Quadratic forms, the Shimura lift, and local Maass forms

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Abstract

The goal of this habilitation thesis is to investigate the interplay between integral quadratic forms and the Shimura lift from half-integral weight modular forms to integral weight modular forms. The relationship which we investigate is two-sided. In one direction, the Shimura lift may be applied to the theta series of (positive definite integral) quadratic forms to obtain results about the number of representations of a natural number n by a quadratic form. These theta functions then have wide-reaching applications to a variety of different areas. The author's Ph.D. thesis [20, 23, 24] was centered around one such application to elliptic curves. Realizing that these methods could also be applied to counting the number of representations by quadratic polynomials, the author then used this theory to show that for every set $S \subset \mathbb{N}$ there is always a finite subset $S_0 \subset S$ for which a sum of triangular numbers f represents S if and only if f represents S_0 [21]. By applying this theory to mixed sums of triangular numbers and squares, the author resolved a conjecture of Sun, disproving the original conjecture but proving a modified version [22]. This was followed up by joint work with Sun [25], where a near classification was determined for which mixed sums represent all but finitely many natural numbers. Returning to the application involving elliptic curves, the author then teamed with Jetchev [19] to prove an equidistribution result about the mod p reduction map from elliptic curves with complex multiplication to supersingular elliptic curves.

In the other direction, the Shimura lift may itself be defined using theta functions through the theory of theta lifts. To be more precise, a two variable indefinite theta function of signature $(2, 1)$, defined in terms of (binary) quadratic forms, was shown by Shintani [35] to be the kernel function for the Shimura lift and its adjoint, the Shintani lift. A holomorphic version of Shintani's kernel for the Shimura lift was given by Kohnen and Zagier [28]. Kohnen and Zagier's kernel function is the generating function for certain functions $f_{k,D}$, previously appearing in [40], which are also naturally defined in terms of binary quadratic forms of discriminant D. Motivated by the theory of half-integral weight harmonic weak Maass forms and its relationship with both integral weight harmonic weak Maass forms and classical modular forms, Bringmann, Kohnen, and the author [4] set out to find a preimage $\mathcal{F}_{1-k,D}$ of $f_{k,D}$ under a natural differential operator from the theory of harmonic weak Maass forms. In doing so, a new modular object, known as a local Maass form, was discovered. Many of the properties of $\mathcal{F}_{1-k,D}$ were then later explained when Bringmann, Viazovska, and the author [5] determined that $\mathcal{F}_{1-k,D}$ may be obtained as a theta lift by a theta function of signature $(1, 2)$ which is closely related to Shintani's theta function.

Altogether, the author has published 12 papers since the Ph.D. (which were not a result of the Ph.D. work) and an additional 3 papers which have been accepted for publication. A complete list of these publications is given below.

Complete list of the publications since the Ph.D.:

(1) W. Bosma, B. Kane, *The aliquot constant,* Q. J. Math **63** (2012), 309–323.

(2) W. Bosma, B. Kane, *The triangular theorem of eight and representation by quadratic polynomials,* Proc. Amer. Math. Soc., to appear, DOI: http://dx.doi.org/10.1090/S0002-9939-2012-11419-4 (online).

(3) K. Bringmann, B. Kane, *New identities involving sums of the tails related to real quadratic fields,* Ramanujan J. **23** (2010), Special issue in honor of G.E. Andrews's 70th birthday, 243–251.

(4) K. Bringmann, B. Kane, *Inequalities for differences of Dyson's rank for all odd moduli,* Math. Res. Lett. **17** (2010), 927–942.

(5) K. Bringmann, B. Kane, *Multiplicative q-hypergeometric series arising from real quadratic fields,* Trans. Amer. Math. Soc. **363** (2011), 2191–2209.

(6) K. Bringmann, B. Kane, R. Rhoades, *Duality and differential operators for harmonic Maass forms,* Dev. Math. **28** (2013), Special volume in memory of Leon Ehrenpreis: From Fourier analysis and number theory to radon transforms and geometry, 85-106.

(7) K. Bringmann, B. Kane, *Inequalities for full rank differences of 2-marked Durfee symbols,* J. Combin. Theory Ser. A **119** (2012), 483–501.

(8) K. Bringmann, B. Kane, *Second order cusp forms and mixed mock modular forms,* Ramanujan J., Special issue in honor of Ismail and Stanton, to appear, DOI: 10.1007/s11139-012-9408-4.

(9) K. Bringmann, B. Kane, P. Guerzhoy, *Mock modular forms as p-adic modular forms,* Trans. Amer. Math. Soc. **364** (2012), 2393–2410.

(10) D. Jetchev, B. Kane, *Equidistribution of Heegner points and ternary quadratic forms*, Math. Ann. **350** (2011), 501–532.

(11) B. Kane, *On two conjectures about mixed sums of squares and triangular numbers*, J. Comb. Number Theory **1** (2009), 77–90.

(12) B. Kane, *Sums of triangular numbers and t-core partitions*, J. Comb. Number Theory **1** (2009), 59–64.

(13) B. Kane, *Representing sets with sums of triangular numbers*, Int. Math. Res. Not. **2009** (2009), no. 17, 3264–3285, doi:10.1093/imrn/rnp053.

(14) B. Kane, *Faber polynomials and Poincaré series,* Math. Res. Lett. **18** (2011), 591–611.

(15) B. Kane, Z.-W. Sun, *On almost universal mixed sums of squares and triangular numbers*, Trans. Amer. Math. Soc. **362** (2010), 6425–6455.

Of the 15 published or accepted papers listed above, 5 of them are relevant to this habilitation thesis. These are given below.

Relevant publications since the Ph.D.:

(i) W. Bosma, B. Kane, *The triangular theorem of eight and representation by quadratic polynomials,* Proc. Amer. Math. Soc., to appear, DOI: http://dx.doi.org/10.1090/S0002-9939-2012-11419-4 (online).

(ii) D. Jetchev, B. Kane, *Equidistribution of Heegner points and ternary quadratic forms*, Math. Ann. **350** (2011), 501–532.

(iii) B. Kane, *On two conjectures about mixed sums of squares and triangular numbers*, J. Comb. Number Theory **1** (2009), 77–90.

(iv) B. Kane, *Representing sets with sums of triangular numbers*, Int. Math. Res. Not. **2009** (2009), no. 17, 3264–3285, doi:10.1093/imrn/rnp053.

(v) B. Kane, Z.-W. Sun, *On almost universal mixed sums of squares and triangular numbers*, Trans. Amer. Math. Soc. **362** (2010), 6425–6455.

There are also two preprints (neither of which are related to the Ph.D.) which are relevant for this habilitation thesis.

Relevant preprints:

(i) K. Bringmann, B. Kane, and W. Kohnen, *Locally harmonic Maass forms and the kernel of the Shintani lift*, submitted for publication, arXiv:1206.1100.

(ii) K. Bringmann, B. Kane, S. Zwegers, *On a completed generating function of locally harmonic Maass forms,* submitted for publication, arXiv:1206.1102.

