Third-Order Asymptotics: Old and New

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Joint works with M. Tomamichel, Y. Sakai and M. Kovačević

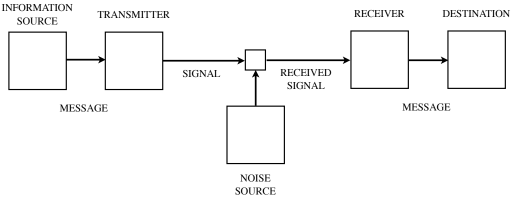
Workshop on Probability and Information Theory
(held at the University of Hong Kong, on 20th August 2019)

Outline

1. Introduction
2. Old Contribution
3. New Contribution

Outline

Introduction: Transmission of Information



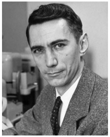


Figure: Shannon's Figure 1

- Information theory $\equiv$ Finding fundamental limits for reliable information transmission

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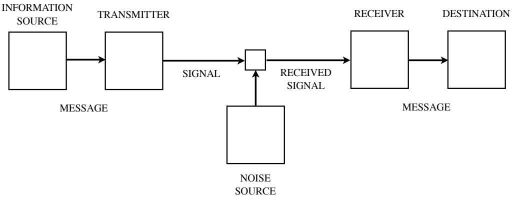


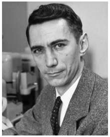
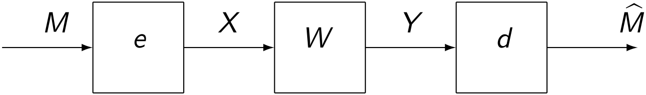


Figure: Shannon's Figure 1

- Information theory $\equiv$ Finding fundamental limits for reliable information transmission
- Channel coding: Concerned with the maximum rate of communication in bits/channel use

Channel Coding (One-Shot)



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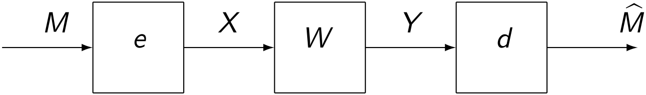

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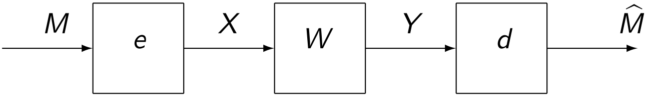

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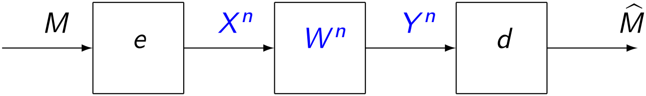
  where M is uniform on $\mathcal{M}$

- Maximum code size at ε-error is

$$M^*(W, \varepsilon) := \sup\left\{m \,\middle|\, \exists \mathcal{C} \;\text{ s.t. }\; m = |\mathcal{M}|,\, p_{\mathrm{err}}(\mathcal{C}) \leq \varepsilon\right\}$$

Channel Coding (n-Shot)



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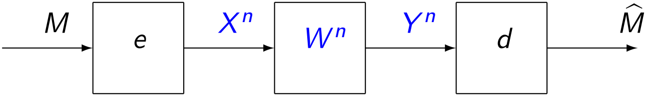

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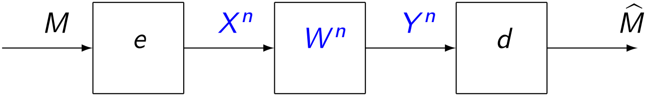

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- For vectors $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{X}^n$ and $\mathbf{y} := (y_1, \ldots, y_n) \in \mathcal{Y}^n$,

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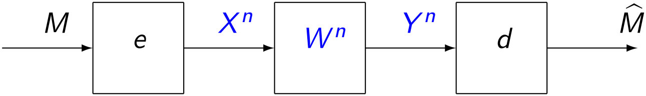

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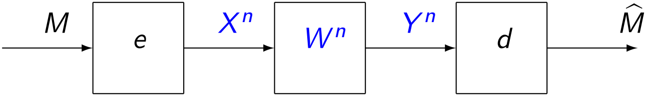

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- Consider both discrete- and continuous-time channels.

Outline

Old Contribution

- Upper bound $\log M^*(W^n, \varepsilon)$ for n large (converse)
- Concerned with the third-order term of the asymptotic expansion
- Going beyond the normal approx terms

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Theorem (Tomamichel-Tan (2013))

For all DMCs with positive ε-dispersion V_ε,

$$\log M^*(W^n, \varepsilon) \le nC + \sqrt{nV_\varepsilon}\Phi^{-1}(\varepsilon) + \frac{1}{2}\log n + O(1)$$

where $\Phi(a) := \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \mathrm{d}x$

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V. Strassen (1964)

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- Requires new converse techniques

Related Work: Third-Order Term

- Recall that we are interested in quantifying the third-order term ρ_n

$$\rho_n = \log M^*(W^n, \varepsilon) - \left[nC + \sqrt{nV_\varepsilon}\Phi^{-1}(\varepsilon) \right]$$

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- ρ_n may be important at very short blocklengths

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- For the BSC [PPV10]

$$\rho_n = \frac{1}{2}\log n + O(1)$$

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- For the AWGN under maximum (or peak) power constraints [PPV10, Tan-Tomamichel (2015)]

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Related Work: Achievability for Third-Order Term

Proposition (Polyanskiy (2010))

Assume that all elements of $\{W(y|x) : x \in \mathcal{X}, y \in \mathcal{Y}\}$ are positive and $C > 0$. Then,

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- Based on the concentration bound [Polyanskiy's thesis]

$$\mathbb{E}\left[\exp\left(\sum_{i=1}^{n} X_i\right) \mathbb{I}\left\{\sum_{i=1}^{n} X_i \geq \gamma\right\}\right] \leq 2\left(\frac{\log 2}{\sqrt{2\pi}} + \frac{12\,T}{\sigma}\right) \frac{\exp(-\gamma)}{\sigma\sqrt{n}}.$$

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- We dispense of this symmetry assumption

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- "A Tight Upper Bound for the Third-Order Asymptotics for Most DMCs" M. Tomamichel and V. Y. F. Tan, IEEE T-IT, Nov 2013

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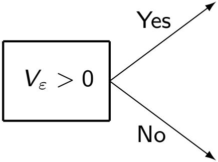
All cases are covered



