# Stochastic Partial Differential Equations

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Partial differential equations (PDEs) have been successful in modelling phenomena in different disciplines such as physics, chemistry and so on. Stochastic partial differential equations (SPDEs) were introduced by adding ``noise'' terms into PDEs, in order to describe phenomena under random influence. SPDEs demonstrates properties that are essentially different from that of PDEs, and are of both theoretical and practical interest for investigation.

Consider the following SPDE driven by a multiplicative noise

$$\mathcal{L}u = u\dot{W}, t > 0, x \in \mathbb{R}^d,$$

with proper initial condition(s).
We are interested in the class of stochastic fractional diffusion equations, in which the operator L may take the form of (fractional) parabolic or hyperbolic operator, and the random influence is modelled by the Gaussian noise. The equation covers a class of fundamental SPDEs including stochastic heat equation stochastic wave equation which have broad application in a variety of disciplines, such as, finance, physics, biology and chemistry.

The SPDE can be understood in two senses, i.e., the Stratonovich equation and the Skorohod equation., depending on the definition of the product in the equation. Different approaches are applied for the two cases. For the study of Stratonovich solution which is heuristically a ``path-wise'' solution, one may combine the Feynman-Kac formula and an approximation argument ([3]). For the Skorohod equation, one may use the Wiener chaos expansion to obtain the existence/uniqueness ([3]).

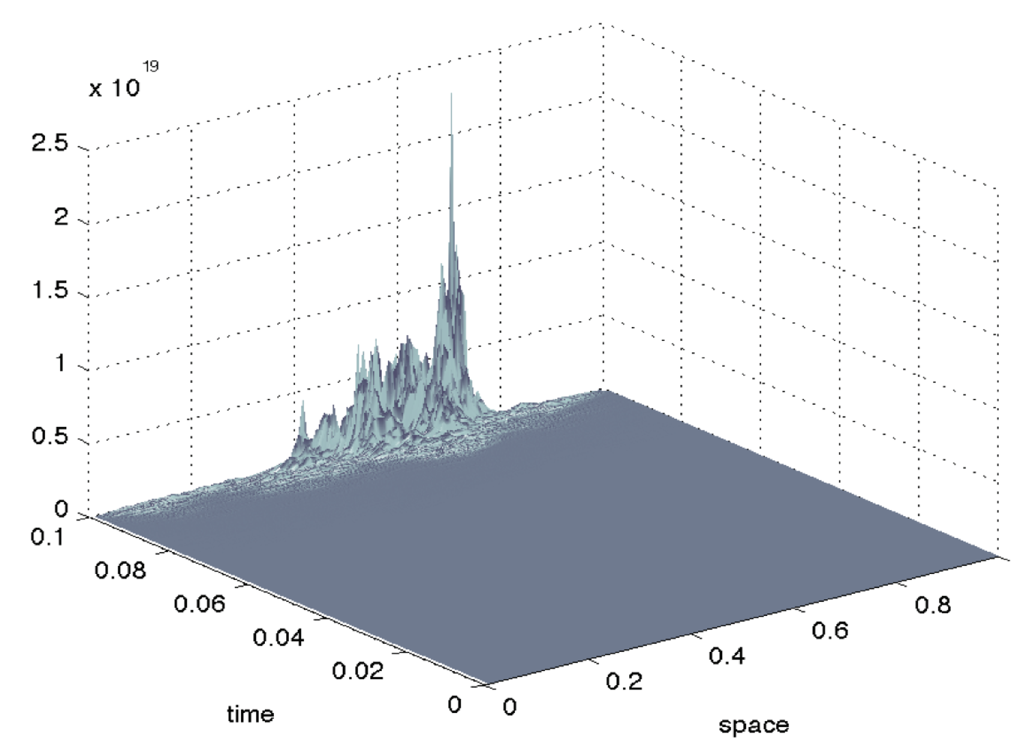
People are interested in the asymptotic behavior of the solutions, which are desirable both in mathematics and in the models. For instance, the ***intermittency*** property ([1]) in the parabolic Anderson model, which explains the phenomena of ***localization*** of electrons in random environment, is closely related to the long-term asymptotics of the moments of the solutions. Therefore, it is desirable to get explicit expressions for the solutions and moments of the solution. When the equation is the fractional stochastic heat equation in the Stratonovich sense, we obtained the following result in [2] which indicates the intermittency property:

$$\lim_{t\to\infty} t^{-\frac{2\alpha-\theta-\alpha\theta_0}{\alpha-\theta_0}} \log \|u(t,x)\|_p = p^{\frac{\alpha}{\alpha-\theta}} \mathbf{M}(\alpha, \theta_0, d, \gamma),$$

