How to Solve Gaussian Interference Channel

Fan Cheng
Shanghai Jiaotong University

Abstract: In this talk, we will introduce the history and current progress on the complete monotonicity conjecture on heat equation/Gaussian distribution (Or Gaussian complete monotonicity conjecture (GCMC)). We will also talk about how to solve Gaussian interference channel by the complete monotonicity of Gaussian distribution.

The second law of thermodynamics stated that the entropy of an isolated system is always nondecreasing. Consider the case of heat equation along the line:

$$\frac{\partial}{\partial t} f(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2} f(x,t) \tag{1}$$

where x denotes the location and t denotes time.

Without loss of generality, assume that the initial condition $f(x,0) \geq 0$. In the literature of mathematical physics, it is well known that the entropy of $f(x,t)$ is increasingly concave in t; i.e.,

$$\frac{\partial}{\partial t} h(f(x,t)) \geq 0 \tag{2}$$

$$\frac{\partial^2}{\partial t^2} h(f(x,t)) \leq 0 \tag{3}$$

A function $f(t)$ is called complete monotone, e.g., $1/t$, if the consecutive derivatives of $f(t)$ alternates in signs.

Conjecture:

$$(-1)^n \frac{\partial^n}{\partial t^n} h(f(x,t)) \leq 0 \tag{4}$$

That is $h(f(x,t))$ is complete monotone in t.

Current progress:

1. In 1966, H. P. McKean [1] considered the problem in studying Boltzmann equation. However, he failed to make any progress. The problem was not accepted as a serious conjecture till 2015 as there is no evidence. Cedric Villani [2] noted this result in his textbook on Boltzmann equation as "super-H" theorem.

2. In 2015, independent of the work of H. P. McKean and Cedric Villani, we [3] proved that up to $n \leq 4$ the conjecture holds.

3. For the general case, the conjecture remains open.

References

[1] H. P. McKean, Jr., "Speed of approach to equilibrium for Kacs caricature of a Maxwellian gas," Arch. Rational Mech. Anal., 21:343-367, 1966.

[2] C. Villani, A review of mathematical topics in collisional kinetic theory.

[3] F. Cheng and Y. Geng, "Higher Order Derivatives in Costa's Entropy Power Inequality," IEEE Trans. Inform. Theory, vol. 61, no. 11, pp. 5892-5905, Nov. 2015.