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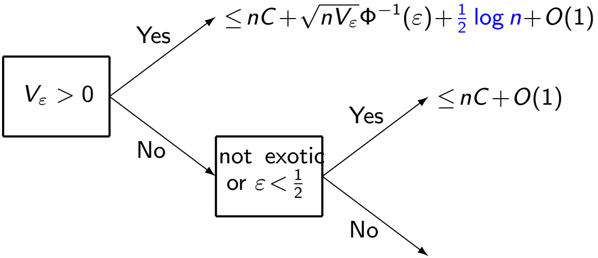

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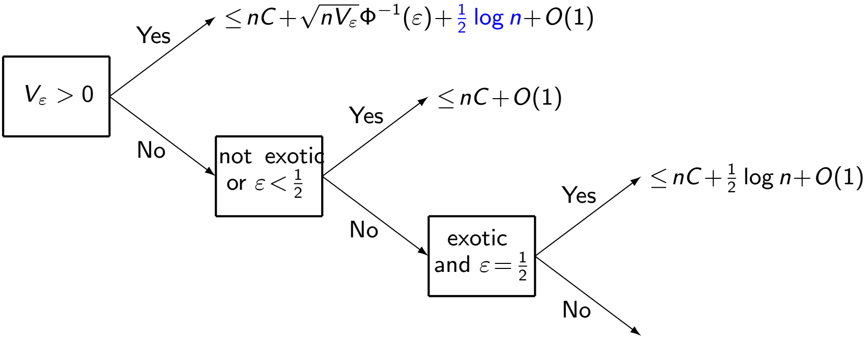

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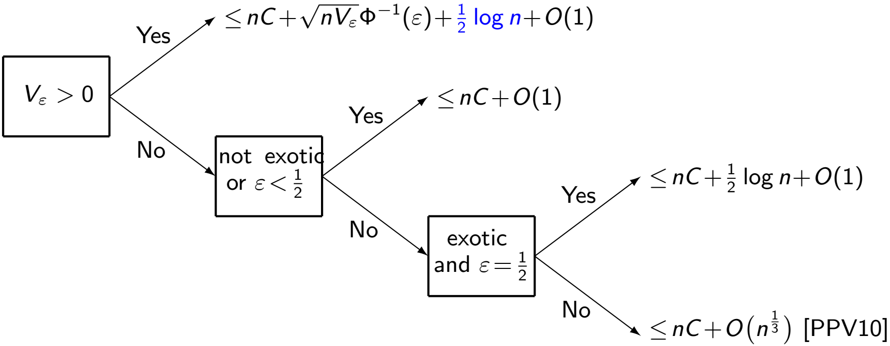

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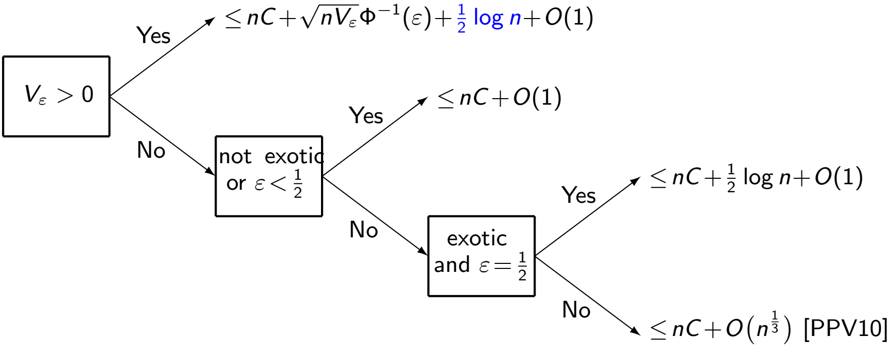

W is exotic if $V_{\max}(W) = 0$ and $\exists\, x_0 \in \mathcal{X}$ such that

$$D(W(\cdot|x_0)\|Q^*) = C, \quad \text{and} \quad V(W(\cdot|x_0)\|Q^*) > 0.$$

Proof Technique for Tight Third-Order Term

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- We need a convenient converse bound for general DMCs
- Information spectrum divergence

$$D_s^{\varepsilon}(P\|Q) := \sup\left\{R \,:\, P\left(\log\frac{P(X)}{Q(X)} \le R\right) \le \varepsilon\right\}$$

"Information Spectrum Methods in Information Theory"
by T. S. Han (2003)

Proof Technique: Information Spectrum Divergence

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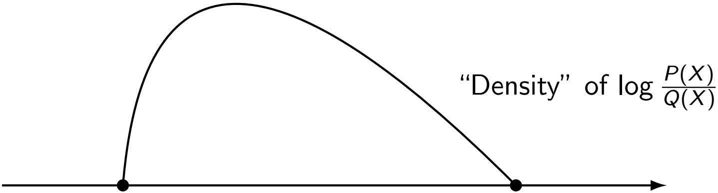

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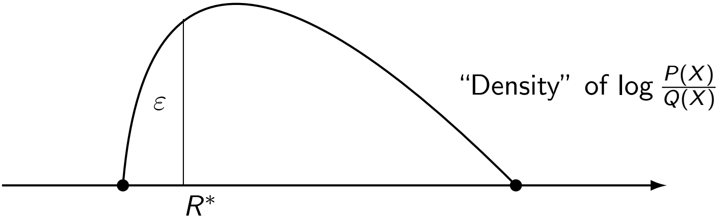

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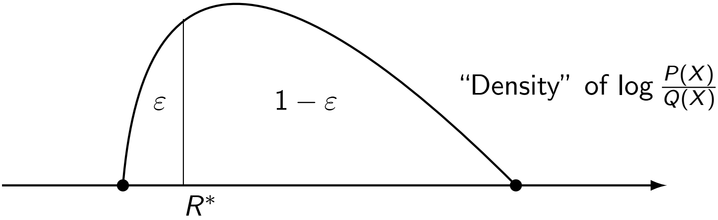

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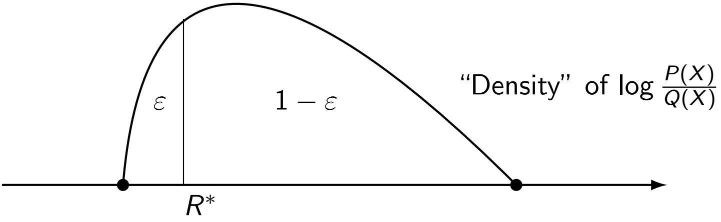

