Uncertainty quantification and model reduction for multiscale problems

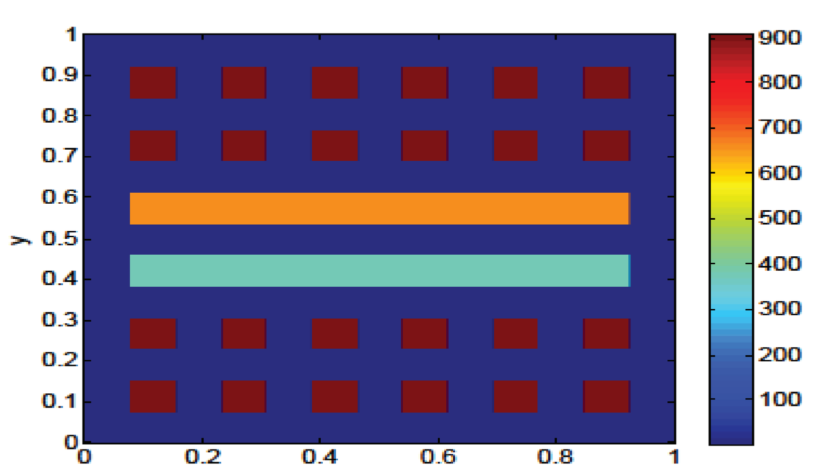
Dr. Z.W. Zhang, Department of Mathematics

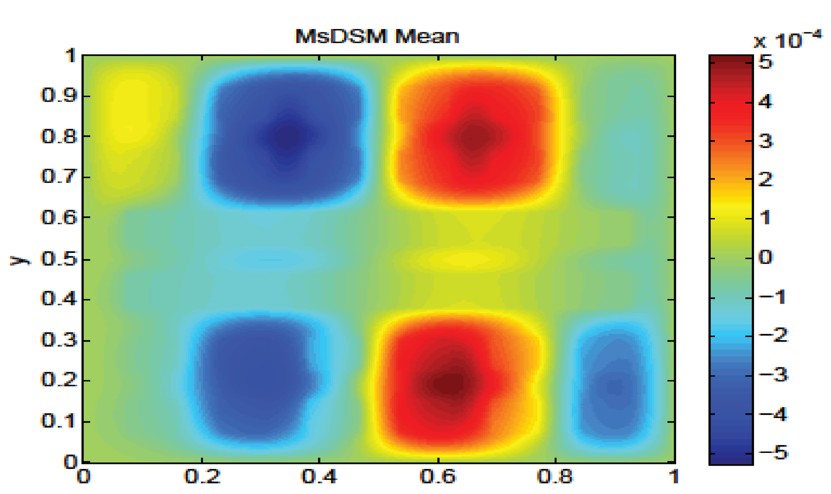
Research Background. Stochastic partial differential equations (SPDEs) with multiscale features can be used to model many complex phenomena, such as composite materials, porous media, and turbulent flows. It is important to analyze uncertainties in the SPDE models and to quantify their influences on prediction values, so we can make better decisions. However, it is very challenging to solve multiscale SPDEs because one not only needs to use a very fine mesh to resolve the small scales of the solution in physical space, but also needs to approximate the solution in stochastic space with high input dimension.

Research Findings. We study the elliptic PDE with multiscale random coefficient,

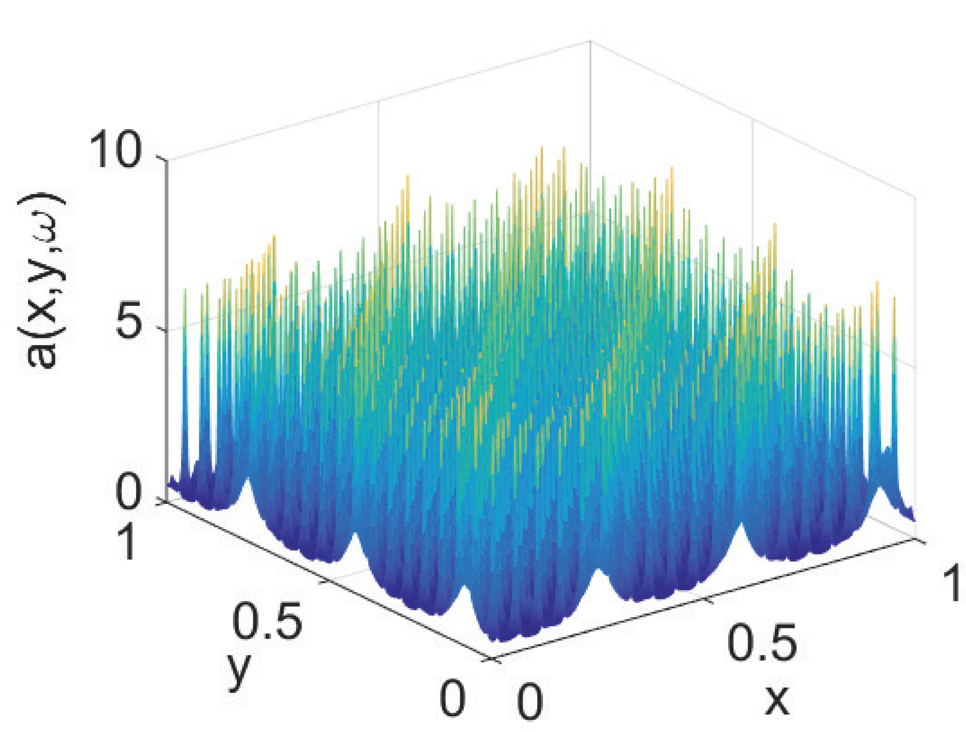
$$-\nabla \cdot (a^{\varepsilon}(x,\omega)\nabla u^{\varepsilon}(x,\omega)) = f(x,\theta)\ ,\ x \in D, \omega \in \Omega\ ,$$