Contents

Chapter 1

The application of the Shimura lift to quadratic forms and quadratic polynomials

1.1 Quadratic forms

The study of *quadratic forms*, homogenous polynomials of degree two, has a long history and has found applications in a variety of different settings. Equivalently, if A is a symmetric $m \times m$ matrix with coefficient in a field K, then there is an associated quadratic form Q defined for $X \in K^m$ by

$$Q(X) = \frac{1}{2} X^T A X.$$

One refers to A as the *Gram matrix* for Q. The *level* of a quadratic form is the smallest $N \in \mathbb{N}$ for which NA^{-1} has even integral coefficients. For $\gamma \in\in \mathrm{SL}_m(K)$, there is an action given by

$$(\gamma Q)(X) = \frac{1}{2} X^T \left(\gamma A \gamma^T\right) X. \tag{1.1}$$

For $K = \mathbb{R}$, Jacobi proved that there exists $\gamma \in \mathrm{SL}_m(\mathbb{R})$ such that γQ is *diagonal,* i.e., the corresponding Gram matrix $\gamma A \gamma^T$ is diagonal. A real quadratic form in m variables is said to have *signature* (r, s) if its diagonalization

$$\gamma A \gamma^T = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix}$$

has r choices of j for which λ_j is positive, s choices of j for which λ_j is negative, and $m-r-s$ choices for which $\lambda_j = 0$. The special case where $(r, s) = (m, 0)$ is called *positive definite.* In this case, for $\tau \in \mathbb{H} := \{x + iy : x, y \in \mathbb{R}, y > 0\}$ and $q := e^{2\pi i\tau}$, the *theta function*

$$\Theta_Q(\tau) := \sum_{X \in \mathbb{Z}^m} q^{Q(X)}$$

converges and is a *modular form* of weight $\frac{m}{2}$, level N and some *Nebentypus* character χ. A modular form of weight κ, level N, and Nebentypus χ is a holomorphic function f on $\mathbb{H}$ which is bounded towards all cusps of $\mathbb{H}/\Gamma_0(N)$ (factored by the usual action of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ given by $\gamma\tau := \frac{a\tau+b}{c\tau+d}$) and satisfies the *modularity* property

$$f\big|_{\kappa}\gamma(\tau) = \chi(d) f(\tau) \tag{1.2}$$

for every $\gamma \in \Gamma_0(N)$. Here

$$f\big|_{\kappa}\gamma(\tau) := j\left(\gamma,\tau\right)^{-2\kappa} f\left(\gamma\tau\right)$$

with

$$j\left(\gamma,\tau\right) := \begin{cases} \sqrt{c\tau+d} & \text{if } \kappa \in \mathbb{Z}, \\ \left(\frac{c}{d}\right)\epsilon_d\sqrt{c\tau+d} & \text{if } \kappa \in \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}, \end{cases} \tag{1.3}$$

where we take the principal branch of the square root, $\left(\frac{c}{d}\right)$ is the Kronecker-Jacobi symbol, and

$$\epsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4}, \\ i & \text{if } d \equiv -1 \pmod{4}. \end{cases}$$

Now consider the case of *integral* quadratic forms, i.e., those which have integral values over $\mathbb{Z}^m$. Defining the *representation number*

$$r_Q(n) := \#\left\{X \in \mathbb{Z}^m : Q(X) = n\right\},$$

one sees that the n-th Fourier coefficient of Θ_Q equals $r_Q(n)$. That is,

$$\Theta_Q(\tau) = \sum_{n\in\mathbb{N}_0} r_Q(n) q^n.$$

The author's Ph.D. thesis [20] was centered around the question of representation numbers for positive definite ternary ($m = 3$) quadratic forms and their applications to elliptic curves, which resulted in the publications [23] and [24].

1.2 Quadratic polynomials

Following the Ph.D. thesis, the author realized that a more general method than that used in [20] could be applied to *quadratic polynomials.* A quadratic polynomial is any multivariable polynomial $R \in K\left[X_1, \ldots, X_m\right]$ which may be written in the form

$$K(X) = Q(X) + L(X) + C, \tag{1.4}$$

where $Q(X)$ is a quadratic form, $L(X)$ is linear, and C is a constant.

The author's interest in quadratic polynomials began with sums of *triangular numbers,* those quadratic polynomials of the form (for $x \in K$)

$$T(x) := \frac{x^2+x}{2}.$$

Fermat famously claimed that every natural number may be written as the sum of at most 3 triangular numbers, 4 squares, 5 pentagonal numbers, ..., and n n-gonal numbers. In 1770, Lagrange proved the statement for squares, while Gauss proved the statement for triangular numbers in 1796, and the full conjecture was resolved by Cauchy in 1813.

Lagrange's theorem began a long and extensive study of *universal quadratic forms,* i.e., those (positive definite) quadratic forms (whose Gram matrix has integral coefficients) which represent every natural number. In search of a simple classification of such forms, Conway and Schneeberger studied the literature and realized that such a quadratic form (in arbitrarily many variables) is universal if and only if it represents every integer less than or equal to 15. Together with Bosma, the author [3] showed that a similar result holds for quadratic polynomials of the form

$$f_b(X) := \sum_{i=1}^{m} b_i T\left(x_i\right), \tag{1.5}$$

with $a_i \in \mathbb{N}$. We call quadratic forms of the type (1.5) *triangular form.*

Theorem 1.1. *If $b_1, \ldots, b_m$ are positive integers, then f_b represents every nonnegative integer if and only if it represents $1, 2, 4, 5,$ and 8.*

However, this result does not generalize to arbitrary quadratic polynomials, as also proven in [3]. That is to say, there is no proper subset S of the natural numbers for which every "reasonable" quadratic polynomial which represents S also represents the natural numbers. To make this statement meaningful, we must first restrict the class of quadratic polynomial to a subclass which we call *normalized totally positive quadratic polynomials.* These are quadratic polynomials f for which $f\left(\mathbb{Z}^m\right) \subseteq \mathbb{N}_0$ and such that there exists some $X \in \mathbb{Z}^m$ for which $f(X) = 0$.

Theorem 1.2. *Suppose that $S \subseteq \mathbb{N}_0$. For every subset $S_0 \subsetneq S$, there exists a normalized totally positive quadratic polynomial f which represents S_0 but does not represent S.*

The proof of Theorem 1.2 is by explicit construction and the quadratic polynomials all belong to a subclass which is only a slight generalization of triangular sums. The proof of Theorem 1.1 is based upon the genus theory of quadratic forms and the idea of *escalator lattices,* which were introduced by Bhargava to give a shorter and more elegant proof of the Conway–Schneeberger 15 Theorem.