If X^n is i.i.d. P, the Berry-Esseen theorem yields

$$D_s^\varepsilon(P^n\|Q^n) = nD(P\|Q) + \sqrt{nV(P\|Q)}\Phi^{-1}(\varepsilon) + O(1)$$

Proof Technique: Symbol-Wise Converse Bound

Lemma (Tomamichel-Tan (2013))

For every channel W, every $\varepsilon \in (0,1)$ and $\delta \in (0, 1-\varepsilon)$, we have

$$\log M^*(W,\varepsilon) \le \min_{Q \in \mathcal{P}(\mathcal{Y})} \max_{x \in \mathcal{X}} D_s^{\varepsilon+\delta}(W(\cdot|x) \| Q) + \log \frac{1}{\delta}$$

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- Choose $\delta = n^{-\frac{1}{2}}$ so $\log \frac{1}{\delta} = \frac{1}{2} \log n$

- Since all $\mathbf{x}$ within a type class result in the same $D_s^{\varepsilon+\delta}$ (if $Q^{(n)}$ is permutation invariant), it's really a max over types $P_{\mathbf{x}} \in \mathcal{P}_n(\mathcal{X})$

Proof Technique: Choice of Output Distribution

$$\log M^*(W^n, \varepsilon) \leq \max_{\mathbf{x} \in \mathcal{X}^n} D_s^{\varepsilon+\delta}(W^n(\cdot|\mathbf{x}) \| Q^{(n)}) + \log \frac{1}{\delta}, \qquad \forall\, Q^{(n)} \in \mathcal{P}(\mathcal{Y}^n)$$

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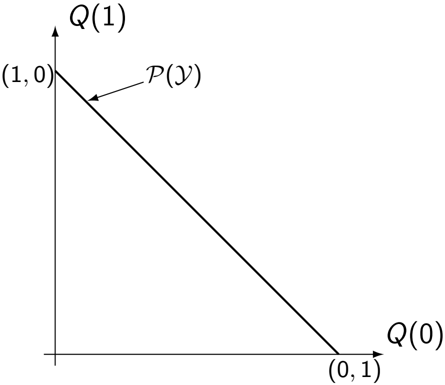

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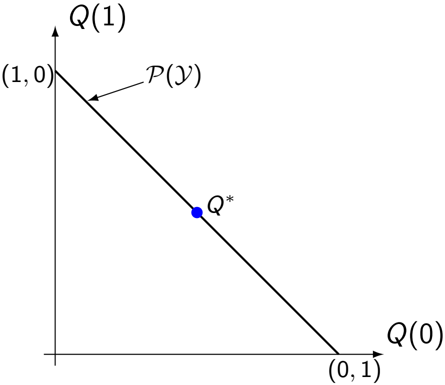

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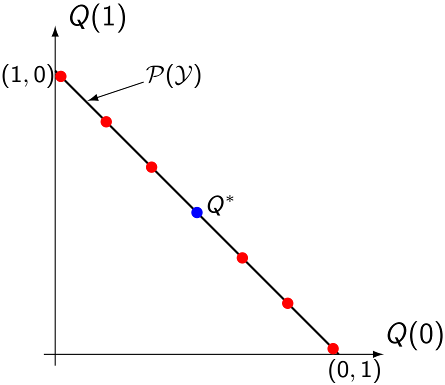

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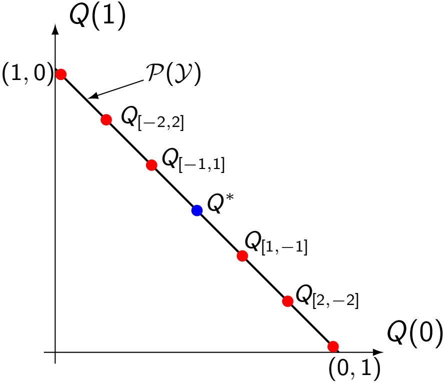

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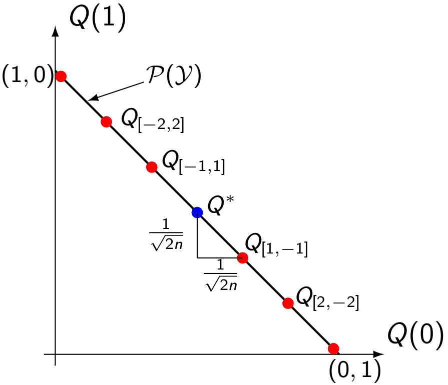

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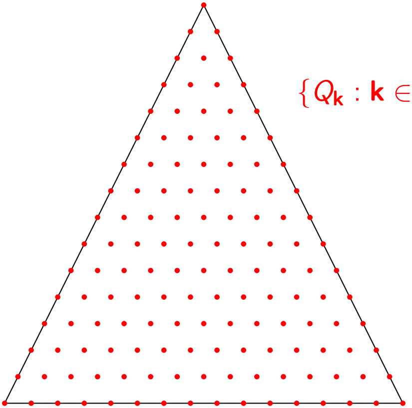

$\{Q_{\mathbf{k}} : \mathbf{k} \in \mathcal{K}\} \subset \mathcal{P}(\mathcal{Y})$

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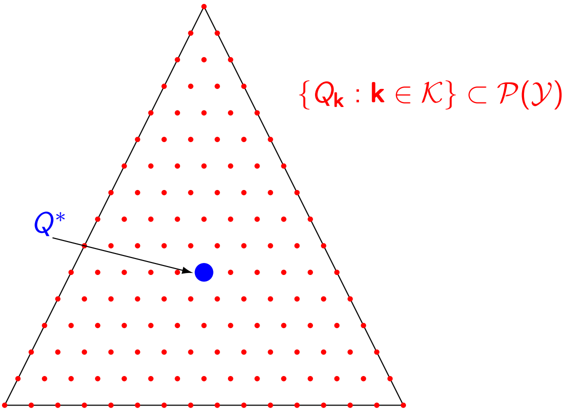