with appropriate boundary condition. Here ε is the smallest-scale parameter.

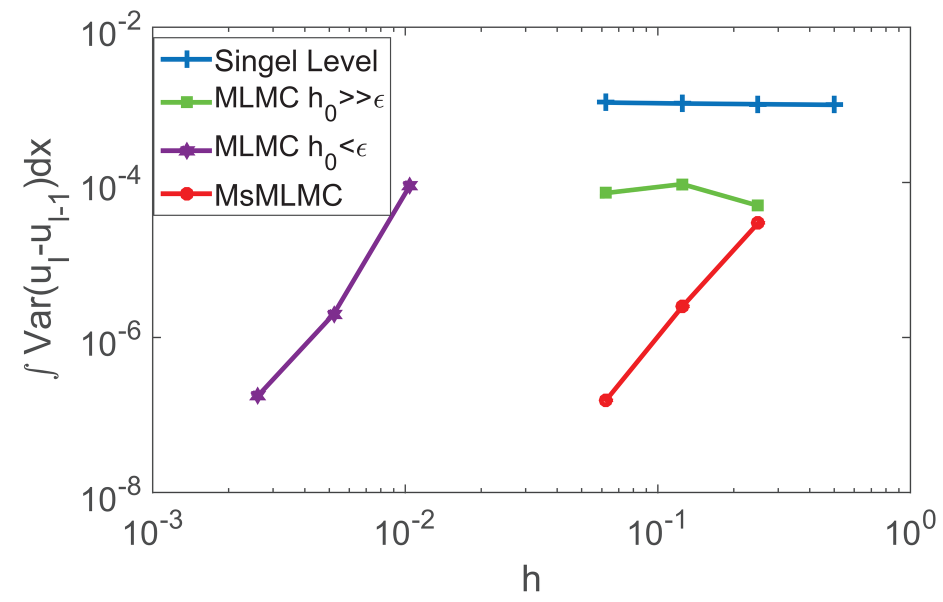




One realization of high-contrast coefficient (top) and mean of solution (bottom)



One realization of multiscale coefficient



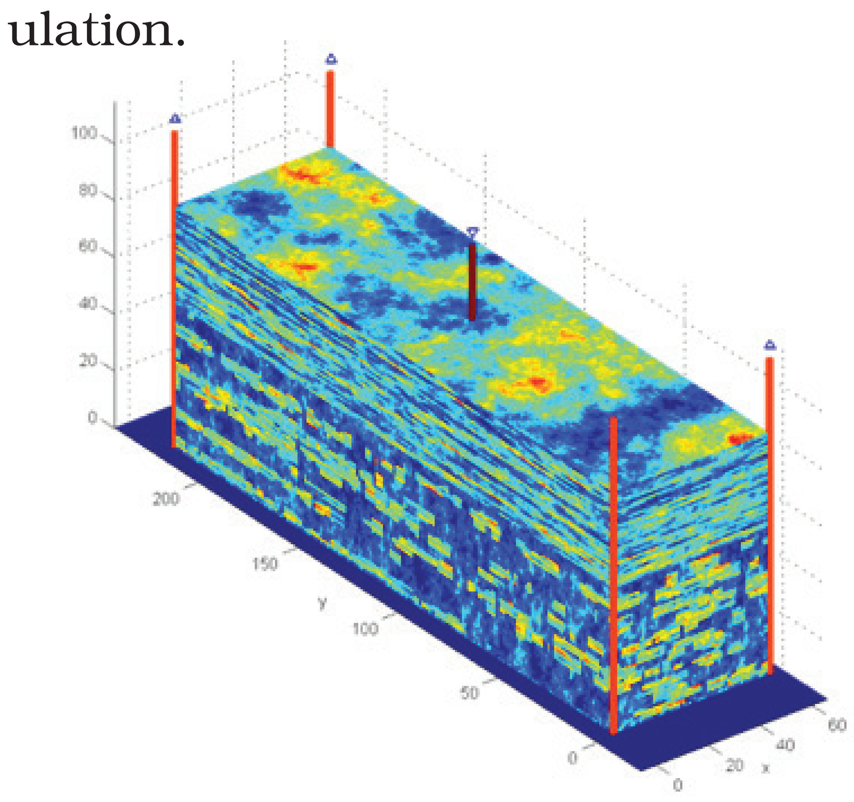
Variance decay properties of different methods