We now explain the essential idea of the theory of escalator lattices. Say that one would like to determine all (integral positive definite) quadratic forms (whose Gram matrix has integral coefficients) which represent every integer from 1 to a fixed natural number n (which is obviously a necessary condition for universal forms), or equivalently, there exist vectors of norm 1 through n in the corresponding lattice. One starts with the zero-dimensional lattice and adds a vector of norm 1 (which must be contained in any lattice given above). One next finds the smallest integer n_0 not represented by the corresponding quadratic form and then adds a new vector of norm n_0 such that the resulting quadratic form is still integral. The process then continues recursively, branching out as a tree because there are multiple choices for the vector of norm n_0. Once all integers from 1 to n are obtained, adding any vector will not change this fact. Therefore, if a quadratic form is universal, then it must contain one of the lattices which are nodes in the resulting tree. The numbers n_0 are precisely the integers which all universal quadratic forms must represent. Finally, one obtains a **finite** tree where the nodes are "probably" universal. The last step is to use some other theory to prove that the nodes are indeed universal. Since any universal quadratic form must contain one of the leaves as a sublattice, the process yields the desired finite subset which is sufficient to determine if a quadratic form is universal. It is important to note here that the key difference between Bhargava's situation and the conditions in Theorem 1.2 is that the breadth of the tree is infinite in the second case. It may be possible to define a statistic on normalized totally positive quadratic polynomials which recovers a finite result like Bhargava's. Indeed, this was the case for the counterexamples constructed in [3].

In [1], Bhargava further realized that one could replace $\mathbb{N}$ with any subset $S \subseteq \mathbb{N}$ and apply the same method. Using this technique, he then explicitly determined the finite subset S_0 when S is chosen to be the set of primes or the set of all odd natural numbers. In [21], the author combined this technique with the arithmetic theory of quadratic forms to show that a similar result holds for sums of triangular numbers.

Theorem 1.3. *Suppose that $S \subseteq \mathbb{N}$. Then there exists a finite subset $S_0 \subseteq S$ such that every triangular sum representing S_0 must also represent S.*

Using escalator lattices, the author then found a conjectural set S_0 when S is the set of all odd natural numbers.

Conjecture 1.4. *A sum of triangular numbers f represents all odd integers if and only if it represents the integers*

$$1, 5, 7, 9, 11, 13, 17, 19, 25, 29, 35, 49.$$

However, this is where Bhargava's theory diverges from that of the author's. While Bhargava's proof in the quadratic forms case was relatively simple, a proof of the conjecture for triangular sums appears to be completely out of reach. The problem is that there are leaves of the corresponding tree at depth 3, such as the case $b = (1, 1, 1)$ proven by Gauss. This was the trigger for the author's initial interest in triangular sums and their interrelation with Bhargava's theory. The resulting theta functions are hence weight $\frac{3}{2}$, and the theory of weight $\frac{3}{2}$ modular forms has a couple of exceptional properties which make it nearly impossible to prove that a depth 3 leaf is truly a leaf, other than an ad hoc solution in special cases.

1.3 The arithmetic theory of quadratic forms

To explain the problems arising for weight $\frac{3}{2}$ modular forms (and ternary quadratic forms), we now take a detour into the arithmetic theory of quadratic forms. Modular forms naturally split into two pieces. The first piece is what is known as a *cusp form,* which obtains its nomenclature from the fact that the function vanishes towards every cusp of $\Gamma_0(N)\backslash\mathbb{H}$. The second part, known as the *Eisenstein series,* is written as an explicit sum. In the simplest case, when $N=1$ and $k\in\mathbb{Z}$, the Eisenstein series is defined by

$$E_k(\tau) := \sum_{\gamma=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{Z})} (c\tau+d)^{-k} = \sum_{\gamma\in\mathrm{SL}_2(\mathbb{Z})} 1\big|_k\gamma(\tau).$$

For $k\geq 3$, the sum converges absolutely and reordering hence implies that E_k satisfies weight k modularity (as in (1.2)). The Fourier coefficients of the Eisenstein series may be explicitly computed and the n-th Fourier coefficient grows (in absolute value) like n^{k-1}.

The first fundamental problem which occurs in weight $\frac{3}{2}$ is that the Fourier coefficients of the Eisenstein series are certain class numbers for imaginary quadratic fields. Although Siegel [37] famously showed that the class number grows like $n^{\frac{1}{2}}$, the constant of proportionality depends upon the location of possible Siegel zeros, and hence the bound is ineffective.

As mentioned above, after subtracting a linear combination of Eisenstein series from a modular form, the difference is a cusp form. The Fourier coefficients of cusp forms generally grow more slowly than the coefficients of the Eisenstein series. Indeed, if $k\in\mathbb{Z}$, then Deligne [10] showed that the Fourier coefficients grow slower than $n^{\frac{k-1}{2}+\varepsilon}$, known previously as the *Ramanujan–Petersson* conjecture. For $\frac{5}{2}\leq k\in\frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}$, bounds of Iwaniec [18] suffice to show that the coefficients grow more slowly than those of Eisenstein series. However, in weight $\frac{3}{2}$, the Ramanujan–Petersson conjecture is false. Specifically, for an *odd character* of modulus D, a character χ such that $\chi(-1)=-1$, the *unary theta function*

$$\sum_{r\in\mathbb{Z}} r\chi(r)q^{Dr^2}$$

is a weight $\frac{3}{2}$ cusp form, but its n-th coefficient clearly grows like $n^{\frac{1}{2}}$. The Ramanujan–Petersson conjecture is believed to be true for the space of cusp forms orthogonal to unary theta functions and results of Duke [13] suffice to show that the coefficients grow more slowly than those of the Eisenstein series. This leads to a further decomposition of cusp forms into two pieces, one formed by unary theta functions and the other orthogonal to unary theta functions. Thus, we have a natural decomposition of the theta function as

$$\Theta = \Theta_E + \Theta_U + \Theta_\perp, \tag{1.6}$$

where Θ_E is a linear combination of Eisenstein series, Θ_U is a linear combination of unary theta functions, and $\Theta_\perp$ is a cusp form which is orthogonal to unary theta functions.

From the point of view of theta functions, the coefficients of the Eisenstein series may be written as a weighted average of the theta series for quadratic forms in the *genus*, those quadratic forms which are locally equivalent at every prime, and the weighted average was shown by Siegel [36] to be the determined by the number of representations at each prime.

The sum $\Theta_E + \Theta_U$ corresponds to a weighted average for a refinement of the genus called the *spinor genus,* and hence Θ_U measures the difference between the weighted averages of the genus and the spinor genus. Using work of Schulze-Pillot [33] involving the spinor norm, one may explicitly compute the coefficients of Θ_U. The coefficients of the cusp form $\Theta_\perp$ measure the difference between the average of the spinor genus and the particular form we are interested in. Since this is small in comparison with the growth of $\Theta_E+\Theta_U$, we consider $\Theta_\perp$ an error term. However, even assuming the best possible bound for the growth of the coefficients of Θ_E, the effective bounds obtained by Duke [13] for the coefficients of $\Theta_\perp$ are insufficient for practical purposes. Hence the combination of the ineffective bounds of Siegel [37], make it impossible to explicitly determine the set of integers not represented by a given form. This is where the difficult occurs in proving that conjectural depth 3 leaves are indeed leaves.

