The Riemann and the Generalised Riemann Integral

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1 The Riemann Integral

1.1 Riemann Integral

Here, we will not provide motivation of the integral, discuss its interpretation as the “area under the graph,” or its applications. We will focus only on the mathematical aspects of the integral.

Definition 1.1.1 (Partition). A partition $P$ of $[a, b]$ is a finite set of points $x_0, x_1, \ldots, x_n$ such that

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$$

We denote partition $P$ by $P := \{[x_{i-1}, x_i]\}_{i=1}^n$ and $[x_{i-1}, x_i]$ is called the $i^{th}$ interval of $P$.

Definition 1.1.2 (Mesh/norm of Partition $P$). We define the norm of $P$ to be

$$||P|| := \max\{x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}\}$$\hspace{1cm} (1.1)

Therefore, we can say that the norm of $P$ is the length of the largest subinterval in the Partition $P$.

Definition 1.1.3 (Tagged Partition). For a partition $P = \{[x_{i-1}, x_i]\}_{i=1}^n$ in $[a, b]$. Let us choose a point $t_i \in [x_{i-1}, x_i]$ for every $i < n$, we call $t_i$ as tags of subinterval $[x_{i-1}, x_i]$. The partition $P$ together with its tags are called tagged partition of interval $[a, b]$ denoted by

$$\hat{P} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$$

Remarks:

1. Tags can be chosen in any way (eg. left endpoint, right endpoint, midpoint, etc)
2. An endpoint of a subinterval can be used as a tag for 2 consecutive subintervals

Definition 1.1.4 (Riemann Sum). Let $f : [a, b] \rightarrow \mathbb{R}$ be a function with tagged partition $\hat{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$. Then we define the Riemann Sum of $f$ to be

$$S(f; \hat{P}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$\hspace{1cm} (1.2)

Using this knowledge of the Partition, now we can define the Riemann Integral

Definition 1.1.5 (Riemann Integral). A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be Riemann Integrable on $[a, b]$ if there exists $L \in \mathbb{R}$ such that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $P$ is a tagged partition of $[a, b]$ with $||\hat{P}|| < \delta$, then

$$\left| S(f; \hat{P}) - L \right| < \epsilon$$

We usually call $L$ as the Riemann Integral of $f$ over $[a, b]$ denoted by

$$L = \int_a^b f \text{ or } \int_a^b f(x)dx$$

We denote the set of all Riemann Integrable functions on $[a, b]$ as $\mathcal{R}[a, b]$.

Remark:

1. Although the definition of the Riemann Integral is similar to the definition of the limit of the Riemann Sum as the norm goes to 0, $\left( \lim_{||\hat{P}|| \to 0} S(f; \hat{P}) = L \right)$, $S(f; \hat{P})$ is not a function of $||\hat{P}||$. So this is a different type of limit from what we have studied before.

Theorem 1.1.6 (Uniqueness Theorem). If $f \in \mathcal{R}[a, b]$, then its integral is unique.
Proof. Assume instead that $L_1$ and $L_2$ both satisfy Definition 1.1.5 with $L_1 \neq L_2$, without loss of generality, assume that $L_2 > L_1$. Then, for all $\epsilon > 0$, there exists $\delta_1 > 0$ such that if a tagged partition $\hat{P}_1$ with norm $||\hat{P}_1|| < \delta_1$, then

$$|S(f; \hat{P}_1) - L_1| < \epsilon$$

Also, there exists $\delta_2 > 0$ such that if a tagged partition $\hat{P}_2$ with norm $||\hat{P}_2|| < \delta_2$, then

$$|S(f; \hat{P}_2) - L_2| < \epsilon$$

Choose $\delta = \min\{\delta_1, \delta_2\} > 0$. Let $\hat{P}$ be a tagged partition with norm $||\hat{P}|| < \delta$.

We then fix $\epsilon = \frac{L_2 - L_1}{2} > 0$. Then from Equation 1.3, we get

$$-\epsilon + L_1 < S(f; \hat{P}) < \epsilon + L_1 \implies \frac{3L_1 - L_2}{2} < S(f; \hat{P}) < \frac{L_1 + L_2}{2}$$

Similarly, Equation 1.4 gives us

$$-\epsilon + L_2 < S(f; \hat{P}) < \epsilon + L_2 \implies \frac{L_1 + L_2}{2} < S(f; \hat{P}) < \frac{3L_2 - L_1}{2}$$

So we have

$$\frac{3L_1 - L_2}{2} < S(f; \hat{P}) < \frac{L_1 + L_2}{2} < S(f; \hat{P}) < \frac{3L_2 - L_1}{2}$$

which is impossible so we have a contradiction that $L_1 \neq L_2$. Therefore, the value of the integral is unique.

The following theorem allows us to form algebraic combinations of integrable function

**Theorem 1.1.7.** Suppose $f, g \in \mathcal{R}[a, b]$, then:

a. If $k \in \mathbb{R}$, then $kf \in \mathcal{R}[a, b]$ and

$$\int_a^b kf = k \int_a^b f$$

b. $f + g \in \mathcal{R}[a, b]$ and

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g$$

c. If $f(x) \leq g(x)$ for all $x \in [a, b]$ then

$$\int_a^b f \leq \int_a^b g$$

**Proof.** Consider $\hat{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ on $[a, b]$, then

$$S(kf; \hat{P}) = \sum_{i=1}^n kf(t_i)(x_i - x_{i-1}) = k \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) = kS(f; \hat{P})$$

Using similar methods, we can conclude that

$$S(f + g; \hat{P}) = S(f; \hat{P}) + S(g; \hat{P})$$

and

$$S(f; \hat{P}) \leq S(g; \hat{P})$$
For part a, as \( f \in \mathcal{R}[a,b] \), then for all \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if a tagged partition \( \hat{P} \) with norm \( ||\hat{P}|| < \delta \), then
\[
\left| S\left(f; \hat{P}\right) - \int_a^b f \right| < \frac{\epsilon}{k}
\]
So
\[
\left| S\left(kf; \hat{P}\right) - k\int_a^b f \right| = kS\left(f; \hat{P}\right) - k\int_a^b f = |k| \left| S\left(f; \hat{P}\right) - \int_a^b f \right| < \frac{|k|\epsilon}{|k|} = \epsilon
\]
Therefore, the Riemann Integral of the function \( kf \) is equal to \( k \int_a^b f \).

For part b, as \( f, g \in \mathcal{R}[a,b] \), then for all \( \epsilon > 0 \), there exists \( \delta_1 > 0 \) such that if a tagged partition \( \hat{P}_1 \) with norm \( ||\hat{P}_1|| < \delta_1 \), then
\[
\left| S\left(f; \hat{P}_1\right) - \int_a^b f \right| < \frac{\epsilon}{2}
\]
Similarly, there exists \( \delta_2 > 0 \) such that if a tagged partition \( \hat{P}_2 \) with norm \( ||\hat{P}_2|| < \delta_2 \), then
\[
\left| S\left(g; \hat{P}_2\right) - \int_a^b g \right| < \frac{\epsilon}{2}
\]
We let \( \delta = \min\{\delta_1, \delta_2\} > 0 \). Since \( ||\hat{P}|| < \delta_1 \) and \( ||\hat{P}|| < \delta_2 \), then
\[
\left| S\left(f; \hat{P}\right) - \int_a^b f \right| < \frac{\epsilon}{2} \text{ and } \left| S\left(g; \hat{P}\right) - \int_a^b g \right| < \frac{\epsilon}{2}
\]
(1.5)
So, by Triangle Inequality
\[
\left| S\left(f + g; \hat{P}\right) - \left( \int_a^b f + \int_a^b g \right) \right| = \left| S\left(f; \hat{P}\right) - \int_a^b f + S\left(g; \hat{P}\right) - \int_a^b g \right|
\leq \left| S\left(f; \hat{P}\right) - \int_a^b f \right| + \left| S\left(g; \hat{P}\right) - \int_a^b g \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]
Therefore, the Riemann Integral of the function \( f + g \) is equal to \( \int_a^b f + \int_a^b g \).

For part c, we use Equation 1.5.
\[
\left| S\left(f; \hat{P}\right) - \int_a^b f \right| < \frac{\epsilon}{2} \Rightarrow \frac{\epsilon}{2} < S\left(f; \hat{P}\right) - \int_a^b f \Rightarrow \int_a^b f < \frac{\epsilon}{2} < S\left(f; \hat{P}\right)
\]
\[
\left| S\left(g; \hat{P}\right) - \int_a^b g \right| < \frac{\epsilon}{2} \Rightarrow S\left(g; \hat{P}\right) - \int_a^b g < \frac{\epsilon}{2} \Rightarrow S\left(g; \hat{P}\right) < \int_a^b g + \frac{\epsilon}{2}
\]
As we know that \( S(f; \hat{P}) \leq S(g; \hat{P}) \), we have
\[
\int_a^b f - \frac{\epsilon}{2} < \int_a^b g + \frac{\epsilon}{2} \Rightarrow \int_a^b f < \int_a^b g + \epsilon
\]
So, we can conclude that \( \int_a^b f \leq \int_a^b g \)

**Theorem 1.1.8.** Let \( h : [a,b] \to \mathbb{R} \) such that \( h(x) = 0 \) except for a finite number of points in \([a,b]\), then \( h \in \mathcal{R}[a,b] \) and \( \int_a^b h = 0 \)

**Proof.** We just need to proof for the case of only 1 point (i.e. \( h(x) = 0 \) except for a point \( c \), then we can use induction to prove for a finite number of points.

For all \( \epsilon > 0 \), let \( \delta < \frac{\epsilon}{2|h(c)|} > 0 \) such that if a tagged partition \( ||\hat{P}|| < \delta \), then
\[
\left| S\left(h; \hat{P}\right) - 0 \right| = \sum_{i=1}^n h(t_i)(x_i - x_{i-1}) \leq 2|h(c)|||\hat{P}|| < 2|h(c)|\delta = \epsilon
\]
Therefore, the function \( h \) is integrable with \( \int_a^b h = 0 \)
Corollary 1.1.9. If \( g \in \mathcal{R}[a,b] \) and if \( f(x) = g(x) \) except for a finite number of points in \([a,b]\), then \( f \) is Riemann integrable and \( \int_a^b f = \int_a^b g \).

Proof. Consider \( h : [a, b] \to \mathbb{R} \) such that \( h(x) = f(x) - g(x) \) then we can use Theorem 1.1.8 to show that 
\[
\int_a^b (f - g) = \int_a^b h = 0 \implies \int_a^b f - \int_a^b g = 0 \implies \int_a^b f = \int_a^b g
\]

Theorem 1.1.10 (Boundedness Theorem on Integrals). If \( f \in \mathcal{R}[a,b] \), then \( f \) is bounded on \([a,b]\).

Remark:
1. The converse \( \) (All bounded functions are Riemann Integrable) is not true, and example is shown in Example 1.2.2

Proof. Assume by contradiction that \( f \in \mathcal{R}[a,b] \) is an unbounded function with \( \int_a^b f = L \). By choosing \( \epsilon = 1 \), there exists \( \delta > 0 \) such that if a tagged partition \( \hat{P} \) with \( ||\hat{P}|| < \delta \), then
\[
\left| S \left( f; \hat{P} \right) - L \right| < 1 \implies \left| S \left( f; \hat{P} \right) \right| < |L| + 1
\]
(1.6)

Then we let \( Q = \{[x_{i-1}, x_i] \}_{i=1}^n \) be a partition with \( ||Q|| < \delta \). Since \( f \) is not bounded on \([a,b]\), then there exists a subinterval \([x_{k-1}, x_k]\) in which \( f \) is not bounded.

Since \( f \) is not bounded for \([x_{k-1}, x_k]\), for all \( M > 0 \), there exists \( x \in [a, b] \) such that \( |f(x)| > M \). Here we fix \( M = |L| + 1 + \sum_{i \neq k} f(t_i)(x_i - x_{i-1}) \) to make the contradiction with Equation 1.6, we need to choose appropriate tags. For subinterval \([x_{i-1}, x_i]\) such that \( i \neq k \), pick \( t_i = x_i \). For \( i = k \) we choose \( t_k \) in \([x_{k-1}, x_k]\) such that
\[
|f(t_k)(x_k - x_{k-1})| > M = |L| + 1 + \sum_{i \neq k} f(t_i)(x_i - x_{i-1})
\]

By using the Triangle Inequality \( |x + y| \geq |x| - |y| \forall x, y \in \mathbb{R} \),
\[
\left| S \left(f; \hat{Q} \right) \right| \geq |f(t_k)(x_k - x_{k-1})| - \sum_{i \neq k} f(t_i)(x_i - x_{i-1}) > |L| + 1
\]

This contradicts Equation 1.6.

\[ \square \]

1.2 Properties of Riemann Integrable Functions

Theorem 1.2.1 (Cauchy Criterion). Let \( f : [a, b] \to \mathbb{R} \). \( f \in \mathcal{R}[a,b] \) if and only if for every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if tagged partitions \( \hat{P} \) and \( \hat{Q} \) of \([a, b]\) with \( ||\hat{P}|| < \delta \) and \( ||\hat{Q}|| < \delta \), then
\[
\left| S \left(f; \hat{P} \right) - S \left(f; \hat{Q} \right) \right| < \epsilon
\]

Proof. \( (\implies) \) As \( f \in \mathcal{R}[a,b] \), let \( \int_a^b f = L \). If we choose tagged partitions \( \hat{P} \) and \( \hat{Q} \) such that \( ||\hat{P}|| < \delta \) and \( ||\hat{Q}|| < \delta \), then
\[
\left| S \left(f; \hat{P} \right) - L \right| < \frac{\epsilon}{2} \text{ and } \left| S \left(f; \hat{Q} \right) - L \right| < \frac{\epsilon}{2}
\]

So, we have
\[
\left| S \left(f; \hat{P} \right) - S \left(f; \hat{Q} \right) \right| = \left| S \left(f; \hat{P} \right) - L + L - S \left(f; \hat{Q} \right) \right| \leq \left| S \left(f; \hat{P} \right) - L \right| + \left| S \left(f; \hat{Q} \right) - L \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]

And so we are done.

