi^s\sqrt{y}}{\Gamma(s)}\sum_{m\neq 0}|m|^{s-1/2}\rho_{\mathfrak{a}}(s,m)K_{s-1/2}(2\pi|m|y)e(mx), \end{aligned}$$

where $\delta_{\mathfrak{a}\infty}=1$ if $\mathfrak{a}=\infty$ and 0 otherwise. As $E_{\mathfrak{a}}(z,\overline{s})=\overline{E_{\mathfrak{a}}(z,s)}$, it follows that

$$\rho_{\mathfrak{a}}(\overline{s},m)=\overline{\rho_{\mathfrak{a}}(s,-m)} \tag{2.1}$$

and also,

$$\rho_j(m)=\epsilon_j\overline{\rho_j(-m)} \tag{2.2}$$

by the property of being an eigenfunction of the reflection operator.

To give a pointwise bound to the Fourier coefficients, we can proceed, as in Sarnak [29, (A.16)], by taking $\{\phi_j\}$ to be the Hecke eigenforms which are assumed from now on. Then we have

$$\rho_j(m)\ll_\varepsilon \frac{(m\mathcal{N}t_j)^\varepsilon}{\sqrt{\mathcal{N}}}\cosh\Big(\frac{\pi t_j}{2}\Big)m^\theta \tag{2.3}$$

and by Blomer [2, Lemma 3.4],

$$\sum_{\mathfrak{a}}\big|\rho_{\mathfrak{a}}(1/2+it,m)\big|^2\ll_\varepsilon \big((|t|+1)m\mathcal{N}\big)^\varepsilon. \tag{2.4}$$

However, in order to make use of the cancellation among the Fourier coefficients, we need the large sieve inequality.

Lemma 2.1 *Let $T,M\geq 1$. Then for any sequence $\{b_m\}$ of complex numbers, we have*

$$\sum_{|t_j|\leq T}\frac{1}{\cosh\pi t_j}\Big|\sum_{m\leq M}b_m\rho_j(m)\Big|^2\ll_\varepsilon M^\varepsilon\big(T^2+M\big)\sum_{m=1}^M|b_m|^2, \tag{2.5}$$

$$\sum_{\mathfrak{a}}\int_{-T}^{T}\Big|\sum_{m\leq M}b_m m^{ir}\rho_{\mathfrak{a}}(\frac{1}{2}+ir,m)\Big|^2\,dr\ll_\varepsilon M^\varepsilon\big(T^2+M\big)\sum_{m=1}^M|b_m|^2. \tag{2.6}$$

See [2, Lemma 3.3] or [6, Theorem 2].

3. Spectral decomposition of a shifted sum

Let $g\in S_l(\Gamma_0(\mathcal{N}))$ or in $S_l(\Gamma_0(\mathcal{N}'))$ be a holomorphic cusp form of weight l with Fourier expansion

$$g(z)=\sum_{n\geq 1}n^{(l-1)/2}\lambda_g(n)e(nz);$$

or let g be a Maass cusp form with Laplace eigenvalue $1/4 + l^2$ and Fourier expansion

$$g(z) = \sqrt{y} \sum_{n \neq 0} \lambda_g(n) K_{il}(2\pi |n| y) e(nx).$$

Assume that ν_1, ν_2, and h are positive integers, and let $s = \sigma + it$ with $\sigma > 1$ and $|t| \leq T$ for any $T \gg 1$. We consider the shifted convolution sums

$$D_g(s, \nu_1, \nu_2, h) = \sum_{\substack{m,n>0 \\ \nu_1 m - \nu_2 n = h}} \lambda_g(n) \overline{\lambda}_g(m) \Big(\frac{\sqrt{\nu_1 \nu_2 mn}}{\nu_1 m + \nu_2 n} \Big)^{l-1} (\nu_1 m + \nu_2 n)^{-s} \tag{3.1}$$

when g is a holomorphic cusp form, and

$$D_g(s, \nu_1, \nu_2, h) = \sum_{\substack{m,n\neq 0 \\ \nu_1 m - \nu_2 n = h}} \lambda_g(n) \overline{\lambda}_g(m) \Big(\frac{\sqrt{\nu_1 \nu_2 |mn|}}{\nu_1 |m| + \nu_2 |n|} \Big)^{2il} (\nu_1 |m| + \nu_2 |n|)^{-s}$$

when g is a Maass form.

We follow the argument in Sarnak [29, Appendix] to give a spectral decomposition of $D_g(s, \nu_1, \nu_2, h)$ for holomorphic g. We will only consider the case of $\Gamma_0(\mathcal{N})$. Write $\Gamma = \Gamma_0(\mathcal{N}\nu_1\nu_2)$ and

$$V(z) = y^l g(\nu_1 z) \overline{g(\nu_2 z)}.$$

Then V is a Γ-invariant function rapidly decreasing at the cusps of Γ, and $V \in L^2(\Gamma \setminus \mathbb{H})$. Define the Poincaré series $U_h(z, s)$ for Γ by

$$U_h(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \Im\mathrm{m}\, (\gamma z)^s e(-h \Re\mathrm{e}\, (\gamma z)),$$

where h is a positive integer, and $e(x) = e^{2\pi i x}$. The standard unfolding method expresses $D_g(s, \nu_1, \nu_2, h)$ in terms of the inner product (see [29, p.444])

$$D_g(s, \nu_1, \nu_2, h) = \frac{(2\pi)^{s+l-1} (\nu_1 \nu_2)^{(l-1)/2}}{\Gamma(s + l - 1)} \langle U_h(\cdot, s), V \rangle. \tag{3.2}$$

Using the square-integrability of V and the fact that the volume of $\Gamma \setminus \mathbb{H}$ is finite, we apply Parseval's identity to deduce that

$$\begin{aligned} \langle U_h(\cdot, s), V \rangle &= \sum_{j \geq 1} \langle U_h(\cdot, s), \phi_j \rangle \overline{\langle V, \phi_j \rangle} \\ &\quad + \frac{1}{4\pi} \sum_{\mathfrak{a}} \int_{-\infty}^{\infty} \langle U_h(\cdot, s), E_{\mathfrak{a}}(\cdot, 1/2 + i\tau) \rangle \overline{\langle V, E_{\mathfrak{a}}(\cdot, 1/2 + i\tau) \rangle}\, d\tau. \end{aligned} \tag{3.3}$$

The first sum begins with $j \geq 1$ since $\langle U_h, \phi_0 \rangle = 0$.

Now we evaluate the inner products. Let us write

$$B_j(s) = (\nu_1\nu_2)^{(l-1)/2}\frac{2^{s+l-3}\pi^{l-1/2}}{\Gamma(s+l-1)}\Gamma\Big(\frac{s-1/2+it_j}{2}\Big)\Gamma\Big(\frac{s-1/2-it_j}{2}\Big), \tag{3.4}$$

$$C_{\mathfrak{a}}(s,\tau) = (\nu_1\nu_2)^{(l-1)/2}\frac{2^{s+l-2}\pi^{l-i\tau}}{\Gamma(s+l-1)}\Gamma\Big(\frac{s-1/2+i\tau}{2}\Big)\Gamma\Big(\frac{s-1/2-i\tau}{2}\Big). \tag{3.5}$$

To evaluate the first factor in each summand on the right side of (3.3), we follow [29, (A12)] and use the formula

$$\int_0^\infty K_\nu(t)t^{\mu-1}\,dt = 2^{\mu-2}\Gamma\Big(\frac{\mu-\nu}{2}\Big)\Gamma\Big(\frac{\mu+\nu}{2}\Big)$$

for $\Re\mathrm{e}\,\mu > |\Re\mathrm{e}\,\nu|$, by Watson [31, p.388(8)]. We have

$$\frac{(2\pi)^{s+l-1}(\nu_1\nu_2)^{(l-1)/2}}{\Gamma(s+l-1)}\langle U_h(\cdot,s),\phi_j\rangle = \frac{\overline{\rho_j(-h)}}{|h|^{s-1/2}}B_j(s),$$

and

$$\frac{(2\pi)^{s+l-1}(\nu_1\nu_2)^{(l-1)/2}}{\Gamma(s+l-1)}\langle U_h(\cdot,s),E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle = \frac{\overline{\rho_{\mathfrak{a}}(1/2+i\tau,-h)}}{|h|^{s-1/2+i\tau}}\frac{C_{\mathfrak{a}}(s,\tau)}{\Gamma(1/2-i\tau)}.$$

Then from (3.2) and (3.3) we have

$$\begin{aligned} & D_g(s,\nu_1,\nu_2,h) \\ = & \sum_{j\ge1}\frac{\overline{\rho_j(-h)}}{|h|^{s-1/2}}B_j(s)\overline{\langle V,\phi_j\rangle} \\ & +\frac{1}{4\pi}\sum_{\mathfrak{a}}\int_{-\infty}^{\infty}\frac{\overline{\rho_{\mathfrak{a}}(1/2+i\tau,-h)}}{|h|^{s-1/2+i\tau}}\frac{C_{\mathfrak{a}}(s,\tau)}{\Gamma(1/2-i\tau)}\overline{\langle V,E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle}\,d\tau. \end{aligned} \tag{3.6}$$

To control the inner products against V, we invoke Good [7], Theorem 1, for the case of $l\ge 4$, and use Krötz and Stanton [21] for other cases. More precisely, we need a modified version of their results, for our function V is different from the form of f there. However, this does not cause big change and we just apply his proof to $f_l(z) = y^kF(z)P_l(z)$ where F and P_l are a cusp form and a Poincaré series for Γ, respectively; see [7, (3.2)] and [7, §4]. The main point one should note is that $g(\nu_1 z)$ and $g(\nu_2 z)$ are cusp forms for Γ, and therefore $g(\nu_2 z)$ is a linear combination of the Poincaré series. Hence,

$$\sum_{t_j\le T}|\langle V,\phi_j\rangle|^2e^{\pi t_j} + \frac{1}{4\pi}\sum_{\mathfrak{a}}\int_{-T}^{T}|\langle V,E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle|^2e^{\pi|\tau|}\,d\tau \ll T^{2l}. \tag{3.7}$$

¿From the fact that $\Gamma(s)$ has no zero but poles at nonpositive integers and the formula

$$|\Gamma(\sigma+it)| \asymp_{A_0} |t|^{\sigma-1/2}e^{-\pi|t|/2}$$

for $|\sigma| \le A_0$ and $|t| \ge 1$, we have

$$\begin{aligned}
&|\Gamma(\sigma+it)^{-1}| \ll_{A_0} (1+|t|)^{1/2-\sigma} e^{\pi|t|/2} \qquad (|\sigma| \le A_0,\ t \in \mathbb{R}),\\
&|\Gamma(\sigma+it)| \asymp_{A_0,\varepsilon} (1+|t|)^{\sigma-1/2} e^{-\pi|t|/2} \qquad (|\sigma| \le A_0,\ |\sigma+it+n| \ge \varepsilon \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for every } n = 0,1,2,\cdots).
\end{aligned} \tag{3.8}$$

It follows that for $s = \sigma + it$ with $|\sigma| \le A_0$ and $|s+2n-1/2 \pm i\tau| \ge \varepsilon$ for all $n \in \{0,1,2,\cdots\}$,

$$\begin{aligned}
&\Gamma\Big(\frac{s-1/2+i\tau}{2}\Big)\Gamma\Big(\frac{s-1/2-i\tau}{2}\Big)\\
&\ll_{A_0,\varepsilon} \big((1+|t-\tau|)(1+|t+\tau|)\big)^{\sigma/2-3/4} e^{-\pi(|t-\tau|+|t+\tau|)/4}.
\end{aligned} \tag{3.9}$$

Therefore from (3.4), (3.8), and (3.9) for $t_j \ge 2T$ and $|t| \le T$, we get

$$\begin{aligned}
B_j(s) &\ll_{l,\nu_1,\nu_2} \Gamma(s+l-1)^{-1}\Gamma\Big(\frac{s-1/2+it_j}{2}\Big)\Gamma\Big(\frac{s-1/2-it_j}{2}\Big)\\
&\ll_{l,\nu_1,\nu_2} \frac{t_j^{\sigma-3/2}}{(1+|t|)^{\sigma+l-3/2}} e^{-\pi t_j/4}.
\end{aligned} \tag{3.10}$$

Similarly from (3.5), (3.8), and (3.9) for $|\tau| \ge 2T$ and $|t| \le T$, we get

$$\begin{aligned}
C_{\mathfrak{a}}(s,\tau) &\ll_{l,\nu_1,\nu_2} \Gamma(s+l-1)^{-1}\Gamma\Big(\frac{s-1/2+i\tau}{2}\Big)\Gamma\Big(\frac{s-1/2-i\tau}{2}\Big)\\
&\ll_{l,\nu_1,\nu_2} \frac{|\tau|^{\sigma-3/2}}{(1+|t|)^{\sigma+l-3/2}} e^{-\pi|\tau|/4}.
\end{aligned} \tag{3.11}$$

To estimate the contribution of large t_js, we divide the interval $[T,\infty)$ dyadically, and use (2.3) and the Cauchy-Schwarz inequality:

$$\begin{aligned}
&\sum_{t_j>2T} \Big|\frac{\rho_j(-h)}{|h|^{s-1/2}} B_j(s)\langle V,\phi_j\rangle\Big|\\
&\le |h|^{1/2-\sigma+\theta+\varepsilon} \sum_{r\ge1} \sum_{j:\,2^rT<t_j\le 2^{r+1}T} t_j^{\varepsilon}|B_j(s)||\langle V,\phi_j\rangle| e^{\pi t_j/2}\\
&\le |h|^{1/2-\sigma+\theta+\varepsilon} \sum_{r\ge1}\Big(\sum_{2^rT<t_j\le 2^{r+1}T} t_j^{\varepsilon}|B_j(s)|^2\Big)^{1/2}\Big(\sum_{2^rT<t_j\le 2^{r+1}T} e^{\pi t_j}|\langle V,\phi_j\rangle|^2\Big)^{1/2}.
\end{aligned} \tag{3.12}$$

For $\sigma > 1$, we have, by (3.10) and Weyl's law,

$$\begin{aligned}
&\sum_{2^rT<t_j\le 2^{r+1}T} t_j^{\varepsilon}|B_j(s)|^2\\
&\ll \big(2^{r+1}T\big)^{2\sigma-3+\varepsilon} e^{-\pi 2^r T/2} \#\{t_j :\ 2^rT < t_j \le 2^{r+1}T\}\\
&\ll \big(2^{r+1}T\big)^{2\sigma-1+\varepsilon} e^{-\pi 2^r T/2} \ll e^{-2^rT}.
\end{aligned}$$

With (3.7), the last line of (3.12) is plainly

$$\begin{aligned} & \sum_{t_j > 2T} \Big| \frac{\rho_j(-h)}{|h|^{s-1/2}} B_j(s) \langle V, \phi_j \rangle \Big| \\ \ll \;& |h|^{1/2-\sigma+\theta+\varepsilon} \sum_{r\geq 1} T^l e^{-2^r T/2} \ll |h|^{1/2-\sigma+\theta+\varepsilon} e^{-T/4}. \end{aligned} \tag{3.13}$$

Similarly by (3.11)

$$\begin{aligned} \int_{2^rT\leq|\tau|\leq 2^{r+1}T} |\tau|^{\varepsilon} |C_{\mathfrak{a}}(s,\tau)|^2 \, d\tau &\ll \int_{2^rT\leq|\tau|\leq 2^{r+1}T} |\tau|^{2\sigma-3+\varepsilon} e^{-\pi|\tau|/2} \, d\tau \\ &\ll \big(2^r T\big)^{2\sigma-2+\varepsilon} e^{-\pi 2^r T/2} \ll e^{-2^r T}. \end{aligned} \tag{3.14}$$

By a dyadic subdivision and using (2.4) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned} & \sum_{\mathfrak{a}} \int_{|\tau|\geq 2T} \Big| C_{\mathfrak{a}}(s,\tau) \frac{\rho_{\mathfrak{a}}(1/2+i\tau,-h)}{|h|^{s-1/2+i\tau}} \frac{\langle V, E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle}{\Gamma(1/2-i\tau)} \Big| \, d\tau \\ \ll \;& |h|^{1/2-\sigma+\varepsilon} \sum_{r\geq 1} \int_{2^rT\leq|\tau|\leq 2^{r+1}T} |\tau|^{\varepsilon} \big| C_{\mathfrak{a}}(s,\tau) \big| e^{\pi|\tau|/2} \big| \langle V, E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle \big| \; d\tau \\ \ll \;& |h|^{1/2-\sigma+\varepsilon} \sum_{r\geq 1} \int_{2^rT\leq|\tau|\leq 2^{r+1}T} \Big(\int_{2^rT\leq|\tau|\leq 2^{r+1}T} |\tau|^{\varepsilon} |C_{\mathfrak{a}}(s,\tau)|^2 \, d\tau \Big)^{1/2} \\ & \times \Big(\int_{2^rT\leq|\tau|\leq 2^{r+1}T} e^{\pi|\tau|} |\langle V, E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle|^2 \, d\tau \Big)^{1/2}. \end{aligned}$$

By (3.7) and (3.14), we have

$$\sum_{\mathfrak{a}} \int_{|\tau|\geq 2T} \Big| C_{\mathfrak{a}}(s,\tau) \frac{\rho_{\mathfrak{a}}(1/2+i\tau,-h)}{|h|^{s-1/2+i\tau}} \frac{\langle V, E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle}{\Gamma(1/2-i\tau)} \Big| \, d\tau \ll |h|^{1/2-\sigma+\varepsilon} e^{-T/4}. \tag{3.15}$$

Using (3.6), (3.13), and (3.15), we conclude that for $\sigma > 1$ and $|t| \leq T$,

$$\begin{aligned} D_g(s,\nu_1,\nu_2,h) &= R_h(s) + \sum_{j:\, 0<t_j\leq 2T} \frac{\overline{\rho_j(-h)}}{|h|^{s-1/2}} B_j(s) \overline{\langle V, \phi_j \rangle} \\ &\quad + \frac{1}{4\pi} \sum_{\mathfrak{a}} \int_{-2T}^{2T} \frac{\overline{\rho_{\mathfrak{a}}(1/2+i\tau,-h)}}{|h|^{s-1/2+i\tau}} \frac{C_{\mathfrak{a}}(s,\tau)}{\Gamma(1/2-i\tau)} \overline{\langle V, E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle} \, d\tau \\ &\quad + O\big(|h|^{1/2-\sigma+\theta+\varepsilon} e^{-T/4} \big) \end{aligned} \tag{3.16}$$

where $R_h(s)$ is the sum over all exceptional eigenvalues and the possible eigenvalue $\lambda = 1/4$,[4]

$$\begin{aligned} R_h(s) &= \frac{(\nu_1\nu_2)^{(l-1)/2} 2^{s+l-3} \pi^{l-1/2}}{\Gamma(s+l-1)} \sum_{1/2\leq s_j\leq 1/2+\theta} \frac{\overline{\rho_j(-h)}}{|h|^{s-1/2}} \\ &\quad \times \Gamma\Big(\frac{s-s_j}{2}\Big) \Gamma\Big(\frac{s-(1-s_j)}{2}\Big) \overline{\langle V, \phi_j \rangle}. \end{aligned} \tag{3.17}$$

[4]We include the possible non-exceptional eigenvalue 1/4 just for technical simplicity.

For later use we record a few notes and estimates. As $R_h(s)$ is a finite sum and

$$\langle V, \phi_j\rangle \ll \|V\|\|\phi_j\| \ll_{\nu_1,\nu_2,g} 1,$$

$R_h(s)$ is analytic in the complex plane except for poles lying on the real axis, due to the two gamma functions. In particular, on the half-plane $\sigma > 0$, there are only finitely many poles at s_j and $1 - s_j$ lying in the interval $[1/2 - \theta, 1/2 + \theta] \subset [0, 1]$. Using (3.8), we have, for $|\sigma| \le A_0$ and $|t| \ge 1$,

$$\Gamma\Big(\frac{s - s_j}{2}\Big)\Gamma\Big(\frac{s - (1 - s_j)}{2}\Big) \ll_{A_0} |t|^{\sigma - 3/2} e^{-\pi|t|/2}$$

as $s_j \in \mathbb{R}$. We get with (2.3), for $|\sigma| \le A_0$ and $|t| \ge 1$,

$$\begin{aligned} R_h(\sigma + it) \ &\ll_{A_0} \ \frac{|\rho_j(-h)}{|h|^{\sigma - 1/2}|\Gamma(\sigma + l - 1 + it)|}\Big|\Gamma\Big(\frac{s - s_j}{2}\Big)\Gamma\Big(\frac{s - (1 - s_j)}{2}\Big)\Big||\langle V, \phi_j\rangle| \\ &\ll \ |h|^{1/2 - \sigma + \theta + \varepsilon}|t|^{-l} \ll |h|^{1/2 - \sigma + \theta + \varepsilon}. \end{aligned} \tag{3.18}$$

Here we have used (3.8) and assumed $l \ge 0$ to cater for the Maass form case.

On the other hand, the functions $B_j(s)$ (when $t_j \ge 0$) and $C_{\mathfrak{a}}(s, \tau)$ are holomorphic in $\sigma > 1/2$, and meromorphic on the whole $\mathbb{C}$. Let $v \in [-2T, 2T]$ where $T \gg 1$. One derives easily with (3.8) and (3.9) that for $-3/2 + \varepsilon \le \sigma \le 3/2$ with $|s - (1/2 \pm iv)| \ge \varepsilon$,

$$\begin{aligned} &\Gamma(s + l - 1)^{-1}\Gamma\Big(\frac{s - 1/2 + iv}{2}\Big)\Gamma\Big(\frac{s - 1/2 - iv}{2}\Big) \\ &\ll (1 + |t|)^{-(\sigma + l - 3/2)}\big((1 + |t - v|)(1 + |t + v|)\big)^{\sigma/2 - 3/4} e^{-\pi(|t-v| + |t+v| - 2|t|)/4} \\ &\ll (1 + |t|)^{-(\sigma + l - 3/2)}\big((1 + ||t| - |v||)(1 + ||t| + |v||)\big)^{\sigma/2 - 3/4} e^{-\pi(|t-v| + |t+v| - 2|t|)/4} \\ &\ll (1 + |t|)^{-(\sigma/2 + l - 3/4)}(1 + \big||t| - |v|\big|)^{\sigma/2 - 3/4}, \end{aligned}$$

since $|t - v| + |t + v| - 2|t| \ge 0$, and $(1 + |t| + |v|)^{\sigma/2 - 3/4} \le (1 + |t|)^{\sigma/2 - 3/4}$ for $\sigma \le 3/2$. Moreover, we have for $\sigma = 1/2 + \varepsilon$ or $\sigma = -1/2$,

$$\begin{aligned} &\int_{|t| \asymp T} \Big|\Gamma(s + l - 1)^{-1}\Gamma\Big(\frac{s - 1/2 + iv}{2}\Big)\Gamma\Big(\frac{s - 1/2 - iv}{2}\Big)\Big|^2 |ds| \\ &\ll \int_{|t| \asymp T} (1 + |t|)^{-(\sigma + 2l - 3/2)}(1 + \big||t| - |v|\big|)^{\sigma - 3/2}\, dt \\ &\ll T^{-(\sigma + 2l - 3/2)} \int_{|t| \ll T} (1 + |t|)^{\sigma - 3/2}\, dt \\ &\ll \begin{cases} T^{1-2l} & \text{for } \sigma = 1/2 + \varepsilon, \\ T^{2-2l} & \text{for } \sigma = -1/2. \end{cases} \end{aligned}$$

The implied constant in the first case depends on ε.

Hence, in case $0 \le t_j \le 2T$ and respectively $|\tau| \le 2T$, we get with (3.4) and (3.5) the crude bounds,

$$B_j(s) \text{ and } C_{\mathfrak{a}}(s,\tau) \ll_{l,\nu_1,\nu_2,\varepsilon} T \tag{3.19}$$

for $-1/2 \le \sigma \le 3/2$ and $|s-(1/2 \pm it_j)| \ge \varepsilon$ or $|s-(1/2 \pm i\tau)| \ge \varepsilon$. Besides, we deduce that

$$\int_{|t|\asymp T} \Big| B_j\Big(\frac{1}{2}+\varepsilon+it\Big)\Big|^2 dt \text{ and } \int_{|t|\asymp T} \Big| C_{\mathfrak{a}}\Big(\frac{1}{2}+\varepsilon+it,\tau\Big)\Big|^2 dt \ll_{l,\nu_1,\nu_2,\varepsilon} T^{1-2l}, \tag{3.20}$$

and

$$\int_{|t|\asymp T} \Big| B_j\Big(-\frac{1}{2}+it\Big)\Big|^2 dt \text{ and } \int_{|t|\asymp T} \Big| C_{\mathfrak{a}}(-\frac{1}{2}+it,\tau\Big)\Big|^2 dt \ll_{l,\nu_1,\nu_2} T^{2-2l}. \tag{3.21}$$

4. Outline of the approach

To fix ideas, we consider the case of Maass Hecke eigenform f for $\Gamma_0(\mathcal{N})$ with Laplace eigenvalue $1/4+k^2$ (but the argument works for holomorphic case as well). Given a holomorphic or Maass cusp form g for $\Gamma_0(\mathcal{N})$, or for $\Gamma_0(\mathcal{N}')$ with $(\mathcal{N},\mathcal{N}')=1$, we can express, via the approximate functional equation, $L(1/2+it, f\otimes g)$ as a rapidly convergent series. We repeat below the argument in [23, Sections 2 and 5] and explain the polynomial growth of the conductor and $|t|$.