By using a method of Ono and Soundararajan [32], the author [21] was able to resolve conjecture 1.4 under the additional assumption of the Generalized Riemann Hypothesis (GRH). In particular, one requires GRH for Dirichlet L-functions and *newforms.* These are modular forms of weight k for $\Gamma_0(N)$ which are *Hecke eigenforms,* that is to say, simultaneously eigenforms for all of the Hecke operators T_p $(p \nmid N)$ defined by

$$f\Big|_k T_p(\tau) := p^{k-1} f(p\tau) + p^{-1} \sum_{r \pmod p} f\left(\frac{\tau+r}{p}\right), \tag{1.7}$$

and are orthogonal to the *old space* spanned by $g(r\tau)$, where $g(\tau)$ is a modular form of weight k for $\Gamma_0(M)$ with $M \mid N$ and $r \mid \frac{N}{M} \neq 1$.

Theorem 1.5. *Assume GRH for Dirichlet L-functions and GRH for L-functions of weight* 2 *newforms. Then conjecture 1.4 is true.*

The first step in the proof of Theorem 1.5 is to explicitly determine the unary theta functions Θ_U occurring in the decomposition (1.6). One then essentially uses GRH for Dirichlet L-series to obtain an explicit bound for the growth of the Eisenstein series using the bounds of Siegel [37] and GRH for L-series of weight 2 newforms to obtain an explicit bound for the growth of the coefficients of weight $\frac{3}{2}$ newforms. The connection between weight 2 and weight $\frac{3}{2}$ newforms is formed through the Shimura lift.

1.4 The Shimura lift

Suppose that g is a weight $k+\frac{1}{2}$ newform for $\Gamma_0(4N)$ and Nebentypus χ which satisfies the Fourier expansion

$$g(\tau) = \sum_{n=1}^{\infty} b(n) q^n.$$

For a squarefree positive integer t, define the *t-th Shimura correspondence* by

$$\mathscr{S}_t(g)(\tau) := \sum_{n=1}^{\infty} \sum_{d|n} d^{k-1}\chi(d)\chi_{(-1)^k t}(d) b\left(\frac{n^2 t}{d^2}\right) q^n.$$

Here $\chi_{(-1)^k t}$ is the Dirichlet character given by the Kronecker–Jacobi symbol

$$\chi_{(-1)^k t}(d) := \left(\frac{(-1)^k t}{d}\right).$$

Shimura [34] proved that $\mathscr{S}_t(g)$ is a weight $2k$ modular form for $\Gamma_0(4N)$ and Nebentypus χ^2. Niwa [31] later used the theory of theta lifts (discussed further in chapter 2) to show that the level may be reduced to $2N$. Kohnen [26, 27] then found a distinguished subspace, known as *Kohnen's plus space,* where the level may be reduced to N. Forms in this space are distinguished by the fact that the coefficients are supported on those n for which $(-1)^k n$ is a discriminant. Kohnen further showed that a linear combination of the Shimura correspondences forms a bijection between the plus space of weight $k+\frac{1}{2}$ and level $4N$ with weight $2k$ modular forms for $\Gamma_0(N)$.

The method of Ono and Sound [32] is based upon a theorem of Waldspurger [39] relating the square of the $|D|$-th coefficient of a half-integral weight newform with the central value of the L-series of its integral weight counterpart under the Shimura lift twisted by the character χ_D. Restricting to Kohnen's plus space, Kohnen and Zagier [28] then determined the constant of proportionality explicitly, implying nonnegativity of the central values of L-series of newforms.

Theorem. *Suppose that D is a fundamental discriminant satisfying $(-1)^k D > 0$, f is a weight $2k$ Hecke eigenform for $\mathrm{SL}_2(\mathbb{Z})$, and g is the corresponding Hecke eigenform of weight $k+\frac{1}{2}$ in Kohnen's plus space satisfying $f = \mathscr{S}_1(g)$. Denote the $|D|$-th Fourier coefficient of g by $c(|D|)$ and the L-series of f twisted by χ_D by $L(f, D, s)$. Then Kohnen and Zagier's formula reads*

$$\frac{c(|D|)^2}{< g, g >} = \frac{(k-1)!}{\pi^k}|D|^{k-\frac{1}{2}}\frac{L(f,D,k)}{< f, f >}. \tag{1.8}$$

Here $< \cdot, \cdot >$ denotes the Petersson inner product, defined in (2.1).

The Shimura correspondences usually send cusp forms to cusp forms, but the exceptional role played by unary theta functions in Section 1.3 may be explained by the fact that unary theta functions are sent to Eisenstein series under the Shimura correspondences. Indeed, if one fixes a squarefree integer t, then the coefficients of g in the square class $t\mathbb{Z}^2$ grow asymptotically like the coefficients of $\mathscr{S}_t(g)$. Although the problem of explicitly determining the nonnegative integer represented by a totally positive ternary quadratic polynomial then becomes infeasible, a determination of the explicit unary theta functions suffices to determine when the quadratic polynomial is *almost universal.* This means that the totally positive quadratic polynomial represents all but finitely many nonnegative integers.

1.5 Almost universal mixed sums of squares and triangular numbers

First note that since $\Theta_E + \Theta_U$ in the decomposition (1.6) is a weighted average of the (nonnegative) number of representations by quadratic forms in the same spinor genus as the quadratic form of interest, if the n-th Fourier coefficient of $\Theta_E + \Theta_U$ is zero, then the form itself cannot represent n, as any such representation would add a positive value to the weighted average. Hence, since the coefficients of the cusp form $\Theta_\perp$ in the decomposition (1.6) grow more slowly than the coefficients of Θ_E and Θ_U, one immediately obtains that the number of representations by quadratic forms in the same spinor genus are equally distributed, and the coefficients of $\Theta_\perp$ may be considered an error term. Therefore, up to finitely many exceptions, a given quadratic form represents the same nonnegative integers as those supported in the Fourier expansion of $\Theta_E + \Theta_U$. Therefore, to determine if a given form is almost universal, one only needs to determine whether or not infinitely many coefficients of $\Theta_E + \Theta_U$ are zero. Determining the coefficients of Θ_E which are zero yields certain congruence conditions which must be satisfied, while work of Schulze-Pillot may be used to explicity determine the coefficients of the unary theta functions Θ_U. The author [22] applied this technique to the following conjecture of Sun about quadratic polynomials which are given as sums of squares and triangular numbers.

Conjecture 1.6. *Let m and n be any nonnegative integers. Then every sufficiently large natural number can be written in any of the following forms (with $x, y, z \in \mathbb{Z}$):*

$$2^m x^2 \quad +2^n y^2 \quad +T(z) \tag{1.9}$$
$$2^m x^2 \quad +2^n T(y) \quad +T(z) \tag{1.10}$$
$$2^m T(x) \quad +2^n T(y) \quad +T(z) \tag{1.11}$$
$$x^2 \quad +2^n \cdot 3y^2 \quad +T(z) \tag{1.12}$$
$$x^2 \quad +2^n \cdot 3T(y) \quad +T(z) \tag{1.13}$$
$$2^n \cdot 3x^2 \quad +2T(y) \quad +T(z) \tag{1.14}$$
$$2^n \cdot 3T(x) \quad +2T(y) \quad +T(z) \tag{1.15}$$
$$2^n \cdot 5T(x) \quad +T(y) \quad +T(z) \tag{1.16}$$
$$2T(x) \quad +3T(y) \quad +4T(z) \tag{1.17}$$
$$2x^2 \quad +3y^2 \quad +2T(z). \tag{1.18}$$

Remark. *This conjecture was obtained empirically by a computer calculation.*

Conjecture 1.6 turns out to be false, as shown by the following specific counterexamples given in [22].