\[ \square \]
(⇐) For every ε > 0, let δ > 0 such that if tagged partitions $\hat{P}$ and $\hat{Q}$ with $||\hat{P}|| < δ$ and $||\hat{Q}|| < δ$, then

$$|S(f;\hat{P}) - S(f;\hat{Q})| < \frac{\epsilon}{2}$$

Let $\hat{P}_n$ be a sequence of tagged partitions with $||\hat{P}_n|| < δ$ for all $n \in \mathbb{N}$. So we choose $m, n \in \mathbb{N}$ and without loss of generality, we assume $m > n$. Clearly, this means that $||\hat{P}_n|| < δ$ and $||\hat{P}_m|| < δ$ which in turn implies

$$|S(f;\hat{P}_n) - S(f;\hat{P}_m)| < \frac{\epsilon}{2}$$

This shows that $S(f;\hat{P}_n)$ is a Cauchy sequence and in turn, converges to some number $L \in \mathbb{R}$. Or in other words,

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : n > N \implies |S(f;\hat{P}_n) - L| < \frac{\epsilon}{2}$$

Finally, to show that $f \in \mathcal{R}[a, b]$ and that $\int_a^b f = L$, given $\epsilon > 0$, we let $\delta > 0$ and $n > N$ such that if a tagged partition $\hat{Q}$ with $||\hat{Q}|| < δ$, then

$$|S(f;\hat{Q}) - L| \leq |S(f;\hat{Q}) - S(f;\hat{P}_n)| + |S(f;\hat{P}_n) - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\square$

**Example 1.2.2.** Show that the Dirichlet function,

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is not Riemann integrable on $[a, b]$ for all $a, b \in \mathbb{R}$

**Proof.** We are going to use the converse of the Cauchy Criterion. Therefore we need to prove that: There exists $\epsilon > 0$ for every $\delta > 0$, there exists tagged partitions $\hat{P}$ and $\hat{Q}$ with $||\hat{P}|| < \delta$ and $||\hat{Q}|| < \delta$ such that $|S(f;\hat{P}) - S(f;\hat{Q})| \geq \epsilon$. Taking $\epsilon = \frac{b-a}{2}$, we let $\hat{P}$ and $\hat{Q}$ be partitions on $[a, b]$

For all $n \in \mathbb{N}$,

$$\hat{P} = \{(x_{i-1}, x_i] : t_i \in \mathbb{Q}\}$$

and

$$\hat{Q} = \{(x_{i-1}, x_i] : t_i \in \mathbb{R} \setminus \mathbb{Q}\}$$

and

$$S(f;\hat{P}) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) = \sum_{i=1}^n (x_i - x_{i-1}) = (x_n - x_n) = (x_n - x_0) = b - a$$

While

$$S(f;\hat{Q}) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) = 0$$

Then,

$$|S(f;\hat{P}) - S(f;\hat{Q})| = |b-a| > \frac{b-a}{2} = \epsilon$$

Therefore, the Dirichlet function is not Riemann integrable on $[a, b]$ for all $a, b \in \mathbb{R}$ $\square$

**Theorem 1.2.3** (Squeeze Theorem). If $f : [a, b] \to \mathbb{R}$, then $f \in \mathcal{R}[a, b]$ if and only if for all $\epsilon > 0$, there exists functions $g, h \in \mathcal{R}[a, b]$ with

$$g(x) \leq f(x) \leq h(x) \forall x \in [a, b]$$

such that

$$\int_a^b (h - g) < \epsilon$$
Proof. (⇒) We can let \( f(x) = g(x) = h(x) \) for all \( x \in [a, b] \) and let \( \epsilon > 0 \)

(⇐) Given \( \epsilon > 0 \), we choose functions \( g, h \in \mathcal{R}[a, b] \) such that

\[
g(x) \leq f(x) \leq h(x) \quad \forall x \in [a, b] \tag{1.7}
\]

such that

\[
\int_a^b (h - g) < \frac{\epsilon}{3} < \epsilon \tag{1.8}
\]

As \( g, h \in \mathcal{R}[a, b] \), there exists \( \delta > 0 \) such that if any tagged partition \( \tilde{P} \) with \( ||\tilde{P}|| \leq \delta \), then

\[
\left| S(g; \tilde{P}) - \int_a^b g \right| < \frac{\epsilon}{3} \quad \text{and} \quad \left| S(h; \tilde{P}) - \int_a^b h \right| < \frac{\epsilon}{3}
\]

It follows that

\[
\int_a^b g - \frac{\epsilon}{3} < S(g; \tilde{P}) \quad \text{and} \quad S(h; \tilde{P}) < \int_a^b h + \frac{\epsilon}{3}
\]

Due to Equation 1.7, we know that \( S(g; \tilde{P}) \leq S(f; \tilde{P}) \leq S(h; \tilde{P}) \). Therefore,

\[
\int_a^b g - \frac{\epsilon}{3} < S(f; \tilde{P}) < \int_a^b h + \frac{\epsilon}{3}
\]

If we choose another tagged partition \( \tilde{Q} \) with \( ||\tilde{Q}|| < \delta \), we also have

\[
\int_a^b g - \frac{\epsilon}{3} < S(f; \tilde{Q}) < \int_a^b h + \frac{\epsilon}{3}
\]

Then

\[
- \left( \int_a^b (h - g) + \frac{2\epsilon}{3} \right) < S(f; \tilde{P}) - S(f; \tilde{Q}) < \int_a^b (h - g) + \frac{2\epsilon}{3}
\]

So,

\[
\left| S(f; \tilde{P}) - S(f; \tilde{Q}) \right| < \int_a^b (h - g) + \frac{2\epsilon}{3} < \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon
\]

This satisfies the Cauchy Criterion (Theorem 1.2.1), therefore \( f \in \mathcal{R}[a, b] \)

\[\square\]

**Theorem 1.2.4.** If \( f : [a, b] \to \mathbb{R} \) is a step function, then \( f \in \mathcal{R}[a, b] \)

**Proof.** Consider the function \( \varphi_A : [a, b] \to \mathbb{R} \) such that

\[
\varphi_A(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{if } x \notin A
\end{cases}
\]

where \( A = [c, d] \) for some \( c, d \in \mathbb{R} \) with \( a < c < d < b \)

Given \( \epsilon > 0 \), let \( \delta > 0 \) such that if a tagged partition with \( ||\tilde{P}|| < \delta \),

\[
S\left( \varphi_A; \tilde{P} \right) = \sum_{i=1}^{n} \varphi(t_i)(x_i - x_{i-1}) = \sum_{x \in A} (x_i - x_{i-1}) = d - c
\]

We claim that the integral of \( \varphi_A(x) \) is \( d - c \). So,

\[
\left| S\left( \varphi_A; \tilde{P} \right) - (d - c) \right| = |(d - c) - (d - c)| = 0 < \epsilon
\]

Therefore, function \( \varphi_A \in \mathcal{R}[a, b] \).

To complete the proof, we write any step function \( f \) as

\[
f(x) = \sum_{i=1}^{k} a_i \varphi_{A_i}(x)
\]

and by using Theorem 1.1.7, we are done.

\[\square\]
Theorem 1.2.5. If \( f : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\), then \( f \in \mathcal{R}[a, b] \)

Proof. As \( f \) is continuous on \([a, b]\), then \( f \) is uniformly continuous over \([a, b]\), i.e.

\[
\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in [a, b] : |x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{b - a}
\]

Let \( P \) be a partition and so as \( f \) is continuous on \([a, b]\), for each subinterval \([x_{i-1}, x_i]\), there exists \( u_i, v_i \in [x_{i-1}, x_i] \) such that for each \( i \)th subinterval, \( f(u_i) \) and \( f(v_i) \) has a minimum value and maximum value respectively on \([x_{i-1}, x_i]\) Then, we define a step function \( \phi \) and \( \Phi \) where

\[
\phi(x) = f(u_i) \text{ and } \Phi(x) = f(v_i) \text{ for all } x \in [x_{i-1}, x_i]
\]

Below is an illustration of the function \( f \), \( \phi \), and \( \Phi \) where the curve is the function \( f \), the step function above \( f \) is \( \Phi \) and the step function below \( f \) is \( \phi \)

Clearly, we have \( \phi(x) \leq f(x) \leq \Phi(x) \) for all \( x \in [a, b] \) So, for all \( \epsilon > 0 \), there exists \( \delta > 0 \), such that if any partition \( \tilde{P} \) with \( ||\tilde{P}|| < \delta \), then

\[
|x_i - x_{i-1}| < \delta \text{ for all } i < n \implies |v_i - u_i| < \delta
\]

Then,

\[
\int_a^b \Phi - \phi = \sum_{i=1}^n (f(v_i) - f(u_i))(x_i - x_{i-1}) < \sum_{i=1}^n \frac{\epsilon}{b - a} (x_i - x_{i-1}) = \frac{\epsilon}{b - a} \left( \sum_{i=1}^n (x_i - x_{i-1}) \right) = \epsilon \frac{(b - a)}{b - a} = \epsilon
\]

Finally, we just apply the squeeze theorem (Theorem 1.2.3) to complete the proof.

\[\square\]

Theorem 1.2.6. If \( f : [a, b] \to \mathbb{R} \) is monotone on \([a, b]\), then \( f \in \mathcal{R}[a, b] \)

As the proof is very similar to Theorem 1.2.5. The proof is omitted.

Theorem 1.2.7 (Additivity Theorem). Let \( f : [a, b] \to \mathbb{R} \) and \( c \in [a, b] \), then \( f \) is Riemann Integrable on \([a, b]\) if and only if \( f \) is Riemann Integrable on \([a, c]\) and \( f \) is Riemann Integrable on \([c, b]\) and so if the condition is true,

\[
\int_a^b f = \int_a^c f + \int_c^b f
\]
Proof. (\(\Leftarrow\)) As \(f \in \mathcal{R}[a,b]\) with \(\int_a^b f = L_1\), then for all \(\epsilon > 0\), there exists \(\delta_1 > 0\) such that if any tagged partition \(\hat{P}_1\) with \(\|\hat{P}_1\| < \delta_1\), then \(|S\left(f; \hat{P}_1\right) - L_1| < \frac{\epsilon}{2}\). Also, as \(f \in \mathcal{R}[c,b]\) with \(\int_c^b f = L_2\), then for all \(\epsilon > 0\), there exists \(\delta_2 > 0\) such that if any tagged partition \(\hat{P}_2\) with \(\|\hat{P}_2\| < \delta_2\), then \(|S\left(f; \hat{P}_2\right) - L_2| < \frac{\epsilon}{2}\). If \(M\) is a bound for \(|f|\), choose \(\delta = \min\{\delta_1, \delta_2, \frac{\epsilon}{2M}\}\), such that if a tagged partition \(\hat{P}\) with \(\|\hat{P}\| < \delta\), we will prove that

\[|S\left(f; \hat{P}\right) - (L_1 + L_2)| < \epsilon\]

(i) Case 1: Let \(\hat{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n\). If \(c\) is a partition point of \(\hat{P}\) (i.e. \(c = x_i\) for some \(i < n\)), we split the partition to \(\hat{P}_1\) and \(\hat{P}_2\) on \([a, c]\) and \([c, b]\) respectively. Then, since we know that \(\|P_1\| < \delta_1\) and \(\|P_2\| < \delta_2\) and \(S\left(f; \hat{P}\right) = S\left(f; \hat{P}_1\right) + S\left(f; \hat{P}_2\right)\),

\[
\left|S\left(f; \hat{P}\right) - (L_1 + L_2)\right| = \left|S\left(f; \hat{P}_1\right) - L_1 + S\left(f; \hat{P}_2\right) - L_2\right|
\leq \left|S\left(f; \hat{P}_1\right) - L_1\right| + \left|S\left(f; \hat{P}_2\right) - L_2\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]

(ii) Case 2: If \(c\) is not a partition point of \(\hat{P}\), then there exists \(k < n\) such that \(c \in [x_{k-1}, x_k]\). We consider a tagged partition \(\hat{P}'\) with one more partition point than \(\hat{P}\) at \(c\), i.e.

\[
\hat{P}' = \{([x_0, x_1], t_1), \ldots, ([x_{k-1}, c], c), ([c, x_k], c), \ldots, ([x_{n-1}, x_n], t_n)\}
\]

So, for all \(\epsilon > 0\), there exists \(\delta > 0\) such that if any partition \(\hat{P}'\) with \(\|\hat{P}'\| < \delta < \frac{\epsilon}{2M}\), then

\[
\left|S\left(f; \hat{P}'\right) - s\left(f; \hat{P}'\right)\right| = \left|f(t_k)(x_k - x_{k-1}) - f(c)(x_k - x_{k-1})\right|
\]

\[
= \left|f(t_k) - f(c)\right|(x_k - x_{k-1}) < 2M(x_k - x_{k-1}) < 2M\left(\frac{\epsilon}{2M}\right) = \epsilon
\]

This satisfies the Cauchy Criterion (Theorem 1.2.1) and so, \(f \in \mathcal{R}[a,b]\)

(\(\Rightarrow\)) As \(f \in \mathcal{R}[a,b]\), given \(\epsilon > 0\), let \(\delta > 0\) such that if tagged partitions \(\hat{P}\) and \(\hat{Q}\) with \(\|\hat{P}\| < \delta\) and \(\|\hat{Q}\| < \delta\), then

\[
\left|S\left(f; \hat{P}\right) - S\left(f; \hat{Q}\right)\right| < \epsilon
\]

We consider \(\hat{P}'\) and \(\hat{Q}'\) on \([a, c]\) with \(\|\hat{P}'\| < \delta\) and \(\|\hat{Q}'\| < \delta\). By adding the same additional partition points and tags in \([c, b]\) to \(\hat{P}'\) and \(\hat{Q}'\), we get \(\hat{P}\) and \(\hat{Q}\) such that \(\|\hat{P}\| < \delta\) and \(\|\hat{Q}\| < \delta\) and so

\[
\left|S\left(f; \hat{P}'\right) - S\left(f; \hat{Q}'\right)\right| = \left|S\left(f; \hat{P}\right) - S\left(f; \hat{Q}\right)\right| < \epsilon
\]

This shows that it fulfills the Cauchy Criterion (Theorem 1.2.1). So \(f\) is Riemann Integrable on \([a, c]\) The same applies to \([c, b]\) and so, we are done.

\[
\square
\]

### 1.3 The Fundamental Theorem of Calculus

In this subsection, we are going to discuss the connection of the derivative and the integral. This subsection will discuss mainly the two forms of the Fundamental Theorem of Calculus.