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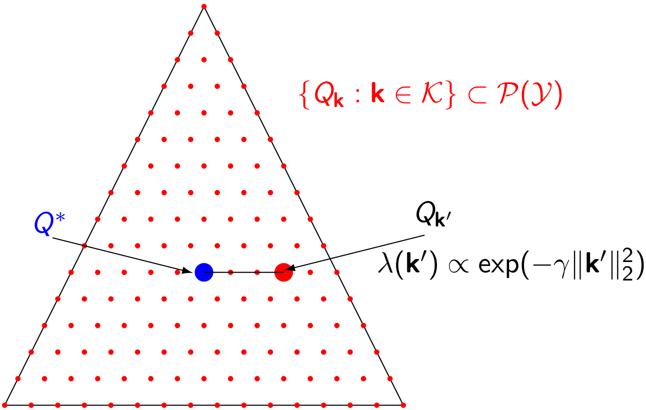

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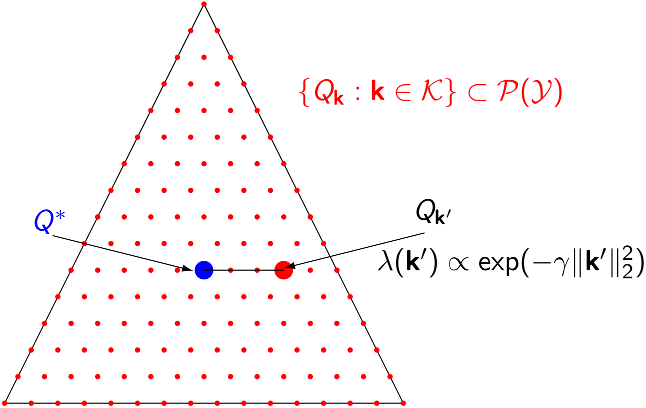

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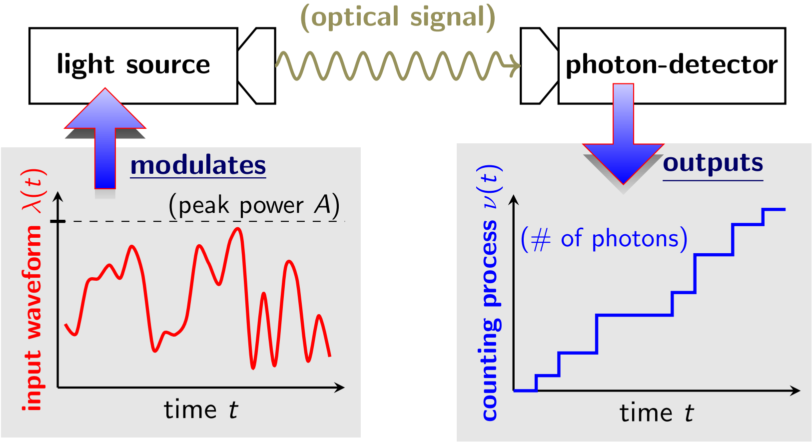
$$\sum_{P\in\mathcal{P}_n(\mathcal{X})} \frac{1}{|\mathcal{P}_n(\mathcal{X})|}(PW)^n(\mathbf{y}).$$

- Serves to take care of "bad input types" (i.e., types $P \in \mathcal{P}_n(\mathcal{X})$ such that PW is far from Q^*)

Outline

Mathematical Model of Poisson Channel (1/3)

Consider the following optical communication:



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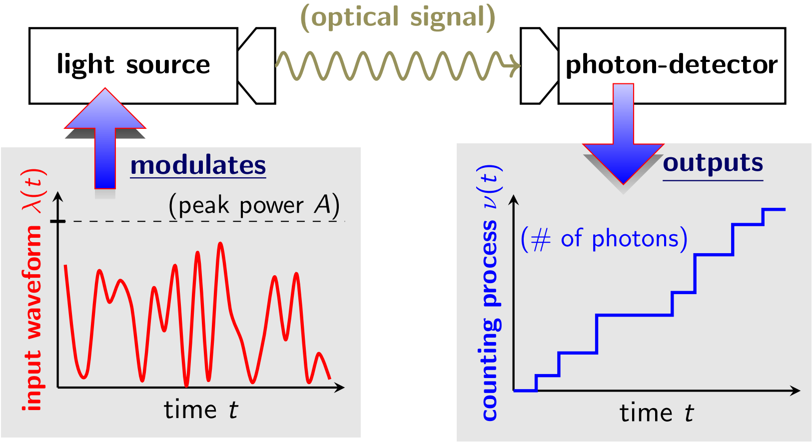
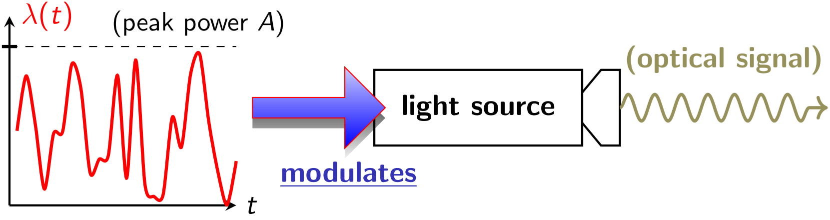


Remark: This is a **continuous-time** channel ($0 \le t < T$).

Mathematical Model of Poisson Channel (2/3)



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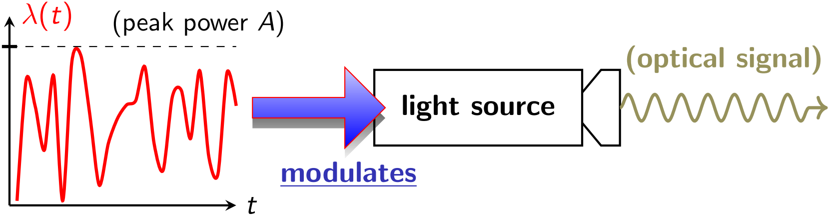


Optical Signal is Modulated by Input Waveform $\lambda(t)$

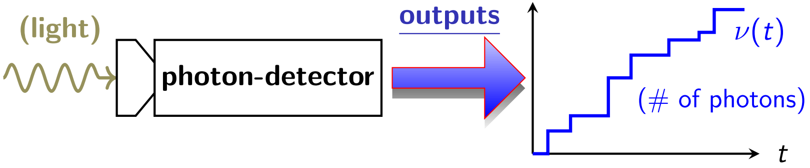
- an integrable function $\lambda(\cdot)$ defined on the time block $[0, T)$;
- with **peak power constraint** **(**$A > 0$**)**:

$$0 \leq \lambda(t) \leq A \qquad \forall t \in [0, T);$$

- with **average power constraint** **(**$0 \leq \sigma \leq 1$**)**:

$$\frac{1}{T} \int_0^T \lambda(t)\,\mathrm{d}t \leq \sigma A.$$

Mathematical Model of Poisson Channel (3/3)