The Rankin-Selberg L-function satisfies the functional equation

$$L(s, f\otimes g) = \epsilon_{f\otimes g} Q_{f\otimes g}^{1/2-s} \gamma(s) L(1-s, f\otimes g)$$

where $\epsilon_{f\otimes g}$ of modulus one is the root number, $Q_{f\otimes g} \le (\mathcal{N}\mathcal{N}')^2$ is the conductor and

$$\gamma(s) = \frac{\prod_{i=1}^4 \Gamma_{\mathbb{R}}(1-s+\mu_i)}{\prod_{i=1}^4 \Gamma_{\mathbb{R}}(s+\mu_i)}$$

comes from the archimedean factors, with $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2)$ and $\mu_i \in \mathbb{C}$. Since g is fixed and the eigenvalue $1/4+k^2$ (or the weight k) of f is large, each parameter $|\mu_i|$ is $\asymp k$, and hence the convexity bound is

$$L(1/2+it, f\otimes g) \ll_{g,\varepsilon} \mathcal{N}^{1/2+\varepsilon}\big(k+|t|\big)^{1+\varepsilon}. \tag{4.1}$$

(In fact, one can show the explicit dependence on g in the convexity.)

In view of (4.1), we may assume $|t| < k^{2/3+\varepsilon}$, for otherwise, $L(1/2+it, f\otimes g) \ll \mathcal{N}^{1/2+\varepsilon}|t|^{3/2}$ and Theorem 1 follows. As in [23, p.1300], we get for $|s| \le k^{1-\varepsilon}$,

$$\begin{aligned}
&\Gamma_{\mathbb{R}}(s+\mu)\\
=\ & \sqrt{2}\pi^{(1-s-\mu)/2}\exp\left(-\frac{s+\mu}{2}+\frac{s+\mu-1}{2}\log\frac{s+\mu}{2}\right)\left(1+O\Big(\frac{1}{|s+\mu|}\Big)\right)\\
=\ & \sqrt{2}\pi^{(1-s-\mu)/2}\exp\left(-\frac{s+\mu}{2}+\frac{s+\mu-1}{2}\log\frac{\mu}{2}\right)\\
&\times\exp\left(\frac{s+\mu-1}{2}\left(\frac{s}{\mu}-\frac{1}{2}\Big(\frac{s}{\mu}\Big)^2+O\Big(\Big(\frac{|s|}{|\mu|}\Big)^3\Big)\right)\right)\left(1+O\Big(\frac{1}{|s+\mu|}\Big)\right)\\
=\ & \sqrt{2}\pi^{(1-s-\mu)/2}\exp\left(-\frac{\mu}{2}+\frac{s+\mu-1}{2}\log\frac{\mu}{2}+\frac{s^2-2s}{4\mu}\right)\left(1+O\Big(\frac{|s|^3}{|\mu|^2}\Big)\right).
\end{aligned}$$

Applying to $\Gamma_{\mathbb{R}}(1-s+\mu)$ and $\Gamma_{\mathbb{R}}(s+\mu)$, it follows that

$$\begin{aligned}
\frac{\Gamma_{\mathbb{R}}(1-s+\mu)}{\Gamma_{\mathbb{R}}(s+\mu)} &= \pi^{s-1/2}\exp\left((\frac{1}{2}-s)\Big(\log\frac{\mu}{2}-\frac{1}{2\mu}\Big)\right)\left(1+O\Big(\frac{|s|^3}{|\mu|^2}\Big)\right)\\
&= \left(\frac{2\pi e^{1/(2\mu)}}{\mu}\right)^{s-1/2}\left(1+O\Big(\frac{|s|^3}{|\mu|^2}\Big)\right).
\end{aligned}$$

Let

$$X=\prod_{i=1}^{4}\Big(\frac{\mu_i}{2\pi e^{1/(2\mu_i)}}\Big)^{1/2}.$$

Then X is a positive real number about the size of k^2 and for $|s|\le k^{1-\varepsilon}$,

$$\gamma(s)=X^{1-2s}\left(1+O\Big(\frac{|s|^3}{k^2}\Big)\right). \tag{4.2}$$

Next, we write $s_0=1/2+it$ (with $|t|\le k^{2/3+\varepsilon}$) and apply the argument in [23, p.1301],

$$\begin{aligned}
&L(s_0, f\otimes g)\\
=\ & \frac{1}{2\pi i}\int_{\Re\mathrm{e}\, s=2} X^s L(s_0+s, f\otimes g)G(s)\,\frac{ds}{s}\\
&+\frac{\epsilon_{f\otimes g}}{2\pi i}\int_{\Re\mathrm{e}\, s=2} Q_{f\otimes g}^{1/2-s_0+s}X^s\gamma(s_0-s)L(1-s_0+s, f\otimes g)G(s)\,\frac{ds}{s}\\
=\ & J_1(X)+J_2(X), \text{say},
\end{aligned}$$

where $G(s)$ is holomorphic in some vertical strip and decays rapidly as $|\Im\mathrm{m}\, s|\to+\infty$.

Due to (4.2) and the decay rate of $G(s)$, we infer, like [23, (2,2)], that

$$\begin{aligned}
J_2(X) =\ & \frac{\epsilon_{f\otimes g}Q_{f\otimes g}^{1/2-s_0}X^{1-2s_0}}{2\pi i}\int_{\Re\mathrm{e}\, s=2} Q_{f\otimes g}^{s}X^s L(\overline{s_0}+s, f\otimes g)G(s)\,\frac{ds}{s}\\
&+O\left(\frac{1}{k^2}\int_{\Re\mathrm{e}\, s=1/2+\varepsilon}\big|Q_{f\otimes g}^{s}X^sL(\overline{s_0}+s, f\otimes g)G(s)\big|\,|s-s_0|^3\,\frac{|ds|}{|s|}\right).
\end{aligned}$$

The integrand in the O-term is $\ll Q_{f\otimes g}^{1/2+2\varepsilon} k^{1+3\varepsilon}(1+|t|)^3 |s^2 G(s)| \ll \mathcal{N}^{1+\epsilon}(1+|t|)^{3/2} k^{2+\epsilon}$, by $|t| \leq k^{2/3+\varepsilon}$. Therefore,

$$J_2(X) \quad = \quad \omega_{f\otimes g}(X,t) \sum_{b\geq 1} \frac{1}{b} \sum_{a\geq 1} \frac{\lambda_f(a)\lambda_g(a)}{\sqrt{a}} \mathrm{V}\left(\frac{ab^2}{X}\right) + O(\mathcal{N}^{1+\epsilon}(1+|t|)^{3/2} k^{\epsilon}), \quad (4.3)$$

where $|\omega_{f\otimes g}(X,t)| = |\epsilon_{f\otimes g} Q_{f\otimes g}^{1/2-s_0} X^{1-2s_0}| = 1$ and

$$\mathrm{V}(y) := \frac{1}{2\pi i} \int_{\Re\mathrm{e}\, s=2} Q_{f\otimes g}^{s} G(s) y^{-s} \frac{ds}{s}.$$

From the estimate of $V(y)$ in [23, p.1302], we have

$$\mathrm{V}(y) \ll_B \left(1 + \frac{|y|}{Q_{f\otimes g}}\right)^{-B}.$$

When $ab^2/X \geq X^{\varepsilon/2}$, we choose $B(\varepsilon) = 2 + 4/\varepsilon$ so that

$$\begin{aligned} \mathrm{V}\left(\frac{ab^2}{X}\right) &\ll Q_{f\otimes g}^{B(\varepsilon)} \left(\frac{X}{ab^2}\right)^2 \left(\frac{X}{ab^2}\right)^{B(\varepsilon)-2} \\ &\ll Q_{f\otimes g}^{B(\varepsilon)} \left(\frac{X}{ab^2}\right)^2 X^{-(B(\varepsilon)-2)\varepsilon/2} \ll \frac{Q_{f\otimes g}^{B(\varepsilon)}}{(ab^2)^2}. \end{aligned}$$

Hence, the tail part is

$$\sum\sum_{ab^2 \geq X^{1+\varepsilon/2}} \frac{1}{b} \frac{\lambda_f(a)\lambda_g(a)}{\sqrt{a}} \mathrm{V}\left(\frac{ab^2}{X}\right) \ll_{\varepsilon} Q_{f\otimes g}^{B(\varepsilon)}. \quad (4.4)$$

Using a partition of unity

$$\sum_{\alpha=-\infty}^{\infty} p\left(\frac{x}{2^{\alpha/2}}\right) = 1, \qquad \text{for } x > 0,$$

with a smooth function p compactly supported in $[1,2]$, the initial section of the double sum over $ab < X^{1+\varepsilon/2}$ in (4.3) can be written as

$$\sum_{\substack{Y=2^{\alpha/2},\, \alpha\geq -1 \\ Y \leq X^{1+\varepsilon/2}}} Y^{-1/2} \sum_{a} \lambda_f(a)\lambda_g(a) \frac{p(a/Y)}{\sqrt{a/Y}} \sum_{b^2 \leq X^{1+\varepsilon/2}/Y} \frac{1}{b} \mathrm{V}\left(\frac{ab^2}{X}\right)$$

Shifting the line of integration to $\Re\mathrm{e}\, s = \varepsilon$ and differentiating r times, we see that for

$y \in (1,2)$,

$$
\begin{aligned}
& \mathrm{V}^{(r)}\left(y\frac{Yb^2}{X}\right) \\
= \quad & \frac{(-1)^r}{2\pi i}\int_{\Re\mathrm{e}\, s=\varepsilon} Q_{f\otimes g}^s G(s)s(s+1)\cdots(s+r-1)y^{-s-r}\left(\frac{X}{Yb^2}\right)^s\frac{ds}{s} \\
\ll \quad & \left(\frac{Q_{f\otimes g}X}{Yb^2}\right)^{\varepsilon} y^{-2-r}\int_{\Re\mathrm{e}\, s=\varepsilon} |G(s)s(s+1)\cdots(s+r-1)|\,\frac{|ds|}{|s|} \\
\ll_{\varepsilon,r} \quad & \left(\frac{Q_{f\otimes g}X}{Yb^2}\right)^{\varepsilon} \ll Q_{f\otimes g}^{\varepsilon}k^{2\varepsilon}b^{-2\varepsilon}.
\end{aligned}
$$

Note that the exponent is independent of r. Therefore, we can write

$$
\frac{p(y)}{\sqrt{y}}\sum_{b^2\le X^{1+\varepsilon/2}/Y}\frac{1}{b}\mathrm{V}\left(y\frac{Yb^2}{X}\right) = Q_{f\otimes g}^{\varepsilon}k^{2\varepsilon}H(y)
$$

where $H(\cdot)$ is smooth function of compact support in $[1,2]$ and its rth derivative is $O_{r,\varepsilon}(1)$.

Consequently, with (4.3) and (4.4), we deduce that

$$
\begin{aligned}
J_2(X) \quad & \ll \sum_{\substack{Y=2^{\alpha/2},\,\alpha\ge -1\\ Y\le X^{1+\varepsilon/2}}} (\mathcal{N}k)^{2\varepsilon}Y^{-1/2}|S_Y(f)| + \mathcal{N}^{2B(\varepsilon)} + \mathcal{N}^{1+\epsilon}(1+|t|)^{3/2}k^{\epsilon} \\
& \ll \mathcal{N}^{2B(\varepsilon)}(1+|t|)^{3/2}k^{\epsilon}\sum_{\substack{Y=2^{\alpha/2},\,\alpha\ge -1\\ Y\le X^{1+\varepsilon/2}}} Y^{-1/2}|S_Y(f)|.
\end{aligned} \tag{4.5}
$$

The function $S_Y(f)$ is defined as

$$
S_Y(f) = \sum_n \lambda_f(n)\lambda_g(n)H\Big(\frac{n}{Y}\Big),
$$

where H is a smooth function of compact support in $[1,2]$. The same argument applies to $J_1(X)$ and leads to sums $S_Y(f)$ as in (4.5), in which the function H, though different from the one in $J_2(X)$, remains smooth and compactly supported in $[1,2]$. (See [23, §2] or [29, §1] as well.)

Let $\{f_j\}$ be an orthonormal basis, consisting of Hecke eigenforms with eigenvalues $1/4+k_j^2$, of the space of Maass cusp forms. The estimation in Theorem 1 is then reduced to

$$
\sum_{K-L\le k_j\le K+L} |S_Y(f_j)|^2 \ll_{\varepsilon} LKY^{1+\varepsilon}
$$

with

$$
K^{1/3+2\varepsilon} \le L \le K^{1-2\varepsilon} \quad \text{and} \quad Y \le K^{2+\varepsilon}, \tag{4.6}
$$

for any arbitrarily small but fixed $\varepsilon > 0$. Or, we may use a smoothly weighted sum and estimate

$$\sum_{f_j}\Big(h\Big(\frac{k_j-K}{L}\Big)+h\Big(-\frac{k_j+K}{L}\Big)\Big)\Big|S_Y(f_j)\Big|^2, \tag{4.7}$$

where $h(t)$ is an even analytic function in $|\mathrm{Im}\ t| \le 1/2$ satisfying $h^{(n)}(t) \ll (1+|t|)^{-N}$ for any $N > 0$ in this region. Thus h is a Schwarz function on $\mathbb{R}$. We also assume that $h(t) \ge 0$ for real t. For example, we may simply take $h(t) = 1/\cosh(t)$.

We remark that the normalization of f_j in (4.7) is different from the normalization $\lambda_f(1) = 1$ as required in the definition of $L(s, f \otimes g)$, but the discrepancy is within k^ε as proved in Hoffstein and Lockhart [12].

Squaring out $|S_Y(f_j)|^2$ and applying the Kuznetsov formula for both cusps being ∞, or Petersson formula for holomorphic f and f_j, the left-hand side of (4.7) leads to the estimates of two types of sums, composing of respectively the diagonal terms and non-diagonal terms. For the Maass case, there is an additional sum over cusps from the Eisenstein part of the Kuznetsov formula, but this sum can be ignored by positivity, see [23, (3.6)].

The main task is to handle the non-diagonal terms

$$T_{K,L}(Y) = \sum_{n,m}\lambda_g(n)\overline{\lambda_g}(m)H\big(\frac{n}{Y}\big)\overline{H}\big(\frac{m}{Y}\big)\sum_{\substack{c\ge1\\ \mathcal{N}|c}}\frac{S(m,n,c)}{c}V_{K,L}\Big(\frac{4\pi\sqrt{mn}}{c}\Big),$$

see [23, §3] (and [29, §4] for holomorphic case). We seek a bound

$$T_{K,L}(Y) \ll_\varepsilon YLK^{1+\varepsilon}$$

for $K^{1/3+2\varepsilon} \le L \le K^{1-2\varepsilon}$ and $Y \le K^{2+\varepsilon}$. Then the function $V_{K,L}(\cdot)$ is replaced by a sum of exponential factors with an negligible error term, see (4.1) and Lemma 4.1 (ii) in [23] or (55) and Proposition 3.1 (ii) in [29]. However, in both treatments, the expansion of $V_{K,L}$ is carried out under the assumption $L \ge K^{1/2}$, forbidding the application here. We shall develop another auxiliary but similar finite series expansion.

At first, we write as in [23, (4.1)] or [29, (55)],

$$V_{K,L}(x) = \frac{1}{2i}\big(W_{K,L}(x)e^{ix} + W_{K,L}(-x)e^{-ix}\big)$$

where

$$W_{K,L}(x) = \int_{\mathbb{R}} e\Big(\frac{tK}{L}+\frac{x}{2\pi}\big(\cosh(\frac{\pi t}{L})-1\big)\Big)\big(h(u)(uL+K)\big)^\wedge(t)\,dt.$$

(The holomorphic case is similar, except that the hyperbolic cosine function will be replaced by the cosine function. See (53) and (54) in [29].) The auxiliary tool we need is the lemma below and its proof is provided in Section 10.

Lemma 4.1 *(i) For $\varepsilon > 0$ and $|x| \le 8\pi L K^{1-\varepsilon}$, $W_{K,L}(x) \ll_{M,\varepsilon} K^{-M}$ for any $M > 0$.*

(ii) For a 5-tuple $\underline{\lambda} = (\mu, \nu, k, \alpha, \beta)$, we set $\lambda = k - \mu + 3\beta - \alpha$ and define

$$
\begin{aligned}
&\widetilde{W}_{\underline{\lambda}}(x) \\
= \; & \frac{L^{2k-2\nu-\alpha} K^{4\beta-\alpha}}{x^{\lambda}} \big(1 + i\,\mathrm{sgn}(x)\big) \frac{L}{\sqrt{|x|}}\, e\big(-\frac{K^2}{\pi x}\big) \\
& \times \left(L \left(u^{2k-\alpha} \frac{d^{2\nu}}{du^{2\nu}} (u h(u)) \right)^{\wedge} \left(\frac{2LK}{\pi x} \right) + K \big(u^{2k-\alpha} h^{(2\nu)}(u) \big)^{\wedge} \left(\frac{2LK}{\pi x} \right) \right).
\end{aligned}
$$

Assume $K^{1/3+2\varepsilon} \le L \le K^{1-2\varepsilon}$ and $LK^{1-\varepsilon} \le |x| \ll K^{2+\varepsilon}$. Then for any fixed $M \ge 1$, we have

$$
W_{K,L}(x) = \sum_{\underline{\lambda} = (\mu,\nu,k,\alpha,\beta)} c_{\underline{\lambda}} \widetilde{W}_{\underline{\lambda}}(x) + O(K^{-M})
$$

where the summation $\sum_{\underline{\lambda}}$ denotes a multiple sum running over $0 \le 3\mu \le \nu \le N_1$, $0 \le k \le N_2$, $0 \le \alpha \le 2k$ and $\alpha \le 4\beta \le N_3$. The coefficients $c_{\underline{\lambda}}$ depend only on the parameters in $\underline{\lambda}$, and N_i $(i = 1, 2, 3)$ denotes some large number depending on ε and M.

For our purpose, x takes the values $\pm 4\pi\sqrt{mn}/c$ and hence $|x| \le 8\pi Y$. Therefore, an immediate consequence of Lemma 4.1 (i) is that for $Y \le LK^{1-\varepsilon}$, all the non-diagonal terms are negligible and (4.7) is done. In what follows we consider the ranges

$$
LK^{1-\varepsilon} \le Y \le K^{2+\varepsilon} \quad \text{and} \quad K^{1/3+2\varepsilon} \le L \le K^{1-2\varepsilon}, \tag{4.8}
$$

where Lemma 4.1 (ii) applies.

Applying Lemma 4.1 in place, we repeat the argument in Sections 4.5-4.8 of [23]. That is, we reduce $T_{K,L}(Y)$ to $\widetilde{T}^{\pm}_{\underline{\lambda}}(Y)$,

$$
T_{K,L}(Y) = \sum_{\underline{\lambda}} c_{\underline{\lambda}} \big(\widetilde{T}^{+}_{\underline{\lambda}}(Y) + \widetilde{T}^{-}_{\underline{\lambda}}(Y) \big) + O(K^{-M})
$$

where M is any suitably large number at our disposal, and

$$
\widetilde{T}^{\pm}_{\underline{\lambda}}(Y) = \sum_{n,m} \lambda_g(n) \overline{\lambda_g}(m) H\big(\frac{n}{Y}\big) \overline{H}\big(\frac{m}{Y}\big) \sum_{\substack{c \ge 1 \\ \mathcal{N} | c}} \frac{S(m,n,c)}{c} e\left(\pm \frac{2\sqrt{mn}}{c} \right) \widetilde{W}_{\underline{\lambda}}\left(\pm \frac{4\pi\sqrt{mn}}{c} \right).
$$

Next, we open the Kloosterman sums and perform a transformation with Voronoi summation formula. Then we insert the series expansion of Bessel functions, see [23, §4.5]. We arrive at

$$
\begin{aligned}
&T_{K,L}(Y)\\
= &\sum_{\eta}\sum_{j}\sum_{\underline{\lambda}} c'_{\eta,j,\underline{\lambda}} \sum_{\substack{c\le Y/(LK^{1-\varepsilon})\\ \mathcal{N}|c}} c^{-2} \sum_{n,r\ge 1} \lambda_g(n)\overline{\lambda_g}(r) H\big(\frac{n}{Y}\big) \sum_{\substack{z \bmod c\\ (z,c)=1}} e\Big(\frac{z(n-r)}{c}\Big)\\
&\times \int_0^\infty \overline{H}\big(\frac{t}{Y}\big) e\Big(\eta_1\frac{2\sqrt{tn}}{c}+\eta_2\frac{2\sqrt{tr}}{c}\Big)\widetilde{W_{\underline{\lambda}}}\Big(\eta_3\frac{4\pi\sqrt{tn}}{c}\Big)\frac{c^{j+1/2}}{(tr)^{j/2+1/4}}\,dt + O(K^{-M})
\end{aligned}
$$

where the summation over $\eta=(\eta_1,\eta_2,\eta_3)$ with $\eta_j=\pm 1$ runs over all of the eight combinations, j ranges from 1 to N_4 with a suitably large $N_4=N_4(\varepsilon,M)$, and the range of $\underline{\lambda}$ is the same as in Lemma 4.1. The coefficient $c'_{\eta,j,\underline{\lambda}}$ depends only on η, j and $\underline{\lambda}$.

Now we change variables from t to w by $t=w^2Y$. Then

$$
\begin{aligned}
&T_{K,L}(Y)\\
= &\sum_{\eta}\sum_{j}\sum_{\underline{\lambda}} c'_{\eta,j,\underline{\lambda}} \sum_{\substack{c\le Y/(LK^{1-\varepsilon})\\ \mathcal{N}|c}} c^{j-3/2} \sum_{n,r\ge 1} \lambda_g(n)\bar{\lambda}_g(r) H\Big(\frac{n}{Y}\Big) \sum_{\substack{z \bmod c\\ (z,c)=1}} e\Big(\frac{z(n-r)}{c}\Big)\\
&\times \int_0^\infty \overline{H}(w^2) e\Big(\eta_1\frac{2w\sqrt{Y}(\eta_1\sqrt{n}+\eta_2\sqrt{r})}{c}\Big)\widetilde{W_{\underline{\lambda}}}\Big(\eta_3\frac{4\pi w\sqrt{Yn}}{c}\Big)\frac{2Y^{3/4-j/2}\,dw}{w^{j-1/2}r^{j/2+1/4}}\\
&+O(K^{-M}).
\end{aligned}
$$

Substituting the formula for $\widetilde{W_{\underline{\lambda}}}$, we then get

$$
T_{K,L}(Y) = \sum_{\eta}\sum_{j}\sum_{\underline{\lambda}} c_{\eta,j,\underline{\lambda}}\widetilde{T}^{(\eta)}_{\underline{\lambda},j}(Y) + O(K^{-M}) \tag{4.9}
$$

where $c_{\eta,j,\underline{\lambda}}$ denotes a coefficient, like $c'_{\eta,j,\underline{\lambda}}$, depending only on the indices and the summand $\widetilde{T}^{(\eta)}_{\underline{\lambda},j}(Y)$ is defined as

$$
\begin{aligned}
\widetilde{T}^{(\eta)}_{\underline{\lambda},j}(Y) \;=\; &\frac{Y^{1/2-(\lambda+j)/2}K^{4\beta-\alpha}}{L^{2\nu+\alpha-2k-1}} \sum_{\substack{c\le Y/(LK^{1-\varepsilon})\\ \mathcal{N}|c}} c^{j+\lambda-1}\\
&\times \sum_{n,r\ge 1} \frac{\lambda_g(n)\lambda_g(r)}{n^{\lambda/2+1/4}r^{j/2+1/4}} H\big(\frac{n}{Y}\big) B^{(\lambda,j)}_{\eta,Y,c}(n,r) \sum_{\substack{z \bmod c\\ (z,c)=1}} e\Big(\frac{z(n-r)}{c}\Big).
\end{aligned}
$$

Here for each $\underline{\lambda}=(\mu,\nu,k,\alpha,\beta)$, we denote by λ the specific expression $k-\mu+3\beta-\alpha$ and write

$$
\begin{aligned}
B^{(\lambda,j)}_{\eta,Y,c}(n,r) \;=\; &2K\int_0^\infty e\Big(\frac{2w\sqrt{Y}(\eta_1\sqrt{n}+\eta_2\sqrt{r})}{c} - \frac{\eta_3 K^2 c}{4\pi^2 w\sqrt{Yn}}\Big)\\
&\times \mathsf{h}_{2\nu,2k-\alpha}\Big(\frac{\eta_3 LKc}{2\pi^2 w\sqrt{Yn}}\Big)\frac{\overline{H}(w^2)}{w^{j+\lambda}}\,dw
\end{aligned} \tag{4.10}
$$

where $\mathsf{h}_{2\nu,2k-\alpha}(t)$ is the smooth and rapidly decaying function,

$$\mathsf{h}_{2\nu,2k-\alpha}(t) = \frac{L}{K}\left(u^{2k-\alpha}\frac{d^{2\nu}}{du^{2\nu}}\big(uh(u)\big)\right)^{\wedge}(t) + \big(u^{2k-\alpha}h^{(2\nu)}(u)\big)^{\wedge}(t).$$

Using the argument in [23, §4.7] (or Lemma 5.1(1) below), we obtain $B^{(\lambda,j)}_{\eta,Y,c}(n,r) \ll K^{-M}$ whenever $|r-n| > Y/2$.