**Definition 1.3.1.** Let \(f : [a, b] \to \mathbb{R}\) be a function. A function \(F : [a, b] \to \mathbb{R}\) is called an antiderivative of \(f\) if \(F'(x) = f(x)\) for all \(x \in [a, b]\)

**Definition 1.3.2.** If \(f \in \mathcal{R}[a,b]\), then the function \(F_a : [a, b] \to \mathbb{R}\) defined by

\[
F_a(x) = \int_a^x f \text{ for } x \in [a, b]
\]

is called the indefinite integral of \(f\) with basepoint \(a\)

Using this definition, we can define the Fundamental Theorem of Calculus.
The Fundamental Theorem of Calculus (First Form). Let $f : [a, b] \to \mathbb{R}$ and $f$ is Riemann Integrable on $[a, b]$. Suppose that there is a function $F : [a, b] \to \mathbb{R}$ such that:

(a) $F$ is continuous on $[a, b]$,

(b) $F'(x) = f(x)$ for most $x \in [a, b]$ except for a finite set of points,

Then, we have $\int_a^b f = F(b) - F(a)$

Remarks:
1. The function $F$ such that $F'(x) = f(x)$ for all $x \in [a, b]$ is called the antiderivative of $x$
2. We can permit a finite number of points $c$ where $F'(c)$ does not exists or $F'(c) \neq f(c)$

Proof. As $f \in R[a, b]$, for all $\epsilon > 0$, there exists $\delta > 0$ such that if any tagged partition $\hat{P}$ with $||\hat{P}|| < \delta$ then

$$\left| S(f; \hat{P}) - \int_a^b f \right| < \epsilon$$

We let $[x_{i-1}, x_i]$ be the $i$th subinterval of P. Then, by applying the Mean Value Theorem on $F$, for each $i < n$, there exists $u_i \in (x_{i-1}, x_i)$ such that

$$\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = F'(u_i) = f(u_i)$$

$$F(x_i) - F(x_{i-1}) = f(u_i)(x_i - x_{i-1})$$

By adding all the $i$th terms, we get

$$F(b) - F(a) = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) = \sum_{i=1}^{n} f(u_i)(x_i - x_{i-1}) \quad (1.9)$$

Then, letting $\hat{P} = \{(x_{i-1}, x_i), u_i\}_{i=1}^{n}$, Equation 1.9 is equal to $S(f; \hat{P})$. So, we can conclude that, $\left| F(b) - F(a) - \int_a^b f \right| < \epsilon$ for all $\epsilon > 0$. But, since we can make $\epsilon > 0$ as small as possible, we infer that $F(b) - F(a) = \int_a^b f$.

Example 1.3.4. Let $a, b \in \mathbb{R}$ such that $a < 0 < b$. Let function $F : [a, b] \to \mathbb{R}$ such that $F(x) = |x|$. Then let $f : [a, b] \to \mathbb{R}$ such that

$$f(x) = \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{if } x = 0 \\
-1 & \text{if } x < 0 
\end{cases}$$

Proof. We can see that $F'(x) = f(x)$ for all $x \in [a, b] \setminus \{0\}$ as $F(x)$ is not differentiable at 0. However, the Fundamental Theorem of Calculus still applies.
We can see that the function $f(x)$ is a step function and so $\int_a^b f = b + a$ and $F(b) - F(a) = |b| - |a| = b + a$ and therefore, $\int_a^b f = F(b) - F(a)$.

**Theorem 1.3.5** (Fundamental Theorem of Calculus (Second Form)). Let $f \in \mathcal{R}[a, b]$, and let $f$ be continuous at $c \in [a, b]$. Then the indefinite integral

$$F(c) = \int_a^c f \text{ for } c \in [a, b]$$

is differentiable at $c$ and $F'(c) = f(c)$.

**Proof.** Suppose $c \in (a, b)$ consider the right hand derivative of $F$ at $c$, since $f$ is continuous at $c$, given $\epsilon > 0$, there exists $\delta > 0$ such that if $c - \delta < x < c + \delta$, then

$$|f(x) - f(c)| < \epsilon \implies f(c) - \epsilon < f(x) < f(c) + \epsilon$$

(1.10)

Let $h$ satisfy $0 < h < \delta$, the Additivity Theorem (Theorem 1.2.7) implies that $f$ is Riemann Integrable on $[a, c]$, $[a, c + h]$ and $[c, c + h]$ and that

$$\int_c^{c+h} f = \int_a^{c+h} f - \int_a^c f = F(c + h) - F(c)$$

Now on $[c, c + h]$, $f$ satisfy Equation 1.10. So we have

$$(f(c) - \epsilon) \cdot h = \int_c^{c+h} (f(x) - \epsilon) \leq F(c + h) - F(c) = \int_c^{c+h} f \leq \int_c^{c+h} (f(x) + \epsilon) = (f(c) + \epsilon) \cdot h$$

$$\implies -\epsilon \leq \frac{F(c + h) - F(c)}{h} - f(c) \leq \epsilon$$

$$\implies \left| \frac{F(c + h) - F(c)}{h} - f(c) \right| \leq \epsilon$$

But, since we can choose $\epsilon > 0$ to be very small, we can infer that

$$\lim_{h \to 0^+} \frac{F(c + h) - F(c)}{h} = F'(c) = f(c)$$

By proving using the similar method for the left hand side, we conclude that $F'(c) = f(c)$.

**1.4 Lebesgue’s Integrability Criterion**

In this subsection, a necessary and sufficient condition for a function to be Riemann Integrable and some application will be given.

Before starting with the Lebesgue’s Integrability Criterion, we start by defining the null set

**Definition 1.4.1** (Lebesgue Outer Measure). Let $A \subseteq \mathbb{R}$ and let $l((a, b)) = b - a$ for all open interval $(a, b)$. Then the Lebesgue Outer Measure of $A$ $\mu^*(A)$ is defined to be

$$\mu^*(A) = \inf \left\{ \sum_{k=1}^{\infty} l(I_k) : \{I_k\} \text{ is a sequence of open intervals in which } A \subseteq \bigcup_{k=1}^{\infty} I_k \right\}$$

Remarks:

1. We can see here that $0 \leq \mu^*(A) \leq \infty$ for all $A \subseteq \mathbb{R}$

**Definition 1.4.2** (Null Set). A set $A \subseteq \mathbb{R}$ is a null set if for all $\epsilon > 0$, $\mu^*(A) \leq \epsilon$

**Example 1.4.3.** All countable sets are null sets.
Proof. Let set $A = \{ x_k : k \in \mathbb{N} \}$ be a countably infinite set. Then, we define the sequence of open interval $\{ I_k \}$ to be

$$I_k = \left( x_k - \frac{\epsilon}{2^k+1}, x_k + \frac{\epsilon}{2^k+1} \right) \text{ for all } k \in \mathbb{N}$$

Then,

$$\sum_{k=1}^{\infty} l(I_k) = \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon$$

As $\mu^*(A)$ is the infimum, we have $\mu^*(A) < \epsilon$ \hfill \Box

Example 1.4.4. The set of rational numbers $\mathbb{Q}$ is a null set.

Proof. We start by denoting that $\mathbb{Q} = \{ r_k : k \in \mathbb{Z} \}$. Then we can use a similar proof with the proof in Example 1.4.3 \hfill \Box

Definition 1.4.5. If $P(x)$ is a statement about the point $x \in [a, b]$, we say that $P(x)$ holds almost everywhere on $I$ if there exists a null set $Z \subset I$ such that $P(x)$ holds for all $x \in I \setminus Z$ or we may write

$$P(x)^\forall x \in I$$

Now, we are ready to introduce the Lebesgue’s Integrability Criterion

Theorem 1.4.6 (Lebesgue’s Integrability Criterion). A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann Integrable if and only if $f$ is continuous almost everywhere on $[a, b]$.

Proof. The proof to this theorem is omitted. \hfill \Box

Example 1.4.7 (Thomae Function). The Thomae Function is integrable on $[0, 1]$. The Thomae Function is defined as

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x \in \mathbb{Q} \text{ such that } x = \frac{p}{q} \text{ in lowest terms and } q > 0 \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Proof. Here $f(x)$ is continuous for all $x \in \mathbb{R} \setminus \mathbb{Q}$ and discontinuous for all $x \in \mathbb{Q}$. Let $a \in \mathbb{Q}$, let $\{ a_n \}$ be a sequence such that $a_n \in \mathbb{R} \setminus \mathbb{Q}$ for all $n \in \mathbb{N}$ and $\{ a_n \} \to a$, then

$$\lim_{n \to \infty} f(a_n) = 0 \neq f(a)$$

$x$ is continuous at $x \in \mathbb{R} \setminus \mathbb{Q}$. By Archimedean Property, let $b \in \mathbb{R} \setminus \mathbb{Q}$, then there exists $n$ such that $\frac{1}{n} < \epsilon$. Consider the interval $(b - 1, b + 1)$, we can say that there is only a finite number of rationals with denominator less than $n$. So we can choose $\delta > 0$ small enough such that $(b - \delta, b + \delta)$ does not contain a rational number with denominator less than $n$.

It follows that $|x - b| < \delta$ where $x \in A$. We have $|f(x) - f(b)| = |f(x)| \leq \frac{1}{n} < \epsilon$. So $f(x)$ is continuous for all $x \in \mathbb{R} \setminus \mathbb{Q}$.

So the set of all discontinuous points is $S = \mathbb{Q} \cap [0, 1]$ and so as the set is countably infinite, then $S$ is a null set and by the Lebesgue Integrability Criterion, the Thomae function is Riemann Integrable on $[0, 1]$. \hfill \Box

Theorem 1.4.8 (Composition Theorem). Let $f \in \mathcal{R}[a, b]$ with $f([a, b]) \subseteq [c, d]$ and let $g : [c, d] \to \mathbb{R}$ be continuous. Then, $g \circ f \in \mathcal{R}[a, b]$.

Remarks:

1. The condition $g$ is continuous can not be dropped

Proof. We can split this proof into two cases

(i) Case 1. If $f$ is continuous at $[a, b]$, then $g \circ f$ is also continuous at $[a, b]$. This implies that $g \circ f$ is Riemann integrable on $[a, b]$

(ii) Case 2. If $f$ has a finite number of discontinuous points in $[a, b]$, then as $f$ satifies the Lebesgue’s Integrability Theorem (Theorem 1.4.6), then the set $S = \{ u : f \text{ is discontinuous at } u \}$ is a null set as it is countable (By example 1.4.3. It follows that the set $S' \subseteq S$ where $S' = \{ u : g \circ f \text{ is discontinuous at } u \}$. Therefore, by Lebesgue’s Integrability Theorem (Theorem 1.4.6), $g \circ f$ is Riemann Integrable on $[a, b]$.
Both cases implies that $g \circ f$ is Riemann integrable on $[a, b]$

**Example 1.4.9.** Let

$$f(x) = \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{if } x = 0 \\
-1 & \text{if } x < 0 
\end{cases}$$

and

$$g(x) = \begin{cases} 
\frac{1}{q} & \text{if } x \in \mathbb{Q} \text{ such that } x = \frac{p}{q} \text{ in lowest terms and } q > 0 \\
0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}
\end{cases}$$

We already know that $f, g \in \mathbb{R}[0, 1]$ Show that $f \circ g(x)$ is not Riemann Integrable on $[0, 1]$

**Proof.** We can see that $g(x) > 0$ for all $x \in \mathbb{Q}$ and $g(x) = 0$ for all $x \in \mathbb{R} \setminus \mathbb{Q}$. So

$$f \circ g(x) = \begin{cases} 
1 & \text{if } x \in \mathbb{Q} \\
0 & x \in \mathbb{R} \setminus \mathbb{Q}
\end{cases}$$

which we know that it is not Riemann Integrable.

**Theorem 1.4.10** (Product Theorem). If $f$ and $g$ is Riemann integrable on $[a, b]$, then the product $fg$ is also Riemann integrable on $[a, b]$.

**Proof.** For each function $f : [a, b] \to \mathbb{R}$, we consider the function $h(x) = x^2$ for $x \in [-M, M]$ where $[-M, M]$ is the bounds of the function $f$. Then by the composition theorem (Theorem 1.4.8), we know that $f^2 = h \circ f$ is Riemann Integrable on $[a, b]$. Therefore, the functions $f^2$, $g^2$, and $(f + g)^2$ are Riemann Integrable on $[a, b]$. To complete the proof, we simply write $fg$ as

$$fg = \frac{1}{2} \left[(f + g)^2 - f^2 - g^2\right]$$

which shows that $fg$ is Riemann Integrable on $[a, b]$.

## 1.5 Substitution Theorem and Integration by Parts

**Theorem 1.5.1** (Substitution Theorem). Let $\alpha : [c, d] \to \mathbb{R}$ be differentiable at $[c, d]$ with its derivative $\alpha'$ being continuous on $[c, d]$. If $f : [a, b] \to \mathbb{R}$ with $a \leq c \leq d \leq b$ and $\alpha([c, d]) \subseteq [a, b]$, then

$$\int_{c}^{d} f(\alpha(x)) \cdot \alpha'(x) \, dx = \int_{\alpha(c)}^{\alpha(d)} f(\alpha) \, d\alpha$$

**Proof.** Consider

$$F(u) = \int_{\alpha(c)}^{u} f(x) \, dx \text{ for } u \in [a, b]$$

By the Fundamental Theorem of Calculus, $F'(u) = f(u)$ Let $H(x) = F(\alpha(x))$ Also by the Fundamental Theorem of Calculus,

$$H(d) = \int_{c}^{d} H'(d) \, dx$$

Then,

$$\int_{\alpha(c)}^{\alpha(d)} f(\alpha) \, d\alpha = F(\alpha(d)) = H(d)$$

To complete the proof, we just need to find $H'(d)$. As $f$ and $\alpha$ are both differentiable on $[a, b]$ and $[c, d]$ respectively, then

$$H'(d) = F'(\alpha(d)) \cdot \alpha'(x) = f(\alpha(x)) \cdot \alpha'(x)$$

Therefore,

$$\int_{c}^{d} f(\alpha(x)) \cdot \alpha'(x) \, dx = \int_{\alpha(c)}^{\alpha(d)} f(\alpha) \, d\alpha$$
**Theorem 1.5.2** (Integration by Parts). Let $F$ and $G$ be differentiable on $[a, b]$, let $f = F’$ and $g = G’$ with $f, g \in \mathcal{R}[a, b]$. Then

$$\int_a^b fG = \left[ FG \right]_a^b - \int_a^b Fg$$

**Proof.** By Product Rule of Differentiation, as $F$ and $G$ are differentiable,

$$(FG)' = F'G + FG' = fG + Fg$$

As all $F, G, f, g \in \mathcal{R}[a, b]$, then $fG, Fg \in \mathcal{R}[a, b]$. So, by using the Fundamental Theorem of Calculus,

$$[FG]_a^b = \int_a^b (FG)' = \int_a^b fG + \int_a^b Fg$$

Therefore,

$$\int_a^b fG = \left[ FG \right]_a^b - \int_a^b Fg$$

\[\square\]

**Theorem 1.5.3** (Equivalence Theorem). A function $f : [a, b] \rightarrow \mathbb{R}$ is Darboux Integrable if and only if it is Riemann Integrable.

