

Sub-additive functionals, information theory, and non-convex optimization

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The Chinese University of Hong Kong

Workshop on Probability and Information Theory

21st August, 2019

Sub-additive functionals, information theory, and non-convex optimization

- Introduction
 - ★ Building blocks
 - ★ How to test the optimality of coding schemes
 - ★ Where do the optimization problems arise?

- Sub-additive functionals
 - ★ Gallager-style proofs of sub-additivity
 - ★ Consequences: inequalities and Gaussian optimality

- Observations and potential future directions

Point-to-point communication

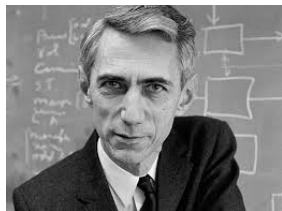


A rate R is achievable if there exists a sequence of encoding/decoding maps so that $\mathbf{P}(M \neq \hat{M}) \rightarrow 0$ as $n \rightarrow \infty$. Capacity, $C(W) := \sup\{R : R \text{ is achievable}\}$.

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Shannon

Random coding can be used to achieve

$$R(W) = \sup_{\mu(x)} I(X; Y)$$

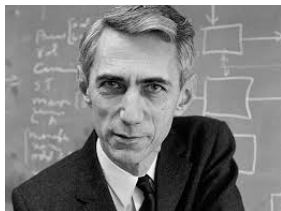
$$\text{where } I(X; Y) := \sum_{x,y} \mu_{X,Y}(x,y) \log \left(\frac{\mu_{X,Y}(x,y)}{\mu_X(x)\mu_Y(y)} \right)$$

$I(X; Y)$: mutual information between X and Y

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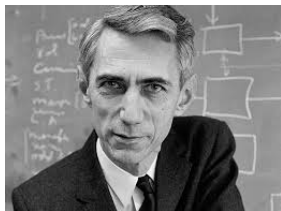
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Question: Is $R(W) = C(W)$? (**YES**) (Shannon '48)

Testing optimality

It is easy (why?) to see that $R(W)$ is optimal if and only if

$$R(W \otimes W) = 2R(W) \quad \forall W.$$

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Sub-additivity

A functional defined over a probability simplex is said to be sub-additive if

$$F_{12}(\mu_{X_1, X_2}) \leq F_1(\mu_{X_1}) + F_2(\mu_{X_2}) \quad \forall \mu_{X_1, X_2}.$$

In above, since W is fixed, $I(X; Y)$ is a functional over μ_X , the space of input distributions.

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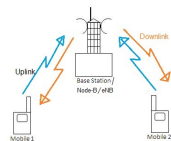
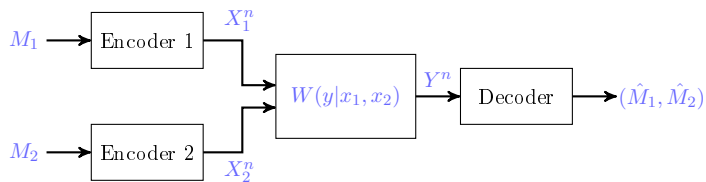
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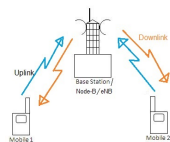
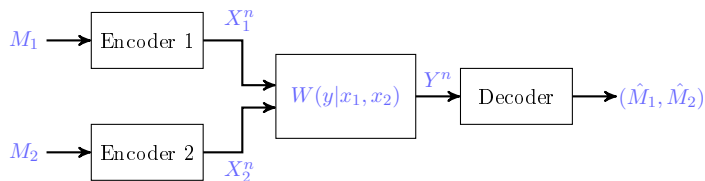
Note: Computing $R(W) = \sup_{p(x)} I(X; Y)$ is relatively easy, since $I(X; Y)$ is a concave function of $p(x)$.

1. Multiple Access Channel (uplink) (Shannon '61)



rfwireless-world

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Ahlswede

Random coding can be used to achieve rate pairs (R_1, R_2) that satisfy

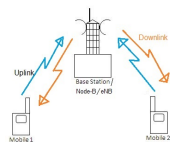
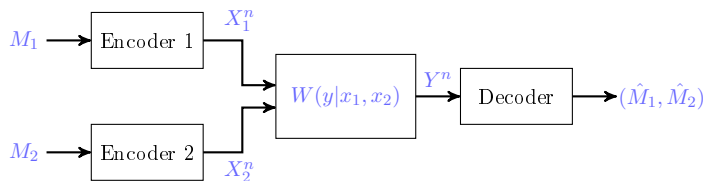
$$R_1 \leq I(X_1; Y|X_2, Q)$$

$$R_2 \leq I(X_2; Y|X_1, Q)$$

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for some $p(q)p(x_1|q)p(x_2|q)$; it suffices to consider $|\mathcal{Q}| \leq 2$. Call this region $\mathcal{R}(W)$.

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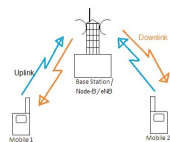
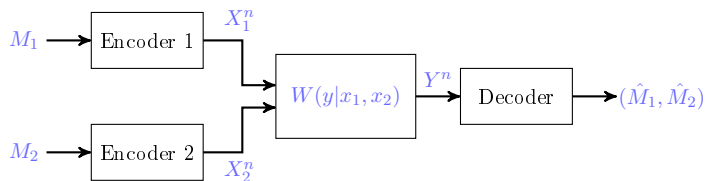
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Question: Is this the capacity (optimal) region?