With the change of variables $h = r-n$ and the well-known formula for the Ramanujan sum (see e.g. [17, (2.26)])

$$\sum_{\substack{z \bmod c\\(z,c)=1}} e\left(\frac{zn}{c}\right) = \sum_{\delta|(c,n)} \mu\left(\frac{c}{\delta}\right)\delta,$$

it follows that

$$\widetilde{T}^{(\eta)}_{\underline{\lambda},j}(Y) \quad = \quad \frac{Y^{1/2-(\lambda+j)/2}K^{4\beta-\alpha}}{L^{2\nu+\alpha-2k-1}} \sum_{\substack{c\le Y/(LK^{1-\varepsilon})\\ \mathcal{N}|c}} c^{j+\lambda}\sum_{\delta|c}\frac{\mu(c/\delta)}{c/\delta}\sum_{\substack{|h|\le Y/2\\ \delta|h}} P(c,h,Y). \quad (4.11)$$

where the last term $P(c,h,Y)$ is a shifted sum,

$$P(c,h,Y) = \sum_{n\ge\max(1,1-h)} \lambda_g(n)\lambda_g(n+h)\frac{H(n/Y)}{n^{\lambda/2+1/4}(n+h)^{j/2+1/4}}B^{(\underline{\lambda},j)}_{\eta,Y,c}(n,n+h).$$

By the inverse Mellin transform, $P(c,h,Y)$ can be expressed as a line integral of $D_g(s,1,1,h)$ against a function as follows:

$$P(c,h,Y) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} D_g(s,1,1,h)\widetilde{G}_{h,c}(s)\,ds \quad (4.12)$$

for any $\sigma > 1$, where

$$\widetilde{G}_{h,c}(s) = (2Y)^s Y^{-(\lambda+j+1)/2}\int_0^\infty G_0(z)\left(z+\frac{h}{2Y}\right)^{s-1} dz. \quad (4.13)$$

Let us take holomorphic cusp form g for illustration. We see from (3.1) that

$$\begin{aligned} G_0(z) \quad = \quad & \left(\frac{2z+h/Y}{\sqrt{z(z+h/Y)}}\right)^{l-1} z^{-(\lambda/2+1/4)}\left(z+\frac{h}{Y}\right)^{-j/2-1/4} \\ & \times H(z)B^{(\underline{\lambda},j)}_{\eta,Y,c}(zY,zY+h), \end{aligned} \quad (4.14)$$

where

$$
\begin{aligned}
& B^{(\lambda,j)}_{\eta,Y,c}(zY, zY+h) \\
= \;& 2K\int_0^\infty e\Big(\frac{2w\sqrt{Y}(\eta_1\sqrt{zY}+\eta_2\sqrt{zY+h})}{c} - \frac{\eta_3 K^2 c}{4\pi^2 wY\sqrt{z}}\Big) \\
& \times \mathsf{h}_{2\nu,2k-\alpha}\Big(\frac{\eta_3 LKc}{2\pi^2 wY\sqrt{z}}\Big)\frac{\overline{H}(w^2)}{w^{j+\lambda}}\,dw \\
= \;& 2Kz^{(j+\lambda-1)/2}\int_0^\infty e\Big(\frac{2wY(\eta_1+\eta_2\sqrt{1+h/(Yz)})}{c} - \frac{\eta_3 K^2 c}{4\pi^2 wY}\Big) \\
& \times \mathsf{h}_{2\nu,2k-\alpha}\Big(\frac{\eta_3 LKc}{2\pi^2 wY}\Big)\overline{H}\Big(\frac{w^2}{z}\Big)\frac{dw}{w^{j+\lambda}}
\end{aligned}
$$

after changing variables from w to $w/\sqrt{z}$.

Then we obtain the following expression for

$$
\begin{aligned}
& \widetilde{G}_{h,c}(s) \\
= \;& 2K(2Y)^\sigma Y^{-(\lambda+j+1)/2}\int_0^\infty\int_0^\infty \left(\frac{2z+h/Y}{\sqrt{z+h/Y}}\right)^{l-1}\left(z+\frac{h}{Y}\right)^{-j/4-1/4}\left(z+\frac{h}{2Y}\right)^{\sigma-1} \\
& \times \mathsf{h}_{2\nu,2k-\alpha}\left(\frac{\eta_3 LKc}{2\pi^2 wY}\right)\overline{H}\left(\frac{w^2}{z}\right)H(z)e(\Phi(z,w,t))\,\frac{dw}{w^{j+\lambda}}\,\frac{dz}{z^{(2l-j+1)/4}},
\end{aligned}
\tag{4.15}
$$

where

$$
\Phi(z,w,t) = \frac{t}{2\pi}\log(2Yz+h) + \frac{2wY(\eta_1+\eta_2\sqrt{1+h/(Yz)})}{c} - \frac{\eta_3 K^2 c}{4\pi^2 wY}. \tag{4.16}
$$

To achieve our goal it remains to evaluate (4.11). A nontrivial estimate of (4.12) will definitely fulfill the purpose to provide a subconvexity bound. This is the main approach of Sarnak [29] and our [23].

5. Estimation of $\widetilde{G}_{h,c}(s)$

The function $\widetilde{G}_{h,c}(s)$ in the integral of $P(c,h,Y)$ is given by an exponential integral. Clearly, the function $\widetilde{G}_{h,c}(s)$ is holomorphic on the whole complex plane by (4.13), as $|h| \le Y/2$ and $z \in [1,2]$. Let A_0 be any fixed positive constant. By (4.15), we have the trivial bound, for any $|\sigma| \le A_0$ and $t \in \mathbb{R}$,

$$
\widetilde{G}_{h,c}(s) \ll_{A_0} KY^{\sigma-(\lambda+j+1)/2} \tag{5.1}
$$

but this estimate is very crude. We show a couple of estimates at different phases. For the values of Y and L we recall the imposed condition (4.8).

Lemma 5.1 *Suppose $s = \sigma + it$ with $-A_0 \le \sigma \le 2$ and $t \in \mathbb{R}$. Let $1 \le c \le Y/(LK^{1-\varepsilon})$ and $|h| \le Y/2$.*

(1) If $\eta_1 = \eta_2$, then for any $M' > 1$,

$$\widetilde{G}_{h,c}(s) \ll_{A_0,M,M',\varepsilon} K^{-M'}. \tag{5.2}$$

(2) Assume $\eta_1 \ne \eta_2$ and $c \le Y/K^{2-\varepsilon}$. If $|h| \ge cK^{\varepsilon}$, then (5.2) holds.

(3) Assume $\eta_1 \ne \eta_2$, $\eta_3 = \mathrm{sgn}(h)\eta_2$, and $c \ge Y/K^{2-\varepsilon}$. Then (5.2) holds. Here the symbol $\mathrm{sgn}(h) = h/|h|$ denotes the sign of h.

(4) Assume $c \ge Y/K^{2-\varepsilon}$. There exists an absolute constant $\delta_0 > 0$ such that (5.2) holds for

$$|h| \le \delta_0 K^2c^2/Y \quad \text{or} \quad |h| \ge \delta_0^{-1}K^2c^2/Y.$$

Proof. All estimates are shown by applying integration by parts to the w-integral in (4.15). We integrate $e(\Phi(z,w,t))\frac{\partial\Phi}{\partial w}$ with respect to w many times and differentiate the rest of the integrand divided by $\frac{\partial\Phi}{\partial w}$. The differentiation of a function like

$$\frac{1}{w^{j+\lambda}}\bar{H}\Big(\frac{w^2}{z}\Big)$$

yields a constant bound. Each differentiation of $\mathsf{h}_{2\nu,2k-\alpha}$ in (4.15) produces a factor

$$-\frac{\eta_3 LKc}{2\pi^2w^2Y} \ll \frac{LKc}{Y}.$$

They overall contribute at most

$$O(1 + LKc/Y).$$

It remains to investigate the size of $\frac{\partial\Phi}{\partial w}$ and its higher derivatives. By (4.16), the derivative of Φ with respect to w is

$$\frac{\partial\Phi}{\partial w} = \frac{2Y}{c}\Big(\eta_1 + \eta_2\sqrt{1 + \frac{h}{Yz}}\Big) + \frac{\eta_3K^2c}{4\pi^2w^2Y},$$

and its higher derivatives satisfy

$$\frac{\partial^r\Phi}{\partial w^r} \ll \frac{K^2c}{Y}.$$

Because H is supported in $[1,2]$, we may confine, by (4.15), the values of z and w lying between 1 and 2.

(1) When $\eta_1 = \eta_2$ and $c \le Y/(LK^{1-\varepsilon})$,

$$\begin{aligned}\left|\frac{\partial \Phi}{\partial w}\right| &= \left|\eta_1 \frac{2Y}{c}\Big(1+\sqrt{1+\frac{h}{Yz}}\Big) + \frac{\eta_3 K^2 c}{4\pi^2 w^2 Y}\right| \\ &\gg \frac{Y}{c}\left(1 - O\Big(\frac{K^2c^2}{Y^2}\Big)\right) \\ &\gg \frac{Y}{c}\left(1 - O\bigg(\Big(\frac{K^\varepsilon}{L}\Big)^2\bigg)\right) \\ &\gg \frac{Y}{c} \gg \frac{K^2 c}{Y}\end{aligned}$$

for $c \le Y/(LK^{1-\varepsilon})$. Therefore, for every $r \ge 0$,

$$\frac{\partial^r}{\partial w^r}\left(\Big(\frac{\partial \Phi}{\partial w}\Big)^{-1}\right) \ll \frac{c}{Y}.$$

Hence each integration by parts produces a saving

$$O\Big(\frac{c}{Y} + \frac{LKc}{Y}\frac{c}{Y}\Big) = O(K^{-\varepsilon}).$$

Doing this repeatedly produces a negligible $O(K^{-M'})$ for arbitrary $M' > 0$, and (5.2) follows by replacing M' by $M' + M_0'$ where the number $M_0' = M_0'(\varepsilon, M)$ (depending on ε and M) is chosen to satisfy $KY^{2-(\lambda+j+1)/2} \ll K^{M_0'}$.

(2) From $\eta_1 \ne \eta_2$, we get

$$\begin{aligned}\frac{\partial \Phi}{\partial w} &= \eta_2 \frac{2Y}{c}\Big(\sqrt{1+\frac{h}{Yz}} - 1\Big) + \frac{\eta_3 K^2 c}{4\pi^2 w^2 Y} \\ &= \operatorname{sgn}(h)\eta_2 \frac{2|h|}{zc}\Big(1+\sqrt{1+\frac{h}{Yz}}\Big)^{-1} + \frac{\eta_3 K^2 c}{4\pi^2 w^2 Y}.\end{aligned} \tag{5.3}$$

Under the assumption in (2), we get

$$\frac{|h|}{c} \ge K^\varepsilon \quad \text{and} \quad \frac{K^2 c}{Y} \le \frac{K^2}{Y}\frac{Y}{K^{2-\varepsilon}} \le K^\varepsilon.$$

With $|h| \le Y/2$ and $1 \le z, w \le 2$, we know that the first term in (5.3) has an absolute value $\ge (\sqrt{6}-2)K^\varepsilon$, while the second term there has an absolute value $\le K^\varepsilon/(4\pi^2)$. Thus, we have for $r \ge 2$ that

$$\frac{\partial^r \Phi}{\partial w^r} \ll \frac{K^2 c}{Y} \ll K^\varepsilon \ll \left|\frac{\partial \Phi}{\partial w}\right|.$$

Every integration by parts produces a factor

$$O\Big(K^{-\varepsilon} + \frac{LKc}{Y}K^{-\varepsilon}\Big) = O\Big(K^{-\varepsilon} + \frac{L}{K^{1-\varepsilon}}K^{-\varepsilon}\Big) = O(K^{-\varepsilon}),$$

by $c \leq Y/K^{2-\varepsilon}$. This case is done.

(3) If $\eta_3 = \operatorname{sgn}(h)\eta_2$, the absolute value of (5.3) is $\gg K^2c/Y$. It follows that

$$\left|\frac{\partial \Phi}{\partial w}\right| \gg \frac{K^2 c}{Y} \gg \frac{\partial^r \Phi}{\partial w^r} \tag{5.4}$$

for $r \geq 2$, and the saving is

$$O\biggl(\Bigl(1 + \frac{LKc}{Y}\Bigr)\frac{Y}{K^2c}\biggr) = O\Bigl(K^{-\varepsilon} + \frac{L}{K}\Bigr) = O(K^{-\varepsilon}).$$

(4) It suffices to consider $\eta_1 \neq \eta_2$ and $\eta_3 \neq \operatorname{sgn}(h)\eta_2$. The proof is similar to the above. By (5.3), we have

$$\left|\frac{\partial \Phi}{\partial w}\right| = \left|\frac{2}{z}\Bigl(1 + \sqrt{1 + \frac{h}{Yz}}\Bigr)^{-1}\frac{|h|}{c} - \frac{K^2 c}{4\pi^2 w^2 Y}\right|$$

Since $|h| \leq Y/2$ and $z, w \in [1, 2]$, there are two absolute constants $B' > A' > 0$ so that

$$A' \leq \frac{2}{z}\left(1 + \sqrt{1 + \frac{h}{Yz}}\right)^{-1} \leq B' \quad \text{and} \quad A' \leq \frac{1}{4\pi^2 w^2} \leq B'.$$

Therefore, we choose a sufficiently small but absolute constant $\delta_0 > 0$ for which

$$\left|\frac{\partial \Phi}{\partial w}\right| \gg \frac{K^2 c}{Y} - \frac{B'}{A'}\frac{|h|}{c} \gg \frac{K^2 c}{Y}\Bigl(1 - \frac{B'}{A'}\delta_0\Bigr) \gg K^2 c/Y$$

whenever $|h|/c \leq \delta_0 K^2 c/Y$, or

$$\left|\frac{\partial \Phi}{\partial w}\right| \gg \frac{|h|}{c} - \frac{B'}{A'}\frac{K^2 c}{Y} \gg \frac{K^2 c}{Y}\Bigl(\delta_0^{-1} - \frac{B'}{A'}\Bigr) \gg K^2 c/Y$$

whenever $|h|/c \geq \delta_0^{-1} K^2 c/Y$.

Thus (5.4) holds in this case. We proceed with the argument in the proof of (3) to conclude (5.2). □

Lemma 5.2 *Suppose $s = \sigma + it$ with $-A_0 \leq \sigma \leq 2$ and $t \in \mathbb{R}$. Assume $1 \leq c \leq Y/(LK^{1-\varepsilon})$ and $1 \leq |h| \leq Y/2$. There exists an absolute positive constant δ_1 such that*

(1) *for all $|t| \geq \delta_1^{-1}|h|/c$,*

$$\widetilde{G}_{h,c}(s) \ll_{A_0, M'} KY^{\sigma - (\lambda + j + 1)/2}|t|^{-M'},$$

(2) *for all $|t| \leq \delta_1 |h|/c$,*

$$\widetilde{G}_{h,c}(s) \ll_{A_0, M'} KY^{\sigma - (\lambda + j + 1)/2}\Bigl(\frac{|h|}{c}\Bigr)^{-M'}$$

for any $M' > 1$. The implied constants depend on A_0 and M' only.

Proof. We take derivative in (4.16) with respective to z,

$$\frac{\partial \Phi}{\partial z} = \frac{t}{2\pi}\frac{1}{z+h/(2Y)} - \eta_2 \frac{w}{z^2\sqrt{1+h/(Yz)}}\frac{h}{c}. \tag{5.5}$$

Note that

$$\frac{1}{2\pi}\frac{1}{z+h/(2Y)} \quad \text{and} \quad \frac{w}{z^2\sqrt{1+h/(Yz)}}$$

are both bounded above and below by some absolute positive constants B' and A', respectively. Hence, there exists some absolute $\delta_1 > 0$ such that for $|t| \geq \delta_1^{-1}|h|/c$,

$$\Big|\frac{\partial \Phi}{\partial z}\Big| \gg |t| - \frac{B'}{A'}\frac{|h|}{c} \gg |t|\big(1 - \frac{B'}{A'}\delta_1\big) \gg |t|,$$

and for $r \geq 2$,

$$\frac{\partial^r \Phi}{\partial z^r} \ll |t| + \frac{|h|}{c} \ll_r |t|.$$

This time the derivative of the remaining part in the z-integral of (4.15) is $O(1)$. Hence, successive integration by parts yields the upper bound $O(|t|^{-M})$ for the z-integral.

Similarly, we get for $|t| \leq \delta_1|h|/c$,

$$\Big|\frac{\partial \Phi}{\partial z}\Big| \gg \frac{|h|}{c} - \frac{B'}{A'}|t| \gg \frac{|h|}{c}\big(1 - \frac{B'}{A'}\delta_1\big) \gg \frac{|h|}{c},$$

and

$$\frac{\partial^r \Phi}{\partial z^r} \ll \frac{|h|}{c}.$$

The above argument yields the desired result. $\square$

In view of Lemma 5.1, the crucial case is $c \geq Y/K^{2-\varepsilon}$ and the non-trivial cases arise when (η_1, η_2, η_3) is respectively of the form $(-\eta, \eta, -\eta)$ for $h > 0$, or $(\eta_1, \eta_2, \eta_3) = (\eta, -\eta, -\eta)$ for $h < 0$ where $\eta = \pm 1$. Also, together with Lemma 5.2, we need to consider only $|h| \asymp K^2c^2/Y$ and $|t| \asymp |h|/c$. To this end, we need a precise form for $\widetilde{G}_{h,c}(s)$. We shall make use of the tools below, which are Lemma 5.1.3 and Lemma 5.5.6 from Huxley [14]. (Note that Lemma 5.5.6 is Theorem 2 in [13] where the proof is provided.)

Lemma 5.3 (Second derivative test) *Let $f(x)$ be real and twice differentiable on the open interval (α, β) with $f''(x) \geq \lambda > 0$ on (α, β). Let $g(x)$ be a real function of bounded total variation, and let V be the sum of its total variation on the closed interval $[\alpha, \beta]$ and the maximum modulus of $g(x)$ on $[\alpha, \beta]$. Then*

$$\Big|\int_\alpha^\beta g(x)e(f(x))\, dx\Big| \leq \frac{4V}{\sqrt{\pi\lambda}}.$$

Lemma 5.4 (Weighted stationary phase integral) *Let* $f \in C^{(4)}([\alpha,\beta])$ *and* $g \in C^{(3)}([\alpha,\beta])$ *be real-valued functions with continuous fourth and third derivatives, respectively. Suppose that there are positive parameters* P, N, Q, U *such that*

$$P \geq \beta - \alpha, \qquad N \geq P/\sqrt{Q},$$

and positive constants C_r *such that, for* $\alpha \leq x \leq \beta$,

$$|f^{(r)}(x)| \leq C_r Q/P^r, \qquad |g^{(s)}(x)| \leq C_s U/N^s,$$

for $1 \leq r \leq 4$ *and* $0 \leq s \leq 3$ *with*

$$f''(x) \geq Q/C_2P^2.$$

Suppose $f'(x)$ *changes sign at a point* $x = \gamma$ *with* $\alpha < \gamma < \beta$. *Then,*

$$\begin{aligned}
&\int_\alpha^\beta g(x)e(f(x))\,dx \\
= \quad & \frac{g(\gamma)e(f(\gamma)+1/8)}{\sqrt{f''(\gamma)}} + \frac{g(\beta)e(f(\beta))}{2\pi i f'(\beta)} - \frac{g(\alpha)e(f(\alpha))}{2\pi i f'(\alpha)} \\
&+ O\Big(\frac{P^4U}{Q^2}\Big(1+\frac{P}{N}\Big)^2\Big(\frac{1}{(\beta-\gamma)^3}+\frac{1}{(\gamma-\alpha)^3}\Big)\Big) + O\Big(\frac{PU}{Q^{3/2}}\Big(1+\frac{P}{N}\Big)^2\Big).
\end{aligned}$$

Remark 3. For a complex-valued g, we split it into $g = g_1 + ig_2$ where both g_1 and g_2 are real. Since $|g_i^{(s)}(x)| \leq |g^{(s)}(x)|$, Lemma 5.4 is also applicable.

Before proving our required formula we introduce some conditions and notation. Assume

$$1 \leq C \leq Y/(LK^{1-\varepsilon}) \ , \ K^\varepsilon \leq K^2C/Y \asymp T \ , \ c \sim C \ , \ |h| \sim CT \ . \tag{5.6}$$

We use the condition $K^2C/Y \geq K^\varepsilon$ to replace $c \geq Y/K^{2-\varepsilon}$. Let us write $J = (\operatorname{sgn} h)CT$ and define for $z \in [1/2, 2]$,

$$\begin{aligned}
\omega(z) &= \frac{1}{2\pi}\frac{Kc}{\sqrt{2|J|Y}}\frac{1}{z}\Bigg(\sqrt{1+z^2\frac{J}{Y}}+1\Bigg)^{1/2}, && (5.7)\\
e_\eta(z,s) &= \int_{1/2}^z \Big(\frac{1}{u^2}+\frac{J}{2Y}\Big)^{s-1} e\Bigg(\eta\frac{K}{\pi}\sqrt{\frac{2|J|}{Y}}u\Bigg(\sqrt{1+u^2\frac{J}{Y}}+1\Bigg)^{-1/2}\Bigg)\,du, && (5.8)\\
\mathcal{F}_{h,c}(z) &= \left(\frac{2}{z^2}+\frac{J}{Y}\right)^{l-1}\left(\frac{1}{z^2}+\frac{J}{Y}\right)^{-(2l+j-1)/4} H\left(\frac{|h|}{|J|z^2}\right) \\
&\quad \times \overline{H}\left(\frac{|J|}{|h|}z^2\omega(z)^2\right)\mathsf{h}_{2\nu,2k-\alpha}\left(\frac{\eta LKc}{2\pi^2\omega(z)Y}\right)\frac{z^{(2l-j-5)/2}}{\omega(z)^{j+\lambda-3/2}} && (5.9)
\end{aligned}$$

where $\eta = \pm 1$ and $\lambda \in \mathbb{R}$. In view of (5.6) and $\operatorname{supp}(H) \subseteq [1,2]$, we have the following.

Remark 4. With the notation above, we have

(i) $Kc/\sqrt{|J|Y} \asymp KC^{1/2}/\sqrt{YT} \asymp KC^{1/2}/\sqrt{K^2C} \asymp 1$,

(ii) $|J|/Y \le K^2C^2/Y^2 \le K^{2\varepsilon}/L^2 = o(1)$,

(iii) $K\sqrt{|J|/Y} \asymp T$,

(iv) $1 \le |h/J| \le 2$,

(v) the support of $H(|h|v/|J|)$ (as a function of v) is contained in $[1/2, 2]$.

Therefore, all three functions are well-defined, and $\mathcal{F}_{h,c}(z)$ is supported inside $[2^{-1/2}, 2^{1/2}]$, for any $|h| \sim CT$.

We want to separate the h and s parts intertwined inside $\widetilde{G}_{h,c}(s)$ in (4.15) into a product of two factors. Let us introduce

$$\begin{aligned} G_0(s;h,c) &= 4\pi(1+i)\frac{|J|}{\sqrt{c}}\left(\frac{2Y}{|J|}\right)^s Y^{-(\lambda+j)/2}\int_0^\infty \mathcal{F}'_{h,c}(z)e_\eta(z,s)\,dz,\\ G_1(s;h,c) &= |h|^{1-s}\widetilde{G}_{h,c}(s) - G_0(s;h,c), \end{aligned} \tag{5.10}$$

where $\mathcal{F}'_{h,c}(z) = \frac{d}{dz}\mathcal{F}_{h,c}(z)$ is also compactly supported in $[2^{-1/2}, 2^{1/2}]$.

The next lemma shows that $G_0(s;h,c)$ is a good approximate to $|h|^{1-s}\widetilde{G}_{h,c}(s)$.

Lemma 5.5 *Let $s = \sigma + it$ with $|\sigma| \le 2$ and $t \in \mathbb{R}$. Assume that (5.6) holds and $(\eta_1, \eta_2, \eta_3) = (-\mathrm{sgn}(h)\eta, \mathrm{sgn}(h)\eta, -\eta)$ where $\eta = \pm 1$. Then both $G_r(s;h,c)$, $r = 0,1$, are holomorphic on $\mathbb{C}$. Moreover, let $0 < a_0 < b_0$ be any fixed absolute constants, then for all $|t| \in (a_0T, b_0T)$,*

$$G_1(s;h,c) \ll K^\epsilon Y^{\sigma-(\lambda+j)/2}(CT)^{1/2-\sigma}T^{-1/2}. \tag{5.11}$$

The implied constant depends on ε only and $0 < \epsilon \ll \varepsilon$; see Notation in Section 1.

Remark 5. Indeed this lemma holds for $-A_0 \le \sigma \le 2$ for any $A_0 > 0$, but the range $|\sigma| \le 2$ will be enough for our applications.

Proof. Recall $J = (\mathrm{sgn}\, h)CT$. By changing variables from z to $|h|z/|J|$, $\widetilde{G}_{h,c}(s)$ in (4.15) is expressed as

$$\begin{aligned} |h|^{1-s}\widetilde{G}_{h,c}(s) \;&=\; 2K|J|\left(\frac{2Y}{|J|}\right)^s Y^{-(\lambda+j+1)/2}\int_0^\infty \Big(\frac{2z+J/Y}{\sqrt{z+J/Y}}\Big)^{l-1}\Big(z+\frac{J}{Y}\Big)^{-j/4-1/4}\\ &\quad\times\Big(z+\frac{J}{2Y}\Big)^{\sigma-1}\int_0^\infty \mathsf{h}_{2\nu,2k-\alpha}\Big(\frac{\eta_3 LKc}{2\pi^2 wY}\Big)\\ &\quad\times\overline{H}\Big(\frac{|J|w^2}{|h|z}\Big)H\Big(\frac{|h|}{|J|}z\Big)e(\phi(z,w,t))\,\frac{dw}{w^{j+\lambda}}\,\frac{dz}{z^{(2l-j+1)/4}}, \end{aligned}$$

where

$$\phi(z,w,t) = \frac{t}{2\pi}\log\Big(z+\frac{J}{2Y}\Big) + \frac{2wY}{c}\left(\eta_1+\eta_2\sqrt{1+\frac{J}{Yz}}\right) - \frac{\eta_3 K^2 c}{4\pi^2 wY} \tag{5.12}$$

and $(\eta_1,\eta_2,\eta_3) = (-\mathrm{sgn}(h)\eta, \mathrm{sgn}(h)\eta, -\eta)$.