**Proof.** $(\Rightarrow)$ Assume $f$ is Darboux Integrable on $[a, b]$. Then, for all $\epsilon > 0$, let $P$ be a partition on $[a, b]$ such that $U(f; P) - L(f; P) < \epsilon$. We then define the step function $\phi$ and $\Phi$ in which

$$\phi(x) = m_k \quad \text{for} \quad x \in [x_{k-1}, x_k) \quad \forall k < n$$

and

$$\Phi(x) = M_k \quad \text{for} \quad x \in [x_{k-1}, x_k) \quad \forall k < n$$

So we have

$$\phi(x) \leq f(x) \leq \Phi(x) \quad \forall x \in [a, b]$$

As $\phi, \Phi$ are step functions, it is Riemann Integrable and moreover,

$$\int_a^b \phi = \sum_{k=1}^n m_k (x_k - x_{k-1}) = L(f; P)$$

and

$$\int_a^b \Phi = \sum_{k=1}^n M_k (x_k - x_{k-1}) = U(f; P)$$

Therefore, we have

$$\int_a^b \Phi - \phi = U(f; P) - L(f; P) < \epsilon$$

and so by squeeze theorem, $f \in \mathcal{R}[a, b]$. As $\int_a^b f = L(f; P) = U(f; P)$ and so if a tagged partition $\tilde{Q}$ with $||\tilde{Q}|| < \delta$, then

$$|S(f; \tilde{Q}) - U(f; P)| < |U(f; P) - L(f; P)| < \epsilon$$

(\(\Leftarrow\)) Assume $f$ is Riemann Integrable on $[a, b]$. By Cauchy criterion, for all $\epsilon > 0$, there exists $\delta > 0$ such that for all tagged partitions $\tilde{P}_M$ and $\tilde{P}_m$, if $||\tilde{P}_M|| < \delta$ and $||\tilde{P}_m|| < \delta$, then

$$|S(f; \tilde{P}_M) - S(f; \tilde{P}_m)| < \frac{\epsilon}{3}$$

We then fix partition $P = \{[x_{i-1}, x_i]\}_{i=1}^n$.

For all $i < n$, consider set $A_i = \{f(x) : x \in [x_{i-1}, x_i]\}$. Let $M_i$ be $\text{sup} A_i$, as $M_i$ is a supremum,

$$\forall \epsilon > 0, \text{ choose } u_i \in [x_{i-1}, x_i] \text{ such that } M_i < f(u_i) + \frac{\epsilon}{3m(x_i - x_{i-1})}$$
This implies,

\[ M_i(x_i - x_{i-1}) < f(u_i)(x_i - x_{i-1}) + \frac{\epsilon}{3n} \]

and so,

\[ U(f; P) < S(f; \hat{P}_M) + \frac{\epsilon}{3} \]

where \( \hat{P}_M = \{(x_{i-1}, x_i, u_i)\}_{i=1}^n \). Similarly, let \( m_i \) be inf \( A_i \),

\[ \forall \epsilon > 0, \text{ choose tags } v_i \in [x_{i-1}, x_i] \text{ such that } m_i > f(v_i) - \frac{\epsilon}{3n(x_i - x_{i-1})} \]

This implies

\[ m_i(x_i - x_{i-1}) > f(v_i)(x_i - x_{i-1}) - \frac{\epsilon}{3n} \]

and so,

\[ L(f; P) > S(f; \hat{P}_m) - \frac{\epsilon}{3} \]

where \( \hat{P}_m = \{(x_{i-1}, x_i, v_i)\}_{i=1}^n \). Therefore,

\[ |U(f; \hat{P}) - L(f; \hat{P})| < |S(f; \hat{P}_M) - S(f; \hat{P}_m) + 2\epsilon| < |S(f; \hat{P}_M) - S(f; \hat{P}_m)| + \frac{2\epsilon}{3} < \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon \]
2 The Generalised Riemann Integral/Henstock-Kurzweil Integral

The Riemann Integral is a powerful tool and has many applications. However, it was found out later that it was not adequate. For example, the Fundamental Theorem of Calculus

\[ \int_a^b F' = F(b) - F(a) \]

does not hold for all differentiable function.

To tackle with these inadequacies, we use a slightly different approach, to arrive at an integral called the Generalised Riemann Integral.

2.1 Definition and Main Properties

**Definition 2.1.1.** A gauge \( \delta \) on \( [a,b] \) is a strictly positive function \( \delta : [a,b] \to (0,\infty) \)

**Definition 2.1.2.** A tagged partition \( \hat{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n \) of \( [a,b] \) is said to be \( \delta \)-fine if

\[ t_i \in [x_{i-1}, x_i] \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)] \text{ for all } i \in \mathbb{N}, i < n \]

Remarks:

1. \( \delta(t_i) \) is dependent on \( t_i \), so for every \( t_i \), it is possible to have a different value of \( \delta \)

The definition of the Generalised Riemann Integral is very similar to the ordinary Riemann Integral and many of the proofs are similar. However, the slight change results in a much larger set of integrable function.

**Definition 2.1.3.** A function \( f : [a,b] \to \mathbb{R} \) is said to be Generalised Riemann Integrable on \( [a,b] \) if there exists a number \( L \in \mathbb{R} \) such that for every \( \epsilon > 0 \), there exists a gauge \( \delta \) on \( [a,b] \) such that if \( \hat{P} \) is a \( \delta \)-fine partition of \( [a,b] \), then

\[ \left| \int(f; \hat{P}) - L \right| < \epsilon \]

Remarks:

1. The use of gauge allow more control on the length of the subinterval than using the norm (we can put smaller subinterval when the function is rapidly increasing and a larger subinterval when the function is nearly constant).

2. The set of all Generalised Riemann Integrable function is denoted by \( \mathcal{R}^*[a,b] \)

3. The Generalised Riemann Integral of \( f \in \mathcal{R}[a,b] \) is also denoted by

\[ \int_a^b f \text{ or } \int_a^b f(x)dx \]

**Theorem 2.1.4** (Uniqueness Theorem). If \( f \in \mathcal{R}^*[a,b] \), then the value of the integral is uniquely defined.

Remarks:

1. The proof below is very similar to the Uniqueness Theorem of the Riemann Integral
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\textbf{Proof.} Assume }L_1\text{ and }L_2\text{ both satisfy Definition 2.1.3 with }L_1 \neq L_2\text{ and without loss of generality, assume }L_2 > L_1\text{. Let }\epsilon > 0\text{, then there exists gauge }\delta_1\text{ such that if }\hat{P}_1\text{ is a }\delta_1\text{-fine partition, then}

\[ |S \left( f; \hat{P}_1 \right) - L_1 | < \epsilon \]

Similarly, there exists }\delta_2\text{ such that if }\hat{P}_2\text{ is a }\delta_2\text{-fine partition, then}

\[ |S \left( f; \hat{P}_2 \right) - L_2 | < \epsilon \]

Define }\delta(t) = \min \{ \delta_1(t), \delta_2(t) \}\text{ for each }t \in [a, b]\text{ so that }\delta\text{ gauge on }[a, b]\text{. If }\hat{P}\text{ is a }\delta\text{-fine partition, the }\hat{\hat{P}}\text{ is both }\delta_1\text{-fine and }\delta_2\text{-fine. So we have}

\[ |S \left( f; \hat{P} \right) - L_1 | < \epsilon \]

\[ |S \left( f; \hat{P} \right) - L_2 | < \epsilon \]

(2.1)

(2.2)

We then let }\epsilon = \frac{L_2 - L_1}{2} > 0\text{. Then from Equation 2.1, we have}

\[ \frac{3L_1 - L_2}{2} < S \left( f; \hat{P} \right) < \frac{L_1 + L_2}{2} \]

and from Equation 2.2, we have

\[ \frac{L_1 + L_2}{2} < S \left( f; \hat{P} \right) < \frac{3L_2 - L_1}{2} \]

We then get

\[ \frac{3L_1 - L_2}{2} < S \left( f; \hat{P} \right) < \frac{L_1 + L_2}{2} < S \left( f; \hat{\hat{P}} \right) < \frac{3L_2 - L_1}{2} \]

which is impossible therefore we have a contradiction that }L_1 \neq L_2\text{ and so the value of the integral is unique. }\square

\textbf{Theorem 2.1.5 (Consistency Theorem).} Let }f : [a, b] \to \mathbb{R}\text{. If }f\text{ is Riemann Integrable on }[a, b]\text{, then }f\text{ is Generalised Riemann Integrable on }[a, b]\text{ and their integrals are the same, i.e.}

\[ \text{Riemann } \int_a^b f = \text{ Generalised Riemann } \int_a^b f \]

\textbf{Proof.} Given }\epsilon > 0\text{, we are going to find an appropriate gauge on }[a, b]\text{. Since }f \in \mathcal{R}[a, b]\text{, there exists }\delta > 0\text{ such that if any tagged partition }\hat{P}\text{ with }||\hat{P}||\text{, then}

\[ |S \left( f; \hat{P} \right) - L | < \epsilon \]

We let a function }\delta' : [a, b] \to (0, \infty)\text{ such that }\delta'(t) = \frac{1}{4}\delta\text{ for all }t \in [a, b]\text{ so that }\delta'\text{ can be a gauge on }[a, b]\text{. If }

\[ \hat{P} = \{ ( [x_{i-1}, x_i], t_i ) \}_{i=1}^n \]

\text{is a }\delta'\text{-fine partition, then}

\[ I_i \subseteq [t_i - \delta'(t_i), t_i + \delta'(t_i)] = \left[ t_i - \frac{1}{4} \delta, t_i + \frac{1}{4} \delta \right] \]

As

\[ 0 < x_i - x_{i-1} - t_i + \frac{1}{4} \delta - t_i + \frac{1}{4} \delta = \frac{1}{2} \delta < \delta \text{ for all } i \in \mathbb{N}, i < n \]

We can see that }\hat{\hat{P}}\text{ also satisfies }||\hat{\hat{P}}|| < \delta\text{ and as }f\text{ is Riemann Integrable, the statement}

\[ |S \left( f; \hat{\hat{P}} \right) - L | < \epsilon \]

must be true. So, we can see that }f\text{ is also Generalised Riemann Integrable. Therefore, by choosing a constant function as a gauge, we can see that all Riemann Integrable function are Generalised Riemann Integrable. }\square

We know that from Example 1.2.2 that the Dirichlet function is not Riemann Integrable, however we are going to prove shortly that it is Generalised Riemann Integrable.
Example 2.1.6. The Dirichlet function

\[ f(x) = \begin{cases} 
1 & \text{if } x \in \mathbb{Q} \\
0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}
\end{cases} \]

is Generalised Riemann Integrable on \([0, 1]\) and \(\int_0^1 f = 0\)

**Proof.** We define the set \(\mathbb{Q} \cap [0, 1]\) as \(Q = \{r_1, r_2, \ldots, r_n\}\) for all \(n \in \mathbb{N}\). Given \(\epsilon > 0\), define \(\delta(r_n) = \frac{\epsilon}{2n+2}\) and \(\delta(x) = 1\) when \(x \in \mathbb{R} \setminus \mathbb{Q}\). So \(\delta\) is a gauge on \([0, 1]\) and if any partition \(\hat{P} = \{(x_{i-1}, x_i)\}_i\) is \(\delta\)-fine, then we have \(x_i - x_{i-1} < 2\delta(t_i)\). As \(f(x) = 0\) if \(x\) is irrational, we only consider the sum of all the rational tags to find \(S(f; \hat{Q})\).

\[ 0 < f(r_n)(x_i - x_{i-1}) = (x_i - x_{i-1}) < \frac{2\epsilon}{2n+1} = \frac{\epsilon}{2n+1} \]

Since at most 2 subinterval can have the same tags, we have

\[ 0 \leq S(f; \hat{Q}) < \sum_{n=1}^{\infty} \frac{2\epsilon}{2n+1} = \sum_{n=1}^{\infty} \frac{\epsilon}{2n} = \epsilon \]

\[ 0 \leq S(f; \hat{Q}) < \epsilon \]

\[ |S(f; \hat{Q}) - 0| < \epsilon \]

Therefore, \(f\) is Generalised Riemann Integrable and \(\int_0^1 f = 0\) \(\square\)

**Theorem 2.1.7.** Suppose \(f, g \in \mathcal{R}^*[a, b]\)

a. If \(k \in \mathbb{R}\), then \(kf \in \mathcal{R}^*[a, b]\) and

\[ \int_a^b kf = k \int_a^b f \]

b. \(f + g \in \mathcal{R}^*[a, b]\) and

\[ \int_a^b (f + g) = \int_a^b f + \int_a^b g \]

c. If \(f(x) \leq g(x)\) for all \(x \in [a, b]\), then

\[ \int_a^b f \leq \int_a^b g \]

**Proof.** As the proof the similar to Theorem 1.1.7, the proof is omitted \(\square\)

Below is the Cauchy Criterion for Generalised Riemann Integral, the proof is similar with the Cauchy Criterion for Riemann Integral (Theorem 1.2.1)

**Theorem 2.1.8** (Cauchy Criterion). A function \(f : [a, b] \to \mathbb{R}\) is Generalised Riemann Integrable if and only if for every \(\epsilon > 0\), there exists a gauge \(\delta\) on \([a, b]\) such that if any tagged \(\delta\)-fine partition \(\hat{P}, \hat{Q}\), then

\[ |S(f; \hat{P}) - S(f; \hat{Q})| < \epsilon \]

**Proof.** \((\Rightarrow)\) As \(f \in \mathcal{R}^*[a, b]\), let \(\int_a^b f\). If we choose a gauge \(\delta'\) such that \(\hat{P}\) and \(\hat{Q}\) are \(\delta'\)-fine partition of \([a, b]\), then

\[ |S(f; \hat{P}) - L| < \frac{\epsilon}{2} \text{ and } |S(f; \hat{Q}) - L| < \frac{\epsilon}{2} \]