with

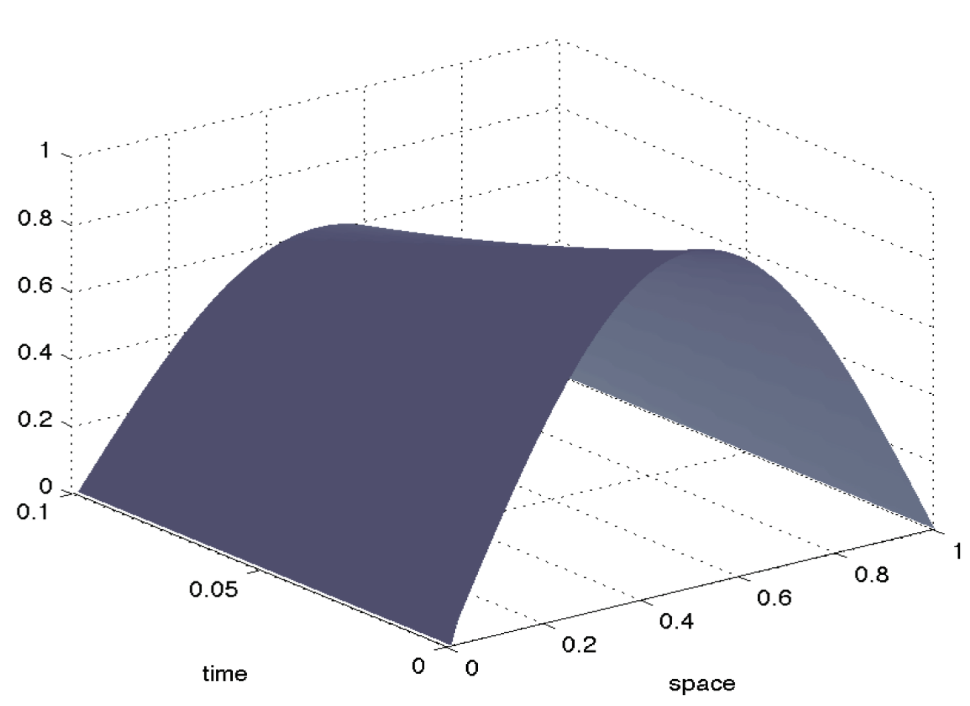
$$\mathbf{M}(\alpha, \theta_0, d, \gamma) = \sup_{g \in \mathcal{A}_{\alpha,d}} \left\{ \frac{1}{2} \int_0^1 \int_0^1 \int_{\mathbb{R}^{2d}} |x-y|^{-\theta} |r-s|^{-\theta_0} g^2(s,x) g^2(r,y)\, dx dy dr ds - \int_0^1 \int_{\mathbb{R}^d} |\xi|^{\alpha} |\widehat{g}(s,\xi)|^2 \, d\xi ds \right\}$$

We have also obtained the precise moment Lyapunov exponent the Skorohod equation solution ([2]), and We are working on this problem for (fractional) stochastic wave equation.

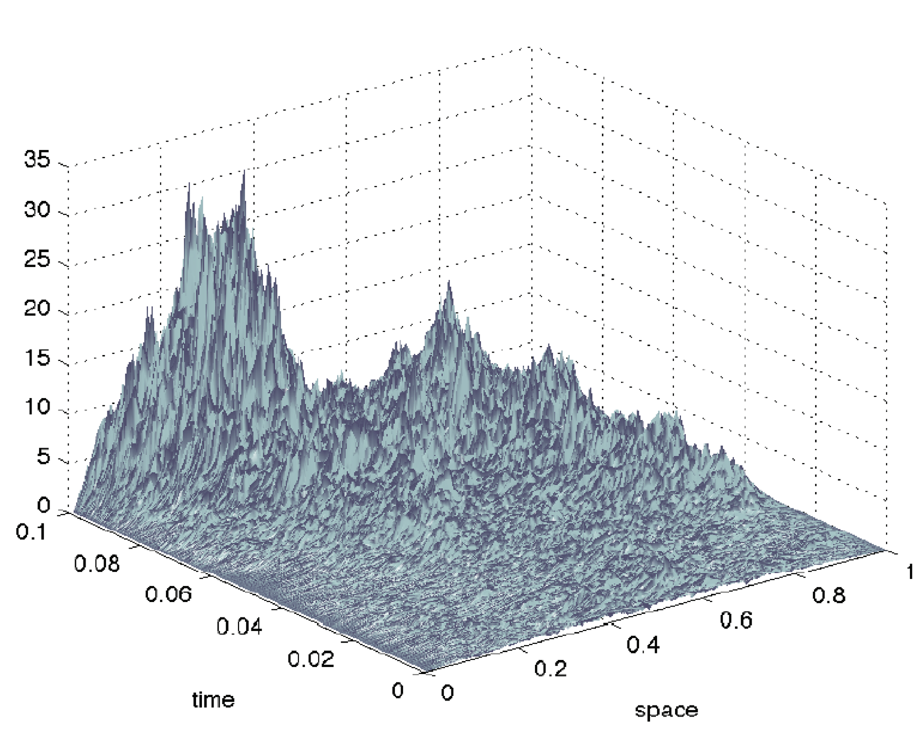



High noise (from [1])

The SPDE is also related to disordered systems such as, the model for directed polymer in random environment, random field Ising model, and the models in the KPZ universality class.




No noise (from [1])




Modest noise (from [1])

**Further Reading**

[1] D. Khoshnevisan, K. Kim. Non-linear noise excitation and intermittency under high disorder. Proc. Amer. Math. Soc. 143 (2015).

[2] X. Chen, Y. Hu, J. Song and X. Song. Temporal asymptotics for fractional parabolic Anderson model. Preprint.

[3] J. Song. On a class of stochastic partial differential equations. Preprint.