We consider the case $h>0$, and thus $J>0$ and $(\eta_1,\eta_2,\eta_3)=(-\eta,\eta,-\eta)$. In other words,

$$\phi(z,w,t) = \frac{t}{2\pi}\log\Big(z+\frac{J}{2Y}\Big) + \eta\left(\frac{2wY}{c}\left(\sqrt{1+\frac{J}{Yz}}-1\right)+\frac{K^2c}{4\pi^2 wY}\right). \tag{5.13}$$

For the purpose of this proof, let us denote by $Aw+B/w$ the last bracket in (5.13), i.e.,

$$A = \frac{2Y}{c}\left(\sqrt{1+\frac{J}{Yz}}-1\right), \quad B = \frac{K^2 c}{4\pi^2 Y}.$$

We will express the w-integral into

$$\int_0^\infty \mathcal{F}(w,z)e(\phi(z,w,t))\,dw \tag{5.14}$$

for z fixed, i.e.,

$$\mathcal{F}(w,z) \quad = \quad \frac{1}{w^{j+\lambda}}\overline{H}\left(\frac{|J|w^2}{|h|z}\right)\mathsf{h}_{2\nu,2k-\alpha}\left(\frac{\eta_3 LKc}{2\pi^2 wY}\right), \tag{5.15}$$

and thus

$$\begin{aligned} |h|^{1-s}\widetilde{G}_{h,c}(s) \quad &= \quad 2K|J|\Big(\frac{2Y}{|J|}\Big)^{s} Y^{-(\lambda+j+1)/2}\int_0^\infty \Big(\frac{2z+J/Y}{\sqrt{z+J/Y}}\Big)^{l-1} \\ &\times \Big(z+\frac{J}{Y}\Big)^{-j/4-1/4}\Big(z+\frac{J}{2Y}\Big)^{\sigma-1}H\Big(\frac{|h|}{|J|}z\Big) \\ &\times \int_0^\infty \mathcal{F}(w,z)e(\phi(z,w,t))\,dw\,\frac{dz}{z^{(2l-j+1)/4}}. \end{aligned} \tag{5.16}$$

Due to the factor $H(|h|z/|J|)$, we may restrict the value of $|h|z/|J|$ to the interval $[1,2]$, for otherwise, the integrand in (4.15) vanishes. Hence the support of $\mathcal{F}(w,z)$ as a function of w satisfies $\mathrm{supp}(\mathcal{F}) \subseteq [1,2]$ from the factor $\overline{H}(|J|w^2/(|h|z))$ and $|h|z/J \in [1,2]$. We can write the integral as

$$\int_{1/2}^{5/2} \mathcal{F}(w,z)e(\phi(z,w,t))\,dw.$$

In order to apply Lemma 5.4, we set $\alpha=1/2$, $\beta=5/2$, $P=\beta-\alpha=2$, $N=1\gg B^{-1/2}$, $Q=B$, and $U=K^{3\varepsilon}$.

It is easy to evaluate the derivatives of ϕ. We observe that the differentiation of $\overline{H}$ and $\mathsf{h}_{2\nu,2k-\alpha}$ with respect to w produces factors $\ll (|J|/(|h|z)) \ll 1$ and $\ll LKc/Y \ll K^{\varepsilon}$, respectively. Therefore, we have for $r \geq 2$ and $s \geq 0$,

$$\frac{\partial\phi}{\partial w} = \eta\Big(A - \frac{B}{w^2}\Big)\ , \quad (-1)^r\frac{\partial^r\phi}{\partial w^r} \asymp \frac{\eta B}{w^{r+1}} \quad \text{and} \quad \frac{\partial^s\mathcal{F}}{\partial w^s} \ll K^{s\varepsilon}\ . \tag{5.17}$$

The conditions in Lemma 5.4 are thus verified. Moreover, we have $\frac{\partial\phi}{\partial w}(\gamma) = 0$ when

$$\gamma = \sqrt{\frac{B}{A}} = \frac{1}{\sqrt{2}}\frac{Kc}{2\pi Y}\Big(\sqrt{1+\frac{J}{Yz}}-1\Big)^{-1/2} \asymp \frac{K}{\sqrt{Y}}\sqrt{\frac{C}{T}}. \tag{5.18}$$

Suppose $\gamma \in [3/4, 9/4]$. Then we apply Lemma 5.4 to get

$$\begin{aligned}
&\int_0^\infty \mathcal{F}(w,z)e\big(\phi(z,w,t)\big)\,dw \\
=\ &\frac{\mathcal{F}(\gamma,z)e(\phi(z,\gamma,t)+1/8)}{\sqrt{\frac{\partial^2\phi}{\partial w^2}(z,\gamma,t)}} \\
&+\frac{\mathcal{F}(5/2,z)e(\phi(z,5/2,t))}{2\pi i\frac{\partial\phi}{\partial w}(z,5/2,t)} - \frac{\mathcal{F}(1/2,z)e(\phi(z,1/2,t))}{2\pi i\frac{\partial\phi}{\partial w}(z,1/2,t)} \\
&+O\Big(\frac{P^4U}{Q^2}\Big(1+\frac{P}{N}\Big)^2\Big(\frac{1}{(5/2-\gamma)^3}+\frac{1}{(\gamma-1/2)^3}\Big)\Big) \\
&+O\Big(\frac{PU}{Q^{3/2}}\Big(1+\frac{P}{N}\Big)^2\Big).
\end{aligned} \tag{5.19}$$

The second and third terms in (5.19) vanish because $\mathcal{F}$ vanishes outside $[1,2]$. The first O-term in (5.19) is absorbed in the second one, because $5/2-\gamma \gg 1$ and $\gamma - 1/2 \gg 1$, and because $P = 2$ and

$$Q^{-1} = B^{-1} \ll \frac{Y}{K^2C} \ll T^{-1}.$$

Therefore, it follows from (5.19) that

$$\begin{aligned}
&\int_0^\infty \mathcal{F}(w,z)e\big(\phi(z,w,t)\big)\,dw \\
=\ &\frac{\mathcal{F}(\gamma,z)e(\phi(z,\gamma,t)+1/8)}{\sqrt{\frac{\partial^2\phi}{\partial w^2}(z,\gamma,t)}} + O\Big(\frac{PU}{Q^{3/2}}\Big(1+\frac{P}{N}\Big)^2\Big) \\
=\ &\frac{\gamma^{3/2}\mathcal{F}(\gamma,z)}{\sqrt{2B}}e\Big(\Psi(z,t)+\frac{1}{8}\Big) + O(K^{3\varepsilon}T^{-3/2}),
\end{aligned} \tag{5.20}$$

where

$$\begin{aligned}
\Psi(z,t) &= \frac{t}{2\pi}\log\left(z+\frac{J}{2Y}\right) + 2\eta\sqrt{AB} \\
&= \frac{t}{2\pi}\log\left(z+\frac{J}{2Y}\right) + \eta\frac{\sqrt{2}K}{\pi}\left(\sqrt{1+\frac{J}{Yz}}-1\right)^{1/2}.
\end{aligned} \tag{5.21}$$

Next we want to show that (5.20) remains valid even when $\gamma \notin [3/4, 9/4]$. To this end, we observe that in this case, the main term in (5.20) is void for $\mathrm{supp}(\mathcal{F}) \subseteq [1,2]$. Therefore, it suffices to show the w-integral is absorbed in the O-term of (5.20). This can be seen easily as follows. For all $w \in \mathrm{supp}(\mathcal{F})$, we have $|\gamma - w| \geq 1/4$ and so

$$\left|\frac{\partial \phi}{\partial w}\right| = \left|A - \frac{B}{w^2}\right| = A\left|1 - (\frac{\gamma}{w})^2\right| \gg A \gg \frac{J}{C} \gg T.$$

With (5.17), we show that (5.14) yields to

$$\int_1^2 \frac{\partial}{\partial w}\left(\left(\frac{\partial \phi}{\partial w}\right)^{-1}\frac{\partial}{\partial w}\left(\mathcal{F}(w,z)\left(\frac{\partial \phi}{\partial w}\right)^{-1}\right)\right) e(\phi(z,w,t))\, dw \ll K^{2\varepsilon}T^{-2}$$

with twice integration by parts. Our claim is then justified.

Now we replace the w-integral in (5.16) by (5.20). The contribution of the main term of (5.20) to (5.16) is

$$\begin{aligned} \widetilde{G}(s;h,c) &= 2K|J|\left(\frac{2Y}{|J|}\right)^s Y^{-(\lambda+j+1)/2}\frac{e(1/8)}{\sqrt{2B}}\int_0^\infty \left(\frac{2z+J/Y}{\sqrt{z+J/Y}}\right)^{l-1}\left(z+\frac{J}{Y}\right)^{-j/4-1/4} \\ &\quad \times \left(z+\frac{J}{2Y}\right)^{\sigma-1} H\left(\frac{|h|}{|J|}z\right)\gamma(z)^{3/2}\mathcal{F}(\gamma(z),z)e(\Psi(z,t))\,\frac{dz}{z^{(2l-j+1)/4}} \qquad (5.22)\end{aligned}$$

with $\gamma(z)$ and $\Psi(z,t)$ defined as in (5.18) and (5.21), respectively. Note that from Remark 4(v), the integral with respect to z in (5.16) is

$$\ll \int_{1/2}^2 \left(\frac{2z+J/Y}{\sqrt{z+J/Y}}\right)^{l-1}\left(z+\frac{J}{Y}\right)^{-j/4-1/4}\left(z+\frac{J}{2Y}\right)^{\sigma-1}\left|H\left(\frac{|h|}{|J|}z\right)\right|\frac{dz}{z^{(2l-j+1)/4}} \ll 1,$$

as each factor is $O(1)$ by $J/Y = o(1)$ according to Remark 4(ii). The error term in (5.20) contributes

$$\begin{aligned} &\ll K|J|\left(\frac{Y}{|J|}\right)^\sigma Y^{-(\lambda+j+1)/2}K^{3\varepsilon}T^{-3/2} \\ &\ll K^{1+\epsilon}Y^{\sigma-(\lambda+j+1)/2}T^{-3/2}(CT)^{1-\sigma} \\ &\ll K^\epsilon Y^{\sigma-(\lambda+j)/2}(CT)^{1/2-\sigma}T^{-1/2}\end{aligned}$$

by $K^2C/Y \ll T$. In other words, we have

$$|h|^{1-s}\widetilde{G}_{h,c}(s) - \widetilde{G}(s;h,c) \ll K^\epsilon Y^{\sigma-(\lambda+j)/2}(CT)^{1/2-\sigma}T^{-1/2}.$$

This accounts for (5.11), once we have verified $\widetilde{G}(s;h,c) = G_0(s;h,c)$ as defined in (5.10). To this end, we change variables from z to z^{-2} in (5.22). ¿From (5.18) we get

$$\gamma(\frac{1}{z^2}) = \frac{1}{2\pi}\frac{Kc}{\sqrt{2JY}}\frac{1}{z}\left(\sqrt{1+z^2\frac{J}{Y}}+1\right)^{1/2} = \omega(z)$$

by (5.7), and the z-integral in (5.22) becomes

$$
\begin{aligned}
&-2\int_0^\infty \left(\frac{2}{z^2}+\frac{J}{Y}\right)^{l-1}\left(\frac{1}{z^2}+\frac{J}{Y}\right)^{-(2l+j-1)/4} H\left(\frac{|h|}{|J|z^2}\right) \\
&\qquad \times z^{l-j/2-5/2}\omega(z)^{3/2}\mathcal{F}(\omega(z),z^{-2})\left(\frac{1}{z^2}+\frac{J}{2Y}\right)^{\sigma-1} e(\Psi(z^{-2},t))\,dz \\
&= \quad -2\int_0^\infty \mathcal{F}_{h,c}(z)\left(\frac{1}{z^2}+\frac{J}{2Y}\right)^{\sigma-1} e(\Psi(z^{-2},t))\,dz
\end{aligned}
$$

by (5.15) and (5.9).

Let $\psi(z,t)=\Psi(z^{-2},t)$. Then by (5.21),

$$
\psi(z,t) \quad = \quad \frac{t}{2\pi}\log\left(\frac{1}{z^2}+\frac{J}{2Y}\right)+\eta\frac{K}{\pi}\sqrt{\frac{2J}{Y}}z\left(\sqrt{1+z^2\frac{J}{Y}}+1\right)^{-1/2}.
$$

As $2K|J|Y^{-1/2}e(1/8)/\sqrt{2B}=2\pi(1+i)|J|/\sqrt{c}$, we see that $\widetilde{G}(s;h,c)$ in (5.22) equals

$$
-4\pi(1+i)\frac{|J|}{\sqrt{c}}\left(\frac{2Y}{|J|}\right)^s Y^{-(\lambda+j)/2}\int_0^\infty \mathcal{F}_{h,c}(z)\left(\frac{1}{z^2}+\frac{J}{2Y}\right)^{\sigma-1} e(\psi(z,t))\,dz. \tag{5.23}
$$

Applying integration by parts, the integral in (5.23) equals

$$
-\int_0^\infty \mathcal{F}'_{h,c}(z)\int_{1/2}^z \left(\frac{1}{u^2}+\frac{J}{2Y}\right)^{\sigma-1} e(\psi(u,t))\,du\,dz.
$$

The inner integral is $e_\eta(z,s)$ by (5.8), so $\widetilde{G}(s;h,c)=G_0(s;h,c)$ and our assertion follows.

When $h<0$, we carry out the same analysis but this time, we choose $\eta=\eta_1$ and $\eta_2=-\eta$. The involved case occurs when

$$
\phi(z,w,t)=\frac{t}{2\pi}\log\left(z+\frac{J}{2Y}\right)+\eta\left(\frac{2wY}{c}\left(1-\sqrt{1+\frac{J}{Yz}}\right)+\frac{K^2c}{4\pi^2wY}\right).
$$

Note that $J<0$ now. The proof will be the same as above. □

Lemma 5.6 *Under the same assumptions and notation as in Lemma 5.5, we have, uniformly in $z\in[1/2,2]$, the following estimates:* (i) $\mathcal{F}'_{h,c}(z)\ll K^\varepsilon$ *and* (ii) $e_\eta(z,s)\ll T^{-1/2}$. *Furthermore, we have*

$$
G_0(s;h,c)\ll_\varepsilon K^\varepsilon Y^{\sigma-(\lambda+j)/2}(CT)^{1/2-\sigma}, \tag{5.24}
$$

and hence,

$$
|h|^{1-s}\widetilde{G}_{h,c}(s)\ll_\varepsilon K^\epsilon Y^{\sigma-(\lambda+j)/2}(CT)^{1/2-\sigma}. \tag{5.25}
$$

Proof. From (5.9), (i) follows by direct computation. In fact, by (i) and (ii) in Remark 4, i.e., $Kc/\sqrt{|J|Y} \asymp 1$ and $J/Y = o(1)$, we see from (5.7) that for $z \in (1/2, 2)$, $|\omega(z)| \asymp 1$ is non-vanishing while its derivative $|\omega'(z)|$ is $O(1)$. The differentiation of $\overline{H}$ and $\mathsf{h}_{2\nu,2k-\alpha}$ produce $O(|h|/|J|)$ and $O(LKc/Y)$, which are respectively $O(1)$ and $O(K^{\varepsilon})$.

To prove (ii), we apply Lemma 5.3 (i.e., the second derivative test). Let us write (5.8) as

$$e_\eta(z, s) = \int_{1/2}^{z} \left(\frac{1}{u^2} + \frac{J}{2Y}\right)^{\sigma-1} e(\psi(u, t))\, du$$

where $\psi(u, t)$ is given by

$$\psi(u, t) \;=\; \frac{t}{2\pi} \log\left(\frac{1}{u^2} + \frac{J}{2Y}\right) + \eta\frac{K}{\pi}\sqrt{\frac{2|J|}{Y}}u\left(\sqrt{1 + u^2\frac{J}{Y}} + 1\right)^{-1/2}.$$

The function $\left(u^{-2} + J/(2Y)\right)^{\sigma-1}$ is positive, monotone and bounded away from zero on $[1/2, 2]$ (with $J/Y = o(1)$ again), and its total variation and maximum are $O(1)$. It remains to evaluate the second derivative $\frac{\partial^2}{\partial u^2}\psi(u, t)$ for $u \in (1/2, 2)$. As $\left(\sqrt{1 + z} + 1\right)^{-1/2}$ is a holomorphic function on $|z| < 1/2$, we can expand it into power series $\sum_{r\geq 0} c_r z^r$ and hence

$$\frac{d^2}{dz^2}\left(z\left(\sqrt{1 + z^2} + 1\right)^{-1/2}\right) = \sum_{r\geq 1}(2r + 1)(2r)c_r z^{2r-1}$$

which converges absolutely on $|z| \leq 1/2$ and is $\ll |z|$. Let us take $z = \sqrt{J/Y}u$ where $\sqrt{J/Y} = i\sqrt{|J|/Y}$ for $J < 0$. As $J/Y = o(1)$, we get with the chain rule,

$$\frac{d^2}{du^2}\left(u\left(\sqrt{1 + u^2\frac{J}{Y}} + 1\right)^{-1/2}\right) = \left(\sqrt{\frac{J}{Y}}\right)^{-1+2} O\left(\sqrt{\frac{|J|}{Y}}\right) \ll \frac{|J|}{Y}$$

for $1/2 < u < 2$. Thus, by $|t| \asymp T$ and Remark 4(ii) and (iii),

$$\frac{\partial^2}{\partial u^2}\psi(u, t) = \frac{t}{\pi}\frac{1 + 3Ju^2/(2Y)}{(u + Ju^3/(2Y))^2} + O\left(K\left(\frac{|J|}{Y}\right)^{3/2}\right) \gg T\left(1 - \frac{K^{2\varepsilon}}{L^2}\right) \gg T.$$

It then follows from Lemma 5.3 that

$$e_\eta(z, s) \ll T^{-1/2}.$$

To get (5.24), we use (5.10), (i), and (ii) with $|J| = CT$. Then

$$G_0(s; h, c) \ll C^{-1/2}|J|^{1-\sigma}Y^{\sigma-(\lambda+j)/2}K^{\varepsilon}T^{-1/2} \ll K^{\varepsilon}Y^{\sigma-(\lambda+j)/2}(CT)^{1/2-\sigma}.$$

Finally (5.25) follows from (5.11) and the fact that $T \geq K^{\varepsilon}$ by (5.6). $\square$

Remark 6. The estimates (i) and (ii) motivate the current form (rather than (5.23)) of the z-integral in $G_0(s;h,c)$. In later applications of the spectral large sieve, we shall use the pointwise bounds for the h and s parts of the integrand, which are now $O(K^{\varepsilon})$ and $O(T^{-1/2})$, respectively. These are only $O(1)$ in (5.23). This will result in a saving of $T^{1/2}$.

6. A reduction process

Now we go back to $P(c,h,Y)$ as given by (4.12). The tail part of the integral in $P(c,h,Y)$ is negligible, because $D_g(s,1,1,h) \ll_{\varepsilon} 1$ for $\sigma = 1+\varepsilon$ and by Lemma 5.2(1),

$$\begin{aligned} &\frac{1}{2\pi}\int_{|t|\gg Y} D_g(\sigma+it,1,1,h)\widetilde{G}_{h,c}(\sigma+it)\,dt \\ \ll\ & KY^{\sigma-(\lambda+j+1)/2}\int_{|t|\gg Y} |t|^{-M'}\,dt \ll KY^{\sigma-(\lambda+j-1)/2-M'}. \end{aligned}$$

Note that this last expression is negligible because $Y \geq LK^{1-\varepsilon}$ by (4.8). Hence we only need to evaluate the integral over $|t| \ll Y$.

Next we separate the situation into seven cases.

(i) $\eta_1 = \eta_2$.

(ii) $\eta_1 \neq \eta_2$, $c \leq Y/K^{2-\varepsilon}$, and $|h| \geq cK^{\varepsilon}$.

(iii) $\eta_1 \neq \eta_2$, $c \leq Y/K^{2-\varepsilon}$, and $|h| \leq cK^{\varepsilon}$.

(iv) $\eta_1 \neq \eta_2$, $c \geq Y/K^{2-\varepsilon}$, and $\eta_3 = \mathrm{sgn}(h)\eta_2$.

(v) $\eta_1 \neq \eta_2$, $c \geq Y/K^{2-\varepsilon}$, $\eta_3 \neq \mathrm{sgn}(h)\eta_2$ (hence $(\eta_1,\eta_2,\eta_3) = (-\mathrm{sgn}(h)\eta, \mathrm{sgn}(h)\eta, -\eta)$), and $|h| \leq \delta_0 K^2c^2/Y$, where δ_0 is the absolute constant given in Lemma 5.1(4).

(vi) $\eta_1 \neq \eta_2$, $c \geq Y/K^{2-\varepsilon}$, $\eta_3 \neq \mathrm{sgn}(h)\eta_2$ (hence $(\eta_1,\eta_2,\eta_3) = (-\mathrm{sgn}(h)\eta, \mathrm{sgn}(h)\eta, -\eta)$), and $|h| \geq \delta_0^{-1}K^2c^2/Y$.

(vii) $\eta_1 \neq \eta_2$, $c \geq Y/K^{2-\varepsilon}$, $\eta_3 \neq \mathrm{sgn}(h)\eta_2$ (hence $(\eta_1,\eta_2,\eta_3) = (-\mathrm{sgn}(h)\eta, \mathrm{sgn}(h)\eta, -\eta)$), and $\delta_0K^2c^2/Y \leq |h| \leq \delta_0^{-1}K^2c^2/Y$.

First we point out that Cases (i), (ii), and (iv) are negligible according to Lemma 5.1(1), (2), and (3), respectively. Cases (v) and (vi) are negligible by Lemma 5.1(4). Therefore, the only nontrivial cases are (iii) and (vii). We can then reduce (4.11) to

$$\widetilde{T}^{(\eta)}_{\underline{\lambda},j}(Y) = \frac{Y^{1/2-(\lambda+j)/2}K^{4\beta-\alpha}}{L^{2\nu+\alpha-2k-1}}\left(\sum{}' + \sum{}''\right) + O(K^{-M})$$

where $\sum'$ and $\sum''$ are sums corresponding to Cases (iii) and (vii), respectively:

$$\sum' = \sum_{\substack{1\le c\le Y/K^{2-\varepsilon}\\ \mathcal{N}|c}} c^{j+\lambda} \sum_{\delta|c} \frac{\mu(c/\delta)}{c/\delta} \sum_{\substack{|h|<cK^{\varepsilon}\\ \delta|h}} P(c,h,Y),$$

$$\sum'' = \sum_{\substack{Y/K^{2-\varepsilon}\le c\le Y/(LK^{1-\varepsilon})\\ \mathcal{N}|c}} c^{j+\lambda} \sum_{\delta|c} \frac{\mu(c/\delta)}{c/\delta} \sum_{\substack{|h|\asymp K^2c^2/Y\\ \delta|h}} P(c,h,Y).$$

The sum $\sum'$ can be treated in a trivial way. When $|h|/c < K^{\varepsilon}$, we have by Lemma 5.2(1) with M'/ε in place of M',

$$\begin{aligned} &\frac{1}{2\pi}\int_{K^{3\varepsilon}<|t|\ll Y} D_g(\sigma+it,1,1,h)\widetilde{G}_{h,c}(\sigma+it)\,dt \\ \ll_{M',\varepsilon}\quad & KY^{\sigma-(\lambda+j+1)/2}\int_{|t|>K^{3\varepsilon}} |t|^{-M'/\varepsilon}\,dt \ll_{M',\varepsilon} K^{2-3M'}Y^{\sigma-(\lambda+j+1)/2}, \end{aligned}$$

which is negligible according to (4.8). On the other hand, by (5.1),

$$\frac{1}{2\pi}\int_{|t|\le K^{3\varepsilon}} D_g(\sigma+it,1,1,h)\widetilde{G}_{h,c}(\sigma+it)\,dt \ll K^{1+3\varepsilon}Y^{(1-\lambda-j)/2+\varepsilon}$$

for $\sigma = 1+\varepsilon$. Hence, $P(c,h,Y) \ll K^{1+3\varepsilon}Y^{(1-\lambda-j)/2+\varepsilon}$ for $|h| < cK^{\varepsilon}$.

Consequently, as $Y/K^{2-\varepsilon} \ll K^{2\varepsilon}$, the sum over c in this range is

$$\begin{aligned} \sum' \quad &\ll \quad \sum_{1\le c\le Y/K^{2-\varepsilon}} c^{j+\lambda}\sum_{\delta|c}\frac{1}{c/\delta}\sum_{\substack{|h|\le cK^{\varepsilon}\\ \delta|h}} K^{1+3\varepsilon}Y^{(1-\lambda-j)/2+\varepsilon} \\ &\ll \quad K^{1+4\varepsilon}Y^{(1-\lambda-j)/2}\sum_{1\le c\le Y/K^{2-\varepsilon}} c^{j+\lambda+\varepsilon}. \end{aligned}$$

Note that $j+\lambda$ may be positive, zero, or negative. Thus,

$$\begin{aligned} \sum' \quad &\ll \quad \delta_{(Y\ge K^{2-\varepsilon})}K^{1+4\varepsilon}Y^{(1-\lambda-j)/2}\max\Big\{1,\Big(\frac{Y}{K^{2-\varepsilon}}\Big)^{j+\lambda}\Big\}\sum_{1\le c\le Y/K^{2-\varepsilon}} c^{\varepsilon} \\ &\ll \quad \delta_{(Y\ge K^{2-\varepsilon})}K^{1+\epsilon}Y^{1/2}\max\Big\{\Big(\frac{1}{\sqrt{Y}}\Big)^{j+\lambda},\Big(\frac{\sqrt{Y}}{K^{2-\varepsilon}}\Big)^{j+\lambda}\Big\}, \end{aligned} \tag{6.1}$$

where $\delta_{(Y\ge K^{2-\varepsilon})} = 1$ if $Y \ge K^{2-\varepsilon}$, and 0 otherwise. Here we used the fact that $Y/K^{2-\varepsilon} \le K^{2\varepsilon}$ by (4.8).

In case of $\sum''$, we have $|h|/c \gg K^2c/Y \ge K^{\varepsilon}$. Clearly from Lemma 5.2, only the integral of $P(c,h,Y)$ over an interval about $|h|/c$ is the critical part.