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Question: Is this the capacity (optimal) region? **(YES)** (Ahlswede '72)

Testing optimality

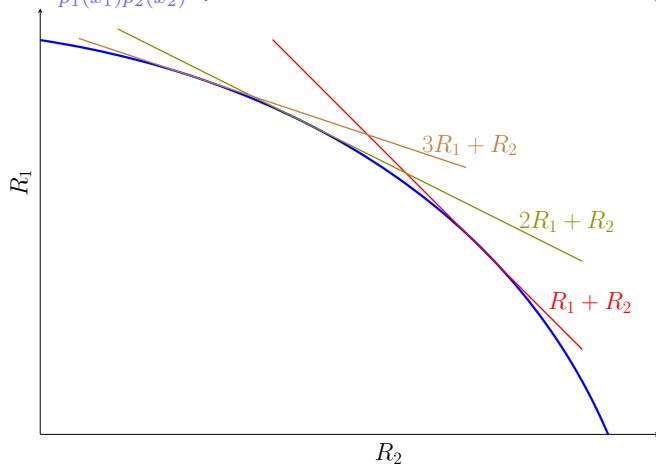
Define, for $\lambda \geq 1$,

$$\begin{aligned} S_\lambda(W) &= \max_{(R_1, R_2) \in \mathcal{R}(W)} \{ \lambda R_1 + R_2 \} \\ &= \max_{p_1(x_1)p_2(x_2)} \{ (\lambda - 1)I(X_1; Y|X_2) + I(X_1, X_2; Y) \} \end{aligned}$$

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Supporting hyperplanes

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As before, $\mathcal{R}(W)$ is optimal if and only if

$$S_\lambda(W \otimes W) = 2S_\lambda(W) \quad \forall W, \lambda \geq 1.$$

The above equality (additivity) follows if the following sub-additivity holds:

$$\begin{aligned} &(\lambda - 1)I(X_{11}, X_{12}; Y_1, Y_2|X_{21}, X_{22}) + I(X_{11}, X_{12}, X_{21}, X_{22}; Y_1, Y_2) \\ &\leq (\lambda - 1)I(X_{11}; Y_1|X_{21}) + I(X_{11}, X_{21}; Y_1) \\ &\quad + (\lambda - 1)I(X_{12}; Y_2|X_{22}) + I(X_{12}, X_{22}; Y_2) \end{aligned}$$

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One can establish this in same way as point-to-point setting.

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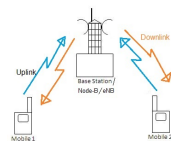
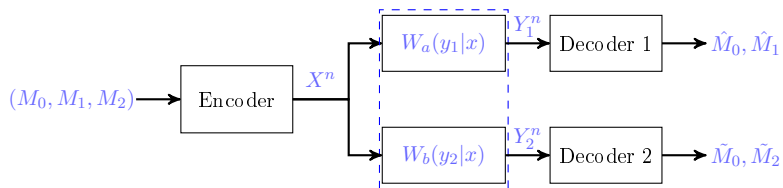
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Note: Computing $S_\lambda(W)$ is relatively easy since

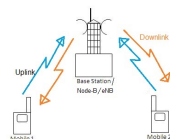
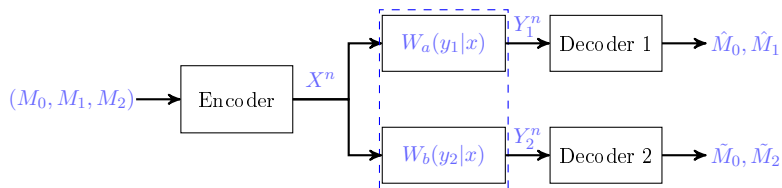
$\{ (\lambda - 1) I(X_1; Y | X_2) + I(X_1, X_2; Y) \}$ is concave in $p_1(x_1), p_2(x_2)$.

2. Broadcast channel (downlink) (Cover '72)



rfwireless-world

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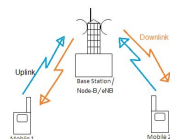
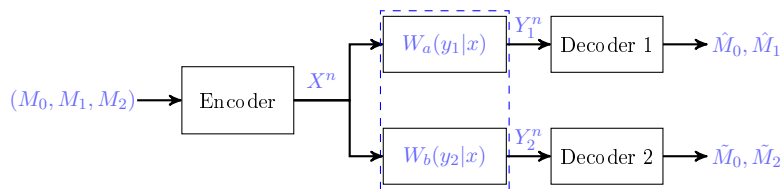
Marton

Superposition coding and random hashing can be used to achieve rate triples (R_0, R_1, R_2) that satisfy

$$\begin{aligned}R_0 &\leq \min\{I(Q; Y_1), I(Q; Y_2)\} \\R_0 + R_1 &\leq I(U, Q; Y_1) \\R_0 + R_2 &\leq I(V, Q; Y_2) \\R_0 + R_1 + R_2 &\leq \min\{I(Q; Y_1), I(Q; Y_2)\} + I(U; Y_1|Q) \\&\quad + I(V; Y_2|Q) - I(U; V|Q)\end{aligned}$$

for some $p(q, u, v, x)$. Call this region $\mathcal{R}(W_a, W_b)$.

