Intrinsic Formalism of The Calculus of Variations

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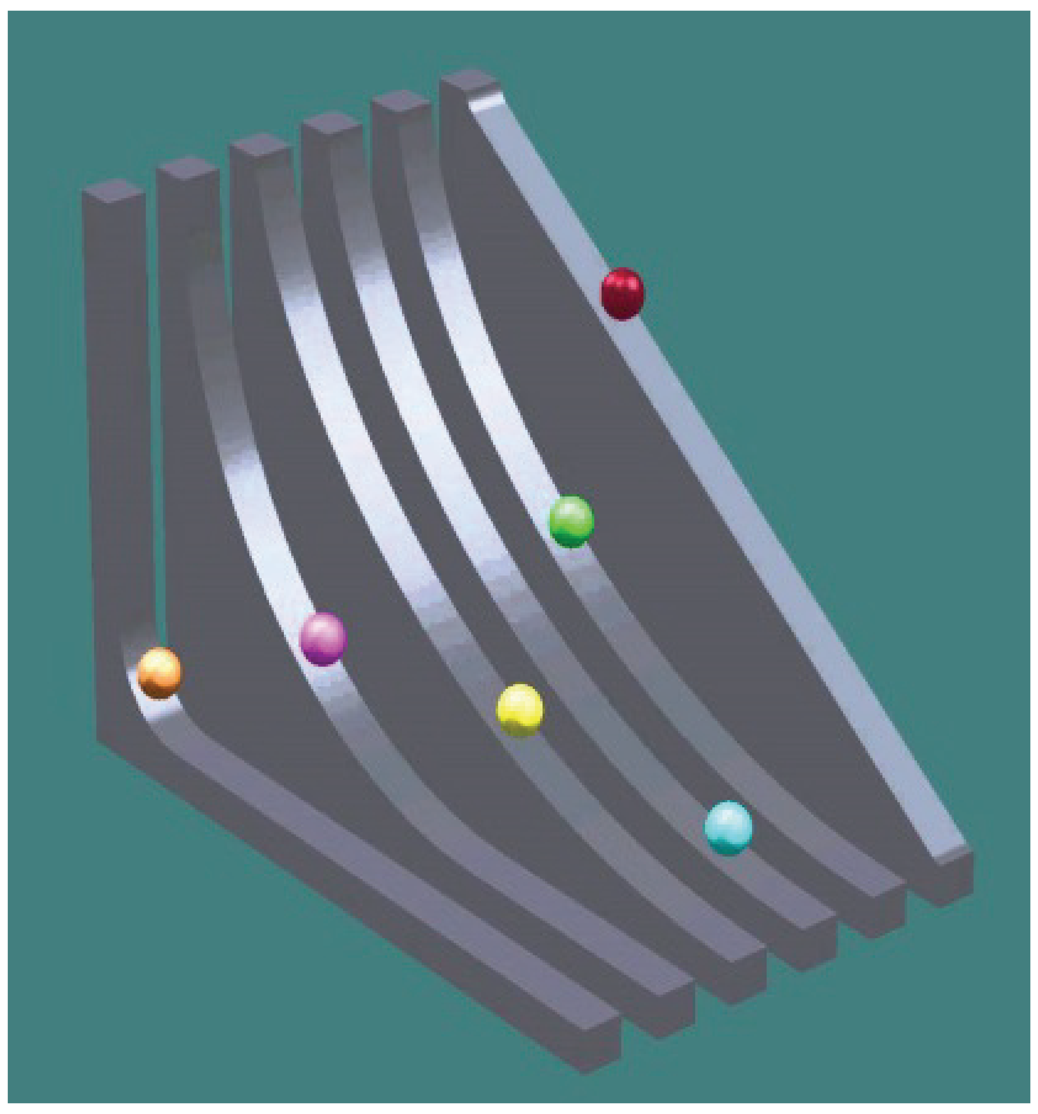
The Calculus of Variations has a history of 300+ years. Although it is a venerable subject in which it seems not much could still be developed, it has serious drawbacks in many practical problems, especially in intrinsic geometric problems.



A Classical Calculus of Variations Problem: Brachistochrone

Consider a very simple example. Take an elastic wire which is free to bend, twist, etc., without tearing, all the time fixing the two end points and the tangent directions at the two end points on a 2-dimensional Riemannian surface *S*. The stable position of the wire would be the position at which the energy stored is a minimum. The energy stored in the elastic wire is

$$\Phi(\gamma) = \int_{\gamma} \kappa^2 \, ds,$$

where γ is the wire parametrized by its arc length s, and $\kappa = \kappa(s)$ is its geodesic curvature. It is well known that κ is a 2nd order invariant, and so in terms of local coordinates, the Euler-Lagrange Equations would be a system of 4^{th} order nonlinear system of differential equations. For such a simple problem, classical approach using local ordinates for which the expression of κ is rather complicated could only lead to partial and somewhat unsatisfactory results.