We set \(\delta(t) = \delta'(t)\) for all \(t \in [a, b]\), so if \(\hat{P}\) and \(\hat{Q}\) are \(\delta\)-fine, then

\[ |S(f; \hat{P}) - S(f; \hat{Q})| \leq |S(f; \hat{P}) - L| + |S(f; \hat{Q}) - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \]

And so we are done.
For every $\epsilon > 0$, there exists a gauge $\delta$ on $[a, b]$ such that if any tagged partition $\hat{P}, \hat{Q}$ are $\delta$-fine, then

$$\left| S\left(f; \hat{P}\right) - S\left(f; \hat{Q}\right) \right| < \frac{\epsilon}{2}$$

Let $\hat{P}_n$ be a sequence of $\delta$-fine partitions. So, we choose $m, n \in \mathbb{N}$ and WLOG, we assume $m > n$. Clearly, $\hat{P}_n$ and $\hat{P}_m$ are $\delta$-fine, which in turn implies

$$\left| S\left(f; \hat{P}_n\right) - S\left(f; \hat{P}_m\right) \right| < \frac{\epsilon}{2}$$

This shows that $S(f; \hat{P}_n)$ is a cauchy sequence and in turn, converges to some number $L \in \mathbb{R}$. Or in other words,

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : n > N \implies \left| S\left(f; \hat{P}_n\right) - L \right| < \frac{\epsilon}{2}$$

Finally, to show that $f \in \mathcal{R}^*[a, b]$ and that $\int_a^b f = L$, given $\epsilon > 0$, we let $\delta$ be a gauge and $n > N$ such that if a tagged partition $\hat{Q}$ is $\delta$-fine, then

$$\left| S\left(f; \hat{Q}\right) - L \right| < \left| S\left(f; \hat{Q}\right) - S\left(f; \hat{P}_n\right) \right| + \left| S\left(f; \hat{P}_n\right) - L \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore, the function $f$ is Generalised Riemann Integrable on $[a, b]$ with integral $\int_a^b f = L$. \hfill \Box

**Theorem 2.1.9** (Squeeze Theorem). Let $f : [a, b] \to \mathbb{R}$. Then $f \in \mathcal{R}^*[a, b]$ if and only if for every $\epsilon > 0$, there exists functions $g, h \in \mathcal{R}^*[a, b]$ with

$$g(x) \leq f(x) \leq h(x) \text{ for all } x \in [a, b]$$

and such that $\int_a^b (h - g) \leq \epsilon$.

**Proof.** The proof is similar to the proof in Theorem 1.2.3 and so will be omitted. \hfill \Box

**Theorem 2.1.10** (Additivity Theorem). Let $f : [a, b] \to \mathbb{R}$, let $c \in [a, b]$. Then $f \in \mathcal{R}^*[a, b]$ if and only if $f$ is Generalised Riemann Integrable on both $[a, c]$ and $[c, b]$ and

$$\int_a^c f = \int_a^c f + \int_c^b f$$

**Proof.** ($\iff$) Suppose $\int_a^c f = L_1$ and $\int_c^b f = L_2$. Then given $\epsilon > 0$, there exists a gauge $\delta_1$ on $[a, c]$ such that if a tagged partition $\hat{P}_1$ is $\delta_1$-fine on $[a, c]$, then

$$\left| S(f_1; \hat{P}_1) - L_1 \right| < \frac{\epsilon}{2}$$

Similarly, there exists a gauge $\delta_2$ on $[c, b]$ such that if a tagged partition $\hat{P}_2$ is $\delta_2$-fine on $[c, b]$, then

$$\left| S\left(f_2; \hat{P}_2\right) - L_2 \right| < \frac{\epsilon}{2}$$

Define gauge $\delta$ on $[a, b]$ by

$$\delta(t) = \begin{cases} \min \{ \delta_1(t), \frac{1}{2}(c - t) \} & \text{if } t \in [a, c) \\ \min \{ \delta_1(c), \delta_2(c) \} & \text{if } t = c \\ \min \{ \delta_2(t), \frac{1}{2}(c - t) \} & \text{if } t \in (c, b] \end{cases}$$

Using this gauge will force the $\delta$-fine partition to have $c$ has a tag for any subinterval containing $c$. We will show that if $\hat{Q}$ is a $\delta$-fine partition of $[a, b]$, then there exists a $\delta_1$-fine partition $\hat{Q}_1$ on $[a, c]$ and a $\delta_2$-fine partition $\hat{Q}_2$ on $[c, b]$ such that

$$S\left(f; \hat{Q}\right) = S\left(f_1; \hat{Q}_1\right) + S\left(f_2; \hat{Q}_2\right) \quad (2.3)$$
(i) Case 1. If \(c\) is a partition point on \(\hat{Q}\), then it belongs to two subinterval of \(\hat{Q}\) and is the tag for both subinterval. Let \(\hat{Q}_1\) be the part of \(\hat{Q}\) in \([a, c]\) and so \(\hat{Q}_1\) is \(\delta_1\)-fine. Similarly, let \(\hat{Q}_2\) be the part of \(\hat{Q}\) in \([c, b]\) and so \(\hat{Q}_2\) is \(\delta_2\)-fine. So then

\[
S \left( f; \hat{Q} \right) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) = \sum_{x_i \in [a, c]} f(t_i)(x_i - x_{i-1}) + \sum_{x_i \in [c, b]} f(t_i)(x_i - x_{i-1})
\]

\[
= S \left( f; \hat{Q}_1 \right) + S \left( f; \hat{Q}_2 \right)
\]

(ii) Case 2. If \(c\) is not a partition point in \(\hat{Q} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}\), then as \(\hat{Q}\) is \(\delta\)-fine, \(c\) is forced to have \(c\) as its tags in one of its intervals, say \([x_{k-1}, x_k]\). Then as

\[
f(c)(x_k - x_{k-1}) = f(c)(c - x_{k-1}) + f(c)(x_k - c),
\]

we can split subinterval \([x_{k-1}, x_k], c\) to \([x_{k-1}, c], c\) and \([c, x_k, c]\) and so using similar methods with case (i), we split \(\hat{Q}\) to \(\hat{Q}_1\), a \(\delta_1\)-fine partition in \([a, c]\) and \(\hat{Q}_2\), a \(\delta_2\)-fine partition in \([c, b]\) and so

\[
S \left( f; \hat{Q} \right) = S \left( f; \hat{Q}_1 \right) + S \left( f; \hat{Q}_2 \right)
\]

Using Equation 2.3, and the triangle inequality,

\[
\left| S \left( f; \hat{Q} \right) - (L_1 + L_2) \right| = \left| S \left( f; \hat{Q}_1 \right) - L_1 + S \left( f; \hat{Q}_2 \right) - L_2 \right|
\]

\[
\leq \left| S \left( f; \hat{Q}_1 \right) - L_1 \right| + \left| S \left( f; \hat{Q}_2 \right) - L_2 \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]

Therefore, we conclude that \(f \in \mathcal{R}^*[a, b]\) and

\[
\int_{a}^{b} f = \int_{a}^{b} f
\]

\((\implies)\) Let \(f \in \mathcal{R}^*[a, b]\) and given \(\epsilon > 0\), there exists a gauge \(\delta\) such that if any tagged partitions \(\hat{P}, \hat{Q}\) are \(\delta\)-fine, then

\[
\left| S \left( f; \hat{P} \right) - S \left( f; \hat{Q} \right) \right| < \epsilon
\]

Then let \(\hat{P}', \hat{Q}'\) be \(\delta\)-fine partitions on \([a, c]\). By adding the same partition points and tags on \([c, b]\) to both \(\hat{P}'\) and \(\hat{Q}'\), we can extend both partitions to \(\delta\)-fine partitions \(\hat{P}\) and \(\hat{Q}\) of \([a, b]\) and

\[
S \left( f; \hat{P} \right) - S \left( f; \hat{Q} \right) = S \left( f; \hat{P}' \right) - S \left( f; \hat{Q}' \right)
\]

So

\[
\left| S \left( f; \hat{P}' \right) - S \left( f; \hat{Q}' \right) \right| = \left| S \left( f; \hat{P} \right) - S \left( f; \hat{Q} \right) \right| < \epsilon
\]

and so the partitions \(\hat{P}'\) and \(\hat{Q}'\) of \([a, c]\) satisfies the Cauchy Criterion. We then repeat for the interval \([c, b]\) and so we have proved that \(f \in \mathcal{R}^*[a, c]\) and \(f \in \mathcal{R}^*[c, b]\). To prove that

\[
\int_{a}^{b} f = \int_{a}^{c} f + \int_{c}^{b} f
\]

we use the fact that \(f \in \mathcal{R}^*[a, c]\) with integral \(L_1\) and that \(f \in \mathcal{R}^*[c, b]\) with integral \(L_2\) and we apply the \(\left( \implies \right)\) part of the proof that \(\int_{a}^{b} f = L_1 + L_2 = \int_{a}^{c} f + \int_{c}^{b} f\)

\(\Box\)

We will now begin with the Fundamental Theorem of Calculus for Generalised Riemann Integral. Notice that the first form is stronger than that of the ordinary Riemann Integral.

**Theorem 2.1.11** (Fundamental Theorem of Calculus (First Form)). Suppose there exists a countable set \(E \subseteq [a, b]\), a function \(f, F : [a, b] \to \mathcal{R}^*\) such that
1) \( F \) is continuous on \([a, b]\).

2) \( F'(x) = f(x) \) for all \( x \in [a, b] \setminus \mathcal{E} \) (i.e. \( F'(x) = f(x) \) for almost all \( x \in [a, b] \) except for a finite number of points contained in set \( \mathcal{E} \)).

Then \( f \in \mathcal{R}^*[a, b] \) and

\[
\int_a^b f = F(b) - F(a)
\]

Remarks:

1. Here, compared to the Fundamental Theorem of Calculus for ordinary Riemann Integral, \( f \in \mathcal{R}^*[a, b] \) is a conclusion instead of a condition.

Proof. We will proof for the case \( \mathcal{E} = \emptyset \), so we assume (2) holds for all \( x \in [a, b] \). Given \( \epsilon > 0 \), we use gauge \( \delta(t) > 0 \) for \( t \in [a, b] \) such that if \( 0 < |z - t| \leq \delta(t) \) for \( z \in [a, b] \), then

\[
\left| \frac{F(z) - F(t)}{z - t} - f(t) \right| < \frac{\epsilon}{2(b - a)}
\]

Therefore, we have

\[
|F(z) - F(t) - f(t)(z - t)| \leq \frac{\epsilon |z - t|}{2(b - a)}
\]

for \( z \in [t - \delta(t), t + \delta(t)] \cap [a, b] \).

Now, for all subinterval \((x_{i-1}, x_i], \{t_i\})\) of \( \delta(t) \)-fine partition, we have \( t_i \in [x_{i-1}, x_i] \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)] \).

If we consider

\[
|F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1})|,
\]

then we get

\[
\begin{align*}
|F(x_i) - F(t_i) - f(t_i)(x_i - t_i) + F(t_i) - F(x_{i-1}) + f(t_i)(t_i - x_{i-1})| \\
\leq |F(x_i) - F(t_i) - f(t_i)(x_i - t_i) + F(x_{i-1}) - F(t_i) - f(t_i)(x_{i-1} - t_i)| \\
\leq \frac{\epsilon |x_i - t_i|}{2(b - a)} + \frac{\epsilon |x_{i-1} - t_i|}{2(b - a)} + \frac{\epsilon (t_i - x_{i-1})}{2(b - a)} = \frac{\epsilon (x_i - x_{i-1})}{2(b - a)}
\end{align*}
\]

So if \( t_i \in [x_{i-1}, x_i] \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)] \), then

\[
|F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1})| \leq \frac{\epsilon (x_i - x_{i-1})}{2(b - a)}
\]

Let partition \( \tilde{P} = \{([x_{i-1}, x_i], \{t_i\})\}_{i=1}^n \) to be \( \delta \)-fine, then \( t_i \in [x_{i-1}, x_i] \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)] \) for all \( i \leq n, i \in \mathbb{N} \).

Then

\[
\sum_{i=1}^n (F(x_i) - F(x_{i-1})) = F(b) - F(a)
\]

So,

\[
\left| S(f; \tilde{P}) - (F(b) - F(a)) \right| = \left| F(b) - F(a) - S(f; \tilde{P}) \right| \leq \sum_{i=1}^n \left| (F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1})) \right| \\
\leq \sum_{i=1}^n \frac{\epsilon (x_i - x_{i-1})}{2(b - a)} = \frac{\epsilon (b - a)}{2(b - a)} = \epsilon
\]

Therefore, we conclude that \( f \in \mathcal{R}^*[a, b] \) and \( \int_a^b f = F(b) - F(a) \).

\[\square\]

**Theorem 2.1.12** (Fundamental Theorem of Calculus(Second Form)). Let \( f \in \mathcal{R}^*[a, b] \). Let \( F \) be the indefinite integral of \( f \) such that

\[
F(x) = \int_a^x f(t)dt \text{ for } x \in [a, b]
\]

Then we have
1) $F$ is continuous on $[a, b]$.

2) There exists a null set $E$ such that if $x \in [a, b] \setminus E$, then $F$ is differentiable at $x$ and $F'(x) = f(x)$ (i.e. $F$ is differentiable at $x$ for almost all $x \in [a, b]$ with $F'(x) = f(x)$ except for a finite number of points contained in set $E$).

3) If $f$ is continuous at $c \in [a, b]$, then $F'(c) = f(c)$

Proof. Proof omitted.

**Theorem 2.1.13** (Substitution Theorem). Let $I = [a, b]$ and $J = [c, d]$. Let $F : I \to \mathbb{R}$ and $\alpha : J \to \mathbb{R}$ be continuous function with $\alpha(J) \subseteq I$.

Suppose there exists $E_f \subset I$ and $E_\alpha \subset J$ such that $f(x) = F'(x)$ for $x \in I \setminus E_f$ and $\alpha'(t)$ exists for $t \in J \setminus E_\alpha$, and that $E = \alpha^{-1}(E_f) \cup E_\alpha$ is countable. If we fix $f(x) = 0$ for all $x \in E_f$ and $\alpha'(t) = 0$ for $t \in E_\alpha$.