We divide dyadically the summation ranges of c and $|h|$. It suffices to evaluate the sum

$$\sum''(C,T) = \sum_{\substack{c\sim C\\ \mathcal{N}|c}} c^{j+\lambda}\sum_{\delta|c}\frac{\mu(c/\delta)}{c/\delta}\sum_{\substack{|h|\sim CT\\ \delta|h}} P(c,h,Y)$$

where

$$1 \le C \le Y/(LK^{1-\varepsilon}) \quad \text{and} \quad T \asymp K^2C/Y \ge K^{\varepsilon}. \tag{6.2}$$

To be specific, the range of T we need is from $\delta_0 K^2C/Y$ to $2\delta_0^{-1}K^2C/Y$. We impose the condition $K^2C/Y \ge K^{\varepsilon}$ because $c \ge Y/K^{2-\varepsilon}$. Together with (6.1), we have

$$\begin{aligned} \widetilde{T}^{(\eta)}_{\underline{\lambda},j}(Y) \ \ll\ & \frac{Y^{-(\lambda+j)/2}K^{4\beta-\alpha}}{L^{2\nu+\alpha-2k}} LY^{1/2} \max_{C,T} \sum{}''(C,T) \\ & + \delta_{(Y\ge K^{2-\varepsilon})} \frac{Y^{-(j+\lambda)/2}K^{4\beta-\alpha}}{L^{2\nu+\alpha-2k}} LYK^{1+\epsilon} \\ & \times \max\Big\{ \Big(\frac{1}{\sqrt{Y}}\Big)^{j+\lambda}, \Big(\frac{\sqrt{Y}}{K^{2-\varepsilon}}\Big)^{j+\lambda} \Big\}, \end{aligned} \tag{6.3}$$

where the maximum takes C and T over the ranges specified above.

Let $c_0 = \delta_1/2$ and $c_0' = 2\delta_1^{-1}$, where δ_1 is given in Lemma 5.2. Since $T/2 \le |h|/c \le 2T$, we see that $|t| \ge c_0'T$ and $|t| \le c_0 T$ imply respectively $|t| \ge \delta_1^{-1}|h|/c$ and $|t| \le \delta_1 |h|/c$. As $|h|/c \gg T \gg K^{\varepsilon}$, we can apply Lemma 5.2(1) and (2), with $(2M' + M_0')/\varepsilon$ in place of M', to get the following estimation uniformly in $-A_0 \le \sigma \le 2$,

$$\begin{aligned} \widetilde{G}_{h,c}(\sigma+it) &\ll KY^{\sigma-(\lambda+j+1)/2}|t|^{-(2M'+M_0')/\varepsilon} \\ &\ll K^{-2M'} && \text{if } |t| \ge c_0'T, \\ \widetilde{G}_{h,c}(\sigma+it) &\ll KY^{\sigma-(\lambda+j+1)/2}(|h|/c)^{-(2M'+M_0')/\varepsilon} \\ &\ll K^{-2M'} && \text{if } |t| \le c_0 T, \end{aligned} \tag{6.4}$$

for any $M' \ge 1$, where the implied constants depend on A_0, M, M', ε. Here $A_0 > 0$ is any fixed number and $M_0' = M_0'(\varepsilon, M)$ is selected so that $K^{-M_0'}$ suppresses $KY^{\sigma-(\lambda+j+1)/2}$. We obtain for $\sigma = 1+\varepsilon$ and for any $M' > 1$,

$$\frac{1}{2\pi}\int_{|t|<c_0T} D_g(\sigma+it,1,1,h)\widetilde{G}_{h,c}(\sigma+it)\,dt \ll K^{-2M'}\int_{|t|<c_0T} dt \ll_{M,\varepsilon} K^{1-2M'},$$

as $T \ll K^{1+\varepsilon}/L \ll K$, and

$$\frac{1}{2\pi}\int_{c_0'T<|t|\ll Y} D_g(\sigma+it,1,1,h)\widetilde{G}_{h,c}(\sigma+it)\,dt \ll K^{-2M'}\int_{c_0'T<|t|\ll Y} dt \ll K^{2+\varepsilon-2M'},$$

by $Y \le K^{2+\varepsilon}$ and $T \gg K^{\varepsilon}$. (The integral over $|t| \gg Y$ was treated at the beginning of this section.)

Thus these integrals are negligible according to (4.8). Then we may write

$$P(c,h,Y) = \frac{1}{2\pi i}\int_{\ell} D_g(s,1,1,h)\widetilde{G}_{h,c}(s)\,ds + O(K^{-M'}) \tag{6.5}$$

for any $M' > 2$, where the implied constant in the O-term depends on M'. Here $\ell = \ell_- \cup \ell_+$ is the (disconnected) path consisting of two disjoint vertical line segments ℓ_- from $1+\varepsilon - ic_0'T$ to $1+\varepsilon - ic_0T$ and ℓ_+ from $1+\varepsilon + ic_0T$ to $1+\varepsilon + ic_0'T$.

Inserting the spectral decomposition (3.16) into (6.5), we deduce that

$$P(c,h,Y) = P_R(c,h,Y) + P_d(c,h,Y) + P_E(c,h,Y) + O(K^{-M'}), \tag{6.6}$$

where

$$\begin{aligned} P_R(c,h,Y) &= \frac{1}{2\pi i}\int_\ell R_h(s)\widetilde{G}_{h,c}(s)\,ds, & (6.7)\\ P_d(c,h,Y) &= \sum_{j:\,0<t_j\le 2T} \overline{\langle V,\phi_j\rangle}\frac{1}{2\pi i}\int_\ell B_j(s)\frac{\overline{\rho_j(-h)}}{|h|^{s-1/2}}\widetilde{G}_{h,c}(s)\,ds, & (6.8)\\ P_E(c,h,Y) &= \frac{1}{4\pi}\sum_{\mathfrak{a}}\int_{-2T}^{2T}\frac{\overline{\langle V,E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle}}{\Gamma(1/2-i\tau)}\,d\tau \\ &\qquad \times\frac{1}{2\pi i}\int_\ell C_{\mathfrak{a}}(s,\tau)\frac{\overline{\rho_{\mathfrak{a}}(1/2+i\tau,-h)}}{|h|^{s-1/2+i\tau}}\widetilde{G}_{h,c}(s)\,ds. & (6.9) \end{aligned}$$

Note that the contribution of the last term in (3.16) is absorbed in $O(K^{-M'})$. In fact,

$$|h|^{1/2-\sigma+\theta+\varepsilon}e^{-T/4}KY^{\sigma-(\lambda+j+1)/2}\int_\ell ds \ll |h|^{1/2-\sigma+\theta+\varepsilon}Te^{-T/4}KY^{\sigma-(\lambda+j+1)/2}. \tag{6.10}$$

As $T \gg K^\varepsilon$, $T \ll K$, and $|h| \le Y/2$, the right side of (6.10) is $\ll$ bounded by the product of powers of K and Y times $e^{-K^\varepsilon/4}$, and hence negligible.

Correspondingly, we now can decompose

$$\textstyle\sum''(C,T) = \sum''_R(C,T) + \sum''_d(C,T) + \sum''_E(C,T) + O(K^{-M'}) \tag{6.11}$$

where for $* = R$, d or E. Explicitly, we have

$$\begin{aligned} \textstyle\sum''_*(C,T) &:= \sum_{\substack{c\sim C\\ \mathcal{N}|c}} c^{j+\lambda}\sum_{\delta|c}\frac{\mu(c/\delta)}{c/\delta}\sum_{\substack{|h|\sim CT\\ \delta|h}} P_*(c,h,Y)\\ &= \sum_{|h|\sim CT}\sum_{\delta|h}\delta\sum_{\substack{c\sim C\\ \mathcal{N}|c,\,\delta|c}} c^{j+\lambda}\frac{\mu(c/\delta)}{c}P_*(c,h,Y) & (6.12)\end{aligned}$$

by interchanging the order of summation.

In the next three sections we will evaluate each $\sum''_*(C,T)$ for $* = R, d, E$ in (6.11). We recall again the values of the parameters due to (4.8) and (6.2),

$$\begin{aligned} LK^{1-\varepsilon} \le Y \le K^{2+\varepsilon}\ , &\quad K^{1/3+2\varepsilon} \le L \le K^{1-2\varepsilon}\ ,\\ 1 \le C \le Y/(LK^{1-\varepsilon})\ , &\quad T \asymp K^2C/Y \ge K^\varepsilon\ , \end{aligned}$$

from whence, we have $C \ll K^{\varepsilon} T$ and both $C, T \ll K^{1+\epsilon}/L$. (The symbol ϵ is explained in Notation at the end of Section 1.)

7. The discrete spectrum $\lambda_j \leq 1/4$

Using the holomorphicity of $R_h(s)$ (in the complex plane omitting $\mathbb{R}$) and $\widetilde{G}_{h,c}(s)$, we apply Cauchy's theorem to shift horizontally the vertical segments $\ell_{\pm}$ to $\Re\mathrm{e}\, s = -A_0$. The integral over horizontal line segments are $\ll K^{-M'}$, by (6.4). Up to an error $O(K^{-M'})$, we have by (3.18), (5.1) and (6.7),

$$\begin{aligned} P_R(c, h, Y) \qquad &\ll_{A_0} |h|^{1/2+A_0+\theta+\varepsilon} T K Y^{-A_0-(\lambda+j+1)/2} \\ &\ll_{A_0} T K^{1+\epsilon} Y^{-A_0-(\lambda+j+1)/2} |h|^{1+A_0}. \end{aligned}$$

Here we have used the trivial bound $\theta = 1/2$ and $|h|^{\varepsilon} \sim (CT)^{\varepsilon} \ll K^{\epsilon}$. Consequently, we deduce from (6.12) that

$$\begin{aligned} {\textstyle\sum}''_R(C, T) \quad &\ll_{A_0} \quad T K^{1+\epsilon} Y^{-A_0-(\lambda+j+1)/2} \sum_{\substack{c \sim C \\ \mathcal{N} | c}} c^{j+\lambda} \sum_{\delta | c} \frac{\delta}{c} \sum_{\substack{|h| \sim CT \\ \delta | h}} |h|^{1+A_0} \\ &\ll_{A_0} \quad T K^{1+\epsilon} Y^{-A_0-(\lambda+j+1)/2} C^{j+\lambda} (CT)^{2+A_0} \sum_{c \sim C} c^{\varepsilon-1} \end{aligned}$$

by evaluating the two inner sums, using $\sum_{\delta | c} 1 \ll c^{\varepsilon}$. Then

$$\begin{aligned} {\textstyle\sum}''_R(C, T) \quad &\ll_{A_0} \quad C^{\varepsilon} K^{1+\epsilon} Y^{1/2} C T^2 \left(\frac{CT}{Y}\right)^{A_0+1} \left(\frac{C}{\sqrt{Y}}\right)^{j+\lambda} \\ &\ll_{A_0} \quad K^{1+\epsilon} Y^{1/2} \left(\frac{K}{L}\right)^3 \left(\frac{CT}{Y}\right)^{A_0+1} \left(\frac{C}{\sqrt{Y}}\right)^{j+\lambda} \end{aligned}$$

by $C, T \ll K^{1+\epsilon}/L$ and absorbing C^{ϵ} and the ϵ-power of K from CT^2 in $K^{1+\epsilon}$. Using $CT/Y \leq K^2 C^2/Y^2 \ll K^{2\varepsilon}/L^2$, we conclude that

$$ {\textstyle\sum}''_R(C, T) \quad \ll_{A_0} \quad K^{1+\epsilon} Y^{1/2} \left(\frac{K^{2\varepsilon+3/(2A_0+5)}}{L}\right)^{2A_0+5} \left(\frac{C}{\sqrt{Y}}\right)^{j+\lambda}. \tag{7.1}$$

For our purpose, it suffices to take $A_0 = 2$ in (7.1).

For the other two $\sum''_*(C, T)$, the path of integration ℓ will be shifted horizontally to left beyond $\Re\mathrm{e}\, s = 1/2$. There are contributions due to the poles of $B_j(s)$ and $C_{\mathfrak{a}}(s, \tau)$. Let us focus firstly on $\sum''_d(C, T)$.

8. The discrete spectrum $\lambda_j > 1/4$

In this case $t_j \in (0, \infty)$. We move $\ell = \ell_- \cup \ell_+$ horizontally to $\ell' = \ell'_- \cup \ell'_+$ whose real part equals $1/2 + \varepsilon$. Note that no pole is encountered during this shifting and the integral over the horizontal line segments is negligible and indeed $\ll K^{l+4-2M'}$, as explained below. Let us write T' for $\pm c_0 T$ or $\pm c'_0 T$. The integration over a horizontal line segment $[1/2 + \varepsilon + iT', 1 + \varepsilon + iT']$ is, by (6.8),

$$\sum_{j:\, 0<t_j \le 2T} \overline{\langle V, \phi_j \rangle} \frac{1}{2\pi i} \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} B_j(s') \frac{\overline{\rho_j(-h)}}{|h|^{s'-1/2}} \widetilde{G}_{h,c}(s')\, d\sigma, \tag{8.1}$$

where $s' = \sigma + iT'$. By (2.3), $\rho_j(-h) \ll |h|^{1/2}(|h|t_j)^{\varepsilon} e^{\pi t_j/2}$ where, again, we have used only $\theta = 1/2$, and by (3.19), the function $B_j(s')$ is $\ll T$. With (3.7) and Cauchy-Schwarz's inequality, it follows that

$$\begin{aligned} &\sum_{0<t_j \le 2T} B_j(s') \overline{\rho_j(-h) \langle V, \phi_j \rangle} \\ \ll\ & T|h|^{1/2}(|h|T)^{\varepsilon} \Bigg(\sum_{0<t_j \le 2T} 1 \Bigg)^{1/2} \Bigg(\sum_{0<t_j \le 2T} |\langle V, \phi_j \rangle|^2 e^{\pi t_j} \Bigg)^{1/2} \\ \ll\ & |h|^{1/2+\varepsilon} T^{l+2+\varepsilon}. \end{aligned}$$

This is crudely $\ll K^{l+4}$ for $|h| \sim CT$ and $C, T \le K$. Hence the integration over horizontal line segments gives a term $O(K^{l+4-2M'})$, by (6.4).

The shifting of path from ℓ to ℓ' yields

$$\begin{aligned} P_d(c,h,Y) &= \sum_{j:\, 0<t_j \le 2T} \overline{\langle V, \phi_j \rangle} \frac{1}{2\pi i} \int_{\ell'} B_j(s) \frac{\overline{\rho_j(-h)}}{|h|^{s-1/2}} \widetilde{G}_{h,c}(s)\, ds + O(K^{l+4-2M'}) \\ &= P_{d,0}(c,h,Y) + P_{d,1}(c,h,Y) + O(K^{l+4-2M'}) \end{aligned}$$

where for $r = 0, 1$,

$$P_{d,r}(c,h,Y) = \sum_{j:\, 0<t_j \le 2T} \overline{\langle V, \phi_j \rangle} \frac{1}{2\pi i} \int_{\ell'} B_j(s) \frac{\overline{\rho_j(-h)}}{|h|^{1/2}} G_r(s;h,c)\, ds \tag{8.2}$$

and $G_0(s;h,c) + G_1(s;h,c) = |h|^{1-s} \widetilde{G}_{h,c}(s)$, as defined in (5.10). We insert the formula before (8.2) for $P_d(c,h,Y)$ into (6.12), and write correspondingly,

$$\textstyle\sum''_d(C,T) = \sum''_d(C,T)_0 + \sum''_d(C,T)_1 + O(K^{-M'}) \tag{8.3}$$

where for $r = 0, 1$,

$$\textstyle\sum_d''(C,T)_r = \displaystyle\sum_{|h|\sim CT} \sum_{\delta|h} \delta \sum_{\substack{c\sim C \\ \mathcal{N}|c,\, \delta|c}} c^{j+\lambda} \frac{\mu(c/\delta)}{c} P_{d,r}(c,h,Y) \tag{8.4}$$

by (8.2) and (6.12). Here M' is any sufficiently large number up to our disposal.

We introduce, for simplicity, the sums

$$S_{h,r}(s) \quad := \quad \sum_{\delta|h} \delta \sum_{\substack{c\sim C \\ \mathcal{N}|c,\, \delta|c}} c^{j+\lambda} \frac{\mu(c/\delta)}{c} G_r(s;h,c) \tag{8.5}$$

for $r = 0, 1$, and remark the following estimates for later uses. By Lemmas 5.5 and Lemma 5.6, we have the upper estimates $G_r(s;h,c) \ll K^\epsilon Y^{\sigma-(\lambda+j)/2}(CT)^{1/2-\sigma}T^{-r/2}$ and thus for $r = 0, 1$,

$$\begin{aligned} S_{h,r}(s) \quad &\ll \quad K^\epsilon Y^{\sigma-(\lambda+j)/2}(CT)^{1/2-\sigma}T^{-r/2} \sum_{\delta|h} \delta \sum_{\substack{c\sim C \\ \delta|c}} c^{j+\lambda-1} \\ &\ll \quad K^\epsilon T^{-r/2} Y^\sigma (CT)^{1/2-\sigma} \left(\frac{1}{\sqrt{Y}}\right)^{j+\lambda} \sum_{\delta|h} \delta \cdot \frac{C^{j+\lambda}}{\delta} \\ &\ll \quad \tau(|h|) K^\epsilon T^{-r/2} Y^\sigma (CT)^{1/2-\sigma} \left(\frac{C}{\sqrt{Y}}\right)^{j+\lambda}, \end{aligned} \tag{8.6}$$

where $\tau(n) = \sum_{1\le d|n} 1$ denotes the divisor function. The case for $r = 1$ is easier, as $S_{h,1}(s)$ has an extra saving $T^{1/2}$ over $S_{h,0}(s)$.

Inserting (8.2) into (8.4), we obtain, after moving the summations over $|h|$, δ and c into the integral, that for $r = 0, 1$,

$$\begin{aligned} \textstyle\sum_d''(C,T)_r \quad &= \quad \frac{1}{2\pi i} \int_{\ell'_- \cup \ell'_+} \sum_{j:\, 0<t_j\le 2T} \overline{\langle V, \phi_j\rangle} B_j(s) \sum_{|h|\sim CT} \frac{\overline{\rho_j(-h)}}{|h|^{1/2}} S_{h,r}(s)\, ds \\ &= \quad \sum_{j:\, 0<t_j\le 2T} \overline{\langle V, \phi_j\rangle} \frac{1}{2\pi i} \int_{\ell'_- \cup \ell'_+} B_j(s) \sum_{|h|\sim CT} \frac{\overline{\rho_j(-h)}}{|h|^{1/2}} S_{h,r}(s)\, ds. \end{aligned} \tag{8.7}$$

We apply the Cauchy-Schwarz inequality to the integral to remove $B_j(s)$ from the sum over t_j. Note that $\Re\mathrm{e}\, s = 1/2 + \varepsilon$ for $s \in \ell'$ and by (3.20),

$$\int_{\ell'_\pm} |B_j(s)|^2\, |ds| \ll T^{1-2l}.$$

The integral in (8.7) is

$$\ll T^{1/2-l} \left(\int_{\ell'_- \cup \ell'_+} \left| \sum_{|h|\sim CT} \frac{\overline{\rho_j(-h)}}{|h|^{1/2}} S_{h,r}(s) \right|^2 |ds| \right)^{1/2}.$$

As the integral in the bracket will appear several times later, we introduce the notation

$$\mathcal{I}_r(\sigma,\rho) = \int_{\ell_-(\sigma)\cup\ell_+(\sigma)} \Big| \sum_{|h|\sim CT} \frac{\overline{\rho(-h)}}{|h|^{1/2}} S_{h,r}(s) \Big|^2 |ds| \tag{8.8}$$

where $\ell_\pm(\sigma)$ denote two vertical line segments $\sigma \pm it$ with $t \asymp T$.

We infer that for $r=1$,

$$\begin{aligned} \textstyle\sum_d''(C,T)_1 &\ll T^{1/2-l} \sum_{j:\, 0<t_j\le 2T} |\langle V,\phi_j\rangle| \mathcal{I}_1\Big(\frac{1}{2}+\varepsilon,\rho_j\Big)^{1/2} \\ &\ll T^{1/2-l}\Big(\sum_{0<t_j\le 2T} |\langle V,\phi_j\rangle|^2 e^{\pi t_j}\Big)^{1/2} \Big(\sum_{0<t_j\le 2T} e^{-\pi t_j} \mathcal{I}_1\Big(\frac{1}{2}+\varepsilon,\rho_j\Big)\Big)^{1/2} \end{aligned} \tag{8.9}$$

by Cauchy-Schwarz's inequality. Now we are in a position to apply Good's estimate (3.7) and the spectral large sieve inequality (2.5) in Lemma 2.1. The first bracket over t_j in (8.9) is $\ll T^{2l}$ by (3.7).

We now evaluate the second bracket in (8.9) for $r=0,1$ by Lemma 2.1, for $|\sigma|\le 2$,

$$\begin{aligned} \sum_{0<t_j\le 2T} e^{-\pi t_j}\mathcal{I}_r(\sigma,\rho_j) &\ll \sum_{0<t_j\le 2T} \frac{1}{\cosh \pi t_j} \int_{\ell_-(\sigma)\cup\ell_+(\sigma)} \Big| \sum_{|h|\sim CT} \frac{\overline{\rho_j(-h)}}{|h|^{1/2}} S_{h,r}(s)\Big|^2 |ds| \\ &\ll \int_{\ell_-(\sigma)\cup\ell_+(\sigma)} \sum_{0<t_j\le 2T} \frac{1}{\cosh \pi t_j} \Big| \sum_{|h|\sim CT} \rho_j(h) \frac{\overline{S_{-h,r}(s)}}{|h|^{1/2}}\Big|^2 |ds|. \end{aligned}$$

By (2.5), we have

$$\sum_{0<t_j\le 2T} e^{-\pi t_j}\mathcal{I}_r(\sigma,\rho_j) \ll (CT)^{\varepsilon}(T^2+CT) \int_{\ell_-(\sigma)\cup\ell_+(\sigma)} \sum_{|h|\sim CT} \frac{|S_{h,r}(s)|^2}{|h|} |ds|.$$

By (8.6), we then get

$$\sum_{0<t_j\le 2T} e^{-\pi t_j}\mathcal{I}_r(\sigma,\rho_j) \ll T^2\cdot T\cdot K^{\epsilon} T^{-r} Y^{2\sigma} (CT)^{1-2\sigma} \Big(\frac{C}{\sqrt{Y}}\Big)^{2(j+\lambda)} \sum_{|h|\sim CT} \frac{\tau(|h|)^2}{|h|},$$

where one factor of $O(T)$ comes from the line integral over $\ell_-(\sigma)\cup\ell_+(\sigma)$. By the estimate $\sum_{1\le h\ll X} \tau(h)^2/h \ll X^{\varepsilon}$,

$$\sum_{0<t_j\le 2T} e^{-\pi t_j}\mathcal{I}_r(\sigma,\rho_j) \ll K^{\epsilon} T^{3-r} Y^{2\sigma} (CT)^{1-2\sigma} \Big(\frac{C}{\sqrt{Y}}\Big)^{2(j+\lambda)}, \tag{8.10}$$

where we have absorbed the ε powers of T and CT into K^{ϵ}.

When $r=1$ and $\sigma=1/2+\varepsilon$, we get

$$\sum_{0<t_j\le 2T} e^{-\pi t_j}\mathcal{I}_1\Big(\frac12+\varepsilon,\rho_j\Big)\ll K^\epsilon T^2 Y\Big(\frac{C}{\sqrt{Y}}\Big)^{2(j+\lambda)}.$$

It follows that

$$\begin{aligned}\textstyle\sum_d''(C,T)_1 &\ll T^{1/2-l}\cdot T^l\cdot K^\epsilon T Y^{1/2}\Big(\frac{C}{\sqrt{Y}}\Big)^{j+\lambda}\\ &\ll K^\epsilon Y^{1/2}T^{3/2}\Big(\frac{C}{\sqrt{Y}}\Big)^{j+\lambda}.\end{aligned}$$

As $T\ll K^{1+\epsilon}/L$, we conclude

$$\textstyle\sum_d''(C,T)_1\ll K^{1+\epsilon}Y^{1/2}\Big(\frac{K}{L^3}\Big)^{1/2}\Big(\frac{C}{\sqrt{Y}}\Big)^{j+\lambda}. \tag{8.11}$$

The above treatment can be applied to $\sum_d''(C,T)_0$ but due to the extra $T^{1/2}$ in $S_{h,0}(s)$, we obtain only the upper estimate

$$O\Bigg(K^{1+\epsilon}Y^{1/2}\frac{K}{L^2}\Big(\frac{C}{\sqrt{Y}}\Big)^{j+\lambda}\Bigg)$$

which only admits the choice $L\ge K^{1/2}$, not sufficient for us. We will carry out a more delicate analysis to avoid the loss from the removal of $B_j(s)$ with Cauchy-Schwarz's inequality. In order to do so, we shift the path ℓ' further to the left.

¿From (3.4) (with $\nu_1=\nu_2=1$), each $B_j(s)$ ($0<t_j\le 2T$) has two simple poles at $s=1/2\pm it_j$ in the strip $-3/2+\varepsilon\le\sigma\le 2$. Clearly, we may assume no t_j equal to c_0T or $c_0'T$ by a small perturbation (of magnitude ε). By the fact that the residue of $\Gamma\big((s-1/2-\nu it_j)/2\big)$ at $s=1/2+\nu it_j$ is 2 for $\nu=\pm$, we obtain

$$\begin{aligned}P_{d,0}(c,h,Y) &= 2^{l-3/2}\pi^{l-1/2}\sum_{\nu=\pm}\\ &\quad\times\sum_{j:\,0<t_j\le 2T}\frac{2^{\nu it_j}\Gamma(\nu it_j)\overline{\langle V,\phi_j\rangle}}{\Gamma(l-1/2+\nu it_j)}\frac{\overline{\rho_j(-h)}}{|h|^{1/2}}G_0\Big(\frac12+\nu it_j;h,c\Big)\\ &\quad+\sum_{j:\,0<t_j\le 2T}\overline{\langle V,\phi_j\rangle}\frac{1}{2\pi i}\int_{\ell''}B_j(s)\frac{\overline{\rho_j(-h)}}{|h|^{1/2}}G_0(s;h,c)\,ds\\ &\quad+O(K^{l+6-2M'})\end{aligned} \tag{8.12}$$

where $\ell''=\ell''_-\cup\ell''_+$ is the union of the two vertical line segments joining $-1/2-ic_0'T$ to $-1/2-ic_0T$, and $-1/2+ic_0T$ to $-1/2+ic_0'T$. The O-term accounts for the contribution of the integration over horizontal line segments, following from the same treatment as in (8.1). This time the integral is taken from $-1/2$ to $1+\varepsilon$, and the factor $1/|h|^{s'-1/2}$ is $\ll CT \ll K^2$ crudely.