Theorem 1.7.

1. *The mixed sum of squares and triangular numbers*

$$x^2 + 16T(y) + T(z)$$

does not represent any natural number of the form $\frac{p^2-17}{8}$*, where* p *is any prime congruent to* 1 *or* 3 *modulo* 8*, and is hence a counterexample to* (1.10).

2. *The sum of triangular numbers*

$$4T(x) + 4T(y) + T(z)$$

and

$$8T(x) + T(y) + T(z)$$

represent precisely the natural numbers not of the form $\frac{a^2-9}{8}$ *and* $\frac{a^2-5}{4}$*, respectively, where* a *is any integer all of whose prime factors are congruent to* 1 *modulo* 4*. Hence both are counterexamples to* (1.11).

3. *The sum of triangular numbers*

$$192T(x) + 2T(y) + T(z)$$

does not represent any natural number of the form $\frac{3p^2-195}{8}$*, where* p *is a prime congruent to* 5 *or* 7 *modulo* 8*, and hence it is a counterexample to* (1.15).

4. *The sum of triangular numbers*

$$160T(x) + T(y) + T(z)$$

does not represent any natural number of the form $\frac{5p^2-162}{8}$*, where* p *is a prime congruent to* 5 *or* 7 *modulo* 8*, and hence it is a counterexample to* (1.16).

However, a revised version of the conjecture, essentially showing that all counterexamples are of the type given in Theorem 1.7, was proven by the author in [22].

Theorem 1.8. *Let* m *and* n *be any nonnegative integers. Then for a sufficiently large natural number* r*, depending on* n *and* m*, the following equations hold*

1.
$$2^m x^2 + 2^n y^2 + T(z) = r.$$

2.
$$2^m x^2 + 2^n T(y) + T(z) = r$$

whenever $8r + 2^n + 1$ *is not a square. This condition is empty when* $n < 3$.

3.
$$2^mT(x) + 2^nT(y) + T(z) = r$$
whenever $8r + 2^n + 2^m + 1$ *is not a square, or when* $n = 0$ *(or, symmetrically,* $m = 0$*) and* $8r + 2^m + 2$ *(*$8r + 2^n + 2$*, respectively) is twice a square.*

4.
$$x^2 + 2^n \cdot 3y^2 + T(z) = r$$

5.
$$x^2 + 2^n \cdot 3T(y) + T(z) = r$$

6.
$$2^n \cdot 3x^2 + 2T(y) + T(z) = r$$

7.
$$2^n \cdot 3T(x) + 2T(y) + T(z) = r$$
whenever $8r + 3 \cdot 2^n + 3$ *is not* 3 *times a square.*

8.
$$2^n \cdot 5T(x) + T(y) + T(z) = r$$
whenever $8r + 5 \cdot 2^n + 2$ *is not 10 times a square.*

9.
$$2T(x) + 3T(y) + 4T(z) = r.$$

10.
$$2x^2 + 3y^2 + 2T(z) = r.$$

The author then teamed with Sun [25] to find a classification of those sums of two squares and one triangular number which are almost universal as well as a near classification of those sums of either one square and two triangular numbers or three triangular numbers which are almost universal. In order to state the theorem, we first need a little notation. We denote the (discrete) p-adic order of $n \in \mathbb{N}$ by $v_p(n)$ and uniquely write $n = 2^{v_2(n)}n'$, with n' odd. Furthermore, we denote the squarefree part of n by $\mathcal{SF}(n)$ and define the relation $n \mathrel{\mathcal{R}} m$ to mean that n is a quadratic residue modulo m, i.e., a is relatively prime to m and $x^2 \equiv a \pmod{m}$ has integral solutions. The classification for two squares and one triangular number is ten given by the following theorem.

Theorem 1.9. *Fix* $a, b, c \in \mathbb{N}$ *with* $\gcd(a, b, c) = 1$ *and* $v_2(a) \geq v_2(b)$*. Then the form*
$$f(x, y, z) := ax^2 + by^2 + cT(z)$$
represents a set of density one (within the natural numbers) if and only if

(i) $-2bc\,\mathcal{R}\,a'$, $-2ac\,\mathcal{R}\,b'$, and $-ab\,\mathcal{R}\,c'$.

(ii) Either $4 \nmid c$, or both $4 \| c$ and $2 \| ab$.

Suppose now that both (i) and (ii) hold. Then f is not almost universal if and only if we have the following (I)–(III).

(I) $2|a$, $4 \nmid c$, $a' \equiv b' \pmod{2^{3-v_2(c)}}$, and

$$\begin{cases} 4 \nmid b \Rightarrow v_2(a) \equiv c \pmod 2, \\ 2 \nmid bc \Rightarrow 8 \mid a \ \&\ 8 \mid (b-c). \end{cases}$$

(II) All prime divisors of $\mathcal{SF}(a'b'c')$ are congruent to 1 modulo 4 if $v_2(a) \equiv v_2(b) \pmod 2$, and congruent to 1 or 3 modulo 8 otherwise.

(III) $2^{3-v_2(c)}(ax^2+by^2)+c'z^2 = \mathcal{SF}(a'b'c')$ has no integral solutions.

The near classification for one square and two triangular numbers is given below.

Theorem 1.10. *Fix $a,b,c \in \mathbb{N}$ with $\gcd(a,b,c)=1$ and $v_2(a) \geq v_2(b)$. Then the form*

$$f(x,y,z) := ax^2 + bT(y) + cT(z)$$

represents a set of density one if and only if

(i) $-bc\,\mathcal{R}\,a'$, $-2ac\,\mathcal{R}\,b'$, and $-2ab\,\mathcal{R}\,c'$.

(ii) Either $4 \nmid b$ or $4 \nmid c$.

Assume now that (i) and (ii) both hold.

(A) When $v_2(b) \notin \{3,4\}$, f is not almost universal if and only if we have the following $(I)-(IV)$.

(I) $4 \nmid b+c$ and $\mathcal{SF}(a'b'c') \equiv (b+c)' \pmod{2^{3-v}}$, where $v := v_2(b+c) < 2$.

(II) All prime divisors of $\mathcal{SF}(a'b'c')$ are congruent to 1 or 3 modulo 8 if $\mathcal{SF}(abc) \equiv b+c \pmod 2$, and congruent to 1 modulo 4 otherwise.

(III) $8ax^2+by^2+cz^2 = 2^v\mathcal{SF}(a'b'c')$ has no integral solutions with y and z odd.

