Zero distributions of derivatives of polynomials : deterministic and random

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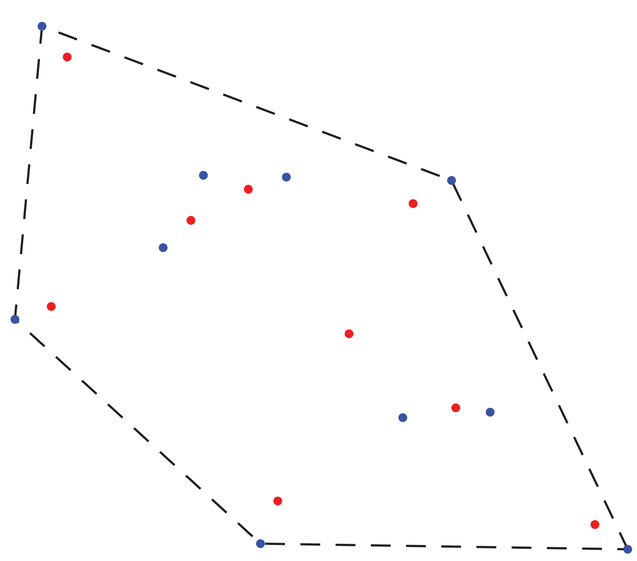
Background

Many physics and engineering problems involve locating the zeros of polynomials and their derivatives.

The zeros of the first order derivative of a polynomial of one variable are called **critical points** and the precise locations of them will provide important information of the polynomial itself. It is also important to know the location of these critical points relative to the location of the zeros.

For example, Gauss (1836) noted that the critical points can be considered as the equilibrium points of the electrical field whose charges were placed at the zeros. The famous Gauss-Lucas Theorem (1879) then says that the critical points must lie in the closed convex hull (smallest convex set) of the zeros.

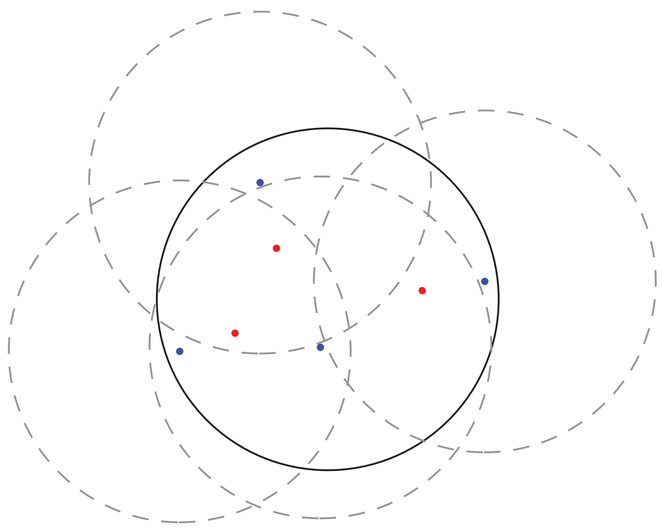


Gauss--Lucas theorem
(blue dots are zeros and red dots are critical points; the dashed lines give the boundary of the convex hull of the zeros)

The Gauss-Lucas Theorem is about the general location of all the critical points relative to all the zeros. A more refined result was conjectured in 1958 by B. Sendov.

Sendov conjecture:

If all the zeros lie in the closed standard unit disk, then for each zero, the closed unit disk centered at that zero must contain at least one critical point.



Sendov conjecture
(blue dots are zeros and red dots are critical points; all the circles are of radius one)

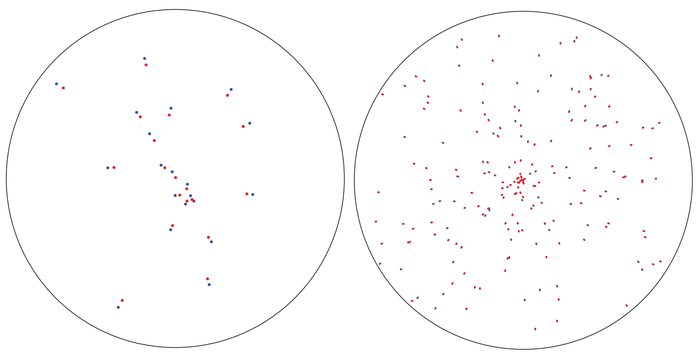
Sendov conjecture remains unsolved for almost 60 years although attempts to verify this conjecture have led to over 100 papers

Borcea's *p*- variance conjecture

In 2011, it was pointed out that the late Borcea has suggested a new approach to Sendov conjecture through the introduction of his *p*-variance conjecture where $p \geq 1$. Borcea noted that Sendov conjecture will follow if his conjecture is true for any particular *p*. So far, very little is known for Borcea's conjecture. In the first part of the project (CERG) HKU 703313P (01/12/2013-30/11/2016), the PI applied techniques in matrix theory to show that Borcea's *p*-variance conjecture is true for some large classes of polynomials.

Strong Pemantle and Rivin conjecture

In the part II of this project, the PI took a probabilistic approach to study the location of the critical points relative to the zero set. This approach was initiated by Rivin and the late Schramm in 2001. But the precise formulation of it was given by Pemantle and Rivin in 2011. Roughly, they conjectured that for a degree *n* polynomial whose zeros are chosen *i.i.d.* from a probability measure *m* supported on a subset of the complex plane, the critical points will converge *in distribution* to *m* as *n* trends to infinity.



Zeros and critical points of polynomials of degree 20 and 100
(blue dots are zeros and red dots are critical points)

The PI studied a stronger conjecture when the mode of convergence is *almost sure* (a.s.) and the critical points are replaced by the zeros of various kinds of derivatives and settled an important case of this strong Pemantle and Rivin conjecture. Again matrix theory (as well as potential theory) plays an important role in the studies of this conjecture.

Finite Blaschke products are the hyperbolic analogue for polynomials on the unit disk. In this project, we also solved the analogous Pemantle and Rivin conjecture for random finite Blaschke products.

Further Reading

[1] P.L. Cheung, **T. W. Ng**, J. Tsai and S. C. P. Yam, Higher order, polar and Sz.-Nagy's generalized derivatives of random polynomials with *i.i.d.* zeros on the unit circle, *Computational Methods and Function Theory*, **15**, (2015) ,159--186.
[2] P.L. Cheung and **T. W. Ng**, Finiteness of fixed equilibrium configurations of point vortices in the plane with background flow, *Nonlinearity* , **27**, (2014), 2445--2463.
[3] P.L. Cheung, **T. W. Ng** and S. C. P. Yam, Critical points of random finite Blaschke products with i.i.d. zeros, *Complex Analysis and Potential Theory, 9th ISAAC Congress*, Cambridge Scientific Publishers, Cambridge, UK, (2014), 23--30.
[4] W.S. Cheung and **T.W. Ng**, A companion matrix approach to the study of zeros and critical points of a polynomial, *Journal of Mathematical Analysis and Its Application*, **319**, (2006), no. 2, 690--707.