We can conclude that $f \in \mathcal{R}^*(\alpha(J))$, that $(f \circ \alpha) \cdot \alpha' \in \mathcal{R}^*(J)$ and that

$$
\int_c^d (f \circ \alpha)\alpha' = (F \circ \alpha)|_c^d = \int_{\alpha(c)}^{\alpha(d)} f
$$

Proof. Since $\alpha$ is continuous on $J$, then the range of $\alpha$, $\alpha(J)$, is a closed interval in $I$. Also, $\alpha^{-1}(E_f)$ is countable, so $E_f \cap \alpha(J) = \alpha^{-1}(E_f)$ is also countable. As $f(x) = F'(x)$ for all $x \in \alpha(J) \setminus E_f$, then the Fundamental Theorem of Calculus implies that $f \in \mathcal{R}^*(\alpha(J))$ and

$$
\int_{\alpha(c)}^{\alpha(d)} f = F|_\alpha \alpha(c) = F(\alpha(d)) - F(\alpha(c))
$$

If $t \in J \setminus E$, then $t \in J \setminus E_\alpha$ and $\alpha(t) \in I \setminus E_f$, then the Chain Rule implies that

$$(F \circ \alpha)'(t) = F'(\alpha(t)) \cdot \alpha'(t) = f(\alpha(t)) \cdot \alpha'(t) \text{ for } t \in J \setminus E$$

Since $E$ is countable, the Fundamental Theorem of Calculus implies that $(f \circ \alpha) \cdot \alpha'(t) \in \mathcal{R}^*(J)$ and

$$
\int_c^d (f \circ \alpha) \cdot \alpha' = F \circ \alpha|_c^d = F(\alpha(d)) - F(\alpha(c))
$$

**Theorem 2.1.14** (Multiplication Theorem). If $f \in \mathcal{R}^*[a, b]$ and if $g : [a, b] \to \mathbb{R}$ is monotone, then $f \cdot g \in \mathcal{R}^*[a, b]$.

Remarks:

1. Note that the product of Generalised Riemann Integrable functions is not necessarily Generalised Riemann Integrable.

Proof. Proof omitted.

**Theorem 2.1.15** (Integration By Parts). Let $F$ and $G$ be differentiable on $[a, b]$, then $F'G \in \mathcal{R}^*[a, b]$ if and only if $FG' \in \mathcal{R}^*[a, b]$. We have

$$
\int_a^b F'G = FG|_a^b - \int_a^b FG'
$$

Proof. Proof omitted.
2.2 Improper and Lebesgue Integral

Consider \( h : [0, 1] \to \mathbb{R} \) such that

\[
h(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}
\]

We can see that it is unbounded on the left end point. But, \( h \) is still belongs to \( \mathcal{R}[\gamma, 1] \) for every \( \gamma \in [0, 1] \). So, we define the improper Riemann Integral of \( h \) on \([0, 1]\) to be

\[
\int_0^1 \frac{1}{\sqrt{x}} \, dx = \lim_{\gamma \to 0^+} \int_{\gamma}^1 \frac{1}{\sqrt{x}} \, dx
\]

We also have the same situation with \( k : [0, 1] \to \mathbb{R} \) with

\[
k(x) = \begin{cases} \sin \left( \frac{1}{x} \right) & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}
\]

Note that here we are not necessarily dealing with Generalised Riemann Integral in these cases.

**Theorem 2.2.1** (Hake’s Theorem). If \( f : [a, b] \to \mathbb{R} \), then \( f \in \mathcal{R}^*[a, b] \) if and only if for all \( \gamma \in [a, b] \),

\[
\lim_{\gamma \to b^-} \int_{a}^{\gamma} f = A \in \mathbb{R}
\]  

(2.4)

In this case, \( \int_a^b f = A \).

**Remarks:**

1. The theorem implies that the Generalised Riemann Integral cannot be extended by taking limits. So if Equation 2.4 holds, \( f \) already belongs to \( \mathcal{R}^*[a, b] \).

2. We can know whether a function is integrable on \([a, b]\) just by looking on subinterval \([a, \gamma]\) with \( \gamma < b \). So this is just another tool to show that a function is Generalised Riemann Integral on \([a, b]\).

3. We can use Equation 2.4 to evaluate the integral of a function.

**Example 2.2.2.** Let \( \sum_{k=1}^{\infty} a_k \) such \( a_k \in \mathbb{R} \) for all \( k \in \mathbb{N} \) be a series converging to \( A \in \mathbb{R} \). Let \( f : [0, 1] \to \mathbb{R} \) such that

\[
\int_0^1 f = \sum_{k=1}^{\infty} a_k = A
\]

and

\[
f(x) = \begin{cases} 2^k a_k & \text{if } c_{k-1} \leq x < c_k \\ 0 & \text{if } x = 1 \end{cases}
\]

where \( c_k = 1 - \frac{1}{2^k} \) for all \( k \in \mathbb{N} \),

then \( f \in \mathcal{R}^*[0, 1] \).

**Proof.** Clearly, we can see that \( f \) on \([0, \gamma]\) for \( \gamma \in [0, 1] \) is a step function and so is integrable. So, by summing the areas,

\[
\int_0^\gamma f = \sum_{k=1}^{n} 2^k a_k (c_k - c_{k-1}) + r_\gamma = \sum_{k=1}^{n} 2^k a_k \left( -\frac{1}{2^k} + \frac{1}{2^{k-1}} \right) + r_\gamma = \sum_{k=1}^{n} 2^k a_k \left( \frac{1}{2^k} \right) + r_\gamma = \sum_{k=1}^{n} a_k + r_\gamma
\]

where \( |r_\gamma| \leq |a_{n+1}| \). But as the series is convergent, \( \lim_{n \to \infty} a_n = 0 \) and so \( r_\gamma \to 0 \) as \( n \) becomes large enough and so,

\[
\lim_{\gamma \to 1^-} \int_0^\gamma f = \lim_{n \to \infty} \sum_{k=1}^{n} a_k = A
\]

So \( f \in \mathcal{R}^*[a, b] \).

**Example 2.2.3.** We will use \( \sum_{k=1}^{\infty} a_k \) and \( f \) from Example 2.2.2.

(a) If \( \sum_{k=1}^{\infty} |a_k| \) converges (\( \sum_{k=1}^{\infty} a_k \) converge absolutely), then \( |f| \) converge.
(b) If $\sum_{k=1}^{\infty} |a_k|$ does not converge ($\sum_{k=1}^{\infty} a_k$ converge conditionally), then $|f|$ does not converge.

Proof. We just apply Example 2.2.2 to $\sum_{k=1}^{\infty} |a_k|$ and $|f|$.

Below, we are going to give the definition of a class of Riemann Integrable function - The Lebesgue Integrable Function.

Definition 2.2.4 (Lebesgue Integrable Functions). Let $f \in R[a,b]$ such that $|f| \in R[a,b]$. Then $f$ is said to be Lebesgue Integrable on $[a,b]$. The collection of all Lebesgue Integrable functions on $[a,b]$ is denoted by $L[a,b]$.

Remark:

1. Note that this is not the standard definition of Lebesgue Integrable Functions. However, by using the set of Generalised Riemann Integrable functions, it is easier to prove if a function is Lebesgue Integrable.

Theorem 2.2.5 (Comparison Test). If $f, g \in R^*[a,b]$ and $|f(x)| \leq g(x)$ for all $x \in [a,b]$, then $f \in L[a,b]$ and

$$\left| \int_a^b f \right| \leq \int_a^b |f| \leq \int_a^b g$$

(2.5)

Proof. We are going to omit the proof for $f \in L[a,b]$ and we are just going to prove Equation 2.5. So we note that

$$-|f| \leq f \leq |f|$$

Theorem 2.1.7 implies that

$$-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|$$

and so

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

Similarly, as $|f(x)| \leq g(x),$

$$\int_a^b |f| \leq \int_a^b g$$

Theorem 2.2.6. If $f, g \in L[a,b]$ and if $c \in \mathbb{R}$, then $c f, f + g \in L[a,b]$ and

$$\int_a^b c f = c \int_a^b f \text{ and } \int_a^b |f + g| \leq \int_a^b |f| + \int_a^b |g|$$

Proof. $|cf(x)| = |c| |f(x)|$ for all $x \in [a,b]$. If $|f| \in R^*[a,b]$, then $cf, |cf| \in R^*[a,b]$. So $cf \in L[a,b]$. By Triangle Inequality, $|f(x) + g(x)| \leq |f(x)| + |g(x)|$ for all $x \in [a,b]$. Since $|f| + |g| \in R^*[a,b]$, by comparison test (Theorem 2.2.5), we have $f + g \in L[a,b]$ and

$$\int_a^b |f + g| \leq \int_a^b (|f| + |g|) = \int_a^b |f| + \int_a^b |g|$$
Theorem 2.2.7. Let $f \in \mathcal{R}^*[a, b]$. Then the following statements are equivalent.

a) $f \in \mathcal{L}[a, b]$

b) There exists $g \in \mathcal{L}[a, b]$ such that $f(x) \leq g(x)$ for all $x \in [a, b]$

c) There exists $h \in \mathcal{L}[a, b]$ such that $h(x) \leq f(x)$ for all $x \in [a, b]$

Proof. (a $\implies$ b) Let $g = f$

(b $\implies$ a) Consider function $g - f$. As $g - f \geq 0$ for all $x \in [a, b]$ and since $g - f \in \mathcal{R}^*[a, b]$, then $g - f \in \mathcal{L}[a, b]$ and so by Definition 2.2.4, $g - f \in \mathcal{L}[a, b]$. Then just apply Theorem 2.2.6 to $f = f + (g - f)$.

Proof for a $\iff$ c uses a similar method (just opposite) and so will be omitted.

Theorem 2.2.8. If $f, g \in \mathcal{L}[a, b]$, then $\max\{f, g\}, \min\{f, g\} \in \mathcal{L}[a, b]$ where

$$\max\{f, g\} = \begin{cases} f(x) & \text{if } f(x) \geq g(x) \text{ for all } x \in [a, b] \\ g(x) & \text{if } g(x) > f(x) \text{ for all } x \in [a, b] \end{cases}$$

and

$$\min\{f, g\} = \begin{cases} f(x) & \text{if } f(x) \leq g(x) \text{ for all } x \in [a, b] \\ g(x) & \text{if } g(x) < f(x) \text{ for all } x \in [a, b] \end{cases}$$

Proof. We can write $\max\{f, g\}$ and $\min\{f, g\}$ as

$$\max\{f, g\} = \frac{1}{2} (f(x) + g(x) + |f(x) - g(x)|)$$

and

$$\min\{f, g\} = \frac{1}{2} (f(x) + g(x) - |f(x) - g(x)|)$$

So by Theorem 2.2.6, $f(x) - g(x) \in \mathcal{L}[a, b]$ and also $|f(x) - g(x)| \in \mathcal{L}[a, b]$. So also by Theorem 2.2.6, $\max\{f, g\} = \frac{1}{2} (f(x) + g(x) + |f(x) - g(x)|)$ and $\min\{f, g\} = \frac{1}{2} (f(x) + g(x) - |f(x) - g(x)|)$ also belong to $\mathcal{L}[a, b]$.

Theorem 2.2.9. Let $f, g, \alpha, \beta \in \mathcal{R}^*[a, b]$. If $f \leq \alpha, g \leq \alpha$ or if $\beta \leq f, \beta \leq g$, then $\max\{f, g\}$ and $\min\{f, g\}$ also belong to $\mathcal{R}^*[a, b]$ respectively.

Proof. Let $f \leq \alpha$ and $g \leq \alpha$, then $\max\{f, g\} \leq \alpha$, then

$$0 \leq |f - g| = 2\max\{f, g\} - f - g \leq 2\alpha - f - g$$

As $2\alpha - f - g \geq 0$ and $2\alpha - f - g \in \mathcal{R}^*[a, b], 2\alpha - f - g \in \mathcal{L}[a, b]$. By comparison test (Theorem 2.2.5), $|f - g| \in \mathcal{L}[a, b]$ and so by Theorem 2.2.6, $\max\{f, g\} \in \mathcal{L}[a, b]$.

The second part of the proof is similar with the proof above and so will be omitted.

Definition 2.2.10 (Seminorm and Distance). If $f \in \mathcal{L}[a, b]$, then we define seminorm of $f$ to be

$$||f|| = \int_a^b |f|$$

If $f, g \in \mathcal{L}[a, b]$, then we define the distance between $f$ and $g$ to be

$$\text{dist}(f, g) = ||f, g|| = \int_a^b |f - g|$$

Theorem 2.2.11 (Properties of Seminorm).

i. $||f|| \geq 0$ for all $f \in \mathcal{L}[a, b]$.

ii. If $f(x) = 0$ for all $x \in [a, b]$, then $||f|| = 0$. 
iii. If \( f \in L[a,b], \) \( c \in \mathbb{R}, \) then \( ||cf|| = |c| \cdot ||f||.\)

iv. Let \( f, g \in L[a,b], \) then \( ||f + g|| \leq ||f|| + ||g||.\)

**Proof.** The proof is trivial and so will be omitted. For part (iv), consider the Triangle Inequality. \( \square \)

**Theorem 2.2.12 (Properties of the Distance Function).**

i. \( \text{dist}(f, g) \geq 0 \) for all \( f, g \in L[a,b] \)

ii. If \( f(x) = g(x) \) for all \( x \in [a,b], \) then \( \text{dist}(f, g) = 0 \)

iii. \( \text{dist}(f, g) = \text{dist}(g, f) \) for all \( f, g \in L[a,b] \)

iv. \( \text{dist}(f, h) \leq \text{dist}(f, g) + \text{dist}(g, h) \) for all \( f, g, h \in L[a,b] \)

**Proof.** Proof is trivial and so is omitted \( \square \)

**Theorem 2.2.13 (Completeness Theorem).** The sequence \( \{f_n\} \) of functions in \( L[a,b] \) converge to a function \( f \in L[a,b] \) if and only if for all \( \epsilon > 0, \) there exists \( M \in \mathbb{R} \) such that if \( m, n \geq M \) then

\[
||f_m - f_n|| = \text{dist}(f_m, f_n) < \epsilon
\]

**Proof.** \( (\Rightarrow) \) As \( \{f_n\} \) converges, then it is a Cauchy Sequence. So, given \( \epsilon > 0, \) there exists \( M \in \mathbb{R}, \) such that if \( m, n \geq M, \) then \( |f_m - f_n| < \frac{\epsilon}{(b-a)u} \) where \( u \) is the maximum value \( f_m - f_n \) can achieve for each \( m, n. \) Then, by Theorem 2.2.6, \( |f_m - f_n| \in L[a,b]. \) So

\[
\text{dist}(f_m, f_n) = \int_a^b |f_m - f_n| < \frac{\epsilon \cdot u \cdot (b-a)}{(b-a) \cdot u} = \epsilon
\]

And so we are done.

