THE UNIVERSITY OF HONG KONG

Institute of Mathematical Research
Department of Mathematics

MINI COURSE

A short course on classical and arithmetic theta lifting

Dr. Jiacheng Xia

The University of Hong Kong

Abstract

In explicit number theory, it is most ideal if one could completely reduce a problem to finite many cases and verify them case by case, for instance under some local–global principle. While this ultimate goal was successfully achieved for certain special problems, such as representing zero by quadratic forms over any number field by the work of Minkowski and Hasse, it is known to be unrealistic in general.

In modern number theory, it is often desirable to find a tool to cover this gap conceptually; and one candidate, first introduced by Euler, is the zeta function and its variants known as the L-functions. Being defined as objects patching together all the information locally, the miracle of L-functions lies in that they are often connected to some global objects of interest.

For instance, in the direction concerning extensions of number fields ("Galois side"), Dirich-let published a proof (1839, also known to Gauss by 1801) of his celebrated class number formula for quadratic fields, which expresses class numbers as special values of his Dirichlet L-functions, up to some explicit constant factors such as the regulator. This prototypi-cal result was later generalized in many further directions, such as to detect the Galois module structure of class groups and beyond (Iwasawa theory, 1959 to this day) and to construct explicit algebraic units and answer Hilbert's twelfth problem for totally real fields (Brumer–Stark conjecture, 2023), where special L-values are all explicitly related to the global objects of interest.

In another direction concerning the rational points of algebraic curves, and more generally, algebraic cycles on varieties ("geometric side"), probably the most celebrated prototypical conjecture is the BSD conjecture (first half of 1960s) which relates the rational points of an elliptic curve over a number field (which is an abelian group of finite rank (1922), called the Mordell–Weil group) to its Hasse–Weil L-function in a miraculous way. Even the coarser rank part of the conjecture predicts that the rank of the Mordell–Weil group should coincide with the order of vanishing at the central point of the Hasse–Weil L-function. The cases for rank at least two are still not known for this conjecture, and two of the most important tools hiterto for the rank zero and one cases are the Gross–Zagier formula (1986) and the Waldspurger formula (1985). Both tools can be viewed from

representation theory as two facets of theta correspondence, and both highlight explicit relations of some globally defined objects to special L-values and L-derivatives.

In this short lecture series, we will start from some historic results, from Hecke (1930s) to Waldspurger (1980s) and Gross–Zagier (1986 with their variants), and build up some key notions such as period integrals and classical theta lifts and see some nonvanishing results which connect these key notions to special L-values. In this part we will combine points of view from both representation theory and classical modular forms.

In the second part, we will investigate how to establish an arithmetic analogue of classical theta lifts and relate their height pairing to special L-derivatives. This in particular includes some details in the known cases for arithmetic surfaces constructed from Shimura curves and further discussions about the higher dimensional cases. In this part we will introduce modular forms for higher rank groups and define generating series of special cycles and some arithmetic intersection theory.

Finally, if time permits, we will outline a proof for some cases of the modularity conjecture for the generating series and motivate some future applications of this rather conceptual picture, such as questions on the "Galois side" or questions arising from arithmetic statistics.

Schedule

Date:	Sep 20, 27 Oct 4, 18, 25 Nov 1, 8, 15, 22, 29, 2024 (Friday)
Time:	15:15pm – 16:15pm
Venue:	Room 320A, Run Run Shaw Building HKU

All are welcome