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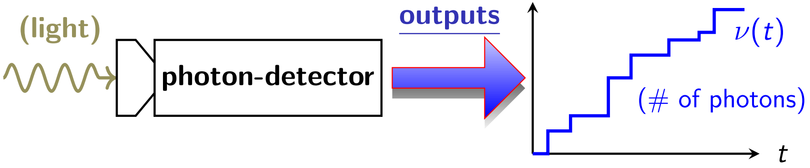

Output is Poisson counting process $\{\nu(t)\}_{0 \le t < T}$

$$\nu(0) = 0 \quad \text{a.s.} \qquad \text{and} \qquad \mathbb{P}\{\nu(t+\tau) - \nu(t) = k\} = \frac{e^{\Lambda}\,\Lambda^k}{k!}$$

for each $t, \tau \in \mathbb{R}_{\ge 0}$ and $k \in \mathbb{Z}_{\ge 0}$, where Λ is given by

$$\Lambda \overset{\text{def}}{=} \int_t^{t+\tau} \big(\lambda(u) + \lambda_0 \big) \, \mathrm{d}u.$$

- **input waveform** (intensity of light) $\lambda : [0, T) \to [0, A]$
- **dark current** (background noise level) $0 \le \lambda_0 < \infty$

Block Coding Scheme for Poisson Channel

- **input alphabet** is the set of waveforms $\lambda(\cdot)$

$$\mathcal{W}(T, A, \sigma) \stackrel{\text{def}}{=} \left\{ \lambda : [0, T) \to [0, A] \;\middle|\; \frac{1}{T} \int_0^T \lambda(t)\, \mathrm{d}t \leq \sigma A \right\},$$

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$$\mathcal{S}(T) \stackrel{\text{def}}{=} \{ g : [0, T) \to \mathbb{Z}_{\ge 0} \mid g(0) = 0 \text{ and } g(t_1) \ge g(t_2), t_1 < t_2 \}$$

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A (T, M, A, σ)-code (ϕ, ψ) for Poisson channel

- encoder $\phi : \{1, 2, \ldots, M\} \to \mathcal{W}(T, A, \sigma)$
- decoder $\psi : \mathcal{S}(T) \to \{1, 2, \ldots, M\}$

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A $(T, M, A, \sigma, \varepsilon)_{\text{avg}}$-code (ϕ, ψ) for Poisson channel

A (T, M, A, σ)-code (ϕ, ψ) is called a $(T, M, A, \sigma, \varepsilon)_{\text{avg}}$-code if

$$\frac{1}{M} \sum_{m=1}^{M} \mathbb{P}\{\psi(\nu) = m \mid \lambda = \phi(m)\} \geq 1 - \varepsilon.$$

Here, λ is the r.v. induced by the encoder ϕ with uniform messages.

Poisson Channel Capacity (1st-Order Asymptotics)

Denote by M^* the max. M s.t. $\exists$ a $(T, M, A, \sigma, \varepsilon)_{\text{avg}}$-code.

Theorem (Kabanov'78; Davis'80; Wyner'88)

$$\log M^* = T\,C^* + \mathrm{o}(T) \qquad (\text{as } T \to \infty),$$

where

$$\begin{cases} C^* \overset{\text{def}}{=} A\left((1-p^*)\,s\log\dfrac{s}{p^*+s} + p^*\,(1+s)\log\dfrac{1+s}{p^*+s}\right), \\ s \overset{\text{def}}{=} \dfrac{\lambda_0}{A} \qquad (\text{ratio of } \textbf{dark current } \lambda_0 \text{ to } \textbf{PPC } A), \\ p^* \overset{\text{def}}{=} \min\{\sigma, p_0\} \qquad (\text{role of } \textbf{CAID}, \text{ where } \sigma \text{ is } \textbf{APC}), \\ p_0 \overset{\text{def}}{=} \dfrac{(1+s)^{1+s}}{s^s\,\mathrm{e}} - s. \end{cases}$$

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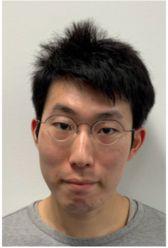


Yuta Sakai

Mladen Kovačević

Poisson Channel Dispersion (2nd-Order Asymptotics)

Theorem (Sakai–Tan–Kovačević'19: `arXiv:1903.10438`)

$$\log M^* = T\,C^* + \sqrt{T\,V^*}\,\Phi^{-1}(\varepsilon) + \rho_T,$$

where the Poisson channel dispersion V^* is given by

$$V^* \overset{\text{def}}{=} A\left((1-p^*)\,s\log^2\frac{s}{p^*+s} + p^*\,(1+s)\log^2\frac{1+s}{p^*+s}\right),$$

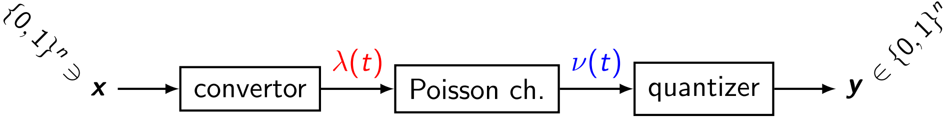
and the third-order term ρ_T satisfies

$$\frac{1}{2}\log T + \mathrm{O}(1) \le \rho_T \le \log T + \mathrm{O}(1) \qquad (\text{as } T \to \infty).$$

Result: 2nd-order term $\sqrt{V^*}\,\Phi^{-1}(\varepsilon)$ and bounds on 3rd-order term ρ_T

Proof Ideas of Second- and Third-Order Asymptotics

In both converse and achievability parts, we shall employ **Wyner's discretization argument** (Wyner'88):



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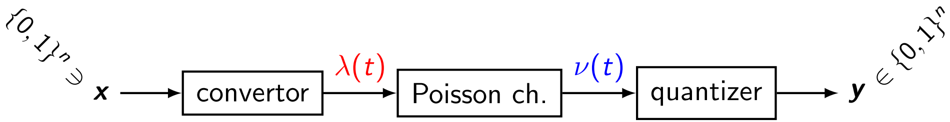