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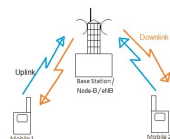
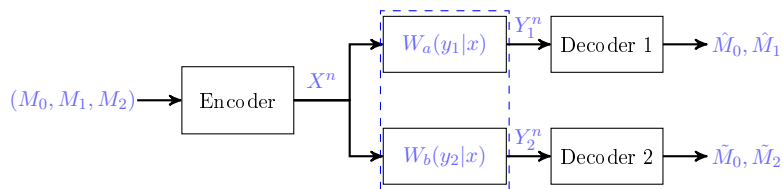
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Question: Is this the capacity (optimal) region? (**Open**) (since Marton '79)

Testing optimality ($R_0 = 0$)

Define, for $\lambda \geq 1$,

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Note: Computing $S_\lambda(W_a, W_b)$ is a **non-convex** optimization problem.

Successes

In spite of the underlying problem being intrinsically non-convex

- $\mathcal{R}(W_a, W_b)$ is optimal on $R_1 = 0$ (or $R_2 = 0$)
 - ★ Degraded message sets: Korner and Marton ('77)
- $\mathcal{R}(W_a, W_b)$ is optimal for some classes of channels
 - ★ Gallager '74, Korner and Marton ('75, '77, '79), Gelfand and Pinsker ('78), Poltyrev ('78), El Gamal ('79, '80)
 - ★ Weingarten and Steinberg and Shamai '06, Nair '10, Geng and Gohari and Nair and Yu '14, Geng and Nair '14
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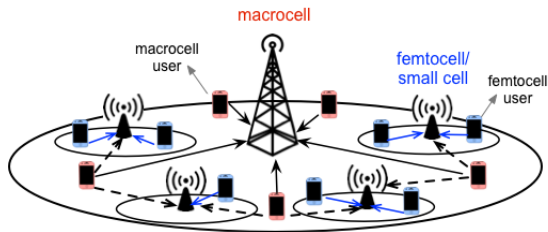
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 - ★ **Cover '72** : development of superposition coding strategy
 - ★ **Gallager '74** : converse to the degraded broadcast channel (sub-additivity)
 - ★ **Weingarten-Steinberg-Shamai '06** : Optimality of $\mathcal{R}(W_a, W_b)$ (on $R_0 = 0$) for Gaussian broadcast channel; developing a family of tight convex relaxations to compute the optimal value of a non-convex optimization problem

Successes

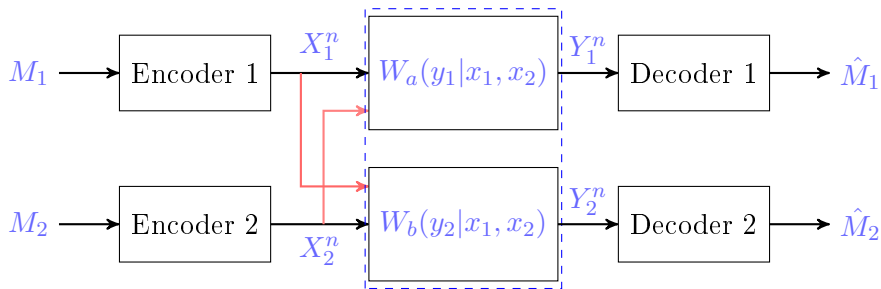
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 - ★ Degraded message sets: Korner and Marton ('77)
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 - ★ Gallager '74, Korner and Marton ('75, '77, '79), Gelfand and Pinsker ('78), Poltyrev ('78), El Gamal ('79, '80)
 - ★ Weingarten and Steinberg and Shamai '06, Nair '10, Geng and Gohari and Nair and Yu '14, Geng and Nair '14
- Novel ideas and techniques were needed to establish these capacity regions
 - ★ **Cover '72** : development of superposition coding strategy
 - ★ **Gallager '74** : converse to the degraded broadcast channel (sub-additivity)
 - ★ **Weingarten-Steinberg-Shamai '06** : Optimality of $\mathcal{R}(W_a, W_b)$ (on $R_0 = 0$) for Gaussian broadcast channel; developing a family of tight convex relaxations to compute the optimal value of a non-convex optimization problem
 - ★ **Geng-Nair '14** : Optimality of $\mathcal{R}(W_a, W_b)$ for Gaussian broadcast channel: Technique for establishing extremality of Gaussian distributions using sub-additivity of functionals

3. Interference Channel (Ahlswede '74)



Credit: www.personal.psu.edu/bxg215/research.html



3. Interference Channel (Ahlsvede '74)



Han



Kobayashi

Superposition coding, message splitting, coded time-sharing can be used to achieve rate pairs (R_1, R_2) that satisfy

$$R_1 < I(X_1; Y_1 | U_2, Q),$$

$$R_2 < I(X_2; Y_2 | U_1, Q),$$

$$R_1 + R_2 < I(X_1, U_2; Y_1 | Q) + I(X_2; Y_2 | U_1, U_2, Q),$$

$$R_1 + R_2 < I(X_2, U_1; Y_2 | Q) + I(X_1; Y_1 | U_1, U_2, Q),$$

$$R_1 + R_2 < I(X_1, U_2; Y_1 | U_1, Q) + I(X_2, U_1; Y_2 | U_2, Q),$$

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$$R_1 + 2R_2 < I(X_2, U_1; Y_2 | Q) + I(X_2; Y_2 | U_1, U_2, Q) + I(X_1, U_2; Y_1 | U_1, Q)$$

for some pmf $p(q)p(u_1, x_1 | q)p(u_2, x_2 | q)$, where $|U_1| \leq |X_1| + 4$, $|U_2| \leq |X_2| + 4$, and $|Q| \leq 7$. Call this region $\mathcal{R}(W_a, W_b)$.