\( (\Leftarrow) \) Proof Omitted. \( \square \)

### 2.3 Infinite Integral

In the elementary calculus course, the usual approach to find infinite integrals is to define the integral on \([a, \infty)\) as improper integral and so

\[
\int_a^\infty f = \lim_{\gamma \to \infty} \int_a^\gamma f
\]

#### 2.3.1 Integral on \([a, \infty)\)

We can define a partition

\[
\hat{P} = \{([x_{0}, x_1], t_1), \ldots, ([x_{n-1}, x_n], t_n), ([x_{n}, \infty], t_{n+1})\}
\]

and

\[
S(f; \hat{Q}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) + f(t_{n+1})(\infty - x_n)
\]

As the final term \( f(t_{n+1})(\infty - x_n) \) is not meaningful, we only consider the first \( n \) turns in which \( n \) is large enough. So, for the integral over \([0, \infty)\) only deal with the first \( n \) subpartition.

\[
\hat{P} = \{([x_0, x_1], t_1), \ldots, ([x_{n-1}, x_n], t_n)\}
\]

Note that the partition \( \hat{P} \) only lack the subinterval \((x_n, \infty), t_{n+1}\) such that

\[
[a, \infty) = \bigcup_{i=1}^{n} [x_{i-1}, x_i] \cup [x_n, \infty)
\]
For \( \hat{P} \) to be meaningful, we put a condition on \( x_n \) in which we define a number \( d^* \) such that \( \hat{P} \) is \((\delta, d^*)\)-fine if
\[
[x_{i-1}, x_i] \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)] \quad \text{for all } t \in \mathbb{N}, t \leq n
\]
and
\[
x_n, \infty \subseteq \left[ \frac{1}{d^*}, \infty \right)
\]

**Definition 2.3.1.** Let \( f : [a, \infty) \to \mathbb{R} \). \( f \) is Generalised Riemann Integrable on \([a, \infty)\) if there exists \( A \in \mathbb{R} \) such that for all \( \epsilon > 0 \) there exists a gauge \( \delta \in [a, \infty) \) such that if \( \hat{P} \) is a \((\delta, d^*)\)-fine tagged partition in \([a, \infty)\), then
\[
|S(f; \hat{P}) - A| \leq \epsilon
\]
If \( f \) suffice the condition above, then \( f \in \mathcal{R}^*[a, \infty) \) and \( \int_a^\infty f = A \)

**Definition 2.3.2.** Let \( f : [a, \infty) \to \mathbb{R} \), \( f \) is Lebesgue Integrable is \( f, |f| \in \mathcal{R}^*[a, \infty) \). If \( f \) suffice the condition, then \( f \in \mathcal{L}[a, \infty) \).

**Theorem 2.3.3** (Hake’s Theorem). If \( f : [a, \infty) \to \mathbb{R} \), then \( f \in \mathcal{R}^*[a, \infty) \) if and only if for all \( \gamma \in (a, \infty) \), \( f : [a, \gamma] \) is Generalised Riemann Integrable on \([a, \gamma]\) and
\[
\lim_{\gamma \to \infty} \int_a^\gamma f = A \in \mathbb{R}
\]
In this case, \( \int_a^\infty f = A \)

**Proof.** Proof Omitted.

**Theorem 2.3.4.** If \( f, g \in \mathcal{R}^*[a, \infty) \), then \( f + g \in \mathcal{R}^*[a, \infty) \) and
\[
\int_a^\infty (f + g) = \int_a^\infty f + \int_a^\infty g
\]

**Proof.** Given \( \epsilon > 0 \), there exists a gauge \( \delta_f \) on \([a, \infty)\) such that if a tagged partition \( \hat{P} \) is \( \delta_f \)-fine, then
\[
|S(f; \hat{P}) - \int_a^\infty f| < \frac{\epsilon}{2}
\]
Similarly, there exist a gauge \( \delta_g \) on \([a, \infty)\) such that if a tagged partition \( \hat{P} \) is \( \delta_g \)-fine, then
\[
|S(g; \hat{P}) - \int_a^\infty g| < \frac{\epsilon}{2}
\]
So, we use the gauge \( \delta(t) = \min \{\delta_f(t), \delta_g(t)\} \) for \( t \in [a, \infty) \) such that if a tagged partition \( \hat{P} \) is \( \delta \)-fine, then
\[
|S(f + g; \hat{P}) - \left( \int_a^\infty f + \int_a^\infty g \right)| \leq |S(f; \hat{P}) - \int_a^\infty f| + |S(g; \hat{P}) - \int_a^\infty g| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]

**Theorem 2.3.5.** Let \( f : [a, \infty) \to \mathbb{R} \) and \( c \in (a, \infty) \), then \( f \in \mathcal{R}^*[a, \infty) \) if and only if \( f \) is Generalised Riemann Integral on \([a, c]\) and \([c, \infty)\). In this case,
\[
\int_a^\infty f = \int_a^c f + \int_c^\infty f
\]

**Proof.** (\( \Leftarrow \)) As \( f \in \mathcal{R}^*[c, \infty) \), using Hake’s Theorem (Theorem 2.3.3) we get for all \( \gamma \in (c, \infty) \), \( f \) is integrable on \([c, \gamma]\) and
\[
\int_c^\infty f = \lim_{\gamma \to \infty} \int_c^\gamma f
\]
We then apply the additivity theorem (Theorem 2.1.10), to \([a, \gamma] = [a, \alpha] \cup [\alpha, \gamma]\) and so

\[
\int_a^\gamma f = \int_a^\alpha f + \int_c^\gamma f
\]

So we have

\[
\int_a^\infty f = \lim_{\gamma \to \infty} \int_a^\gamma f = \int_a^\infty f + \lim_{\gamma \to \infty} \int_c^\gamma f = \int_a^\infty f + \int_c^\infty f
\]

(\(\implies\)) We can get the proof just by working backwards from the first part of the proof.

\[\Box\]

**Example 2.3.6.** Let \(\alpha > 1\) and \(f_a(x) = \frac{1}{x^\alpha}\) for \(x \in [1, \infty)\). We show that \(f_a \in \mathcal{R}^*[1, \infty)\).

**Proof.** Let \(y \in (1, \infty)\), then \(f_a\) is continuous on \([1, \gamma]\) and so \(f \in \mathcal{R}^*[1, \gamma]\). By Fundamental Theorem of Calculus, we also have

\[
\int_1^\gamma \frac{1}{x^\alpha} \, dx = \left[ \frac{1}{1 - \alpha} x^{1 - \alpha} \right]_1^\gamma = \frac{1}{\alpha - 1} \left[ 1 - \frac{1}{\gamma^{\alpha - 1}} \right]
\]

Then, Hake’s Theorem implies that

\[
\int_1^\infty \frac{1}{x^\alpha} \, dx = \lim_{\gamma \to \infty} \int_1^\gamma \frac{1}{x^\alpha} \, dx = \lim_{\gamma \to \infty} \left( \frac{1}{\alpha - 1} \left( 1 - \frac{1}{\gamma^{\alpha - 1}} \right) \right) = \frac{1}{\alpha - 1} \quad \text{when} \quad \alpha > 1
\]

Then by Fundamental Theorem of Calculus (Theorem 2.1.12), we can show that \(f_a \in \mathcal{R}^*[1, \infty)\). \(\Box\)

**Example 2.3.7.** Let \(f(x) = \begin{cases} \sin x & \text{if } x \in (0, \infty) \\ \frac{1}{x} & \text{if } x = 0 \end{cases}\)

Consider the Generalised Riemann Integrability of \(\int_0^\infty f(x) \, dx\).

**Proof.** Since \(f(x)\) is continuous on \([0, \gamma]\) for all \(\gamma \in (0, \infty), f \in \mathcal{R}^*[0, \gamma]\). To prove that \(\int_0^\infty f(x) \, dx\) has a limit as \(\gamma\) reaches \(\infty\), we let \(0 < \beta < \gamma\) and use integration by parts.

\[
\int_0^\gamma f(x) \, dx = \int_0^\beta f(x) \, dx + \int_\beta^\gamma f(x) \, dx = \int_\beta^\gamma \sin x \, dx = \cos x \bigg|_\beta^\gamma - \int_\beta^\gamma \cos x \frac{x}{x^2} \, dx
\]

Since \(|\cos x| \leq 1\) and \(|\sin x| \leq 1\), we are going to prove that the above terms approach to 0 as \(\beta < \gamma\) goes to \(\infty\). Consider the Cauchy sequence \(\{ \frac{1}{n} \}\). As it is a Cauchy sequence, we know that

For all \(\epsilon > 0\), there exists \(M \in \mathbb{R}\) such that if \(M < \beta < \gamma\), then \(\left| \frac{1}{\gamma} - \frac{1}{\beta} \right| < \frac{\epsilon}{2}\)

Then, for all \(\epsilon > 0\), there exists \(M \in \mathbb{R}\) such that if \(M < \beta < \gamma\), then

\[
\left| \frac{\cos x}{x} \bigg|_\beta^\gamma - \int_\beta^\gamma \frac{\cos x}{x^2} \, dx \right| = \left| \frac{\cos x}{x} \bigg|_\beta^\gamma + \int_\beta^\gamma \frac{\cos x}{x^2} \, dx \right| \leq \left| \frac{1}{\gamma} \right| + \int_\beta^\gamma \frac{1}{x^2} \, dx \leq \frac{2}{\gamma} - \frac{1}{\beta} < \frac{\epsilon}{2}
\]

Therefore, the Cauchy Criterion applies and and Hake’s Theorem implies that \(f \in \mathcal{R}^*[0, \infty)\). \(\Box\)

**Theorem 2.3.8 (Fundamental Theorem).** Let \(E \subset [a, \infty)\) such that \(E\) is countable and that \(f, F : [a, \infty) \to \mathbb{R}\) such that

1) \(F\) is continuous on \([a, \infty)\) and \(\lim_{x \to \infty} F(x)\) exists.

2) \(F'(x) = f(x)\) for all \(x \in (a, \infty), x \in E\)

Then \(f \in \mathcal{R}^*[a, \infty)\) and

\[
\int_a^\infty f = \lim_{x \to \infty} F(x) - F(a) \tag{2.6}
\]

**Proof.** If \(\gamma \in [a, \infty)\), we apply the Fundamental Theorem to interval \([a, \gamma]\) to conclude that \(f \in \mathcal{R}^*[a, \gamma]\) and \(\int_a^\gamma f = F(\gamma) - F(a)\). We then use Hake’s Theorem to show that \(f \in \mathcal{R}^*[a, \infty)\) and we have Equation 2.6. \(\Box\)
2.3.2 Integrals on $(\infty, b]$

In a similar way, we just consider a tagged partition significant enough so that we don’t have to deal with the left end point.

Let $b \in \mathbb{R}$ and $f : (-\infty, b] \to \mathbb{R}$ such that $f \in \mathcal{R}^*(\infty, b]$ then given $\epsilon > 0$ there exists a gauge on $(-\infty, b]$ consisting of a number $d_\epsilon > 0$ and a strictly positive function $(-\infty, b]$.

So a tagged partition $\tilde{P}$ is $(d_\epsilon, \delta)$-fine if

$$(-\infty, b] = (-\infty, x_0) \cup \bigcup_{i=1}^{n} [x_{i-1}, x_i]$$

So that

$$[x_{i-1}, x_i] \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)] \text{ for all } i \leq n, i \in \mathbb{N}$$

and

$$(-\infty, x_0] \subseteq \left(-\infty, -\frac{1}{d_\epsilon}\right)$$

2.3.3 Integral on $(\infty, \infty)$

Let $f : (-\infty, \infty) \to \mathbb{R}$ such that $f \in \mathcal{R}^*(\infty, \infty)$. Given $\epsilon > 0$, there exists a gauge on $(-\infty, \infty)$ containing strictly positive function on $(-\infty, \infty)$ and $d_\epsilon, d^\ast > 0$. So a tagged subpartition $\tilde{P} = \{(x_{i-1}, x_i), t_i\}_{i=1}^{n}$ is $(d_\epsilon, \delta, d^\ast)$-fine if

$$(-\infty, \infty) = (-\infty, x_0) \cup \bigcup_{i=1}^{n} [x_{i-1}, x_i] \cup [x_n, \infty)$$

such that

$$[x_{i-1}, x_i] \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)] \text{ for all } i \in \mathbb{N}, i \leq n$$

and

$$(-\infty, x_0] \subseteq \left(-\infty, -\frac{1}{d_\epsilon}\right) \text{ and } [x_n, \infty) \subseteq \left[\frac{1}{d^\ast}, \infty\right)$$

**Theorem 2.3.9** (Hake’s Theorem). If $f : (-\infty, \infty) \to \mathbb{R}$, then $f \in \mathcal{R}^* (-\infty, \infty)$ if and only if for all $\beta < \gamma \in (-\infty, \infty)$, $f \in \mathcal{R}^*[\beta, \gamma]$ and

$$\lim_{\gamma \to -\infty} \int_{\beta}^{\gamma} f = C \in \mathbb{R}$$

**Theorem 2.3.10** (Fundamental Theorem). Let $E \subseteq (-\infty, \infty)$ and $E$ is countable and $f, F : (-\infty, \infty) \to \mathbb{R}$ such that

a) $F$ is continuous $(-\infty, \infty)$ and $\lim_{x \to \pm \infty} F(x)$ exists.

b) $F'(x) = f(x)$ for all $x \in (-\infty, \infty)$ with $x \notin E$

Then $f \in \mathcal{R}^* (-\infty, \infty)$ and

$$\int_{-\infty}^{\infty} f = \lim_{x \to -\infty} F(x) - \lim_{y \to -\infty} F(y)$$

**Theorem 2.3.11** (Comparison Test for Improper Integral). Let $g$ be a non-negative function continuous on $(a, b)$, let $f$ be a function continuous on $(a, b)$. Suppose

$$|f(x)| \leq g(x) \text{ for all } x \in (a, b)$$

If improper integral $\int_{a}^{b} g$ exists, then $\int_{a}^{b} f$ also exists

Remark:

1. We allow $b = \infty$ and/or $a = -\infty$

2. This theorem is analogous to comparison test of series
Proof. Proof Omitted.

**Example 2.3.12** (The Gamma Function). The gamma function \( \Gamma : (0, \infty) \to \mathbb{R} \) in which

\[
\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx
\]

is an generalization of the factorial function such that it extends to the real number. Show that it is Generalised Riemann Integrable, that the gamma function satisfies the equation :

\[
\Gamma(t + 1) = t\Gamma(t)
\]

and so deduce that \( \Gamma(t + 1) = t! \) for all \( t \in \mathbb{N} \)

**Proof.** We first start by splitting the integral into two parts by Additivity Theorem

\[
\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx = \int_0^1 x^{t-1}e^{-x}dx + \int_1^\infty x^{t-1}e^{-x}dx
\]

For the first integral \( \int_0^1 x^{t-1}e^{-x}dx \), we note that

\[
0 \leq x^{t-1}e^{-x} \leq x^{-1} \quad \text{for all} \quad x, t \in \mathbb{R}
\]

We also know that \( x^{t-1} \in \mathcal{R}^*[0, 1] \) using Hake’s Theorem.

\[
\int_0^1 x^{t-1}dx = \lim_{\gamma \to 0^+} \int_0^1 x^{t-1}dx = \lim_{\gamma \to 0^+} \left[ \frac{x^t}{t} \right] = \lim_{\gamma \to 0^+} \left[ \frac{1}{t} - \frac{\gamma^t}{t} \right] = \frac{1}{t} \quad \text{if} \quad t > 0
\]

Therefore, by the Comparison Test for Improper Integrals (Theorem 2.3.11), \( \int_0^1 x^{t-1}e^{-x}dx \) exists.