Converse Part

- symbol-wise meta converse bound (Tomamichel–Tan'13)
- **novel choice of output distribution** (projected ϵ-net)

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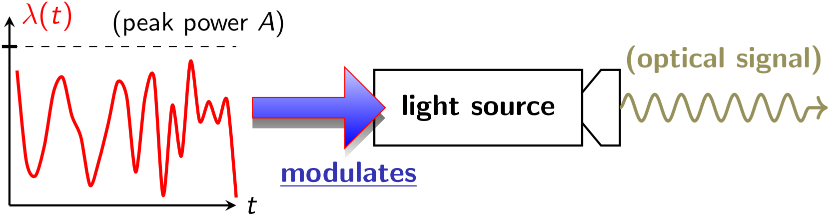
- **novel choice of output distribution** (projected ϵ-net)

Achievability Part

- random coding union bound (PPV'10) with cost constraint
- some other techniques to handle the continuous nature (e.g., logarithmic Sobolev inequality)

Wyner's Discretization Part I: Input Restriction



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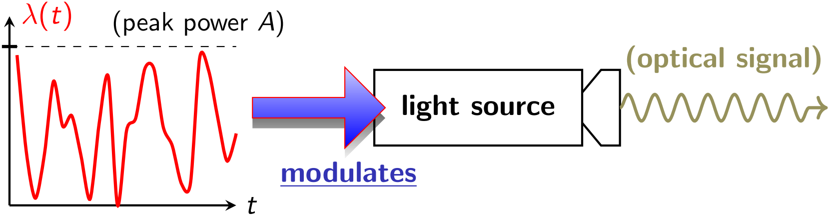
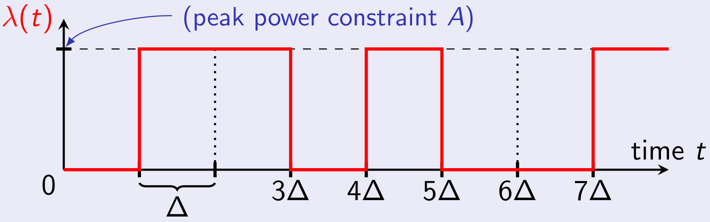


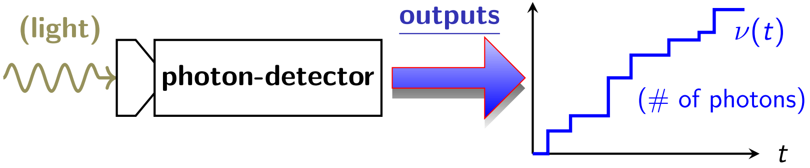
Discretization of $\{\lambda(t)\}_{0\le t<T}$ into n Blocks (here, $\Delta = T/n$)

input waveform $\lambda(t)$ is restricted to be **square**, e.g.,



That is, we may think of $\lambda(t)$ as **a binary sequence** $\{x_k\}_{k=1}^{n}$.

Wyner's Discretization Part II: Output Quantization



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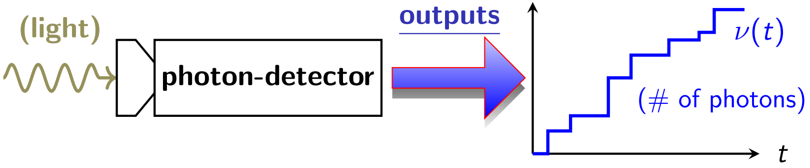

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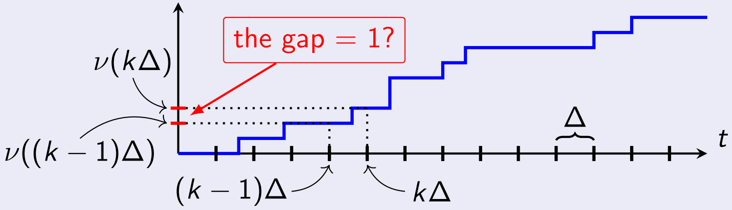
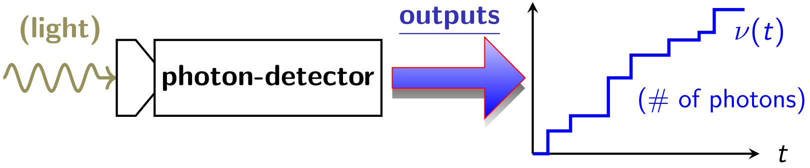
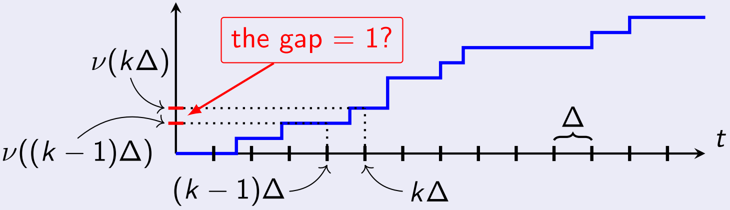


Wyner's Discretization Part II: Output Quantization



Discretization of $\{\nu(t)\}_{0\le t<T}$ into n Blocks (here, $\Delta = T/n$)



Poisson counting process $\nu(t)$ is quantized as $\{y_k\}_{k=1}^n$:

$$y_k \overset{\text{def}}{=} \begin{cases} 0 & \text{if } \nu(k\Delta) - \nu((k-1)\Delta) \neq 1, \\ 1 & \text{if } \nu(k\Delta) - \nu((k-1)\Delta) = 1. \end{cases}$$

Overall Diagram of Wyner's Discretization

- input sequence $\boldsymbol{x} = (x_1, \ldots, x_n) \in \{0, 1\}^n$
  (which is converted to a square wave $\lambda(t)$: )
- output sequence $\boldsymbol{y} = (y_1, \ldots, y_n) \in \{0, 1\}^n$
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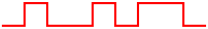
  (which is converted to a square wave $\lambda(t)$: )
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Discretized channel $W_n^n : \{0,1\}^n \to \{0,1\}^n$

$$W_n^n(\boldsymbol{y} \mid \boldsymbol{x}) \stackrel{\text{def}}{=} \prod_{i=1}^{n} W_n(y_i \mid x_i),$$

where the single-letter channel $W_n : \{0,1\} \to \{0,1\}$ **depends** on n.

Remark: the discretization error is negligible as $n \to \infty$ (next slide).