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for some pmf $p(q)p(u_1, x_1|q)p(u_2, x_2|q)$, where $|U_1| \leq |X_1| + 4$, $|U_2| \leq |X_2| + 4$, and $|Q| \leq 7$. Call this region $\mathcal{R}(W_a, W_b)$.

Question: Is this the capacity region?

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for some pmf $p(q)p(u_1, x_1|q)p(u_2, x_2|q)$, where $|U_1| \leq |X_1| + 4$, $|U_2| \leq |X_2| + 4$, and $|Q| \leq 7$. Call this region $\mathcal{R}(W_a, W_b)$.

Question: Is this the capacity region?

Had been **open** (since Han and Kobayashi '81)

Successes

In spite of the underlying problem being intrinsically non-convex

- $\mathcal{R}(W_a, W_b)$ is optimal for some classes of channels
 - ★ Carleial '75, Sato '81, El Gamal and Costa ('81 and '86)
- $\mathcal{R}(W_a, W_b)$ is close to optimal for Gaussian Interference channel
 - ★ Etkin and Tse and Wang ('09)
- Novel ideas and mathematical results came out from the investigation of optimality
 - ★ Concavity of entropy power (Costa '85)
 - ★ Genie based approach to prove sub-additivity (El Gamal and Costa '81, Kramer '02)

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 - ★ Concavity of entropy power (Costa '85)
 - ★ Genie based approach to prove sub-additivity (El Gamal and Costa '81, Kramer '02)
- $\mathcal{R}(W_a, W_b)$ is not optimal in general (Nair, Xia, Yazdanpanah '15)

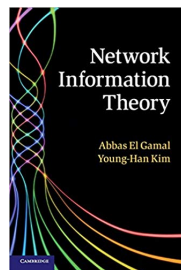
Broadcast and interference channels are far too important

- To let non-convexity dissuade us
- To not investigate simple classes that require new ideas

A class of open problems

A sub-collection of the 15 open problems listed in Chaps. 5-9.

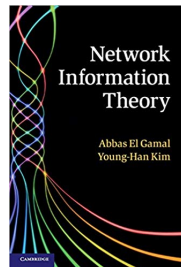
- 5.1 What is the capacity region of less noisy broadcast-channels with four or more receivers?
(two-receiver: Korner-Marton '76, three-receiver: Nair-Wang '10)
- 5.2 What is the capacity region of more capable broadcast-channels with three or more receivers?
(two-receiver: El Gamal '79)
- 6.1 What is the capacity region of the Gaussian Interference channel with weak interference?
(strong-interference: Sato '79; mixed-interference corner-points: Sato' 81, Costa'85; weak-interference corner-points: rate-sum (partial): three-groups '09)
- 6.4 Is the Han-Kobayashi inner bound tight in general for interference channels?
- 8.2 Is superposition coding optimal for the general 3-receiver broadcast channel with one message to all three receivers and another message to two receivers?
- 8.3 What is the sum-capacity of the binary skew-symmetric broadcast channel?
- 8.4 Is the Marton inner bound tight in general for broadcast channels?
- 9.2 Can the converse for the Gaussian broadcast channel be proved directly by optimizing the Nair-El Gamal outer bound?
- 9.3 What is the capacity region of the 2-receiver Gaussian broadcast channel with common message?



A class of open problems

My reformulations of a few of them.

- 5.1 Is superposition coding optimal for less-noisy broadcast channels with four or more receivers?
- 5.2 Is superposition coding optimal for more-capable broadcast channels with three or more receivers?
- 6.1 Is the Han-Kobayashi scheme with Gaussian signaling tight for the Gaussian Interference channel with weak interference?
- 6.4 Is the Han-Kobayashi inner bound tight in general for interference channels?
- 8.2 Is superposition coding optimal for the general 3-receiver broadcast channel with one message to all three receivers and another message to two receivers?
- 8.3 Does the Marton inner bound achieve the sum-capacity of the binary skew-symmetric broadcast channel?
- 8.4 Is the Marton inner bound tight in general for broadcast channels?
- 9.2 Can the converse for the Gaussian broadcast channel be proved directly by optimizing the Nair-El Gamal outer bound?
- 9.3 Does the Marton inner bound achieve the capacity region of the 2-receiver Gaussian broadcast channel with common message?



The common theme to these (reformulated) questions

Common theme

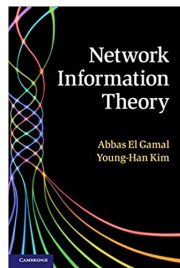
Is a candidate rate region optimal?