We insert (8.12) into (8.4) for $P_{d,0}(c,h,Y)$ and move inwards the summations over $|h|$, δ and c. We obtain the decomposition

$$\textstyle\sum_d''(C,T)_0 = \sum_d''(C,T)_0^+ + \sum_d''(C,T)_0^- + \sum_d''(C,T)_0^\ell + O(K^{-M'}) \tag{8.13}$$

where for $\nu = \pm$,

$$\begin{aligned}\textstyle\sum_d''(C,T)_0^\nu &= 2^{l-3/2}\pi^{l-1/2} \sum_{j:\,0<t_j\le 2T} \frac{2^{\nu it_j}\Gamma(\nu it_j)\overline{\langle V,\phi_j\rangle}}{\Gamma(l-1/2+\nu it_j)} \\ &\qquad \times \sum_{|h|\sim CT} \frac{\overline{\rho_j(-h)}}{|h|^{1/2}} S_{h,0}\Big(\frac{1}{2}+\nu it_j\Big), \qquad (8.14)\\ \textstyle\sum_d''(C,T)_0^\ell &= \frac{1}{2\pi i}\int_{\ell''} \sum_{j:\,0<t_j\le 2T} \overline{\langle V,\phi_j\rangle} B_j(s) \sum_{|h|\sim CT} \frac{\overline{\rho_j(-h)}}{|h|^{1/2}} S_{h,0}(s)\,ds. \qquad (8.15)\end{aligned}$$

See (8.5) for the definition of $S_{h,0}(\cdot)$.

We handle $\sum_d''(C,T)_0^\ell$ by adopting the treatment for $\sum_d''(C,T)_1$. We apply the Cauchy-Schwarz inequality to the integral in (8.15) to get rid of $B_j(s)$. In this case $\Re\mathrm{e}\, s = -1/2$, we have by (3.21),

$$\int_{\ell''} |B_j(s)|^2\,|ds| \ll T^{2-2l}.$$

We infer with the notation in (8.8) that

$$\begin{aligned}\textstyle\sum_d''(C,T)_0^\ell &\ll T^{1-l} \sum_{j:\,0<t_j\le 2T} |\langle V,\phi_j\rangle| \mathcal{I}_0\Big(-\frac{1}{2},\rho_j\Big)^{1/2} \\ &\ll T^{1-l}\Big(\sum_{0\le t_j\le 2T} |\langle V,\phi_j\rangle|^2 e^{\pi t_j}\Big)^{1/2}\Big(\sum_{0\le t_j\le 2T} e^{-\pi t_j}\mathcal{I}_0\Big(-\frac{1}{2},\rho_j\Big)\Big)^{1/2}\end{aligned}$$

by Cauchy-Schwarz's inequality on the sum. Then, we apply Good's estimate (3.7) to the first factor, and apply the estimate in (8.10), which incorporates the spectral large sieve, to the second factor with $\sigma = -1/2$ and $r = 0$. It follows that

$$\begin{aligned}\textstyle\sum_d''(C,T)_0^\ell &\ll TK^\epsilon Y^{-1/2} CT^{5/2}\Big(\frac{C}{\sqrt{Y}}\Big)^{j+\lambda} \\ &= K^\epsilon Y^{1/2}\frac{CT}{Y}T^{5/2}\Big(\frac{C}{\sqrt{Y}}\Big)^{j+\lambda} \\ &\ll K^{1+\epsilon}Y^{1/2}\Big(\frac{K}{L^3}\Big)^{3/2}\Big(\frac{C}{\sqrt{Y}}\Big)^{j+\lambda}, \qquad (8.16)\end{aligned}$$

by $CT/Y \ll K^\epsilon/L^2$ and $T \ll K^{1+\epsilon}/L$. Comparing with (8.11), (8.16) yields an acceptable bound for $L \geq K^{1/3}$.

Next we evaluate $\sum_d''(C,T)_0^\nu$ for $\nu = \pm$, as given in (8.14) with $S_{h,0}$ given in (8.5) and $G_0(s;h,c)$ given in (5.10). To this end, we apply the explicit formula in Lemma 5.5 to unwind the h and t_j parts in $G_0(1/2+\nu it_j;h,c)$, in order to utilize the large sieve inequality. (In the cases of $\sum_d''(C,T)_1$ and $\sum_d''(C,T)_0^\ell$, this step is not necessary due to the extra averaging over t.) Invoking the integral representation of $G_0(s;h,c)$ in (5.10) and noting that the integrand is supported in $[2^{-1/2}, 2^{1/2}] \subset [1/2, 2]$, we can write

$$\begin{aligned} G_0\Big(\frac{1}{2}+\nu it_j;h,c\Big) &= 4\pi(1+i)\sqrt{2Y|J|}Y^{-(\lambda+j)/2} \\ &\quad \times \int_{1/2}^{2} \frac{1}{c^{1/2}}\mathcal{F}'_{h,c}(z)\Big(\frac{2Y}{|J|}\Big)^{\nu it} e_\eta\Big(z,\frac{1}{2}+\nu it\Big)\,dz, \end{aligned}$$

where $\mathcal{F}_{h,c}(z)$ and $e_\eta(z,s)$ are given as in (5.9) and (5.8), respectively. Then we can express (8.5) as

$$S_{h,0}\Big(\frac{1}{2}+\nu it\Big) = 4\pi(1+i)\sqrt{2Y|J|}\,Y^{-(\lambda+j)/2}\int_{1/2}^{2} \mathcal{S}_h(z)\Big(\frac{2Y}{|J|}\Big)^{\nu it} e_\eta\Big(z,\frac{1}{2}+\nu it\Big)\,dz \quad (8.17)$$

for $s = 1/2+\nu it$, where

$$\mathcal{S}_h(z) = \sum_{\delta|h}\delta \sum_{\substack{c\sim C \\ \mathcal{N}|c,\,\delta|c}} c^{j+\lambda}\frac{\mu(c/\delta)}{c^{3/2}}\mathcal{F}'_{h,c}(z). \quad (8.18)$$

Let us also introduce the function

$$\mathcal{B}(z,\nu\tau) = \frac{2^{\nu i\tau}\Gamma(\nu i\tau)}{\Gamma(l-1/2+\nu i\tau)}\Big(\frac{2Y}{|J|}\Big)^{\nu i\tau} e_\eta\Big(z,\frac{1}{2}+\nu i\tau\Big). \quad (8.19)$$

The symbol η is not shown up in $\mathcal{B}(z,\nu t)$ because its value has no effective in the estimation below.

The key feature of $\mathcal{S}_h(z)$ and $\mathcal{B}(z,\nu\tau)$ is their respective independence from the spectral parameter t_j and the parameter h, which enables the application of Good's inequality and the large sieve inequality. But at first we need some estimates. From Lemma 5.6 (i), it is clear that

$$\mathcal{S}_h(z) \ll K^\epsilon C^{j+\lambda-1/2}\sum_{\delta|h}\delta\sum_{\substack{c\sim C \\ \delta|c}} c^{-1} \ll \tau(|h|)K^\epsilon C^{j+\lambda-1/2}. \quad (8.20)$$

Moreover, we have by (3.8),

$$\frac{2^{\nu i\tau}\Gamma(\nu i\tau)}{\Gamma(l-1/2+\nu i\tau)} \ll |\tau|^{1/2-l},$$

for $\tau \geq \varepsilon$, and then with Lemma 5.6 (ii), we infer that for $|\tau| \asymp T$,

$$\mathcal{B}(z, \nu\tau) \ll T^{-l}. \tag{8.21}$$

In view of (8.14) and (8.17), we have

$$\begin{aligned}
&{\textstyle\sum_d''}(C,T)_0^\nu \\
= \;& 2^{l+1}\pi^{l+1/2}(1+i)\sqrt{Y|J|}\,Y^{-(\lambda+j)/2} \\
&\times \int_{1/2}^{2} \sum_{j\colon 0<t_j\leq 2T} \frac{2^{\nu i t_j}\Gamma(\nu i t_j)\overline{\langle V,\phi_j\rangle}}{\Gamma(l-1/2+\nu i t_j)}\Big(\frac{2Y}{|J|}\Big)^{\nu i t_j} e_\eta\Big(z, \frac{1}{2}+\nu i t_j\Big) \sum_{|h|\sim CT} \frac{\overline{\rho_j(-h)}}{|h|^{1/2}}\mathcal{S}_h(z)\,dz \\
= \;& 2^{l+1}\pi^{l+1/2}(1+i)\sqrt{Y|J|}\,Y^{-(\lambda+j)/2} \\
&\times \int_{1/2}^{2} \sum_{j\colon 0<t_j\leq 2T} \mathcal{B}(z,\nu t_j)\overline{\langle V,\phi_j\rangle} \sum_{|h|\sim CT} \frac{\overline{\rho_j(-h)}}{|h|^{1/2}}\mathcal{S}_h(z)\,dz
\end{aligned} \tag{8.22}$$

by (8.19). We insert the factor $e^{\pi t_j/2}(\cosh(\pi t_j))^{-1/2}$ ($\asymp 1$) and apply Cauchy-Schwarz's inequality to the integrand,

$$\begin{aligned}
&\sum_{j\colon 0<t_j\leq 2T} \mathcal{B}(z,\nu t_j)\overline{\langle V,\phi_j\rangle} \sum_{|h|\sim CT} \frac{\overline{\rho_j(-h)}}{|h|^{1/2}}\mathcal{S}_h(z) \\
\ll \;& \Big(\sum_{j\colon 0<t_j\leq 2T} |\mathcal{B}(z,\nu t_j)|^2|\langle V,\phi_j\rangle|^2 e^{\pi t_j}\Big)^{1/2} \\
&\times\Big(\sum_{j\colon 0<t_j\leq 2T} \frac{1}{\cosh(\pi t_j)}\Big|\sum_{|h|\sim CT} \frac{\overline{\rho_j(-h)}}{|h|^{1/2}}\mathcal{S}_h(z)\Big|^2\Big)^{1/2}.
\end{aligned} \tag{8.23}$$

The first bracket is very small, for we have, by (3.7) and (8.21),

$$\begin{aligned}
&\sum_{j\colon 0<t_j\leq 2T} |\mathcal{B}(z,\nu t_j)|^2|\langle V,\phi_j\rangle|^2 e^{\pi t_j} \\
\ll \;& (\log T)\sup_{X\leq T} X^{-2l} \sum_{j\colon |t_j|\sim X} |\langle V,\phi_j\rangle|^2 e^{\pi t_j} \ll T^\varepsilon.
\end{aligned} \tag{8.24}$$

The second bracket is treated by the large sieve inequality (2.5) and (8.20),

$$\begin{aligned}
&\sum_{j\colon 0<t_j\leq 2T} \frac{1}{\cosh(\pi t_j)}\Big|\sum_{|h|\sim CT} \frac{\overline{\rho_j(-h)}}{|h|^{1/2}}\mathcal{S}_h(z)\Big|^2 \\
\ll \;& (CT)^\varepsilon(T^2+CT)\sum_{|h|\sim CT}\frac{|\mathcal{S}_h(z)|^2}{|h|} \\
\ll \;& T^2\cdot K^\epsilon C^{2(j+\lambda)-1}\sum_{|h|\sim CT}\frac{\tau(|h|)^2}{|h|} \\
\ll \;& K^\epsilon T^2 C^{2(j+\lambda)-1},
\end{aligned} \tag{8.25}$$

where we again used the fact that $\sum_{1\le h\ll X}\tau(h)^2/h \ll X^{\varepsilon}$.

Using (8.24) and (8.25), we get from (8.23) that

$$\sum_{j:\,0<t_j\le 2T}\mathcal{B}(z,\nu t_j)\overline{\langle V,\phi_j\rangle}\sum_{|h|\sim CT}\frac{\overline{\rho_j(-h)}}{|h|^{1/2}}\mathcal{S}_h(z)\ll K^{\epsilon}T^{1+\varepsilon}C^{j+\lambda-1/2}.$$

This in turn gives us an estimate of (8.22) for $\nu=\pm$,

$$\begin{aligned}\textstyle\sum''_d(C,T)_0^{\nu} &\ll \sqrt{Y|J|}\,Y^{-(\lambda+j)/2}\cdot T^{\varepsilon}\cdot K^{\epsilon}TC^{(j+\lambda)-1/2}\\ &\ll K^{1+\epsilon}Y^{1/2}\Big(\frac{K}{L^3}\Big)^{1/2}\Big(\frac{C}{\sqrt{Y}}\Big)^{j+\lambda},\end{aligned}\tag{8.26}$$

by $|J|=TC$ and $T\ll K^{1+\epsilon}/L$.

By (8.3) and (8.13), the estimation of $\sum''_d(C,T)$ is reduced to estimations for $\sum''_d(C,T)_1$, $\sum''_d(C,T)_0^{\ell}$, $\sum''_d(C,T)_0^{+}$, and $\sum''_d(C,T)_0^{-}$. These are established respectively in (8.11), (8.16) and (8.26). We therefore conclude that

$$\textstyle\sum''_d(C,T)\ll K^{1+\epsilon}Y^{1/2}\Big(\frac{K}{L^3}\Big)^{1/2}\Big(\frac{C}{\sqrt{Y}}\Big)^{j+\lambda}\tag{8.27}$$

for $L\ge K^{1/3}$.

9. The continuous spectrum

Finally we evaluate $\sum''_E(C,T)$ in (6.11). The treatment is very similar to the case of discrete spectrum. However, when we shift the path of integration ℓ' beyond $\Re\mathrm{e}\, s=1/2$, a technical problem arises as we cannot avoid the poles of $C_{\mathfrak{a}}(s,\tau)$ (at $s=1/2\pm i\tau$, see (3.5)) to appear on the horizontal line segments. (Note that c_0 or c_0' may lie in $[-2,2]$.) To this end, we put $I=[c_0T/2,2c_0'T]$ and divide the τ-integral in the definition of $P_E(c,h,Y)$ in (6.9) into two parts:

$$P_E(c,h,Y) \;=\; P_{E,O}(c,h,Y)+P_{E,I}(c,h,Y)$$

according as $|\tau|\in[0,2T]\setminus I$ or $|\tau|\in[0,2T]\cap I$, and write correspondingly,

$$\textstyle\sum''_E(C,T) \;=\; \sum''_E(C,T)_O+\sum''_E(C,T)_I.\tag{9.1}$$

Explicitly, we have

$$\begin{aligned}P_{E,O}(c,h,Y) &= \frac{1}{4\pi}\sum_{\mathfrak{a}}\int_{|\tau|\in[0,2T]\setminus I}\frac{\overline{\langle V,E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle}}{\Gamma(1/2-i\tau)}\\ &\quad\times\frac{1}{2\pi i}\int_{\ell}C_{\mathfrak{a}}(s,\tau)\frac{\overline{\rho_{\mathfrak{a}}(1/2+i\tau,-h)}}{|h|^{s-1/2+i\tau}}\widetilde{G}_{h,c}(s)\,ds\,d\tau,\end{aligned}\tag{9.2}$$

$$
\begin{aligned}
P_{E,I}(c,h,Y) &= \frac{1}{4\pi}\sum_{\mathfrak{a}}\int_{|\tau|\in[0,2T]\cap I}\frac{\overline{\langle V, E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle}}{\Gamma(1/2-i\tau)}\\
&\qquad\times\frac{1}{2\pi i}\int_{\ell}C_{\mathfrak{a}}(s,\tau)\frac{\overline{\rho_{\mathfrak{a}}(1/2+i\tau,-h)}}{|h|^{s-1/2+i\tau}}\widetilde{G}_{h,c}(s)\,ds\,d\tau, \qquad (9.3)
\end{aligned}
$$

$$
{\textstyle\sum}''_E(C,T)_O = \sum_{|h|\sim CT}\sum_{\delta|h}\delta\sum_{\substack{c\sim C\\ \mathcal{N}|c,\,\delta|c}}c^{j+\lambda}\frac{\mu(c/\delta)}{c}P_{E,O}(c,h,Y), \qquad (9.4)
$$

$$
{\textstyle\sum}''_E(C,T)_I = \sum_{|h|\sim CT}\sum_{\delta|h}\delta\sum_{\substack{c\sim C\\ \mathcal{N}|c,\,\delta|c}}c^{j+\lambda}\frac{\mu(c/\delta)}{c}P_{E,I}(c,h,Y). \qquad (9.5)
$$

Let $\mathcal{R}$ be the rectangle with vertices at $-1/2+ic_0T$, $-1/2+ic_0'T$, $1+\varepsilon+ic_0T$ and $1+\varepsilon+ic_0'T$, and $\mathcal{R}'$ be its mirror image about the real axis. When $|\tau|\notin I$ (and thus $\tau\notin[c_0T,c_0'T]$), the point $1/2+i\tau$ always lies outside $\mathcal{R}\cup\mathcal{R}'$. Hence, the shifting of ℓ to $\ell''=\ell''_-\cup\ell''_+$ (where $\sigma=-1/2$ and $c_0T\le|t|\le c_0'T$ for $\sigma+it\in\ell''$) does not cross any pole of $C_{\mathfrak{a}}(s,\tau)$.

We move the path ℓ in the inner integral of $P_{E,O}(c,h,Y)$ in (9.2) to ℓ'' with a cost of producing a term $O(K^{l+4-2M'})$, by (6.4) and the same fashion of treatment in (8.1). As in (8.7), we move the sums over $|h|$, δ and c in (9.4) inside the integrals of $P_{E,O}(c,h,Y)$. Let us abbreviate

$$
\rho_{\mathfrak{a},\tau}(-h)=|h|^{i\tau}\rho_{\mathfrak{a}}(1/2+i\tau,-h). \qquad (9.6)
$$

Then from (9.4) we get

$$
\begin{aligned}
{\textstyle\sum}''_E(C,T)_O &= \frac{1}{4\pi}\sum_{\mathfrak{a}}\int_{|\tau|\in[0,2T]\setminus I}\frac{\overline{\langle V, E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle}}{\Gamma(1/2-i\tau)}\,d\tau\\
&\quad\times\frac{1}{2\pi i}\int_{\ell''_-\cup\ell''_+}C_{\mathfrak{a}}(s,\tau)\sum_{h\sim CT}\frac{\overline{\rho_{\mathfrak{a},\tau}(-h)}}{|h|^{1/2}}\\
&\quad\times\sum_{\delta|h}\delta\sum_{\substack{c\sim C\\ \mathcal{N}|c,\,\delta|c}}c^{j+\lambda}\frac{\mu(c/\delta)}{c}|h|^{1-s}\widetilde{G}_{h,c}(s)\,ds+O(K^{-M'}). \qquad (9.7)
\end{aligned}
$$

Now we recall $|h|^{1-s}\widetilde{G}_{h,c}(s)=\sum_{r=0,1}G_r(s;h,c)$ and

$$
\sum_{\delta|h}\delta\sum_{\substack{c\sim C\\ \mathcal{N}|c,\,\delta|c}}c^{j+\lambda}\frac{\mu(c/\delta)}{c}|h|^{1-s}\widetilde{G}_{h,c}(s)=S_{h,0}(s)+S_{h,1}(s),
$$

by (8.5). Therefore we can rewrite (9.7) as

$$\textstyle\sum''_E(C,T)_O = \frac{1}{4\pi}\sum_{r=0,1}\sum_{\mathfrak{a}}\int_{|\tau|\in[0,2T]\setminus I}\frac{\overline{\langle V, E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle}}{\Gamma(1/2-i\tau)}\,d\tau$$
$$\times\frac{1}{2\pi i}\int_{\ell''_-\cup\ell''_+}C_{\mathfrak{a}}(s,\tau)\sum_{h\sim CT}\frac{\overline{\rho_{\mathfrak{a},\tau}(-h)}}{|h|^{1/2}}S_{h,r}(s)\,ds+O(K^{-M'}). \quad (9.8)$$

When $s\in\ell''$ and $|\tau|\le 2T$, we have $s=-1/2+it$ with $|t|\asymp T$ and thus

$$\int_{\ell''}|C_{\mathfrak{a}}(s,\tau)|^2\,|ds|\ll T^{2-2l}$$

by (3.21). By the Cauchy-Schwarz inequality, the s-integral in (9.8) is

$$\ll\left(\int_{\ell''}|C_{\mathfrak{a}}(s,\tau)|^2\,|ds|\right)^{1/2}\left(\int_{\ell''}\Big|\sum_{h\sim CT}\frac{\overline{\rho_{\mathfrak{a},\tau}(-h)}}{|h|^{1/2}}S_{h,r}(s)\Big|^2\,|ds|\right)^{1/2}.$$

It follows that

$$\textstyle\sum''_E(C,T)_O \ll T^{1-l}\sum_{r=0,1}\sum_{\mathfrak{a}}\int_{|\tau|\in[0,2T]\setminus I}|\langle V,E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle|$$
$$\times e^{\pi|\tau|/2}\mathcal{I}_r\Big(-\frac{1}{2},\rho_{\mathfrak{a},\tau}\Big)^{1/2}\,d\tau \quad (9.9)$$

by (8.8). Note that the factor $e^{\pi|\tau|/2}$ comes up from $\Gamma(1/2-i\tau)^{-1}$, by (3.8). Relaxing the range of integration and using Cauchy-Schwarz's inequality, we obtain

$$\textstyle\sum''_E(C,T)_O \ll T^{1-l}\sum_{r=0,1}\left(\sum_{\mathfrak{a}}\int_{-2T}^{2T}|\langle V,E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle|^2e^{\pi|\tau|}\,d\tau\right)^{1/2}$$
$$\times\left(\sum_{\mathfrak{a}}\int_{-2T}^{2T}\mathcal{I}_r\Big(-\frac{1}{2},\rho_{\mathfrak{a},\tau}\Big)\,d\tau\right)^{1/2}.$$

As before, we apply Good's estimate and the large sieve inequality. The first bracket is $\ll T^{2l}$ by (3.7).

The second bracket is evaluated along the line of argument in (8.10), as follows. Interchanging the order of integrations, we have

$$\sum_{\mathfrak{a}}\int_{-2T}^{2T}\mathcal{I}_r(\sigma,\rho_{\mathfrak{a},\tau})\,d\tau=\int_{\ell_-(\sigma)\cup\ell_+(\sigma)}|ds|\sum_{\mathfrak{a}}\int_{-2T}^{2T}\Big|\sum_{h\sim CT}\frac{\overline{\rho_{\mathfrak{a},\tau}(-h)}}{|h|^{1/2}}S_{h,r}(s)\Big|^2\,d\tau.$$

Using the spectral large sieve (2.6), we get

$$\sum_{\mathfrak{a}}\int_{-2T}^{2T}\mathcal{I}_r(\sigma,\rho_{\mathfrak{a},\tau})\,d\tau \ll (CT)^{\varepsilon}(T^2+CT)\int_{\ell_-(\sigma)\cup\ell_+(\sigma)}\sum_{|h|\sim CT}\frac{|S_{h,r}(s)|^2}{|h|}\,|ds|$$
$$\ll K^{\epsilon}T^{3-r}Y^{2\sigma}(CT)^{1-2\sigma}\left(\frac{C}{\sqrt{Y}}\right)^{2(j+\lambda)} \quad (9.10)$$

by (8.6), where $r = 0$ or 1. Taking $\sigma = -1/2$ in (9.10), we conclude that

$$\begin{aligned} \textstyle\sum_E''(C,T)_O &\ll T^{1-l}\cdot T^l\cdot K^\epsilon T^{3/2}Y^{-1/2}(CT)\Big(\frac{C}{\sqrt{Y}}\Big)^{(j+\lambda)} \\ &\ll K^{1+\epsilon}Y^{1/2}\Big(\frac{K}{L^3}\Big)^{3/2}\Big(\frac{C}{\sqrt{Y}}\Big)^{j+\lambda}, \end{aligned} \tag{9.11}$$

for $CT/Y \ll K^\epsilon/L^2$ and $T \ll K^{1+\epsilon}/L$.

It remains to estimate the sum $\sum_E''(C,T)_I$ corresponding to $P_{E,I}(c,h,Y)$ given by (9.5) and (9.3). We shall follow the argument in treating $\sum_d''(C,T)$ in §8. To bypass the technical problem mentioned at the beginning, we elongate the two components $\ell_\pm$ of ℓ. More specifically, we take $c_1 = c_0/4$ and $c_1' = 4c_0'$, and let $\tilde{\ell}_-$ and $\tilde{\ell}_+$ be the straight line segments joining from $1+\varepsilon - c_1' iT$ to $1+\varepsilon - c_1 iT$ and from $1+\varepsilon + c_1 iT$ to $1+\varepsilon + c_1' iT$, respectively. Also, we denote by $\tilde{\ell} = \tilde{\ell}_- \cup \tilde{\ell}_+$ their union.

The newly fill-in is indeed negligible because by (6.4), $\widetilde{G}_{h,c}(s) \ll K^{-2M'}$ for $s \in \tilde{\ell}\setminus\ell$, whose imaginary part is outside $[c_0T, c_0'T]$, while the remnant in the integral is at most

$$\begin{aligned} &\int_{\tilde{\ell}\setminus\ell}\int_{|\tau|\in[0,2T]\cap I}\left|C_{\mathfrak{a}}(s,\tau)\frac{\overline{\langle V, E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle}}{\Gamma(1/2-i\tau)}\frac{\overline{\rho_{\mathfrak{a}}(1/2+i\tau,-h)}}{|h|^{s-1/2+i\tau}}\right| d\tau\,|ds| \\ &\ll |h|^{-1/2}\int_{\tilde{\ell}\setminus\ell}\big|C_{\mathfrak{a}}(s,\tau)\big|\,|ds|\int_{|\tau|\le 2T}\big|\langle V, E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle\big|e^{\pi|\tau|/2}\,d\tau \end{aligned} \tag{9.12}$$

by (2.4) and (3.8). Applying (3.19) to the first integral on the right side of (9.12) and applying Cauchy-Schwarz's inequality to the second integral, we conclude that (9.12) is bounded by

$$\ll |h|^{-1/2}T^2\Big(\int_{|\tau|\le 2T}\big|\langle V, E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle\big|^2 e^{\pi|\tau|}\,d\tau\Big)^{1/2}\Big(\int_{|\tau|\le 2T}d\tau\Big)^{1/2} \ll T^{5/2+l}$$

by (3.7). Note that the number of summands in the sum over cusps is independent of K. The added portion is thus $O(K^{l+5/2-2M'})$.