Wyner's Discretization Well-Approximates Poisson Channel

Denote by

- $M^*_{\text{Poisson}}(\varepsilon)$: fundamental limit of Poisson channel
- $M^*(W_n^n, \varepsilon)$: fundamental limit of discretized channel W_n^n

Lemma (Wyner'88)

There exist a sequence $\epsilon_n = \mathrm{o}(1)$ and a subsequence $\{n_k\}_{k=1}^{\infty}$ s.t.

$$M^*_{\text{Poisson}}(\varepsilon) = M^*(W_{n_k}^{n_k}, \varepsilon + \epsilon_{n_k}) \qquad (\forall k \geq 1).$$

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Therefore, we observe that

$$\log M^*_{\text{Poisson}}(\varepsilon) \leq \limsup_{n \to \infty} \log M^*(W_n^n, \varepsilon + \epsilon_n),$$

implying that it suffices to examine **the RHS** in the converse part.

Meta Converse Bound and Output Distribution

Apply the symbol-wise meta converse (Tomamichel–Tan'13):

$$\log M^*(W_n^n, \varepsilon + \epsilon_n) \leq \max_{\boldsymbol{x} \in \{0,1\}^n} D_{\mathrm{s}}^{\varepsilon+\epsilon_n+\eta}(\underbrace{W_n^n(\cdot \mid \boldsymbol{x})}_{\text{discretized Poisson channel}} \| Q^{(n)}) + \log \frac{1}{\eta}$$

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$$Q^{(n)}(\boldsymbol{y}) = \frac{1}{3} \prod_{i=1}^{n} P^*_{[-\kappa]} W_n(y_i) + \frac{1}{3} \prod_{i=1}^{n} P^*_{[\kappa]} W_n(y_i) + \frac{1}{3F} \sum_{\substack{m=-\infty: \\ 0 \leq p^* + m/T \leq 1}}^{\infty} \mathrm{e}^{-\gamma m^2/T} \prod_{i=1}^{n} P^*_{[m/T]} W_n(y_i)$$

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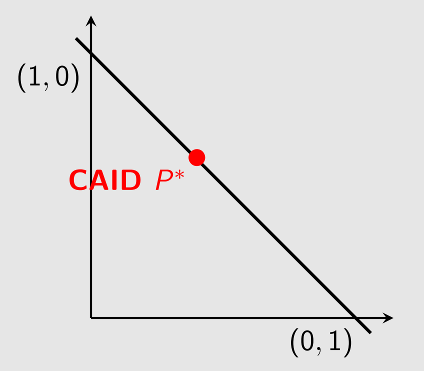
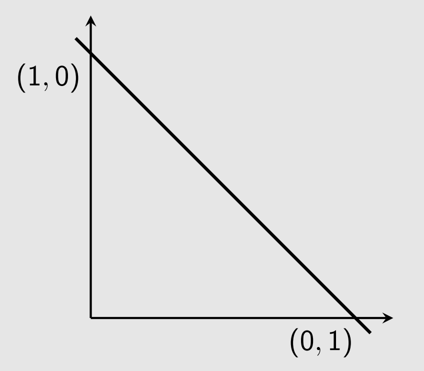
- **third term** is the main part of **our novel construction**
- **first** and **second** terms are to apply Lipschitz properties

ϵ-Net Argument: Tomamichel–Tan's Original Choice

Consider a binary-input binary-output channel $W : \{0,1\} \to \{0,1\}$.



[input probab. simplex]



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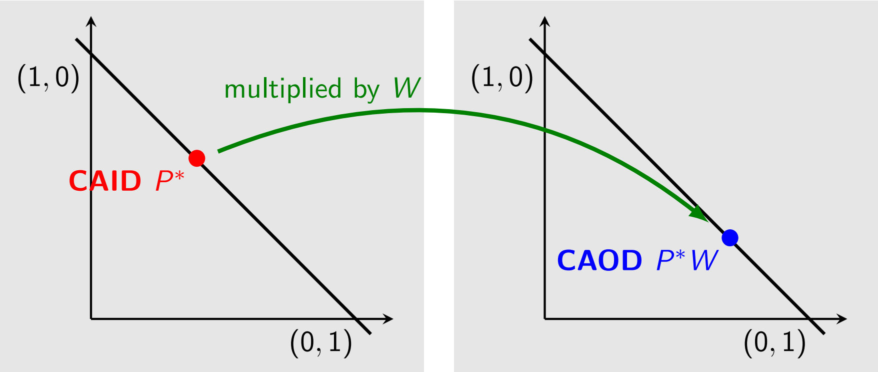

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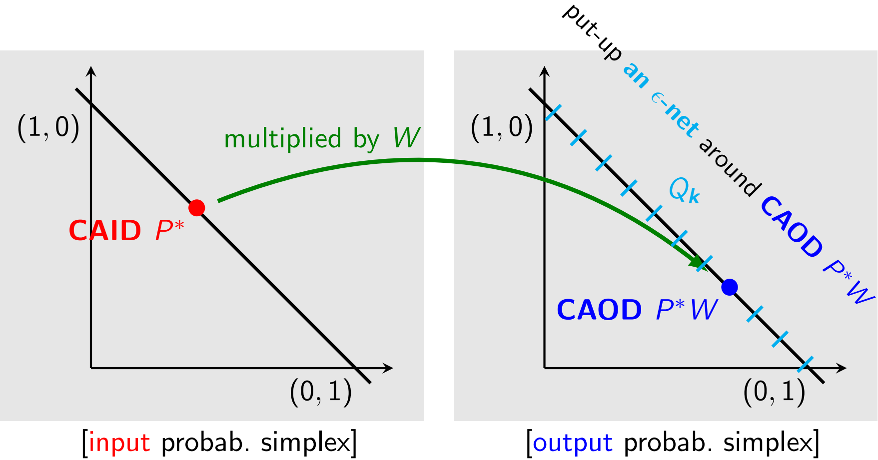