Idea for testing optimality:

- $S_\lambda(W \otimes W) \stackrel{?}{=} 2S_\lambda(W)$
- Determine sub-additivity of an associated **non-convex** functional

Status of the open problems

- 5.1 Is superposition coding optimal for less-noisy broadcast channels with four or more receivers?(OPEN)
- 5.2 Is superposition coding optimal for more-capable broadcast channels with three or more receivers?
(NO: Nair-Xia '12)
- 6.1 Is the Han-Kobayashi scheme with Gaussian signaling tight for the Gaussian Interference channel with weak interference?(OPEN) (YES: corner points using ideas in measure transportation by Polyanskiy-Wu '15)
- 6.4 Is the Han-Kobayashi inner bound tight in general for interference channels?
(NO: Nair-Xia-Yazdanpanah '15)
- 8.2 Is superposition coding optimal for the general 3-receiver broadcast channel with one message to all three receivers and another message to two receivers?
(NO: Nair-Yazdanpanah '17)
- 8.3 Does the Marton inner bound achieve the sum-capacity of the binary skew-symmetric broadcast channel?(OPEN)
- 8.4 Is the Marton inner bound tight in general for broadcast channels?(OPEN)
- 9.2 Can the converse for the Gaussian broadcast channel be proved directly by optimizing the Nair-El Gamal outer bound?(YES: Geng-Nair '14)
- 9.3 Does the Marton inner bound achieve the capacity region of the 2-receiver Gaussian broadcast channel with common message?(YES: Geng-Nair '14)



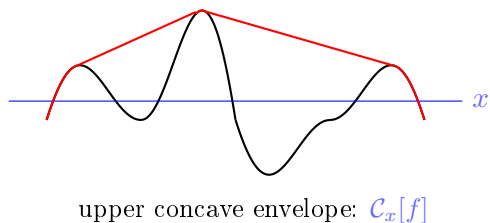
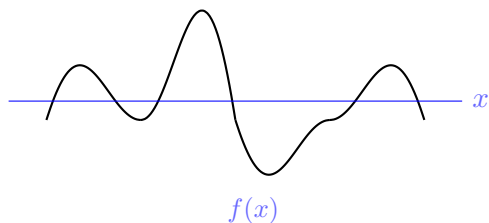
Part II: Sub-additive functionals

- Two examples of sub-additive functionals
 - ★ Gallager-style proofs of sub-additivity
 - ★ Consequences: inequalities and Gaussian optimality

- Family of non-convex optimization problems
 - ★ Relation to problems of interest in other fields
 - ★ Unifying observations and some conjectures

The "Lieb-Stam" functional (j.w.: Anantharam and Jog '19)

Upper-concave envelope



Not easy to compute (in general)

The "Lieb-Stam" functional (j.w.: Anantharam and Jog '19)

Let $X \sim \mu_X$ and $Y \sim \nu_Y$

$$\begin{aligned} F_{LS}(\mu_X, \nu_Y) &:= \sup_{U: X-U-Y} ah(X|U) + bh(Y|U) - h(X + Y|U) \\ &= \mathcal{C}[ah(X) + bh(Y) - h(X + Y)]|_{\mu_X, \nu_Y} \end{aligned}$$

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Proposition

The "Lieb-Stam" functional is sub-additive, i.e.

$$F_{LS}(\mu_{X_1 X_2}, \nu_{Y_1 Y_2}) \leq F_{LS}(\mu_{X_1}, \nu_{Y_1}) + F_{LS}(\mu_{X_2}, \nu_{Y_2})$$

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"Gallager-style" proof of sub-additivity

$$\begin{aligned} &ah(X_1, X_2|U) + bh(Y_1, Y_2|U) - h(X_1 + Y_1, X_2 + Y_2|U) \\ &= ah(X_1|U) + bh(Y_1|U) - h(X_1 + Y_1|U) \\ &\quad + ah(X_2|U, X_1) + bh(Y_2|U, Y_1) - h(X_2 + Y_2|U, X_1 + Y_1) \\ &\leq ah(X_1|U) + bh(Y_1|U) - h(X_1 + Y_1|U) \\ &\quad + ah(X_2|U, X_1, Y_1) + bh(Y_2|U, X_1, Y_1) - h(X_2 + Y_2|U, X_1, Y_1) \\ &\leq F_{LS}(\mu_{X_1}, \nu_{Y_1}) + F_{LS}(\mu_{X_2}, \nu_{Y_2}). \end{aligned}$$

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One can also define, more generally, for a channel (Markov operator) $W(z|x, y)$

$$\begin{aligned} F_{LS}^W(\mu_X, \nu_Y) &:= \sup_{U: X-U-Y} ah(X|U) + bh(Y|U) - h(Z|U) \\ &= \mathcal{C}[ah(X) + bh(Y) - h(Z)]|_{\mu_X, \nu_Y} \end{aligned}$$

One can establish similarly that

$$F_{LS}^{W_1 \otimes W_2}(\mu_{X_1 X_2}, \nu_{Y_1 Y_2}) \leq F_{LS}^{W_1}(\mu_{X_1}, \nu_{Y_1}) + F_{LS}^{W_2}(\mu_{X_2}, \nu_{Y_2})$$

Korner-Marton functional (j.w.: Geng '14)

Let $X \sim \mu_X$ $W_a(y|x)$ and $W_b(z|x)$ be two channels and $\lambda \geq 1$.