For the second integral \( \int_1^\infty x^{t-1}e^{-x}dx \), consider \( g(x) = x^2 \), \( f(x) = x^{t+1}e^{-x} \). We note that

\[
\lim_{x \to \infty} g(x) = \lim_{x \to \infty} x^{t+1}e^{-x} = 0
\]

As the function \( g(x) \) converges, it is bounded. So there exists a number \( M \in \mathbb{R} \) such that

\[
g(x) = x^{t+1}e^{-x} \leq M \quad \text{for all} \quad x \geq 1
\]

which implies that

\[
f(x) = x^{t-1}e^{-x} \leq Mx^{-2}
\]

It is also easy to prove that the function \( Mx^{-2} \in \mathcal{R}^*[1, \infty) \) and so the rest of the proof follows from the first integral and Hake’s Theorem.

By Hake’s Theorem, as the gamma function is Generalised Riemann Integrable, given \( \epsilon > 0 \), the gamma function is Generalised Riemann Integrable on \([0, \gamma]\) and

\[
\Gamma(t + 1) = \int_0^\infty x^{t-1}e^{-x}dx = \lim_{\gamma \to \infty} \int_0^\gamma x^{t-1}e^{-x}dx
\]

Using Integration by Parts,

\[
\lim_{\gamma \to \infty} \int_0^\gamma x^{t-1}e^{-x}dx = \lim_{\gamma \to \infty} \left( -x^t e^{-x} \bigg|_0^\gamma + t \int_0^\gamma x^{t-1}e^{-x}dx \right) = \lim_{\gamma \to \infty} t \int_0^\gamma x^{t-1}e^{-x}dx
\]

Applying Hake’s Theorem again, we get

\[
\Gamma(t + 1) = \int_0^\infty x^{t-1}e^{-x}dx = t \int_0^\infty x^{t-1}e^{-x}dx = t\Gamma(t)
\]

and so we are done with the first part.

To prove the second part, we first compute \( \Gamma(1) \) using Hake’s Theorem and the Fundamental Theorem of Calculus

\[
\Gamma(1) = \int_0^\infty e^{-x}dx = \lim_{\gamma \to \infty} \int_0^\gamma e^{-x}dx = \lim_{\gamma \to \infty} [-e^{-x}]_0^\gamma = \lim_{\gamma \to \infty} (-e^{-\gamma} + e^0) = 1
\]

Therefore,

\[
\Gamma(t + 1) = t(t-1)\ldots(2)(\Gamma(1)) = t!
\]

\[\square\]
2.4 Convergence Theorems

We now turn from functions to sequence of functions in terms of Generalised Riemann Integral.

**Theorem 2.4.1** (Uniform Convergence Theorem). Let \( \{f_n\} \) be a sequence in \( \mathcal{R}^*[a,b] \) and suppose \( \{f_n\} \) converges uniformly on \( [a,b] \) to \( f \). Then \( f \in \mathcal{R}^*[a,b] \) and

\[
\int_a^b f = \lim_{k \to \infty} \int_a^b f_k
\]

Remarks:
1. A sequence of bounded function \( \{f_n\} \) such that \( f_n : A \to \mathbb{R} \) where \( A \subseteq \mathbb{R} \) for all \( n \in \mathbb{N} \) converges uniformly on \( A_0 \subseteq A \) to \( f : A_0 \to \mathbb{R} \) is for all \( \epsilon > 0 \), there exists \( N \in \mathbb{N} \) such that if \( n > N \), then

\[
|f_n(x) - f(x)| < \epsilon \quad \text{for all} \quad x \in A_0
\]

2. Sequence \( \{x_n\} \) is a Cauchy sequence if for all \( \epsilon > 0 \), there exists \( N \in \mathbb{N} \) such that if \( n > m > N \), then

\[
|a_n - a_m| < \epsilon
\]

3. The theorem does not apply for infinite interval

Proof. As \( \{f_k\} \) converge uniformly, given \( \epsilon > 0 \), there exists \( N \in \mathbb{N} \) such that if \( n > N \), then

\[
|f_n - f(x)| < \frac{\epsilon}{2 \cdot (b - a)} \quad \text{for all} \quad x \in [a,b]
\]

Similarly, if \( m > N \), then

\[
|f_m(x) - f(x)| < \frac{\epsilon}{2 \cdot (b - a)} \quad \text{for all} \quad x \in [a,b]
\]

So we get

\[
|f_n(x) - f_m(x)| = |f_n(x) - f(x) + f(x) - f_m(x)| < |f_n(x) - f(x)| + |f_m(x) - f(x)| < \frac{\epsilon}{b - a} \quad \text{for all} \quad x \in [a,b]
\]

This implies

\[
\frac{\epsilon}{b - a} < f_n(x) - f_m(x) < \frac{\epsilon}{b - a} \quad \text{for all} \quad x \in [a,b]
\]

Then by Theorem 2.1.7,

\[
\int_a^b \frac{\epsilon}{b - a} < \int_a^b (f_n - f_m) < \int_a^b \frac{\epsilon}{b - a} \quad \Rightarrow \quad -\epsilon < \int_a^b f_n - \int_a^b f_m < \epsilon \quad \Rightarrow \quad \left| \int_a^b f_n - \int_a^b f_m \right| < \epsilon
\]

Therefore, the sequence \( \{f_n\} \) is a Cauchy Sequence in \( \mathbb{R} \) and so converges to some number \( A \in \mathbb{R} \).

So, now we just need to show that \( f \in \mathcal{R}^*[a,b] \) with integral \( \int_a^b f = A \). For all \( \epsilon > 0 \), we use \( N \in \mathbb{N} \) as above. If \( \hat{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n \) is a tagged partition of \([a,b]\) and if \( n \geq N \), then

\[
\left| \int_{[x_{i-1}, x_i]} f - f(t_i) \right| = \left| \sum_{i=1}^n (f_n(t_i) - f(t_i)) (x_i - x_{i-1}) \right| 
\]

\[
\leq \sum_{i=1}^n |f_n(t_i) - f(t_i)| (x_i - x_{i-1}) < \sum_{i=1}^n \epsilon(x_i - x_{i-1}) = \epsilon(b - a)
\]

Fix \( m \geq N \) such that \( |f_m - A| < \epsilon \), let \( \delta \) be a gauge on \([a,b]\) such that if \( \hat{P} \) is \( \delta \)-fine, then

\[
\left| \int_{[x_{i-1}, x_i]} f - f(t_i) \right| < \epsilon
\]

Then we have

\[
\left| S(f, \hat{P}) - A \right| \leq \left| S(f, \hat{P}) - S(f, \hat{P}) \right| + \left| S(f, \hat{P}) - \int_a^b f \right| + \left| \int_a^b f - A \right| < \epsilon(b - a) + \epsilon + \epsilon = \epsilon(b - a + 2)
\]

As we can set \( \epsilon \) to be as small as we want, we conclude that \( f \in \mathcal{R}^*[a,b] \) and \( \int_a^b f = A \).
Another way of obtaining \( \lim_{k \to \infty} \int_{a}^{b} f_n \) is using the Equi-integrability Theorem discussed below

**Definition 2.4.2 (Equi-integrability Hypothesis).** The sequence \( \{f_k\} \) in \( \mathcal{R}^*[a, b] \) is said to be equi-integrable if for all \( \epsilon > 0 \), there exists a gauge \( \delta \in [a, b] \) such that if \( \hat{P} \) is a \( \delta \)-fine partition on \([a, b]\) and \( n \in \mathbb{N} \), then
\[
\left| S\left( f_n; \hat{P} \right) - \int_{a}^{b} f_n \right| < \epsilon
\]

**Theorem 2.4.3 (Equi-integrability Theorem).** If \( \{f_n\} \in \mathcal{R}^*[a, b] \) is equi-integrable on \([a, b]\) and if
\[
f(x) = \lim_{n \to \infty} f_n(x) \text{ for all } x \in [a, b],
\]
then \( f \in \mathcal{R}^*[a, b] \) and
\[
\int_{a}^{b} f = \lim_{n \to \infty} \int_{a}^{b} f_n
\]

*Proof.* For all \( \epsilon > 0 \), by Equi-integrability Hypothesis (Definition 2.4.2), there exists a gauge \( \delta \) on \([a, b]\) such that if \( \hat{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{N} \) on \([a, b]\) is \( \delta \)-fine, we have
\[
\left| S\left( f_n; \hat{P} \right) - \int_{a}^{b} f_n \right| < \epsilon \text{ for all } n \in [a, b]
\]
Since \( \hat{P} \) has finite number of tags and \( f(t) = \lim_{n \to \infty} f_n(t) \) for all \( t \in [a, b] \), there exists \( N \in \mathbb{N} \) such that if \( m, n \geq N \), then
\[
\left| S\left( f_n; \hat{P} \right) - S\left( f_m; \hat{P} \right) \right| \leq \sum_{i=1}^{N} |f_n(t_i) - f_m(t_i)| \cdot (x_i - x_{i-1}) \leq \epsilon(b - a) \tag{2.9}
\]
If we take the limit as \( m \to \infty \) in Equation 2.9, we get
\[
\left| S\left( f_n; \hat{P} \right) - S\left( f; \hat{P} \right) \right| \leq \epsilon(b - a) \text{ for all } n \geq N
\]
Also, using \( m, n \geq N \), Definition 2.4.2 and Equation 2.9, we get
\[
\left| \int_{a}^{b} f_n - \int_{a}^{b} f_m \right| = \left| \int_{a}^{b} f_n - S\left( f_n; \hat{P} \right) + S\left( f_n; \hat{P} \right) - S\left( f_m; \hat{P} \right) + S\left( f_m; \hat{P} \right) - \int_{a}^{b} f_m \right| \\
\leq \left| S\left( f_n; \hat{P} \right) - \int_{a}^{b} f_n \right| + \left| S\left( f_n; \hat{P} \right) - S\left( f_m; \hat{P} \right) \right| + \left| S\left( f_m; \hat{P} \right) - \int_{a}^{b} f_m \right| \leq \epsilon + \epsilon(b - a) + \epsilon = \epsilon(2 + b - a)
\]
Since we can set \( \epsilon > 0 \) to be as small as we want, \( \left\{ \int_{a}^{b} f_n \right\} \) is a Cauchy sequence converging to some number \( A \in \mathbb{R} \). As \( \left\{ \int_{a}^{b} f_n \right\} \) converges, we write for all \( \epsilon > 0 \), there exists \( N \in \mathbb{N} \) such that if \( n > N \), then
\[
\left| \int_{a}^{b} f_n - A \right| \leq \epsilon
\]
As we know that sequence \( \left\{ \int_{a}^{b} f_n \right\} \) converge to some number \( A \in \mathbb{R} \), we just need to show that \( f \in \mathcal{R}^*[a, b] \) with its integral equal to the same number \( A \). So, given \( \epsilon > 0 \), there exists a gauge \( \delta \) on \([a, b]\) such that if \( \hat{P} \) is \( \delta \)-fine on \([a, b]\) and \( n \geq N \), then
\[
\left| S\left( f; \hat{P} \right) - A \right| = \left| S\left( f; \hat{P} \right) - S\left( f_n; \hat{P} \right) + S\left( f_n; \hat{P} \right) - S\left( f_m; \hat{P} \right) + S\left( f_m; \hat{P} \right) - \int_{a}^{b} f_n + \int_{a}^{b} f_n - A \right| \\
\leq \left| S\left( f; \hat{P} \right) - S\left( f_n; \hat{P} \right) \right| + \left| S\left( f_n; \hat{P} \right) - \int_{a}^{b} f_n \right| + \left| \int_{a}^{b} f_n - A \right| \leq \epsilon(b - a) + \epsilon + \epsilon = \epsilon(2 + b - a)
\]
As we can set \( \epsilon \) to be as small as we want, we conclude that \( f \in \mathcal{R}^*[a, b] \) with \( \int_{a}^{b} f = A \)
Theorem 2.4.4 (Monotone Convergence Theorem). Let \( \{f_k\} \) be a monotone sequence of function in \( \mathcal{R}^*[a,b] \) such that \( f(x) = \lim_{n \to \infty} f_n(x) \) almost everywhere in \( [a,b] \), then \( f \in \mathcal{R}^*[a,b] \) if and only if \( \{\int_a^b f_n\} \) is bounded in \( \mathbb{R} \) and if the condition is satisfied,

\[
\int_a^b f = \lim_{n \to \infty} \int_a^b f_n
\]

Proof. Proof Omitted

Theorem 2.4.5 (Dominated Convergence Theorem). Let \( \{f_n\} \) be a sequence in \( \mathcal{R}^*[a,b] \). Let \( f(x) = \lim_{n \to \infty} f_n(x) \) almost everywhere on \( [a,b] \). If there exists \( g, h \in \mathcal{R}^*[a,b] \) such that \( g(x) \leq f_n(x) \leq h(x) \) for almost every \( x \in [a,b] \), then \( f \in \mathcal{R}^*[a,b] \) and

\[
\int_a^b f = \lim_{n \to \infty} \int_a^b f_n
\]

Moreover, if \( g, h \in \mathcal{L}[a,b] \), then \( f_n, f \in \mathcal{L}[a,b] \) and \( \|f_k - f\| = \int_a^b |f_k - f| \to 0 \)

Proof. Proof Omitted.
3 Reference