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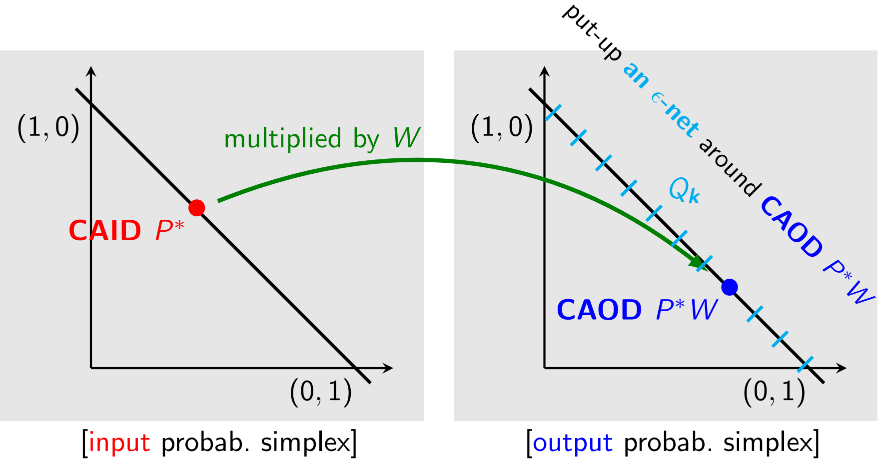

Use a convex combination of **the ϵ-net**: $\sum_{\boldsymbol{k}} \mu(\boldsymbol{k}) Q_{\boldsymbol{k}}^n$.

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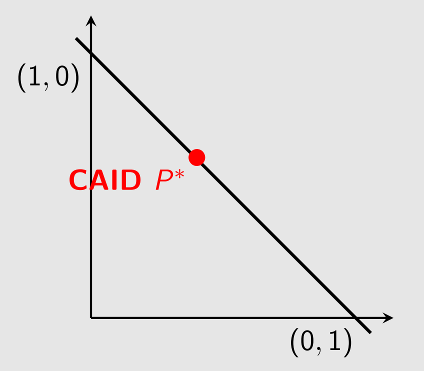
Consider a (single-letter) discretized channel $W_n : \{0,1\} \to \{0,1\}$.



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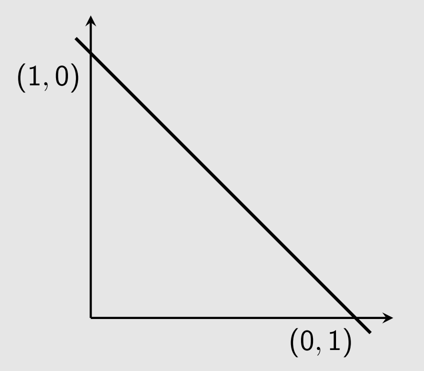

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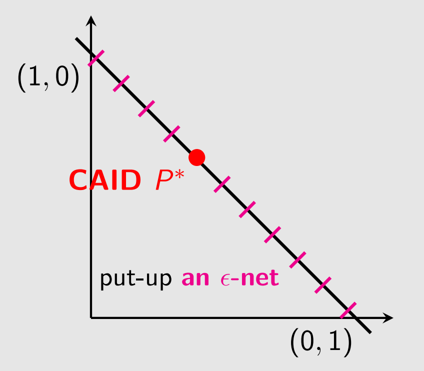

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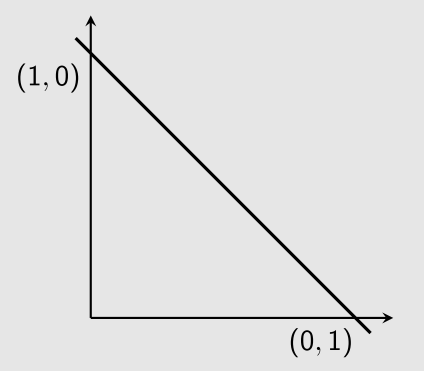

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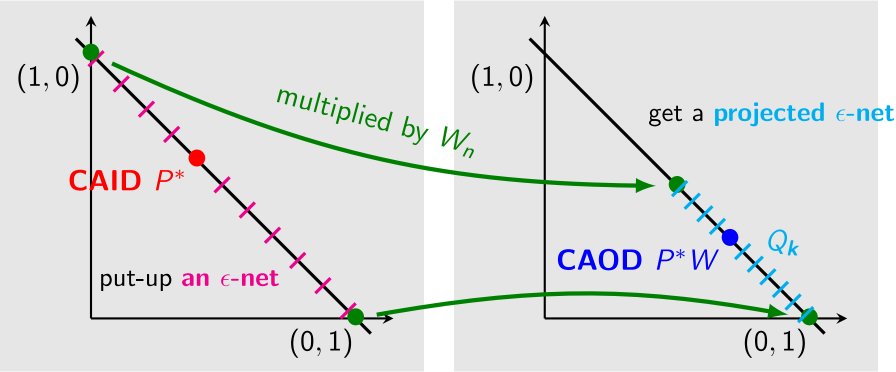

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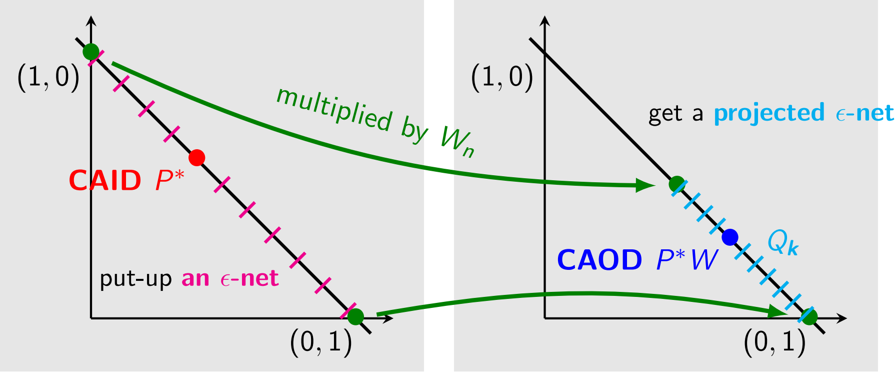

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Why This Choice of Output Distn. and not TT13?

- Recall that we chose

$$\text{Third Term of } Q^{(n)}(\boldsymbol{y}) = \frac{1}{3F} \sum_{\substack{m=-\infty: \\ 0 \le p^* + m/T \le 1}}^{\infty} \mathrm{e}^{-\gamma m^2/T} \prod_{i=1}^{n} P^*_{[m/T]} W_n(y_i)$$

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- Tomamichel-Tan's construction in the output distn. space cannot handle the non-stationary W_n^n.

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- Second- and third-order asymptotics for the Poisson channel

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- Check out `arXiv:1903.10438`.