Details of the Research. In this project, we have developed two effective multiscale model reduction methods to solve multiscale SPDEs.

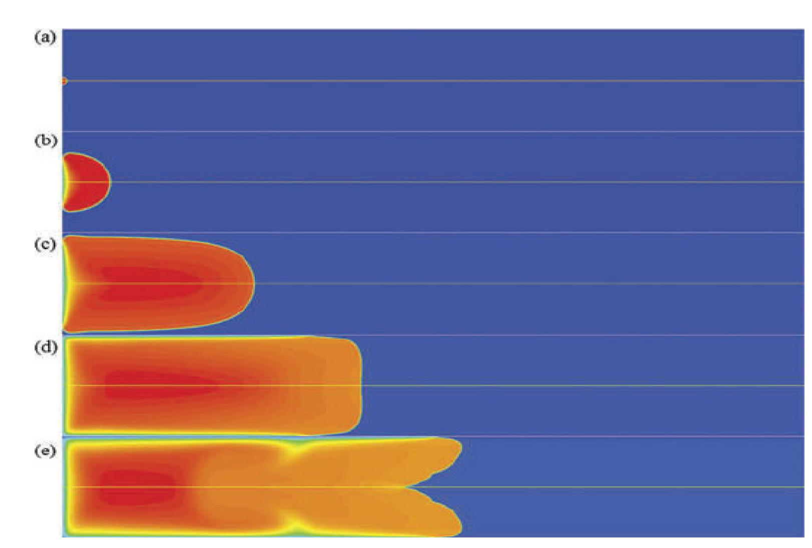
In our multiscale date-driven stochastic method (MsDSM). We first derive an upscaled stochastic equation that can be well-resolved on a coarse grid. Then, we construct a set of data-driven stochastic basis functions under which the stochastic solutions enjoy a compact representation for a broad range of forcing functions. By applying the model reduction in both the physical and stochastic dimensions, the MsDSM offers considerable saving over traditional methods.

The multilevel Monte Carlo (MLMC) method is an effective variance reduction method in solving SPDEs. For the multiscale problem, however, it is still very expensive since the variance decay property holds only if the coarsest grid resolves the smallest-scale feature. To overcome this difficulty, we develop a multiscale multilevel Monte Carlo (MsMLMC) method, we construct a small number of reduced basis functions within each coarse grid, which can be used to approximate the multiscale finite element basis functions efficiently. Since these multiscale basis functions contain the multiscale information of the solution, we can apply the MLMC to multiscale SPDEs starting from a relatively coarse grid, without requiring the coarsest grid to resolve the smallest-scale of the solution.

Research Plans. I aim to develop efficient computational methods for more challenging problems, i.e., stochastic convection dominated and multiscale flow problems. Moreover, I will provide rigorous mathematical analysis for the convergence and stability of these new methods. Finally, I will conduct many numerical simulations to assess and analyze the accuracy and efficiency of the proposed methods, such as the two-phase immiscible flow in heterogeneous porous medium which is frequently used in oil reservoir simulation, and flame front propagation in the combustion simulation.



Reservoir simulation Courtesy of Stanford SUPRI-B (SPE10)



Flame propagation Courtesy of Prof. Indrek S. Wichman

The proposed project will attack the fundamental challenge by exploring the intrinsic low dimensional structures of the SPDE solutions and constructing some problem-dependent basis functions to solve SPDEs. The successful completion of this research project will represent a huge advancement in the uncertainty quantification (UQ) field and have a broad scientific impact in a long term.

Further Reading.

[1] Z. Zhang, T. Hou, M. Ci, A multiscale data-driven stochastic method for elliptic PDEs with random coefficients, Multiscale Model. Simul. 2015, 13: 173-204.

[2] M. Ci, M. Giles, T. Hou, and Z. Zhang, A multiscale multilevel Monte Carlo method for elliptic PDEs with random coefficients. Submitted to SIAM/ASA JUQ.