Donaldson-Thomas theory and stability conditions

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1. Yang-Mills theory

Yang-Mills theory is a gauge theory on 4-dimensional space-time with the Lagrangian

$$L_A = ||F_A||^2 \tag{0.1}$$

where F_A is the curvature tensor associated to a connection A on a principal $SU(N)$-bundle. The Euler-Lagrangian equation for the above Lagrangian is

$$d_A^* F_A = 0. \tag{0.2}$$

The solutions to equation (0.2) are called instantons. To understand the properties of instantons is a core problem in geometry and theoretical physics. If we take the gauge group to be $U(1)$, the classical Maxwell equation for the electric-magnetic fields can be derived from the above equation. Therefore, Yang-Mills theory can be viewed as a non-abelian generalization of the Maxwell theory.

Mathematicians consider various generalizations of Yang-Mills theory using analogous functional to (0.1) defined on Kahler manifold of dimension n. Its global minimums correspond to the stable holomorphic vector bundles. If the Kahler manifold M is of dimension three and admits a complex volume form (Calabi-Yau threefold), then the holomorphic vector bundles are indeed the critical points of another functional:

$$S = \int_M \mathrm{tr}(A \wedge \bar{\partial} A + \frac{2}{3} A \wedge A \wedge A) \wedge \Omega^{3,0} \tag{0.3}$$

where A is a $(0,1)$-form valued in the adjoint bundle and $\Omega^{3,0}$ is the complex volume form on M.

2. Geometry of moduli spaces

On a Calabi-Yau threefold M, the moduli space of holomorphic vector bundles can be heuristically identified with the gauge equivalence classes of critical points of (0.3). The first question asked by algebraic geometers about this moduli space is:

Question: Can we find a polynomial function such that the moduli space is realized as its set of critical points, at least locally.

This is a meaningful question because the functional (0.3) is defined on an infinite dimensional space and is not so helpful for most calculations.

Using the techniques in derived category, we are able to provide an affirmative answer to the above question for toric Calabi-Yau threefolds, i.e. Calabi-Yau threefolds which admit good action by algebraic tori.

Theorem: [1][2] Let M be a toric Calabi-Yau threefold. Any bounded family of coherent sheaves with compact support on M is the critical set of a polynomial function modulo a finite dimensional gauge group.

Another fundamental question is whether the moduli space admits a spin structure. In the classical Yang-Mills theory on the 4-dimensional space-time, the moduli space of instantons admits a symplectic structure. The moduli space of holomorphic vector bundles on a Calabi-Yau threefold admits a -1-shifted symplectic structure. The spin structure is a topological implication of the existence of the -1-shifted symplectic structure. The existence of the spin structure for a large class of Calabi-Yau threefolds has been proved in [3].

3. Donaldson-Thomas invariants

Let M be a compact Calabi-Yau threefold. Given a class β in the topological K-theory $K_0(X)$, we denote the moduli space of coherent sheaves on X with class β by $\mathcal{M}(X,\beta)$. The substack consisting of Gieseker semi-stable sheaves is denoted by $\mathcal{M}^{ss}(X,\beta)$. If we assume that all the semi-stable sheaves are stable, then $\mathcal{M}^{ss}(X,\beta)$ has a projective coarse moduli scheme. In a very rough sense, the Donaldson-Thomas invariants on X with class β is the virtual Euler number of the moduli space $\mathcal{M}^{ss}(X,\beta)$. The Donaldson-Thomas invariants share many properties with the Donaldson invariants of 4-manifolds. First, they are deformation invariants. This means $DT_\beta(X)$ is invariant under a small perturbation of complex structure on X. Second, the Donaldson-Thomas invariants obey certain transformation law under the variation of the stability conditions called the *wall-crossing formula*.

4. Application to knot theory

The space of all stability conditions admits a chamber decomposition, where "chambers" are separated by a countable collections of codimension one sub manifolds called "walls". The invariants $DT_\beta(X)$ are constant in a fixed chamber but maybe different in different chambers. The wall-crossing formula is the transformation law of $DT_\beta(X)$ between different chambers.

One example where the Donaldson-Thomas invariants are extensively studied is when X is the neighborhood of a contractible rational curve in a threefold. The simplest case is when X is the total space of a vector bundle $\mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$ where C is a smooth rational curve. In [4], we setup a relation between $DT_\beta(X)$ and the HOMFLY polynomial of algebraic knots. Let C_0 be an affine plane curve with isolated singularity p. Embed C_0 in the central fiber of the above vector bundle such that p lies on the zero section. The *link* of C_0 is the intersection of a small sphere centered at p, with the curve C_0. If C_0 is uni-branched then the link is a knot. Such knots are called *algebraic knots*. The HOMFLY polynomial is a 2-variable knot polynomial generalizing the famous Jones polynomial. The relations between Jones polynomial and quantum field theory (Chern-Simons theory) has been explored by Witten.

The Donaldson-Thomas theory can be viewed as a holomorphic analogue of the Chern-Simons theory. We consider the moduli space $\mathcal{M}^{ss}(X, C_0, \beta)$ consisting of semi-stable sheaves supported on the curve C_0. In this example, the stability conditions are parametrized by a complex number t. We denote its associated Donaldson-Thomas invariants by $DT^t_\beta(X)$. Using the result joint with E. Diaconescu and Y. Soibelman, D. Maulik proved the following beautiful theorem

Theorem: [5] When $t \to \infty$, the generating series of $DT^t_\beta(X)$ is the HOMFLY polynomial of the link of C_0.

5. Contraction algebra

A general contractible rational curve in a Calabi-Yau threefold can have normal bundle of types other than $\mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$. We consider a contractible smooth rational curve in a Calabi-Yau threefold X, which contracts to a Gorenstein variety Y. Van den Bergh constructed a tilting bundle for the derived category of coherent sheaves on X. A *tilting* bundle $\mathcal{U}$ is a vector bundle such that $\mathbf{R}\mathrm{Hom}(\mathcal{U}, \ )$ defines an equivalence $D^b(X) \cong D^b(\text{mod-}A)$ where A is the endomorphism algebra $End_X(\mathcal{U})$. Denote the push forward of $\mathcal{U}$ along the map $X \to Y$ by N. The *contraction algebra* A_{con} is defined to be the endomorphism algebra of N in the derived category of singularities of Y.

From the point of view of noncommutative deformation theory, contraction algebra is the structure sheaf of the noncommutative moduli space of the contractible rational curves. In my joint paper with Toda [6], we prove that the contraction algebra recovers the Donaldson-Thomas invariants of X.

Theorem: [6] Let X be a Calabi-Yau threefold with a contractible smooth rational curve C. Denote the homology class of C by β. The Donaldson-Thomas invariants $DT_\beta(X)$ can be recovered from A_{con}.

The result indicates that the contraction algebra is an enhancement of the Donaldson-Thomas invariants. This algebra turns out to be substantial in the study of birational geometry.

6. References

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