(IV)

$$\begin{cases} v_2(b) \leq 1 \Rightarrow v_2(a)-v_2(b) \in \{2,4,6,\ldots\}, \\ v_2(b) = 2 \Rightarrow v_2(a) \in \{1,3,5,,\ldots\}, \\ v_2(b) \in \{5,7,\ldots\} \Rightarrow (4 \mid a \text{ or } 2 \mid c), \\ v_2(b) \in \{6,8,\ldots\} \Rightarrow (2 \mid a \text{ or } a \equiv c \pmod 8). \end{cases}$$

(B) In the case $v_2(b) \in \{3,4\}$*, if* f *is not almost universal, then the above (I)–(III) hold and also*

$$\begin{cases} v_2(b) = 3 \Rightarrow (4 \mid a \text{ or } 2 \mid c) \\ v_2(b) = 4 \Rightarrow (2 \mid a \text{ or } a \equiv c \pmod 8). \end{cases}$$

Moreover, provided (I)–(III) in part (A) and the condition $2 \nmid v_2(a)$*,* f *is not almost universal if* $v_2(b) = 4$*, or* $v_2(a) \geq v_2(b) = 3$ *and* $b' \equiv c' \pmod 8$*.*

The near classification for the sum of three triangular numbers is given in the next theorem.

Theorem 1.11. *Fix* $a, b, c \in \mathbb{N}$ *with* $\gcd(a,b,c) = 1$ *and* $v_2(a) \geq v_2(b) \geq v_2(c) = 0$*. Then the form*

$$f(x,y,z) := aT(x) + bT(y) + cT(z)$$

represents a set of density one if and only if

$$-bc\,\mathcal{R}\,a', \ -ac\,\mathcal{R}\,b', \text{ and } -ab\,\mathcal{R}\,c'. \tag{1.19}$$

Assume now that (1.19) *is satisfied.*

(A) If f *is not almost universal, then we have the following (I)–(IV).*

(I) $4 \nmid a+b+c$ *and* $\mathcal{SF}(a'b'c') \equiv (a+b+c)' \pmod{2^{3-v}}$*, where* $v = v_2(a+b+c) < 2$*.*

(II) All prime divisors of $\mathcal{SF}(a'b'c')$ *are congruent to* 1 *modulo* 4 *if* $\mathcal{SF}(abc) \equiv a+b+c \pmod 2$*, and congruent to* 1 *or* 3 *modulo* 8 *otherwise.*

(III) $ax^2 + by^2 + cz^2 = 2^v \mathcal{SF}(a'b'c')$ *has no integral solutions with* x, y, z *all odd.*

(IV)

$$\begin{cases} v_2(b) \leq 1 \Rightarrow v_2(a) - v_2(b) \in \{3,5,7,\ldots\}, \\ v_2(b) = 2 \Rightarrow v_2(a) \in \{2,4,6,\ldots\}. \end{cases}$$

(B) The form f *is not almost universal under (I)–(III) in part (A), and the following condition stronger than (IV):*

$$\begin{cases} v_2(b) \leq 1 \Rightarrow v_2(a) - v_2(b) \in \{5,7,\ldots\}, \\ v_2(b) \in \{2,4\} \Rightarrow v_2(a) \in \{4,6,\ldots\}, \\ v_2(b) = 3 \Rightarrow (v_2(a) \in \{6,8,\ldots\} \ \& \ b' \equiv c \pmod 8). \end{cases}$$

Remark. *The classification was later completed by Chan and Oh [9] (showing that the necessary condition in Theorem 1.11 (A)(IV) is indeed sufficient) in the case of three triangular numbers and Chan and Haensch [8] (showing that the necessary condition in Theorem 1.10 (B) is indeed sufficient) in the case of one square and two triangular numbers.*

1.6 Equidistribution of Heegner points

In this section, we return to the question of reductions of CM elliptic curves to supersingular elliptic curves. If one takes the solutions to the equation for an elliptic curve C with complex multiplication (CM) by an $\mathcal{O}_d$ (meaning that the endomorphisms are isomorphic to $\mathcal{O}_d$, with d a negative discriminant) and applies the reduction map red_ℓ at a prime ℓ which is not split in $\mathbb{Q}\left(\sqrt{D}\right)$, then a result of Deuring [11] shows that $\mathrm{red}_\ell(C)$ is supersingular (meaning that its endomorphisms are isomorphic to an order of a quaternion algebra). If one further defines a pair of a CM elliptic curve and a subgroup of order N which are stabilized under the endomorphisms, then one obtains Heegner points on $\Gamma_0(N)$, which we denote $\Gamma_{d,N}$, and one may define a reduction map to *supersingular points on* $\Gamma_0(N)$ (denoted here by $X_0(N)^{\mathrm{SS}}$), namely, pairs of supersingular elliptic curves and subgroups of order N which are stabilized. The endomorphisms of supersingular points turn out to be Eichler orders of level N (the intersection of two maximal orders) of the unique (definite) quaternion algebra ramified precisely at ℓ and ∞. For a supersingular point $s \in X_0(N)^{\mathrm{SS}}$, we denote the number of units of this quaternion algebra by w_s.

The question addressed by Jetchev and the author in [19] is how large the preimage of the reduction map is for a fixed supersingular point. Special cases of this had been previously studied by Vatsal [38], Michel [30], and Elkies–Ono–Yang [14]. To clarify the difference between these results, we must consider how the discriminant d varies. In Michel's and Elkies–Ono–Yang's work, the discriminant is fundamental, while Vatsal fixes a fundamental discriminant D and lets d vary as $d = Dp^{2r}$ with $p \neq \ell$ prime and $r \in \mathbb{N}$. When $d = Dc^2$ for a fundamental discriminant D, then one refers to c as the *conductor*. In our result, we allow $d \to \infty$ to vary both in the fundamental discriminant as well as in the conductor, but restrict the conductor to be relatively prime to ℓ.

Theorem 1.12. *Suppose that ℓ is a prime, $d_1, d_2, \dots$ is a sequence of discriminants with $\lim_{r\to\infty} d_r = \infty$, ℓ is not split in $\mathbb{Q}(d_r)$, and ℓ does not divide the conductor of d_r. Then*

$$\lim_{r\to\infty} \frac{\#\left\{x \in \Gamma_{d_r,N} : \mathrm{red}_\ell(x) = s\right\}}{\#\Gamma_{d_r,N}} = \frac{1/w_s}{\sum_{s'\in X_0(N)^{SS}} 1/w_{s'}}.$$

As in the above examples, the main difficulty in showing Theorem 1.12 is that the unary theta functions may grow as quickly as the Eisenstein series. Hence, the main step in the proof involves showing that certain theta functions are orthogonal to unary theta functions.