After replacing ℓ by $\tilde{\ell}$, we follow closely the argument in the discrete spectrum case. We shift the path $\tilde{\ell}$ to $\tilde{\ell}'$ (with $\sigma = 1/2+\varepsilon$), and then split $|h|^{1-s}\widetilde{G}_{h,c}(s)$ into $G_0(s;h,c)$ and $G_1(s;h,c)$ as defined in (5.10). The horizontal line segments produce an admissible error $O(K^{l+4-2M'})$ by the same arguments as at the beginning of §8. This leads to the evaluation, for $r = 0, 1$,

$$\begin{aligned} \textstyle\sum_E''(C,T)_{I,r} &= \frac{1}{4\pi}\sum_{\mathfrak{a}}\int_{|\tau|\in[0,2T]\cap I}\frac{\overline{\langle V, E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle}}{\Gamma(1/2-i\tau)}\,d\tau \\ &\quad\times\frac{1}{2\pi i}\int_{\tilde{\ell}'}C_{\mathfrak{a}}(s,\tau)\sum_{h\sim CT}\frac{\overline{\rho_{\mathfrak{a},\tau}(-h)}}{|h|^{1/2}}S_{h,r}(s)\,ds + O(K^{-M'}) \end{aligned} \tag{9.13}$$

where $\rho_{\mathfrak{a},\tau}(-h)$ and $S_{h,r}(s)$ are defined as in (9.6) and (8.5) respectively. Note that

$$\textstyle\sum_E''(C,T)_I = \sum_E''(C,T)_{I,0} + \sum_E''(C,T)_{I,1} + O(K^{-M'}). \tag{9.14}$$

We note $\Re\mathrm{e}\, s = 1/2+\varepsilon$ for $s \in \tilde{\ell}'$ in (9.13) and by (3.20),

$$\int_{\tilde{\ell}'} |C_{\mathfrak{a}}(s,\tau)|^2 \, |ds| \ll T^{1-2l}.$$

We deduce from $\Gamma(1/2-i\tau)^{-1} \ll e^{\pi|\tau|/2}$ and (8.8) that

$$\begin{aligned}\textstyle\sum_E''(C,T)_{I,1} \quad &\ll \quad T^{1/2-l} \sum_{\mathfrak{a}} \int_{-2T}^{2T} \big|\langle V, E_{\mathfrak{a}}(\cdot, 1/2+i\tau)\rangle\big| \\ &\qquad \times e^{\pi|\tau|/2} \mathcal{I}_1\Big(\frac{1}{2}+\varepsilon, \rho_{\mathfrak{a},\tau}\Big)^{1/2} \, d\tau, \end{aligned} \tag{9.15}$$

with a relaxation of the range of integration, and then

$$\begin{aligned}\textstyle\sum_E''(C,T)_{I,1} \quad &\ll \quad T^{1/2-l} \left(\sum_{\mathfrak{a}} \int_{-2T}^{2T} \big|\langle V, E_{\mathfrak{a}}(\cdot, 1/2+i\tau)\rangle\big|^2 e^{\pi|\tau|} \, d\tau \right)^{1/2} \\ &\qquad \times \left(\sum_{\mathfrak{a}} \int_{-2T}^{2T} \mathcal{I}_1\Big(\frac{1}{2}+\varepsilon, \rho_{\mathfrak{a},\tau}\Big) \, d\tau \right)^{1/2}\end{aligned}$$

by Cauchy-Schwarz's inequality. Invoking Good's result (3.7) and (9.10), we derive

$$\begin{aligned}\textstyle\sum_E''(C,T)_{I,1} \quad &\ll \quad T^{1/2-l} \cdot T^l \cdot K^{\epsilon} T Y^{1/2} \left(\frac{C}{\sqrt{Y}}\right)^{j+\lambda}. \\ &\ll \quad K^{1+\epsilon} Y^{1/2} \left(\frac{K}{L^3}\right)^{1/2} \left(\frac{C}{\sqrt{Y}}\right)^{j+\lambda}. \end{aligned} \tag{9.16}$$

We shift the path $\tilde{\ell}'$ in $\sum_E''(C,T)_{I,0}$ horizontally to $\tilde{\ell}''$ whose real part is $-1/2$. Due to $C_{\mathfrak{a}}(s,\tau)$, two poles come up at $s = 1/2 \pm i\tau$, and the residues are

$$2^{l-1/2}\pi^l \frac{2^{\pm i\tau}\pi^{-i\tau}\Gamma(\pm i\tau)}{\Gamma(l-1/2\pm i\tau)}. \tag{9.17}$$

This is not big, actually, $\ll |\tau|^{1/2-l}$ by (3.8).

There is no pole lying on the horizontal line segments because the end-points of $\tilde{\ell}$ have imaginary parts $\pm c_0 T/4$ or $\pm 4c_0' T$ but $c_0T/2 \le |\tau| \le 2c_0'T$. These horizontal line segments again produce an admissible error of $O(K^{l+4-2M'})$. Like in $\sum_d''(C,T)_0$ in (8.13), we split $\sum_E''(C,T)_{I,0}$ into three parts,

$$\textstyle\sum_E''(C,T)_{I,0} = \sum_E''(C,T)_{I,0}^{+} + \sum_E''(C,T)_{I,0}^{-} + \sum_E''(C,T)_{I,0}^{\ell} + O(K^{-M'}), \tag{9.18}$$

where for $\nu = \pm$ the contribution from the residue in (9.17) is

$$\textstyle\sum''_E(C,T)^{\nu}_{I,0} = 2^{l-5/2}\pi^{l-1}\sum_{\mathfrak{a}}\int_{|\tau|\in[0,2T]\cap I}\frac{2^{\nu i\tau}\pi^{-i\tau}\Gamma(\nu i\tau)\overline{\langle V, E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle}}{\Gamma(l-1/2+\nu i\tau)\Gamma(1/2-i\tau)} \times \sum_{h\sim CT}\frac{\overline{\rho_{\mathfrak{a},\tau}(-h)}}{|h|^{1/2}}S_{h,0}\Big(\frac{1}{2}+\nu i\tau\Big)\,d\tau, \tag{9.19}$$

and the new line integral on $\tilde{\ell}''$ is

$$\textstyle\sum''_E(C,T)^{\ell}_{I,0} = \frac{1}{4\pi}\sum_{\mathfrak{a}}\int_{|\tau|\in[0,2T]\cap I}\frac{\overline{\langle V, E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle}}{\Gamma(1/2-i\tau)}\,d\tau \times\frac{1}{2\pi i}\int_{\tilde{\ell}''}C_{\mathfrak{a}}(s,\tau)\sum_{h\sim CT}\frac{\overline{\rho_{\mathfrak{a},\tau}(-h)}}{|h|^{1/2}}S_{h,0}(s)\,ds.$$

¿From (3.21), we have

$$\int_{\tilde{\ell}''}|C_{\mathfrak{a}}(s,\tau)|^2\,|ds| \ll T^{2-2l},$$

and thus,

$$\textstyle\sum''_E(C,T)^{\ell}_{I,0} \ll T^{1-l}\sum_{\mathfrak{a}}\int_{-2T}^{2T}\big|\langle V, E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle\big|e^{\pi|\tau|/2}\mathcal{I}_1\Big(-\frac{1}{2},\rho_{\mathfrak{a},\tau}\Big)^{1/2}\,d\tau$$

by arguments similar to (9.9) and (9.15). Again, by Cauchy-Schwarz's inequality, (3.7) and (9.10) for $\sigma=-1/2$ and $r=0$, it follows

$$\begin{aligned}\textstyle\sum''_E(C,T)^{\ell}_{I,0} &\ll T^{1-l}T^lK^{\epsilon}T^{3/2}Y^{-1/2}(CT)\Big(\frac{C}{\sqrt{Y}}\Big)^{j+\lambda}\\ &\ll K^{1+\epsilon}Y^{1/2}\Big(\frac{K}{L^3}\Big)^{3/2}\Big(\frac{C}{\sqrt{Y}}\Big)^{j+\lambda}\end{aligned}\tag{9.20}$$

as in (9.11).

Now let us turn to (9.19). Using $\mathcal{S}_h(z)$ and $e_\eta(z,s)$ as defined in (8.18) and (5.8), respectively, we can insert (8.17) into (9.19) and get

$$\textstyle\sum''_E(C,T)^{\nu}_{I,0} = 2^{l-5/2}\pi^{l-1}\sum_{\mathfrak{a}}\int_{|\tau|\in[0,2T]\cap I}\frac{2^{\nu i\tau}\pi^{-i\tau}\Gamma(\nu i\tau)\overline{\langle V, E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle}}{\Gamma(l-1/2+\nu i\tau)\Gamma(1/2-i\tau)} \times\sum_{h\sim CT}\frac{\overline{\rho_{\mathfrak{a},\tau}(-h)}}{|h|^{1/2}}\,d\tau\;4\pi(1+i)\sqrt{2Y|J|}\,Y^{-(\lambda+j)/2} \times\int_{1/2}^{2}\mathcal{S}_h(z)\Big(\frac{2Y}{|J|}\Big)^{\nu it}e_\eta\Big(z,\frac{1}{2}+\nu it\Big)\,dz.$$

Then we use $\mathcal{B}(z,\nu\tau)$ as defined in (8.19) to rewrite

$$\begin{aligned}\textstyle\sum''_E(C,T)^{\nu}_{I,0} &= (2\pi)^l(1+i)\sqrt{Y|J|}\,Y^{-(\lambda+j)/2}\int_{1/2}^{2}dz\sum_{\mathfrak{a}}\int_{|\tau|\in[0,2T]\cap I}\mathcal{B}(z,\nu\tau)\\ &\quad\times\frac{\pi^{-i\tau}\overline{\langle V,E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle}}{\Gamma(1/2-i\tau)}\sum_{h\sim CT}\frac{\overline{\rho_{\mathfrak{a},\tau}(-h)}}{|h|^{1/2}}\mathcal{S}_h(z)\,d\tau. \end{aligned}\tag{9.21}$$

Applying Cauchy-Schwarz's inequality with $\Gamma(1/2-i\tau)^{-1}\ll e^{\pi|\tau|/2}$, the integrand of the z-integral in (9.21) is

$$\begin{aligned}&\ll \left(\sum_{\mathfrak{a}}\int_{|\tau|\in[0,2T]\cap I}|\mathcal{B}(z,\nu\tau)|^2|\langle V,E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle|^2e^{\pi|\tau|}\,d\tau\right)^{1/2}\\ &\quad\times\left(\sum_{\mathfrak{a}}\int_{|\tau|\in[0,2T]\cap I}\left|\sum_{h\sim CT}\frac{\overline{\rho_{\mathfrak{a},\tau}(-h)}}{|h|^{1/2}}\mathcal{S}_h(z)\right|^2d\tau\right)^{1/2}. \end{aligned}\tag{9.22}$$

When $|\tau|\in I$ (i.e. $|\tau|\asymp T$), we can apply the estimate (8.21) so that the first bracket in (9.22) is

$$\ll T^{-2l}\sum_{\mathfrak{a}}\int_{|\tau|\in[0,2T]\cap I}|\langle V,E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle|^2e^{\pi|\tau|}\,d\tau\ll 1$$

by (3.7). From the spectral large sieve inequality (2.6), the second bracket in (9.22) is

$$\begin{aligned}&\ll \sum_{\mathfrak{a}}\int_{-2T}^{2T}\left|\sum_{h\sim CT}\frac{\overline{\rho_{\mathfrak{a},\tau}(-h)}}{|h|^{1/2}}\mathcal{S}_h(z)\right|^2d\tau\\ &\ll (CT)^{\varepsilon}(T^2+CT)\sum_{h\sim CT}\frac{|\mathcal{S}_h(z)|^2}{|h|}\\ &\ll T^2K^{\epsilon}C^{2(j+\lambda)-1}, \end{aligned}$$

by (8.20), following an argument similar to (8.25). Consequently, as $|J|=CT$ and $T\ll K^{1+\epsilon}/L$, we get from (9.22) that

$$\begin{aligned}\textstyle\sum''_E(C,T)^{\nu}_{I,0} &\ll \sqrt{Y|J|}\,Y^{-(\lambda+j)/2}TK^{\epsilon}C^{(j+\lambda)-1/2}\\ &\ll K^{\epsilon}\sqrt{Y}T^{3/2}\left(\frac{C}{\sqrt{Y}}\right)^{j+\lambda}\\ &\ll K^{1+\epsilon}Y^{1/2}\left(\frac{K}{L^3}\right)^{1/2}\left(\frac{C}{\sqrt{Y}}\right)^{j+\lambda}. \end{aligned}\tag{9.23}$$

Consequently by (9.20) and (9.23), (9.18) becomes

$$\textstyle\sum''_E(C,T)_{I,0}\ll K^{1+\epsilon}Y^{1/2}\left(\frac{K}{L^3}\right)^{1/2}\left(\frac{C}{\sqrt{Y}}\right)^{j+\lambda}$$

for $L^3 \geq K$. With (9.16), we get the same bound for (9.14). Finally by (9.11) and (9.1), we infer that

$$\sum\nolimits_E''(C,T) \ll K^{1+\epsilon} Y^{1/2} \Big(\frac{K}{L^3}\Big)^{1/2} \Big(\frac{C}{\sqrt{Y}}\Big)^{j+\lambda} \tag{9.24}$$

for $L^3 \geq K$.

10. The proof of Theorem 1 for holomorphic g

In view of our estimates for $\sum_R''(C,T)$, $\sum_d''(C,T)$, and $\sum_E''(C,T)$ (with the choice $A_0 = 2$) in (7.1), (8.27), and (9.24), we deduce from (6.11) that

$$\begin{aligned} \sum\nolimits''(C,T) &\ll K^{1+\epsilon} Y^{1/2} \Big(\Big(\frac{K}{L^3}\Big)^{1/2} + \Big(\frac{K^{1+6\varepsilon}}{L^3}\Big)^3 \Big) \max_C \Big(\frac{C}{\sqrt{Y}}\Big)^{j+\lambda} \\ &\ll K^{1+\epsilon} Y^{1/2} \Big(\Big(\frac{1}{\sqrt{Y}}\Big)^{j+\lambda} + \Big(\frac{\sqrt{Y}}{LK^{1-\varepsilon}}\Big)^{j+\lambda} \Big) \end{aligned} \tag{10.1}$$

under the conditions

$$\begin{aligned} &LK^{1-\varepsilon} \leq Y \leq K^{2+\varepsilon} , \quad K^{1/3+2\varepsilon} \leq L \leq K^{1-2\varepsilon} , \\ &1 \leq C \leq Y/(LK^{1-\varepsilon}) , \quad T \asymp K^2 C/Y \geq K^{\varepsilon} . \end{aligned}$$

Note that λ and even $j + \lambda$ may be negative. Inserting the last estimate in (10.1) into (6.3), we get

$$\begin{aligned} \widetilde{T}^{(\eta)}_{\underline{\lambda},j}(Y) \ll\ & \frac{Y^{-(\lambda+j)/2} K^{4\beta-\alpha}}{L^{2\nu+\alpha-2k}} LYK^{1+\epsilon} \Big(\Big(\frac{1}{\sqrt{Y}}\Big)^{j+\lambda} + \Big(\frac{\sqrt{Y}}{LK^{1-\varepsilon}}\Big)^{j+\lambda} \Big) \\ & + \delta_{(Y \geq K^{2-\varepsilon})} \frac{Y^{-(j+\lambda)/2} K^{4\beta-\alpha}}{L^{2\nu+\alpha-2k}} LYK^{1+\epsilon} \\ & \times \max\Big\{ \Big(\frac{1}{\sqrt{Y}}\Big)^{j+\lambda}, \Big(\frac{\sqrt{Y}}{K^{2-\varepsilon}}\Big)^{j+\lambda} \Big\}. \end{aligned} \tag{10.2}$$

Observing

$$\frac{1}{\sqrt{Y}} \leq \frac{\sqrt{Y}}{K^{2-\varepsilon}} \leq \frac{\sqrt{Y}}{LK^{1-\varepsilon}}$$

for $Y \geq K^{2-\varepsilon}$, we conclude that the term corresponding to $(\sqrt{Y}/K^{2-\varepsilon})^{j+\lambda}$ in the second bracket on the right side of (10.2) can be absorbed by other terms. Therefore

$$\begin{aligned} \widetilde{T}^{(\eta)}_{\underline{\lambda},j}(Y) &\ll \frac{Y^{-(\lambda+j)/2} K^{4\beta-\alpha}}{L^{2\nu+\alpha-2k}} LYK^{1+\epsilon} \Big(\Big(\frac{1}{\sqrt{Y}}\Big)^{j+\lambda} + \Big(\frac{\sqrt{Y}}{LK^{1-\varepsilon}}\Big)^{j+\lambda} \Big) \\ &\ll YLK^{1+\epsilon} \frac{K^{4\beta-\alpha}}{L^{2\nu+\alpha-2k}} \Big(\Big(\frac{1}{Y}\Big)^{j+\lambda} + \Big(\frac{1}{LK^{1-\varepsilon}}\Big)^{j+\lambda} \Big) \\ &\ll YLK^{1+\epsilon} \frac{K^{4\beta-\alpha}}{L^{2\nu+\alpha-2k}} \Big(\Big(\frac{1}{Y}\Big)^{k-\mu+3\beta-\alpha} + \Big(\frac{1}{LK^{1-\varepsilon}}\Big)^{k-\mu+3\beta-\alpha} \Big) \end{aligned} \tag{10.3}$$

as $j \geq 0$ and $\lambda = k - \mu + 3\beta - \alpha$.

Since $\nu \geq 3\mu$, $0 \leq \alpha \leq 2k$ and $\alpha \leq 4\beta$ (see Lemma 4.1), we see that

$$\begin{aligned}\frac{K^{4\beta-\alpha}}{L^{2\nu+\alpha-2k}}\left(\frac{1}{Y}\right)^{k-\mu+3\beta-\alpha} &= \left(\frac{L^2}{Y}\right)^k\left(\frac{Y^\mu}{L^{2\nu}}\right)\left(\frac{K^4}{Y^3}\right)^\beta\left(\frac{Y}{KL}\right)^\alpha \\ &\leq \left(\frac{L^2}{Y}\right)^{k-\alpha/2}\left(\frac{Y}{L^6}\right)^\mu\left(\frac{K^4}{Y^3}\right)^{\beta-\alpha/4}\left(\frac{1}{Y^{1/4}}\right)^\alpha \\ &\leq \left(\frac{L}{K^{1-\varepsilon}}\right)^{k-\alpha/2}\left(\frac{K^{1/3+\varepsilon}}{L}\right)^{6\mu+3(\beta-\alpha/4)}\end{aligned}$$

as $K^{1-\varepsilon}L \leq Y \leq K^{2+\varepsilon}$. This last quantity is less than 1 for our choice of L. Similarly, we have

$$\begin{aligned}\frac{K^{4\beta-\alpha}}{L^{2\nu+\alpha-2k}}\left(\frac{1}{LK^{1-\varepsilon}}\right)^{k-\mu+3\beta-\alpha} &= \left(\frac{L}{K^{1-\varepsilon}}\right)^k\left(\frac{(LK^{1-\varepsilon})^\mu}{L^{2\nu}}\right)\left(\frac{K^4}{(LK^{1-\varepsilon})^3}\right)^\beta K^{-\varepsilon\alpha} \\ &\leq \left(\frac{L}{K^{1-\varepsilon}}\right)^k\left(\frac{K^{1-\varepsilon}}{L^5}\right)^\mu\left(\frac{K^{1/3+\varepsilon}}{L}\right)^{3\beta}\end{aligned}$$

which is also less than 1 for the same range of L. From (10.3) we thus obtain $\widetilde{T}^{(\eta)}_{\underline{\lambda},j}(Y) \ll YLK^{1+\epsilon}$, and consequently by (4.9), we prove

$$\sum_{K-L\leq k_j\leq K+L} |S_Y(f_j)|^2 \ll YLK^{1+\epsilon},$$

for $K^{1/3+2\varepsilon} \leq L \leq K^{1-2\varepsilon}$ and $Y \leq K^{2+\varepsilon}$. Note that the multiple sum in (4.9) produces only a constant multiple. Writing $\epsilon = C_0\varepsilon$ for some constant $C_0 > 0$, we can replace ε by $\varepsilon/(C_0+2)$. Then (4.7) holds true whenever $K^{1/3+\varepsilon} \leq L \leq K^{1-\varepsilon}$. □

11. The proof of Theorem 1 for Maass g

The proof of Theorem 1 for a holomorphic form g is accomplished in the last section. Now we indicate the necessary changes in the proof for Maass form g. In this case the function $D_g(s,\nu_1,\nu_2,h)$ is defined as

$$D_g(s,\nu_1,\nu_2,h) = \sum_{\substack{m,n\neq 0\\ \nu_1m-\nu_2n=h}} \lambda_g(n)\overline{\lambda}_g(m)\Big(\frac{\sqrt{\nu_1\nu_2|mn|}}{\nu_1|m|+\nu_2|n|}\Big)^{2il}(\nu_1|m|+\nu_2|n|)^{-s}.$$

Its spectral decomposition will follow from the line of argument in Section 3.

We consider the inner $\langle U_h(\cdot,s),V\rangle$ where $V(z)=g(\nu_1 z)\overline{g(\nu_2 z)}$. From the proof of Theorem A.2 in [29], we have for $\Re\mathrm{e}\, s>1$,

$$\begin{aligned}
&\frac{(\nu_1\nu_2)^{-1/2}2^{3-s-2il}\pi^s\Gamma(s)}{\Gamma(s/2+il)\Gamma(s/2-il)\Gamma(s/2)^2}\langle U_h(\cdot,s),V\rangle \\
=\ &\sum_{\substack{m,n\neq 0\\ \nu_1 m-\nu_2 n=h}}\lambda_g(n)\overline{\lambda}_g(m)\Big(\frac{\sqrt{\nu_1\nu_2|mn|}}{\nu_1|m|+\nu_2|n|}\Big)^{2il}(\nu_1|m|+\nu_2|n|)^{-s} \\
&\times F\left(\frac{s}{2}+il,\frac{1}{2}+il;\frac{s}{2}+\frac{1}{2};\Big(\frac{|\nu_1 m|-|\nu_2 n|}{|\nu_1 m|+|\nu_2 n|}\Big)^2\right),
\end{aligned}\tag{11.1}$$

where F is the hypergeometric function. Let us write $s=\sigma+it$ and denote

$$\begin{aligned}
D_g^F(s,\nu_1,\nu_2,h)\ =\ &\sum_{\substack{m,n\neq 0\\ \nu_1 m-\nu_2 n=h}}\lambda_g(n)\overline{\lambda}_g(m)\Big(\frac{\sqrt{\nu_1\nu_2|mn|}}{\nu_1|m|+\nu_2|n|}\Big)^{2il}(\nu_1|m|+\nu_2|n|)^{-s} \\
&\times\left\{1-F\left(\frac{s}{2}+il,\frac{1}{2}+il;\frac{s}{2}+\frac{1}{2};\Big(\frac{|\nu_1 m|-|\nu_2 n|}{|\nu_1 m|+|\nu_2 n|}\Big)^2\right)\right\}.
\end{aligned}$$

The bracket $\{\cdots\}$ in the last line is

$$\ll\Big(\frac{|\nu_1 m|-|\nu_2 n|}{|\nu_1 m|+|\nu_2 n|}\Big)^2\ll\frac{|h|^2}{(|\nu_1 m|+|\nu_2 n|)^2}$$

for bounded σ, say $|\sigma|\le 2$. By the simple inequalities $|ab|\le|a|^2+|b|^2$ and $2(|a|+|a+b|)\ge|a|+|b|$, we get

$$\begin{aligned}
D_g^F(s,\nu_1,\nu_2,h)\ &\ll\ |h|^2\sum_{\substack{m,n\neq 0\\ \nu_1 m-\nu_2 n=h}}\frac{|\lambda_g(n)\lambda_g(m)|}{(\nu_1|m|+\nu_2|n|)^{\sigma+2}} \\
&\ll\ |h|^2\sum_{n\neq 0}\frac{|\lambda_g(n)|^2}{(\nu_2|n|+|h|)^{\sigma+2}}+|h|^2\sum_{m\neq 0}\frac{|\lambda_g(m)|^2}{(\nu_1|m|+|h|)^{\sigma+2}} \\
&\ll_{\nu_1,\nu_2}\ |h|^{-\sigma}\sum_{|n|\le|h|}|\lambda_g(n)|^2+|h|^2\sum_{|n|>|h|}\frac{|\lambda_g(n)|^2}{|n|^{\sigma+2}}.
\end{aligned}$$

The estimate $\sum_{|n|\le X}|\lambda_g(n)|^2\ll X$ (see [17, (8.7)]) implies that $D_g^F(s,\nu_1,\nu_2,h)$ converges absolutely for $\sigma\ge -1+\varepsilon$ and satisfies

$$D_g^F(s,\nu_1,\nu_2,h)\ll|h|^{1-\sigma}.\tag{11.2}$$

Besides, we have

$$D_g(s,\nu_1,\nu_2,h)+D_g^F(s,\nu_1,\nu_2,h)=\frac{(\nu_1\nu_2)^{-1/2}2^{3-s-2il}\pi^s\Gamma(s)}{\Gamma(s/2+il)\Gamma(s/2-il)\Gamma(s/2)^2}\langle U_h(\cdot,s),V\rangle\tag{11.3}$$

by (11.1).