$$\begin{aligned} F_{KM}^{\lambda, W_a, W_b}(\mu_X) &:= \sup_{U: U-X-(Y,Z)} I(X; Y|U) - \lambda I(X; Z|U) \\ &= \mathcal{C}[I(X; Y) - \lambda I(X; Z)]|_{\mu_X} \end{aligned}$$

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"Gallager-style" proof of sub-additivity

$$\begin{aligned} &I(X_1, X_2; Y_1, Y_2|U) - \lambda I(X_1, X_2; Z_1, Z_2|U) \\ &= I(X_1; Y_1|U, Z_2) - \lambda I(X_1; Z_1|U, Z_2) \\ &\quad + I(X_2; Y_2|U, Y_1) - \lambda I(X_2; Z_2|U, Y_1) - (\lambda - 1)I(Y_1; Z_2|U) \\ &\leq F_{KM}^{\lambda, W_a, W_b}(\mu_{X_1}) + F_{KM}^{\lambda, \hat{W}_a, \hat{W}_b}(\mu_{X_2}). \end{aligned}$$

Remarks: Sub-additive functionals

One can extract from

- outer bounds (channel coding)
- lower bounds (source coding)

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- Proofs of sub-additivity are almost exclusively "Gallager-style"
- The "art" has been in determining which functional is sub-additive
 - ★ Lift to a higher-dimensional space (Genie approach)
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Wish List: It would be nice to have a "repository" of sub-additive functionals

- like OEIS but much smaller magnitude obviously

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Next: Gaussian Optimality from sub-additivity

Gaussian optimality via sub-additivity (Geng-Nair '14)

Korner-Marton functional - extremal distribution

Maximize, for $\lambda > 1$, the value of the functional

$$\mathcal{C}_{\mu_X}[h(AX + Z) - \lambda h(BX + Z)]$$

over $X : E(XX^T) \preceq K$, where A, B are invertible matrices and $Z \sim \mathcal{N}(0, I)$.

We will see that the maximum value is

$$h(AX_* + Z) - \lambda h(BX_* + Z),$$

where $X_* \sim \mathcal{N}(0, K')$ for some $K' \preceq K$.

Gaussian optimality via sub-additivity (Geng-Nair '14)

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where $X_* \sim \mathcal{N}(0, K')$ for some $K' \preceq K$.

Lemma: $\mathcal{C}_{\mu_X}[h(AX + Z) - \lambda h(BX + Z)]$ is sub-additive.

Proof: For any μ_{X_1, X_2}

$$\begin{aligned} & h(AX_1 + Z_1, AX_2 + Z_2|U) - \lambda h(BX_1 + Z_1, BX_2 + Z_2|U) \\ &= h(AX_1 + Z_1|U, AX_2 + Z_2) - \lambda h(BX_1 + Z_1|U, AX_2 + Z_2) \\ & \quad + h(AX_2 + Z_2|U, BX_1 + Z_1) - \lambda h(BX_2 + Z_2|U, BX_1 + Z_1) \\ & \quad - (\lambda - 1)I(AX_2 + Z_2; BX_1 + Z_1|U) \end{aligned}$$

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Proof: For any μ_{X_1, X_2}

$$\begin{aligned} & \mathcal{C}_{\mu_{X_1, X_2}}[h(AX_1 + Z_1, AX_2 + Z_2) - \lambda h(BX_1 + Z_1, BX_2 + Z_2)] \\ & \leq \mathcal{C}_{\mu_{X_1}}[h(AX_1 + Z_1) - \lambda h(BX_1 + Z_1)] \\ & \quad + \mathcal{C}_{\mu_{X_2}}[h(AX_2 + Z_2) - \lambda h(BX_2 + Z_2)] \end{aligned}$$

Gaussian optimality via sub-additivity (Geng-Nair '14)

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Gaussian optimality: ctd..

Let $(U_{\dagger}, X_{\dagger})$ be a maximizer, i.e.

$$V = \max_{\mu_X} \mathcal{C}_{\mu_X} [h(AX + Z) - \lambda h(BX + Z)] = h(AX_{\dagger} + Z|U_{\dagger}) - \lambda h(BX_{\dagger} + Z|U_{\dagger}).$$

Let (X_a, U_a) and (X_b, U_b) be i.i.d. according to $(U_{\dagger}, X_{\dagger})$.

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Setting $U = (U_a, U_b)$, $X_+ = \frac{X_a + X_b}{\sqrt{2}}$ and $X_- = \frac{X_a - X_b}{\sqrt{2}}$ the proof of sub-additivity yields

$$\begin{aligned} 2V &= \mathcal{C}_{\mu_{X_1, X_2}} [h(AX_1 + Z_1, AX_2 + Z_2) - \lambda h(BX_1 + Z_1, BX_2 + Z_2)] \Big|_{(\mu_{X_+}, \mu_{X_-})} \\ &\leq \mathcal{C}_{\mu_{X_1}} [h(AX_1 + Z_1) - \lambda h(BX_1 + Z_1)] \Big|_{\mu_{X_+}} \\ &\quad + \mathcal{C}_{\mu_{X_2}} [h(AX_2 + Z_2) - \lambda h(BX_2 + Z_2)] \Big|_{\mu_{X_-}} \\ &\quad - (\lambda - 1) I(AX_- + Z_2; BX_+ + Z_1 | U_a, U_b) \\ &\leq V + V \end{aligned}$$

Gaussian optimality: ctd..