Alternatively, if we consider the Frenét frames associated to the curve γ as a curve N in the Euclidean motion group $\boldsymbol{E}(n)$, then N is an integral curve of a canonical left-invariant exterior differential system I with an independence condition $\omega \in \Omega^1(\boldsymbol{E}(n))$. Conversely, every integral curve of I with independence condition ω in $\boldsymbol{E}(n)$ arises from a curve $\gamma \subset \boldsymbol{R}^n$ in such a manner. Hence Φ can be thought of as an invariant functional on integral curves of (I, ω) :

$$\begin{array}{ccccc} \boldsymbol{E}(n) & \supset & N & & \\ \downarrow & & \downarrow & \searrow \Phi & \\ \boldsymbol{R}^n & \supset & \gamma & \text{----->} & \boldsymbol{R} \end{array}$$

With this, it is seen that an intrinsic formalism should be in place which should work effectively for intrinsic problems.

By utilizing the theory and techniques of Exterior Differential Systems, a new formalism for functionals whose domain of definition consists of integral manifolds of an exterior differential system is obtained by Phillip Griffiths and Wing-Sum Cheung. This new formalism is, while in greater generality than customary, particularly effective in intrinsic geometric problems, and it sheds new light on even the classical Lagrange Problem.

Let X be a manifold, I an exterior differential system on X generated by some 1-forms $\{\theta_\alpha : \alpha = 1, \ldots, m\}$, and ω an independence n-form. Denote by $\mathcal{I}(I, \omega)$ the collection of integral manifolds of (I, ω). Without loss of generality, assume that the Lagrange n-form is given by $\varphi = L\omega$ for some function L, and so the functional to be extremized is

$$\Phi(N) = \int_N \varphi, \qquad N \in \mathcal{I}(I, \omega).$$

Here, for the case $n = 1$, Frobenius Theorem guarantees the existence of integral curves, but for the case $n > 1$, the situation is much more complicated, as there is no general C^∞ existence theorem for integral manifolds. So we have to at least assume real analyticity and the subtle condition of involutivity for which Cartan-Kähler Theorem applies.

Under this setting, two immediate questions are to determine those variations that are admissible, and the condition for an integral manifold N to give an extremal of Φ among all admissible variations. The variational equations are found to be

$$v \lrcorner \, d\theta + d(v \lrcorner \, \theta) \equiv 0 \mod N$$

for all $v \in C^\infty(N, TX)$ and all $\theta \in I$, and the condition for an integral manifold N to give an extremal of Φ is the following equations which we shall call the Euler-Lagrange Equations

$$v \lrcorner \, d(\varphi + \lambda_\alpha^i \theta^\alpha \wedge \omega_i) \equiv 0 \mod N$$

for all $v \in C^\infty(N, TX)$, where

$$\omega_i = (-1)^{i-1} \omega^1 \wedge \ldots \wedge \widehat{\omega^i} \wedge \ldots \wedge \omega^n ,$$

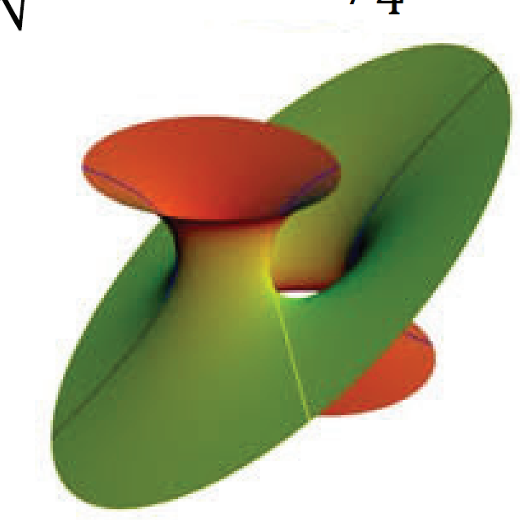
and the λ_α^i 's are functions to be determined.

To illustrate the effectiveness of this intrinsic formalism, we reconsider the aforesaid example of energy stored in an elastic wire. The Euler-Lagrange Equation is easily found to be

$$\frac{d^2\kappa}{ds^2} + \left(\frac{\kappa^3}{2} + \kappa R\right) = 0 ,$$

which is a simple 2^{nd} order differential equation in κ. In particular, if R is a constant, the invariant $\kappa(s)$ is completely solved in terms of an elliptic integral

$$\int_{\kappa_0}^{\kappa(s)} \frac{dt}{\sqrt{c - t^2 R - t^4/4}} = s + c_1 .$$



A typical Calculus of Variations Problem: Minimal Surface

This intrinsic formalism of the Calculus of Variations is in a relatively young stage and is still being developed. It is a huge project and there is still a lot of work to be done, including for example the study of Jacobi equation, conjugate points, sufficient conditions, the Cartan Form, etc.