Chapter 2

The Shimura lift and local Maass forms

2.1 The Shimura lift as a theta lift

The goal of this section is to investigate the connection between quadratic forms and the Shimura lift in the other direction. In particular, the Shimura lift may be defined as a *theta lift.* For a two variable indefinite theta function Θ which satisfies weight $\kappa \in \frac{1}{2}\mathbb{Z}$ modularity (as in (1.2)) for $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ in the first variable, the theta lift of function f satisfying weight κ modularity is defined via the *Petersson inner product*

$$\langle f, \Theta\rangle := \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]} \int_{\mathcal{F}} f(z)\Theta(z,\tau)y^{\kappa-2}dxdy, \tag{2.1}$$

where $z = x+iy$ (this notation will be used throughout), the index of Γ in $\mathrm{SL}_2(\mathbb{Z})$ is given by $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$ and $\mathcal{F}$ is any fundamental domain for Γ. Shintani [35] showed that the Shimura lift may be written as a theta lift and discovered the adjoint Shintani lift from integral to half-integral weight modular forms in the process. We consider here the definition of Θ given by Shintani, but slightly modify the definition so that Θ is in Kohnen's plus space for the second variable. We first make a couple of definitions. Let $\mathcal{Q}_D$ denote the set of all binary quadratic forms $Q = [a, b, c]$ of discriminant $D = b^2 - 4ac$ and further define the abbreviation

$$Q_z := \frac{1}{y}\left(a|z|^2 + bx + c\right). \tag{2.2}$$

Denoting $\tau = u + iv$ and the real and imaginary parts of z as above, for $k \in 2\mathbb{Z}$ our modification of Shintani's theta function is then given by

$$\Theta(z,\tau) := y^{-2k}v^{\frac{1}{2}} \sum_{\substack{D\in\mathbb{Z}\\ Q\in\mathcal{Q}_D}} Q(z,1)^k e^{-4\pi Q_z^2 v} e^{2\pi i D\tau}.$$

The theta series Θ is modular of weight $2k \in \mathbb{Z}$ for $\mathrm{SL}_2(\mathbb{Z})$ in the z variable and weight $k+\frac{1}{2}$ for $\Gamma_0(4)$ in Kohnen's plus space in the τ variable. For a weight $k+\frac{1}{2}$ cusp form f, the theta lift

$$\Phi_k(f)(z) := \langle f, \Theta(z,\cdot)\rangle$$

is proportional to the Shimura lift $\mathscr{S}_1$ via

$$\Phi_k(f) = \frac{\mathscr{S}_1(f)}{6 \cdot 2^k}.$$

Due to this identity, one calls $6 \cdot 2^k \Theta$ a *theta kernel* for the Shimura lift. Kohnen and Zagier [28] later found a holomorphic (in both variables) theta kernel for the Shimura lift.

2.2 A holomorphic kernel for the Shimura lift

In this section, we consider a holomorphic theta kernel for the Shimura lift, which was defined by Kohnen and Zagier [28]. While investigating the Doi–Naganuma lift [12] in [40], for a discriminant $D > 0$ and $k \in 2\mathbb{N}$, Zagier noticed that the function

$$f_{k,D}(z) := \frac{D^{k-\frac{1}{2}}}{\binom{2k-2}{k-1}\pi} \sum_{Q \in \mathcal{Q}_D} Q(z,1)^{-k} \tag{2.3}$$

is modular of weight $2k$ for $\mathrm{SL}_2(\mathbb{Z})$. Working together with Kohnen, these were then packaged into a generating function

$$\Omega_k(z,\tau) := \sum_{D \in \mathbb{N}} f_{k,D}(z) e^{2\pi i D\tau}.$$

They then proved [28] that with respect to the τ variable, Ω_k is a cusp form of weight $k + \frac{1}{2}$ in Kohnen's plus space. They further showed that for a cusp form f of weight $k + \frac{1}{2}$

$$\langle f, \Omega(-\overline{z}, \cdot)\rangle = \frac{(-1)^{\frac{k}{2}}\pi}{2^{3k-2}} \binom{2k-2}{k-1} \mathscr{S}_1(f)(z).$$

This chapter is centered around understanding the functions $f_{k,D}$ from the point of view of harmonic weak Maass forms. In particular, there is a natural differential operator

$$\xi_{\kappa,z} := 2iy^{\kappa} \overline{\frac{\partial}{\partial \overline{z}}} \tag{2.4}$$

which sends harmonic weak Maass forms of weight κ to weakly holomorphic modular forms, i.e., those meromorphic modular forms all of whose poles lie at the cusps, of weight $2 - \kappa$.

2.3 Harmonic weak Maass forms

The study of harmonic weak Maass forms has grown in interest in the past few years due to the connections to Ramanujan's *mock theta functions.* These are certain functions, defined by Ramanujan in his last letter to Hardy, which Ramanujan claimed "entered into mathematics as beautifully as the ordinary theta functions" but were not themselves theta functions.

However, he did not give a thorough definition of these objects and only gave some defining characteristics of 17 such functions. Their role within the field of modular and automorphic forms remained a mystery until it was shown by Zwegers [41] and Bringmann and Ono [6] that the mock theta functions are the "holomorphic parts" of harmonic weak Maass forms.

The theory of harmonic weak Maass forms was developed by Bruinier and Funke [7], laying the framework to understand these functions and their connections with classical modular forms. One first defines the weight κ *hyperbolic Laplacian*

$$\Delta_\kappa := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+i\kappa y\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right) = -\xi_{2-\kappa}\circ\xi_\kappa.$$

A *harmonic weak Maass form* of weight κ for Γ is a real analytic function $\mathcal{F}:\mathbb{H}\to\mathbb{C}$ satisfying the following properties.

1. $\mathcal{F}|_\kappa\gamma(\tau)=\mathcal{F}(\tau)$ for every $\gamma\in\Gamma$,
2. $\Delta_\kappa(\mathcal{F})=0$,
3. $\mathcal{F}$ has at most linear exponential growth at $i\infty$.

Each harmonic weak Maass form $\mathcal{F}$ naturally splits into a holomorphic part and a non-holomorphic part. Consider the special case that $\xi_\kappa(\mathcal{F})$ is a cusp form f. The *non-holomorphic Eichler integral of f* is defined by

$$f^*(\tau) := (2i)^{\kappa-1}\int_{-\overline{\tau}}^{i\infty}\frac{f^c(z)}{(z+\tau)^\kappa}dz, \tag{2.5}$$

where $f^c(\tau) := \overline{f(-\overline{\tau})}$ is the cusp form whose Fourier coefficients are the conjugates of the coefficients of f. Then the *non-holomorphic part* of $\mathcal{F}$ is defined to be f^* and the *holomorphic part* $\mathcal{F}-f^*$ indeed turns out to be holomorphic on $\mathbb{H}$.

The aforementioned connection between Ramanujan's mock theta functions is then formed by showing that there exists $\mathcal{F}$ for which the holomorphic part equals the mock theta function. In these cases, the harmonic weak Maass forms satisfy weight $\frac{1}{2}$ modularity and the image under the ξ-operator is a weight $\frac{3}{2}$ unary theta function.