Let $(U_{\dagger}, X_{\dagger})$ be a maximizer, i.e.

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Therefore: we get that conditioned on (U_a, U_b) : $X_+ \perp X_-$.

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Note: Thus, conditioned on (U_a, U_b) :

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Note: There are some similarities with work of Lieb and Barthe (90s)

They also use rotations (but not information measures and its algebra)

Lieb-Stam functional - extremal distribution

Maximize, for $\lambda > 1$, the value of the functional

$$\mathcal{C}_{\mu_X, \nu_Y} [ah(X) + bh(Y) - h(X + Y)]$$

over $X : E(XX^T) \preceq K_a$, and $Y : E(YY^T) \preceq K_b$.

In a very similar fashion, the maximum value is

$$ah(X_*) + bh(Y_*) - h(X_* + Y_*),$$

where $X_* \sim \mathcal{N}(0, K')$ and $Y_* \sim \mathcal{N}(0, K^\dagger)$.

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Remark: We consider a more general functional that yields EPI as one extreme and Brascamp-Lieb inequality as the other

<http://arxiv.org/abs/1901.06619>

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Corollary: Entropy-Power inequality

Observe: Sub-additivity results holds even in discrete spaces and with any channel $W(z|x, y)$.

Question: Can this be used to get new extremal inequalities (e.g. discrete EPIs)?

Another functional - (Courtade '17)

Let $X \sim \mu_X$ and Z be standard Gaussian independent of X

$$g(\mu_X) := \sup_{\substack{U, Y: (U, X) \perp Z, \\ I(X; Y|U, X+Z)=0}} \lambda h(X|U) + (1 - \lambda)h(Z) - h(X + Z|U) \\ + I(X; Y|U) - \lambda I(X + Z; Y|U)$$

Theorem

The functional $g_\epsilon(X)$ is sub-additive.

Proof: "Gallager-type"

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Theorem

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Proof: "Gallager-type"

Corollary: Courtade's "strengthened-EPI"

$$2^{2(h(X+Z)-I(X;Y))} \geq 2^{2(h(X)-I(X+Z;Y))} + 2^{2h(Z)}$$

Open problem: Marton's region for the broadcast channel?

Is the following functional sub-additive or is there an example where it is super-additive?

Let $W_a(y|x)$ and $W_b(z|x)$ be given channels, $\alpha \in [0, 1]$, and $\lambda \geq 1$.

$$C_{\mu_X} \left[(\lambda - \alpha)H(Y) - \alpha H(Z) + \max_{p(u,v|x)} \{ \lambda I(U; Y) + I(V; Z) - I(U; V) \} \right]$$

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Remarks:

- Conjectured to be sub-additive (Anantharam-Gohari-Nair '13)
- To evaluate the concave envelope
 - ★ Suffices to consider (U, V) : $|U| + |V| \leq |X| + 1$.
 - ★ We did not get any contradiction to sub-additivity for binary input broadcast channels
- Can prove sub-additivity when $\alpha = 0$ or $\alpha = 1$.

A specific family of non-convex optimization problems

Shows up: Testing the optimality of coding schemes

Testing optimality (usually) reduces to testing sub-additivity of

$$C_{\nu_X} \left[\sum_{S \subseteq [n]} \alpha_S H(X_S) \right], \alpha_S \in \mathbb{R}.$$

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Using Fenchel duality this is same as (Anantharam, Gohari, Nair '13)

$$G_1(\gamma_1) := \max_{\mu_{\mathbf{X}}} \sum_{S \subseteq [n]} \alpha_S H(X_S) - \mathbb{E}(\gamma_1(\mathbf{X}))$$

$$G_2(\gamma_2) := \max_{\mu_{\mathbf{X}}} \sum_{S \subseteq [n]} \alpha_S H(X_S) - \mathbb{E}(\gamma_2(\mathbf{X}))$$

$$G_{12}(\gamma_1, \gamma_2) := \max_{\mu_{\mathbf{X}_1, \mathbf{X}_2}} \sum_{S \subseteq [n]} \alpha_S H(X_{1S}, X_{2S}) - \mathbb{E}(\gamma_1(\mathbf{X}_1)) - \mathbb{E}(\gamma_2(\mathbf{X}_2))$$

Is $G_{12}(\gamma_1, \gamma_2) = G_1(\gamma_1) + G_2(\gamma_2) \forall \gamma_1, \gamma_2$?

i.e. Is the maximizer of G_{12} a product distribution?

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Are there other fields where the same family shows up?

Hypercontractivity

Studied in functional analysis, cs theory, etc.

Definition

$(X, Y) \sim \mu_{XY}$ is (p, q) -hypercontractive for $1 \leq q \leq p$ if

$$\|Tg\|_p \leq \|g\|_q \quad \forall g(Y)$$

where T is the Markov operator characterized by $\mu_{Y|X}$

Here $\|Z\|_p = E(|Z|^p)^{\frac{1}{p}}$.

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There is a lot of interest in evaluating hypercontractivity parameters for distributions.

