THE UNIVERSITY OF HONG KONG
Institute of Mathematical Research
Department of Mathematics

Workshop on Crystal Bases, Cluster Algebras, and Poisson Geometry

November 29 – December 3, 2018

Room 210, Run Run Shaw Building, HKU

Thursday, November 29	
10:00 – 11:00	**Toshiki Nakashima**, Sophia University, Japan ***Lecture 1***: *Basics of Crystal Bases and Geometrical Crystals*
Coffee / Tea Break	
11:30 – 12:30	**Ivan Ip**, The Hong Kong University of Science and Technology, Hong Kong *Cluster Realization and Tensor Product Decomposition of Positive Representations*
Friday, November 30	
10:00 - 11:00	**Toshiki Nakashima**, Sophia University, Japan ***Lecture 2***: *Basics of Crystal Bases and Geometrical Crystals*
Coffee / Tea Break	
11:30 – 12:30	**Victor Mouquin**, Shanghai Jiaotong University, China *Quantization of a Poisson Structure on Products of Principal Affine Spaces*
Lunch Break	
14:30 – 15:30	**Yanpeng Li**, University of Geneva, Switzerland *Poisson Geometry, Crystals, and Integrable Systems*
Coffee / Tea Break	
16:00 – 17:00	**Jeremy Lane**, University of Geneva, Switzerland *Scaled Ginzburg-Weinstein Isomorphisms and Concentration of Volumes in K/T*
Workshop Dinner	
Monday, December 3	
11:30 – 12:30	**Toshiki Nakashima**, Sophia University, Japan ***Lecture 3***: *Basics of Crystal Bases and Geometrical Crystals*

Organizers: Zheng Hua & JiangHua Lu

Titles & Abstracts

Workshop on Crystal Bases, Cluster Algebras, and Poisson Geometry
Nov. 29 - Dec. 3, 2018, HKU

Toshiki Nakashima (Three lectures), Sophia University, Japan

Basics of Crystal Bases and Geometrical Crystals

The theory of crystal base has been initiated by Masaki Kashiwara. Nowadays, it is widely known and applied to many areas in mathematics and physics. In the former half of this lecture, we will introduce the theory of crystal bases from the beginning, which includes the following topics: representation theory of quantum groups; Kashiwara operators; definiton of crystal base; crystal graphs; existence theorem; tensor products of crystal bases; tableaux realizations; crystal base of the negative part of the quantum group; polyhedral realizations.

In the latter half of the lectures, we shall introduce theory of geometric crystals, which has been invented by A.Berenstein and D.Kazhdan. It is a kind of geometric analogue of crystal base theory, indeed, which is defined on some algebraic variety and each Kashiwara operator will be realized as some rational action of the 1-dimensional multiplicative group. Geometric crystal is not a simple and superficial analogue of crystal base, but it holds certain functorial correspondence to crystal base, which is called "tropicalization". In the lecture, the following topics will be treated: definition of geometric crystal; product structure of geometric crystals; positive structure of geometric crystals; tropicalization of geometric crystals; monomial realization of "crystal bases"; geometric crystal on Schubert variety; decorated geometric crystals.

If time permits, we also present some relations between crystal bases and cluster algebras.

Ivan Ip, The Hong Kong University of Science and Technology, Hong Kong

Cluster Realization and Tensor Product Decomposition of Positive Representations

For each simple Lie algebra $\mathfrak{g}$, a cluster realization of the corresponding Drinfeld-Jimbo quantum groups $U_q(\mathfrak{g})$ has been found via the positive representations, where an embedding into a quantum torus algebra $X_\mathfrak{g}$ is described by certain quiver diagram. Using this new realization, we discuss its application towards the proof of the tensor product decomposition of the positive representations of split real quantum groups restricted to the Borel part, as well as the full decomposition in type A_n found recently by Schrader-Shapiro related to the spectral decomposition of the Hamiltonians of certain open Coxeter-Toda system.

Victor Mouquin, Shanghai Jiaotong University, China

Quantization of a Poisson Structure on Products of Principal Affine Spaces

The irreducible representations of a complex semisimple Lie group G are all encoded in its principal affine space G/N, and the decomposition of tensors products of irreps of G is intimately linked to the geometry of the diagonal action of G on several copies of G/N. This classical theory has a natural Poisson version. In this talk, we show that the standard multiplicative Poisson structure π_{st} on G induces a Poisson bracket on $(G/N)^m$ which is graded and for which the natural diagonal action of G is Poisson. Its deformation quantization is a graded $U_h(\mathfrak{g})$-module algebra which is locally factored in the sense of Etingof and Kazhdan.

Yanpeng Li, University of Geneva, Switzerland

Poisson Geometry, Crystals, and Integrable Systems

For the Poisson-Lie dual K^* of a compact semisimple Lie group K, we construct a Poisson manifold $\mathrm{PT}(K^*) = \mathcal{C} \times T$ with a constant Poisson structure (here $\mathcal{C}$ is a certain polyhedral cone and T is a torus). The manifold $\mathrm{PT}(K^*)$ carries natural completely integrable systems with action-angle variables. In the case of $K = SU(n)$, one of these integrable systems is the Gelfand-Cetlin completely integrable system. Our main result is a one-to-one correspondence between generic symplectic leaves in $\mathrm{PT}(K^*)$ and generic coadjoint orbits in $\mathrm{Lie}(K)^*$ preserving symplectic volumes of the leaves. This observation gives hope to construct a dense Darboux chart in $\mathrm{Lie}(K)^*$ modeled on $\mathrm{PT}(K^*)$.

Our proof is based on the comparison of the Poisson-Lie dual G^* of $G = K^{\mathbb{C}}$ and of its Langlands dual $G^{\vee}$. We show that the integral cone defined by the cluster structure and the Berenstein-Kazhdan potential on the double Bruhat cell $G^{\vee;w_0,e} \subset G^{\vee}$ is isomorphic to the integral Bohr-Sommerfeld cone defined by the Poisson structure on $\mathrm{PT}(K^*)$. This statement implies the equality of volumes of symplectic leaves.

This is a joint work with A. Alekseev, A. Berenstein, B. Hoffman, and J. Lane.

Jeremy Lane, University of Geneva, Switzerland

Scaled Ginzburg-Weinstein Isomorphisms and Concentration of Volumes in K/T

This talk is a continuation of the talk by Y. Li. It is based on joint work with A. Alekseev, B. Hoffman, and Y. Li. See arXiv:1804.01504 and arXiv:1808.06975.

A theorem of Ginzburg and Weinstein (1993) says that for K a compact, connected, and simply connected semisimple Lie group, the Poisson manifolds $\mathfrak{k}^*$ and K^* are Poisson isomorphic (when equipped with their natural Poisson structures). A scaled Ginzburg-Weinstein isomorphism is a Poisson isomorphism of $\mathfrak{k}^*$ and K^* when the Poisson structure on K^* is scaled by a factor of s.

One can compose scaled Ginzburg-Weinstein isomorphisms with certain scaled cluster coordinates introduced in the previous talk (using the same scaling parameter for both maps). It is conjectured that the limit as s goes to infinity of this composition is a Poisson isomorphism from an open dense subset of $\mathfrak{k}^*$ to the Poisson manifold $\mathrm{PT}(K^*)$.

This talk will present progress towards this conjecture. In particular, we present a result on concentration of volumes in the Poisson homogeneous space K/T.