The right-hand side above is essentially the same as the right-side of (3.2). Indeed, we have, by (3.8),

$$\frac{(\nu_1\nu_2)^{-1/2}2^{3-s-2il}\pi^s\Gamma(s)}{\Gamma(s/2+il)\Gamma(s/2-il)\Gamma(s/2)^2} \ll (\nu_1\nu_2)^{-1/2}|t|^{3/2-\sigma}e^{\pi|t|/2} \asymp \left|\frac{(\nu_1\nu_2)^{-1/2}}{\Gamma(s-1)}\right| \tag{11.4}$$

for $|\Im\mathrm{m}\, s| \geq 1$. But for the gamma factors in (11.3), we have to be careful when $\Re\mathrm{e}\, s \leq -1$, because there are poles at $s = -1, -3, \cdots$. We shall see why these poles have no influence in our proof. We apply the argument in (3.3)-(3.16) to obtain

$$\begin{aligned} D_g(s,\nu_1,\nu_2,h) &= -D_g^F(s,\nu_1,\nu_2,h) + \widetilde{R}_h(s) + \sum_{j:\,0<t_j\leq 2T} \frac{\overline{\rho_j(-h)}}{|h|^{s-1/2}}\widetilde{B}_j(s)\overline{\langle V,\phi_j\rangle} \\ &\quad + \frac{1}{4\pi}\sum_{\mathfrak{a}}\int_{-2T}^{2T}\frac{\overline{\rho_{\mathfrak{a}}(1/2+i\tau,-h)}}{|h|^{s-1/2+i\tau}}\frac{\widetilde{C}_{\mathfrak{a}}(s,\tau)}{\Gamma(1/2-i\tau)}\overline{\langle V,E_{\mathfrak{a}}(\cdot,1/2+i\tau)\rangle}\,d\tau \\ &\quad + O\big(|h|^{1/2-\sigma+\theta+\varepsilon}e^{-T/4}\big) \end{aligned} \tag{11.5}$$

where $\widetilde{R}_h(s)$, $\widetilde{B}_j(s)$ and $\widetilde{C}_{\mathfrak{a}}(s,\tau)$ are defined as in (3.17), (3.4) and (3.5) with only a small change, that is, the factor

$$\frac{(2\pi)^{s+l-1}(\nu_1\nu_2)^{(l-1)/2}}{\Gamma(s+l-1)}$$

in their definitions is replaced by the expression on the left side of (11.4). It should be noted that we need a substitute of Good's estimate in (3.7), which is available in [21].

The reduction process (in Section 6) up to (6.5) makes use of the absolutely convergence of $D_g(s,1,1,h)$ for $\sigma > 1$. It is plainly valid for the Maass form case. Then we insert (11.5), instead of (3.16), into (6.5); consequently, instead of (6.6), we have

$$P(c,h,Y) = P_F(c,h,Y) + P_{\widetilde{R}}(c,h,Y) + P_{\widetilde{d}}(c,h,Y) + P_{\widetilde{E}}(c,h,Y) + O(K^{-M'})$$

where the three $P_{\widetilde{*}}(c,h,Y)$ are defined analogously as $P_*(c,h,Y)$ in (6.6) with $R_h(s)$, $B_j(s)$ and $C_{\mathfrak{a}}(s,\tau)$ being replaced by $\widetilde{R}_h(s)$, $\widetilde{B}_j(s)$ and $\widetilde{C}_{\mathfrak{a}}(s,\tau)$. The function $P_F(s,h,Y)$ is defined as

$$P_F(c,h,Y) = -\frac{1}{2\pi i}\int_{\ell} D_g^F(s,1,1,h)\widetilde{G}_{h,c}(s)\,ds.$$

We recall that $\ell = \ell_- \cup \ell_+$ where $\ell_\pm$ are straight line segments joining $1+\varepsilon-ic_0'T$ to $1+\varepsilon-ic_0T$ and $1+\varepsilon+ic_0T$ to $1+\varepsilon+ic_0'T$ respectively. Hence, $\sum''(C,T)$ takes the decomposition

$$\textstyle\sum''(C,T) = \sum_F''(C,T) + \sum_{\widetilde{R}}''(C,T) + \sum_{\widetilde{d}}''(C,T) + \sum_{\widetilde{E}}''(C,T) + O(K^{-M'})$$

where $\sum''_{\widetilde{*}}(C,T)$ is defined as in (6.12) with a corresponding change in $P_*(c,h,Y)$.

By (6.4), we shift horizontally the path ℓ to real part equal to $-1+\varepsilon$, on which $D_g^F(s,1,1,h)$ is still absolutely convergent. Therefore, we infer that

$$\begin{aligned} P_F(c,h,Y) &\ll \int_{c_0T}^{c_0'T} |D_g^F(-1+\varepsilon+it,1,1,h)||\widetilde{G}_{h,c}(-1+\varepsilon+it)|\,dt + |h|^{2+\varepsilon}T^{-M'} \\ &\ll TK^{\epsilon}Y^{-1-(\lambda+j)/2}(CT)^{3/2} \end{aligned}$$

by (5.25) in Lemma 5.6 and (11.2). From (6.12), we get that

$$\begin{aligned} \textstyle\sum''_F(C,T) &\ll TK^{\epsilon}Y^{-1-(\lambda+j)/2}(CT)^{3/2} \sum_{|h|\sim CT} \sum_{\delta|h} \delta \sum_{\substack{c\sim C \\ \mathcal{N}|c,\, \delta|c}} c^{j+\lambda-1} \\ &\ll TK^{\epsilon}Y^{-1}(CT)^{5/2}\left(\frac{C}{\sqrt{Y}}\right)^{j+\lambda} \\ &\ll K^{1+\epsilon}Y^{1/2}\left(\frac{K^2}{L^6}\right)\left(\frac{C}{\sqrt{Y}}\right)^{j+\lambda} \end{aligned}$$

as $T \ll K^2C/Y \le K^{1+\varepsilon}/L$, $CT \le YK^{2\varepsilon}/L^2$ and $Y \le K^{2+\varepsilon}$. This can be absorbed in (10.1).

It remains to check that the estimation of $\sum''_*(C,T)$ in §§7-9 is valid for $\sum''_{\widetilde{*}}(C,T)$ with $* = R, d, E$. In the evaluation, we need to shift the path ℓ horizontally to the left. Although the new factor (11.4) contains poles, they all lie on the real axis. As ℓ does not cross the real axis, there will be no contribution coming up from these poles. Moreover, the same asymptotic behavior shown in (11.4) verifies the applicability of our argument. This concludes that (10.1) holds true for the case of Maass g, and therefore, Theorem 1 follows. $\square$

12. Proof of Lemma 4.1

This section is devoted to prove Lemma 4.1, which consists of two parts.

Part (i). Let us write

$$W_{K,L}(x) = K\int_{\mathbb{R}} e(\phi(t))\big(h(u)(u\frac{L}{K}+1)\big)^{\wedge}(t)\,dt,$$

where $\phi(t) = tK/L + x\big(\cosh(\pi t/L)-1\big)/(2\pi)$. The Fourier transform $\big(h(u)(uL/K+1)\big)^{\wedge}(t)$ decays rapidly, and more accurately, is $\ll |t|^{-M}$ for any $M \ge 1$ and $|t| \gg 1$. Introducing a smooth partition $\varphi_1(t)+\varphi_2(t) \equiv 1$ where φ_1 is supported on $[-2,2]$ and $\varphi_1 \equiv 1$ on $[-1,1]$, we infer that

$$W_{K,L}(x) = K\int_{\mathbb{R}} \varphi_1\left(\frac{t}{K^{\varepsilon/2}}\right)e(\phi(t))\big(h(u)(u\frac{L}{K}+1)\big)^{\wedge}(t)\,dt + O(K^{-M}). \tag{12.1}$$

If $|x| \le 8\pi L K^{1-\varepsilon}$ and $|t| \le K^{\varepsilon/2}$, then the phase $\phi(t)$ satisfies

$$\phi'(t) = \frac{K}{L} + \frac{x}{2L}\sinh(\frac{\pi t}{L}) \gg \frac{K}{L} - O\Big(\frac{|x|K^{\varepsilon/2}}{L^2}\Big) \gg \frac{K}{L},$$

and $\phi^{(r)}(t) \ll |x|/L^2 < K/L$ for $r \ge 2$. The derivative of the integrand without the factor $e(\phi(t))$ is $O(1)$, hence successive integration by parts shows that the integral in (12.1) is $O(K^{-M})$. This completes part (i).

Part (ii). To prove it, we follow the line of argument in [23, §§4.1-4.4]. We apply the Taylor expansion to $\cosh(\pi t/L)$ and expand the exponential factor, like (4.2) in [23], but this time we keep the terms up to t^4, as follows. Recall $LK^{1-\varepsilon} \le |x| \ll K^{2+\varepsilon}$ and $K^{1/3+2\varepsilon} \le L \le K^{1-2\varepsilon}$. As $x/L^6 = o(1)$ and the Fourier transform of $(h(u)(uL+K)$ is rapidly decaying, we have

$$\begin{aligned} W_{K,L}(x) \;\; &= \sum_{0\le 3\mu\le\nu\le N} c_{\mu,\nu}\frac{x^\mu}{L^{2\nu}}\int_{\mathbb{R}} e\Big(\frac{tK}{L}+\frac{\pi x t^2}{4L^2}\Big)e\Big(\frac{\pi^3 x t^4}{48L^4}\Big)t^{2\nu}\big(h(u)(uL+K)\big)^\wedge(t)\,dt \\ &\quad + O\left(\frac{|x|^{N/3}}{L^{2N}}\int_{\mathbb{R}}|t|^{2N}\left|\big(h(u)(uL+K)\big)^\wedge(t)\right|\,dt\right). \end{aligned}$$

Now, as $|x| \ll K^{2+\varepsilon}$ and $L \ge K^{1/3+2\varepsilon}$, the O-term is

$$\ll K\Big(\frac{K^{2+\varepsilon}}{L^6}\Big)^{N/3} \ll K^{1-\varepsilon N} \ll K^{-M},$$

by choosing a suitable $N = N_1(\varepsilon, M)$.

Temporarily we write

$$g(t) = e\Big(\frac{\pi^3 x t^4}{48L^4}\Big)t^{2\nu}\big(h(u)(uL+K)\big)^\wedge(t).$$

By Parseval's theorem, we obtain

$$\int_{\mathbb{R}} g(t)e\Big(\frac{tK}{L}+\frac{\pi x t^2}{4L^2}\Big)\,dt = \frac{L}{\sqrt{\pi|x|}}\big(1+i\,\mathrm{sgn}(x)\big)e(-\frac{K^2}{\pi x})\int_{\mathbb{R}}\widehat{g}(u)e\Big(\frac{2uLK}{\pi x}-\frac{u^2L^2}{\pi x}\Big)\,du.$$

Since for all $u \in \mathbb{R}$,

$$e^{iu} = \sum_{0\le k\le N}\frac{(iu)^k}{k!} + O_N\big(|u|^N\big),$$

we expand the exponential factor $e(-u^2L^2/(\pi x))$ to obtain

$$\begin{aligned} \int_{\mathbb{R}}\widehat{g}(u)e\Big(\frac{2uLK}{\pi x}-\frac{u^2L^2}{\pi x}\Big)\,du \;\; &= \sum_{0\le k\le N}\frac{1}{k!}\left(-\frac{2iL^2}{x}\right)^k\int_{\mathbb{R}} u^{2k}\widehat{g}(u)e\Big(\frac{2uLK}{\pi x}\Big)\,du \\ &\quad + O\bigg(\Big(\frac{L^2}{x}\Big)^N\int_{\mathbb{R}}|u|^{2N}\big|\widehat{g}(u)\big|\,du\bigg). \end{aligned} \tag{12.2}$$

It is plain that for $r \geq 0$,

$$g^{(r)}(t) \ll_r K\left(1+\left(\frac{x}{L^4}\right)^r\right)\frac{1}{1+t^2},$$

and $\widehat{g}(u) = (2\pi i u)^{-r}\widehat{g^{(r)}}(u)$ for $u \neq 0$. Thus, we have

$$\widehat{g}(u) \ll_r K\left(1+\left(\frac{x}{L^4}\right)^r\right)\frac{1}{1+|u|^r}.$$

We shall take $r = 2N+2$. By $K^{1/3+2\varepsilon} \leq L \leq K^{1-2\varepsilon}$ and $LK^{1-\varepsilon} \leq |x| \ll K^{2+\varepsilon}$, the O-term in (12.2) is

$$\begin{aligned}
&\ll_N \quad K\left(\frac{L^2}{x}\right)^N\left(1+\left(\frac{x}{L^4}\right)^{2N+2}\right)\int_{\mathbb{R}}\frac{|u|^{2N}}{1+|u|^{2N+2}}\,du \\
&\ll \quad K\left(\frac{L^2}{x}\right)^N + K\left(\frac{x}{L^4}\right)^2\left(\frac{x}{L^6}\right)^N \\
&\ll \quad K\left(\frac{L}{K^{1-\varepsilon}}\right)^N + K\left(\frac{x}{L^4}\right)^2\left(\frac{K^{2+\varepsilon}}{L^6}\right)^N \\
&\ll \quad K^{1-\varepsilon N}\left(1+\left(\frac{x}{L^4}\right)^2\right) \ll K^{-M},
\end{aligned}$$

with a suitable $N = N_2(\varepsilon, M)$. Therefore,

$$\begin{aligned}
W_{K,L}(x) \quad &= \sum_{0\leq 3\mu\leq\nu\leq N_1} c_{\mu,\nu}\frac{x^\mu}{L^{2\nu}}\frac{L}{\sqrt{\pi|x|}}\big(1+i\,\mathrm{sgn}(x)\big)e\left(-\frac{K^2}{\pi x}\right) \\
&\times \sum_{0\leq k\leq N_2}\frac{1}{k!}\left(-\frac{2iL^2}{x}\right)^k\int_{\mathbb{R}} u^{2k}\widehat{g}(u)e\left(\frac{2uLK}{\pi x}\right)du + O(K^{-M}). \qquad (12.3)
\end{aligned}$$

The integral in (12.3) equals

$$(2\pi i)^{-2k}g^{(2k)}\left(\frac{2LK}{\pi x}\right)$$

By definition of g and Leibniz's theorem, we have

$$g^{(2k)}(t) = \sum_{0\leq\alpha\leq 2k}\binom{2k}{\alpha}\frac{d^\alpha}{dt^\alpha}e\left(\frac{\pi^3 x t^4}{48L^4}\right)\times\frac{d^{2k-\alpha}}{dt^{2k-\alpha}}t^{2\nu}\big(h(u)(uL+K)\big)^\wedge(t)$$

where $\binom{n}{r}$ denotes the binomial coefficient. Expanding in power series and differentiating termwisely, the derivative of the exponential factor is

$$\begin{aligned}
\frac{d^\alpha}{dt^\alpha}e\left(\frac{\pi^3 x t^4}{48L^4}\right) \quad &= \quad \frac{d^\alpha}{dt^\alpha}\sum_{\beta=0}^{\infty}\frac{1}{\beta!}\left(\frac{i\pi^4}{24}\right)^\beta\left(\frac{x}{L^4}\right)^\beta t^{4\beta} \\
&= \quad \sum_{4\beta\geq\alpha}\frac{(4\beta)!}{(4\beta-\alpha)!\beta!}\left(\frac{i\pi^4}{24}\right)^\beta\left(\frac{x}{L^4}\right)^\beta t^{4\beta-\alpha}.
\end{aligned}$$

Next we show that the tail is negligible about $t = 2LK/(\pi x)$. Indeed, as $K^4/x^3 \ll K^{1+3\varepsilon}/L^3 \ll K^{-3\varepsilon}$, we have for $N \geq 8N_2$,

$$\begin{aligned} & \sum_{\beta > N} \frac{(4\beta)!}{(4\beta-\alpha)!\beta!} \left(\frac{\pi^4}{24}\right)^\beta \left(\frac{x}{L^4}\right)^\beta \left(\frac{2LK}{\pi x}\right)^{4\beta-\alpha} \\ = & \left(\frac{\pi x}{2LK}\right)^\alpha \sum_{\beta > N} \frac{(4\beta)!}{(4\beta-\alpha)!\beta!} \left(\frac{2}{3}\right)^\beta \left(\frac{K^4}{x^3}\right)^\beta \\ \ll & \left(\frac{\pi x}{2LK}\right)^\alpha \left(\frac{K^4}{x^3}\right)^N \sum_{4\beta \geq \alpha} \frac{(4\beta)!}{(4\beta-\alpha)!\beta!} \left(\frac{2}{3}\right)^\beta \\ \ll_\alpha & \left(\frac{x}{LK}\right)^\alpha K^{-3\varepsilon N}. \end{aligned}$$

Note that the last sum over $4\beta \geq \alpha$ yields the constant

$$\frac{d^\alpha}{dt^\alpha}\bigg|_{t=1} e^{2t^4/3}.$$

Thus

$$\begin{aligned} \frac{d^\alpha}{dt^\alpha} e\left(\frac{\pi^3 x t^4}{48L^4}\right)\bigg|_{t=2LK/(\pi x)} = & \sum_{\alpha \leq 4\beta \leq 4N} \frac{(4\beta)!}{(4\beta-\alpha)!\beta!} \left(\frac{i\pi^4}{24}\right)^\beta \left(\frac{x}{L^4}\right)^\beta \left(\frac{2LK}{\pi x}\right)^{4\beta-\alpha} \\ & + O\left(\left(\frac{x}{LK}\right)^\alpha K^{-3\varepsilon N}\right). \end{aligned}$$

As

$$t^{2\nu}\big(h(u)(uL+K)\big)^\wedge(t) = \frac{1}{(2\pi i)^{2\nu}}\big(\frac{d^{2\nu}}{du^{2\nu}} h(u)(uL+K)\big)^\wedge(t),$$

all its rth derivatives at $t = 2LK/(\pi x)$ are $O(K)$ and the O-constant depends only on ν and r for a given h. Together with

$$\frac{d^{2k-\alpha}}{dt^{2k-\alpha}} t^{2\nu}\big(h(u)(uL+K)\big)^\wedge(t) = (-2\pi i)^{2k-\alpha}\left(u^{2k-\alpha}\frac{d^{2\nu}}{du^{2\nu}}\big(h(u)(uL+K)\big)\right)^\wedge(t),$$

the integral in (12.3) equals

$$\int_{\mathbb{R}} u^{2k}\widehat{g}(u) e\left(\frac{2uLK}{\pi x}\right) du = \sum\nolimits_{\alpha,\beta} + O\left(\left(1 + \left(\frac{x}{LK}\right)^{2k}\right) K^{-3\varepsilon N}\right) \tag{12.4}$$

where with some suitable coefficients $c_{k,\alpha,\beta}$,

$$\begin{aligned} & \sum\nolimits_{\alpha,\beta} \\ = & \sum_{0 \leq \alpha \leq 2k} \sum_{\alpha \leq 4\beta \leq N} c_{k,\alpha,\beta} \left(\frac{x}{LK}\right)^\alpha \left(\frac{K^4}{x^3}\right)^\beta \left(u^{2k-\alpha}\frac{d^{2\nu}}{du^{2\nu}}\big(h(u)(uL+K)\big)\right)^\wedge \left(\frac{2LK}{\pi x}\right). \end{aligned}$$

Hence, we may select $N = N_3(\varepsilon, M, N_2)$ so that the cumulative error in (12.3) caused by the O-term in (12.4) is $O(K^{-M})$. Then, we conclude

$$\begin{aligned} W_{K,L}(x) \quad = \quad & \sum_{0\le 3\mu\le\nu\le N_1} \sum_{0\le k\le N_2} \sum_{0\le\alpha\le 2k} \sum_{\alpha\le 4\beta\le N} c_{\mu,\nu,k,\alpha,\beta} \frac{x^\mu}{L^{2\nu}} \left(\frac{L^2}{x}\right)^k \left(\frac{x}{LK}\right)^\alpha \left(\frac{K^4}{x^3}\right)^\beta \\ & \times \big(1 + i\,\mathrm{sgn}(x)\big) \frac{L}{\sqrt{\pi|x|}} e\left(-\frac{K^2}{\pi x}\right) \left(u^{2k-\alpha} \frac{d^{2\nu}}{du^{2\nu}} \big(h(u)(uL+K)\big)\right)^\wedge \left(\frac{2LK}{\pi x}\right) \\ & + O(K^{-M}), \end{aligned}$$

where the summands are $c_{\underline{\lambda}} \widetilde{W}_{\underline{\lambda}}(x)$, with a little but straightforward computation. This ends the proof. □

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References

[1] J. Bernstein and A. Reznikov, Analytic continuation of representations and estimates of automorphic forms, *Ann. Math.*, **150** (1999), 329-352.

[2] V. Blomer, Shifted convolution sums and subconvex bounds for automorphic L-functions, *Internat. Math. Res. Notices*, **73** (2004), 3905-3926.

[3] V. Blomer, Rankin-Selberg L-functions on the critical line, *Manuscripta Math.*, **117** (2005), 111-133.

[4] V. Blomer, G. Harcos, and P. Michel, A Burgess-like subconvexity bound for twisted L-functions, to appear in *Forum math.*

[5] J.B. Conrey and H. Iwaniec, The cubic moment of central values of automorphic L-functions, *Ann. Math.*, **151** (2000), 1175-1216.

[6] J.-M. Deshouillers and H. Iwaniec, Kloosterman Sums and Fourier Coefficients of Cusp Forms, *Invent. Math.*, **70** (1982), 219-288.

[7] A. Good, Cusp Forms and Eigenfunctions of the Laplacian, *Math. Ann.*, **255** (1981), 523-548.

[8] A. Good, Beiträge zur Theorie der Dirichletreihen, die Spitzenformen zugeordnet sind, *J. Number Theory*, **13** (1981), 18-65.

[9] A. Good, The square mean of Dirichlet series associated with cusp forms, *Mathematika*, **29** (1982), 278-295.

[10] A. Good, The convolution method for Dirichlet series, in: D. Hejhal, P. Sarnak, A. Terras (Eds.), *The Selberg trace formula and related topics (Brunswick, Maine, 1984)*, Amer. Math. Soc. Contemporary Mathematics, Vol. 53, Amer. Math. Soc., Providence, RI, 1986, 207-214.

[11] G. Harcos and P. Michel, The subconvexity problem for Rankin-Selberg L-functions and equidistribution of Heegner points II, *Invent. Math.*, **163** (2006), 581-655.

[12] J. Hoffstein and P. Lockhart, Coefficients of Maass forms and the Siegel zero, *Ann. Math.*, **140** (1994), 161-181.

[13] M.N. Huxley, On stationary phase integrals, *Glasgow Math. J.*, **36** (1994), 355-362.

[14] M.N. Huxley, *Area, lattice points, and exponential sums*, London Mathematical Society Monographs, Oxford University Press, Oxford, 1996.

[15] A. Ivic, *The Riemann zeta-function The theory of the Riemann zeta-function with applications*, John Wiley & Sons, New York 1985.

[16] A. Ivic, On sums of Hecke series in short intervals, *J. Théor. Nombres Bordeaux*, **13** (2001), 453-468.

[17] H. Iwaniec, *Spectral Methods of Automorphic Forms*, 2nd edition, Amer. Math. Soc. and Revista Matemática Iberoamericana, Providence, 2002.

[18] M. Jutila and Y. Motohashi, Uniform bounds for Hecke L-Functions, *Acta Math.*, **195** (2005), 61-115.

[19] H. Kim and P. Sarnak, Appendix 2: Refined estimates towards the Ramanujan and Selberg conjectures, *J. Amer. Math. Soc.*, **16** (2003), 175-181.

[20] E. Kowalski, P. Michel and J. VanderKam, Rankin-Selberg L-functions in the level aspect, *Duke Math. J.*, **114** (2002), 123-191.

[21] B. Krötz and R. J. Stanton, Holomorphic extensions of representations (I): automorphic functions, *Ann. Math.*, **159** (2004), 641-724.

[22] Y.-K. Lau, J. Liu, and Y. Ye, Subconvexity bounds for Rankin-Selberg L-functions for congruence subgroups, to appear in *J. Number Theory.*

[23] J. Liu and Y. Ye, Subconvexity for Rankin-Selberg L-functions of Maass forms, *Geom. Funct. Anal.*, **12** (2002), 1296-1323.

[24] J. Liu and Y. Ye, Petersson and Kuznetsov trace formulas, in: *Lie Groups and Automorphic Forms*, Amer. Math. Soc. and International Press, Providence, 2006, 147-168.

[25] T. Meurman, On the order of the Maass L-function on the critical line, in: K. Györy, G. Halász (Eds.), *Number Theory, Vol. I (Budapest, 1987)*, Colloquium of Mathematical Society Janos Bolyai, Vol. 51, North-Holland, Amsterdam, 1990, 325-354.

[26] P. Michel, The subconvexity problem for Rankin-Selberg L-functions and equidistribution of Heegner points, *Ann. Math.*, **160** (2004), 185-236.

[27] Z. Peng, Zeros and central values of automorphic L-functions, Ph.D thesis, Princeton University, Princeton, 2001.

[28] P. Sarnak, Integrals of products of eigenfunctions, *Internat. Math. Res. Notices*, **6** (1994), 251-260.

[29] P. Sarnak, Estimates for Rankin-Selberg L-functions and Quantum Unique Ergodicity, *J. Funct. Anal.*, **184** (2001), 419-453.

[30] P. Sarnak, *Lecture at the Ohio State University*, March, 2003.

[31] G. N. Watson, *A treatise on the theory of Bessel functions*, 2nd edition, Cambridge University Press, Cambridge, 1966.

[32] Y. Ye, The fourth power moment of automorphic L-functions for $GL(2)$ over a short interval, *Trans. Amer. Math. Soc.*, **358** (2006), 2259-2268.

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