Theorem (Nair '14)

$(X, Y) \sim \mu_{XY}$ is (p, q) -hypercontractive for $1 \leq q \leq p$ if and only if

$$\begin{aligned} & \mathcal{C}_{\nu_{X,Y}} \left[H(X, Y) - \left(1 - \frac{1}{p}\right)H(X) - \frac{1}{q}H(Y) \right] \Big|_{\mu_{X,Y}} \\ & = H(X, Y) - \left(1 - \frac{1}{p}\right)H(X) - \frac{1}{q}H(Y) \end{aligned}$$

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Hypercontractivity parameters satisfies a property called **tensorization** :

If $(X_1, Y_1) \perp (X_2, Y_2)$ are both (p, q) -hypercontractive, then $((X_1, X_2), (Y_1, Y_2))$ is also (p, q) -hypercontractive

Gets around the curse of dimensionality.

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Rather immediate that sub-additivity, i.e.

$$\begin{aligned} & \mathcal{C}_{\mu_{X_1 Y_1 X_2 Y_2}} [H(X_1 Y_1 X_2 Y_2) - \lambda_1 H(X_1 X_2) - \lambda_2 H(Y_1 Y_2)] \\ & \leq \mathcal{C}_{\mu_{X_1 Y_1}} [H(X_1 Y_1) - \lambda_1 H(X_1) - \lambda_2 H(Y_1)] + \mathcal{C}_{\mu_{X_2 Y_2}} [H(X_2 Y_2) - \lambda_1 H(X_2) - \lambda_2 H(Y_2)] \end{aligned}$$

is equivalent to tensorization of hypercontractivity parameters

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This (serendipitous) rediscovery of the link between hypercontractivity and information measures and these equivalent characterizations is spurring a lot of work

Consequences

Computation of hypercontractivity parameters is considered hard

- X is uniform and $\mu_{Y|X}$ is binary symmetric channel
 - ★ (Bonami-Beckner inequality '70s, Borrell '82)
- (X, Y) Jointly Gaussian (Gross '75)

Evaluation of achievable regions is of similar difficulty as determining hypercontractivity (same family and similar terms)

For testing optimality of schemes we had to develop tools for evaluating achievable regions for certain channels

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Yes, we can.

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Other techniques we used to solve these non-convex problems:

- Identify a lower dimensional manifold that contains all the stationary points
- Analyze the function directly on this manifold or
- Use properties of the points on this manifold to deduce sub-additivity

An Observation

Reminder: Family of functionals that showed up in network information theory

$$\sum_{S \subseteq [n]} \alpha_S H(X_S), \quad \alpha_S \in \mathbb{R}.$$

Usually, one is interested in testing the sub-additivity of

$$C_{\mu_X}[\alpha_S H(X_S)].$$

This is **equivalent** to testing a **global tensorization** property.

Definition

A functional $\sum_{S \subseteq [n]} \alpha_S H(X_S)$ is said to satisfy **global tensorization** if a **product distribution maximizes** $G_{12}^\mu(\gamma_1, \gamma_2)$ for all γ_1, γ_2 , where

$$G_{12}^\mu(\gamma_1, \gamma_2) := \sum_{S \subseteq [n]} \alpha_S H(X_{1S}, X_{2S}) - E(\gamma_1(\mathbf{X}_1)) - E(\gamma_2(\mathbf{X}_2))$$

An Observation

Definition

A functional $\sum_{S \subseteq [n]} \alpha_S H(X_S)$ is said to satisfy **local tensorization** if the product of local maximizers of $G^{\mu_1}(\gamma_1)$ and $G^{\mu_2}(\gamma_2)$ is a local maximizer of $G_{12}^{\mu}(\gamma_1, \gamma_2)$ for all γ_1, γ_2 , where

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Observation (Conjecture)

For functionals in this family global tensorization holds if and only if local tensorization holds

Note: Similarity to testing concavity using a local (second derivative) condition

For some of the remaining open problems (mentioned earlier), we can establish local-tensorization

- Marton's inner bound for binary input broadcast channels
- Gaussian Z-interference channel

Therefore, if the Conjecture is true, then we would establish the capacity region for these settings

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- Marton's inner bound for binary input broadcast channels
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Therefore, if the Conjecture is true, then we would establish the capacity region for these settings

Question: How may these two phenomena be connected?

A possible answer is suggested by our computations in various examples

- Construct a path in the space of functionals and track the global maximizers

Optimization based approaches

Optimization based approaches have been game changers

First jump: Linear programming to convex optimization

Semi-definite program based algorithm design and analysis

- Compressive sensing
- Phase recovery
- Clustering
- Image processing

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New Jump: Convex optimization to **specific families** of non-convex optimization

Studies on these families are already making impact in

- Machine learning and AI (Singular Value Decomposition)
- Graphical models and Statistical Physics based approaches (sum of energy and entropy terms)
- **Communication networks** (linear combination of entropies of subsets)

Acknowledgements (Rogues gallery)



Amin Gohari



Max Costa



Abbas El Gamal



Yanlin Geng



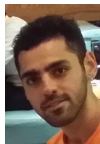
Vincent Wang



Salman Beigi



Lingxiao Xia



Babak Yazdanpanah



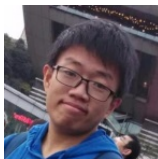
Janos Korner



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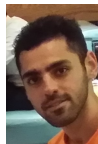
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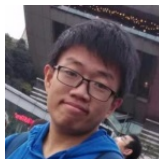
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Venkat Anantharam

THANK YOU