When the weight $\kappa=2-2k$ is an even (negative) integer, there is also another natural differential map from the space of harmonic weak Maass forms to the space of weakly holomorphic modular forms given by $\mathcal{D}^{2k-1}$, where

$$\mathcal{D} := \frac{1}{2\pi i}\frac{\partial}{\partial z}. \tag{2.6}$$

This complements the operator ξ_{2-2k} in the sense that when ξ sends a harmonic weak Maass form to a cusp form, $\mathcal{D}^{2k-1}$ sends the form to the space of weakly holomorphic modular forms which are orthogonal to cusp forms.

Bruinier and Funke [7] proved that the map ξ is surjective, but the kernel is given by weakly holomorphic modular forms. Hence, it is unclear how to find a canonical preimage. Investigating this question and its implications to the Shimura lift led to the discovery of *locally harmonic Maass forms* by Bringmann, Kohnen, and the author [4].

2.4 Local Maass forms and the Shimura lift

In this section, we consider the question of finding a natural preimage of $f_{k,D}$ under ξ_{2-2k}. By natural, we mean that we would like the function to essentially be defined in the same way as $f_{k,D}$. Hence, the function should be defined in terms of binary quadratic forms, it should satisfy weight $2-2k$ modularlity, and should be annihilated by Δ_{2-2k}. It turns out that one may obtain such a natural preimage if the last condition in further relaxed. This yields a new modular object which we call a *local Maass form.* For $\kappa \in 2\mathbb{Z}$, $\lambda \in \mathbb{C}$, and a measure zero set E, we call a function $\mathcal{F}$ a weight κ *local Maass form* with eigenvalue λ and exceptional set E if $\mathcal{F}$ satisfies the following:

1. For every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, one has $\mathcal{F}|_{\kappa}\gamma = \mathcal{F}$
2. For every $\tau \notin E$ there exists a neighborhood around τ for which $\mathcal{F}$ is real analytic and
$$\Delta_{\kappa}(\mathcal{F})(\tau) = \lambda\mathcal{F}(\tau).$$
3. For $\tau \in E$ one has
$$\mathcal{F}(\tau) = \frac{1}{2}\lim_{r\to 0^+}\left(\mathcal{F}\left(\tau+ir\right)+\mathcal{F}\left(\tau-ir\right)\right).$$
4. The function $\mathcal{F}$ exhibits at most polynomial growth as $v \to \infty$.

In the special case that the eigenvalue λ is zero, we call the function a *locally harmonic Maass form.* In order to find a preimage of $f_{k,D}$ under ξ_{2-2k}, Bringmann, Kohnen, and the author [4] then defined

$$\mathcal{F}_{1-k,D}\left(z\right) := \frac{1}{12\psi(1)}\left(4\pi D\right)^{\frac{3}{4}-\frac{k}{2}}\sum_{Q\in\mathcal{Q}_D}\operatorname{sgn}\left(Q_z\right)Q\left(z,1\right)^{k-1}\psi\left(\frac{Dy^2}{\left|Q\left(z,1\right)\right|^2}\right),$$

where
$$\psi(v) := \beta\left(v; k-\frac{1}{2}; \frac{1}{2}\right)$$
is a special case of the incomplete β-function defined for $r, s \in \mathbb{C}$ by
$$\beta(v;s,r) := \int_0^v u^{s-1}(1-u)^{r-1}du.$$

We define the exceptional set

$$E_D := \left\{\tau = x+iy \in \mathbb{H} : \exists a,b,c \in \mathbb{Z},\ b^2-4ac = D,\ a\left|\tau\right|^2+bx+c = 0\right\}.$$

Bringmann, Kohnen, and the author [4] then proved the following theorem.

Theorem 2.1. *Suppose that $k>1$ is even and $D>0$ is a non-square discriminant. Then the function $\mathcal{F}_{1-k,D}$ is a weight $2-2k$ locally harmonic Maass form with exceptional set E_D for $\mathrm{SL}_2(\mathbb{Z})$. Moreover, for $z\notin E_D$*

$$\xi_{2-2k}\left(\mathcal{F}_{1-k,D}\right)(z)=\frac{2^{2k-3}}{3}(4\pi D)^{\frac{3}{4}-\frac{k}{2}}f_{k,D}(z).$$

$$\mathcal{D}^{2k-1}\left(\mathcal{F}_{1-k,D}\right)(z)=\frac{(2k-2)!}{24(2\pi)^{2k-1}}(4\pi D)^{\frac{3}{4}-\frac{k}{2}}f_{k,D}(z).$$

Remark. *In light of the complementary nature of ξ_{2-2k} and $\mathcal{D}^{2k-1}$ described after definition* (2.6)*, it is somewhat surprising that the image under both ξ_{2-2k} and $\mathcal{D}^{2k-1}$ are both cusp forms. Moreover, it is worth noting that although $\mathcal{F}_{1-k,D}$ is not itself real analytic, its image under each of these differential operators may be (complex) analytically continued.*

Understanding the image of $\mathcal{F}_{1-k,D}$ under both of these differential operators allows one to obtain a Fourier-type expansion for $\mathcal{F}_{1-k,D}$. In addition to the non-holomorphic Eichler integral defined in (2.5), we require the *holomorphic Eichler integral of a weight $2k$ cusp form $f=\sum_{n=1}^{\infty}a_nq^n$* given by

$$\mathcal{E}_f(\tau):=\sum_{n=1}^{\infty}\frac{a_n}{n^{2k-1}}q^n. \tag{2.7}$$

Using the (known) fact that f^* is a preimage of f under ξ_{2-2k} and $\mathcal{E}_f$ is a preimage under $\mathcal{D}^{2k-1}$ together with the fact that f^* is annihilatd by $\mathcal{D}^{2k-1}$ and $\mathcal{E}_f$ is annihilated by ξ_{2-2k}, Bringmann, Kohnen, and the author [4] obtained the following expansion.

Theorem 2.2. *Suppose that $k>1$, $D>0$ is a non-square discriminant, and $\mathcal{C}$ is one of the connected components of $\mathbb{H}\setminus E_D$. Then there exists a polynomial $P_{\mathcal{C}}$ of degree at most $2k-2$ such that for all $\tau\in\mathcal{C}$,*

$$\mathcal{F}_{1-k,D}(\tau)=P_{\mathcal{C}}(\tau)+\frac{2^{2k-3}}{3}(4\pi D)^{\frac{3}{4}-\frac{k}{2}}f^*_{k,D}(\tau)-(4\pi D)^{\frac{3}{4}-\frac{k}{2}}\frac{(2k-2)!}{24(2\pi)^{2k-1}}\mathcal{E}_{f_{k,D}}(\tau).$$

Theorem 2.2 was then used to reprove a theorem of Kohnen and Zagier [29] showing that the periods of $f_{k,D}$ are rational.

Bruinier suggested to consider the relationship between locally harmonic Maass forms and theta lifts, which in the case that $k=1$ was independently studied by his student Hövel [17].

2.5 Local Maass forms and theta lifts

In addition to the theta function Θ, Bringmann, Viazovska, and the author [5] used the theta function

$$\Theta^*(z,\tau):=v^k\sum_{\substack{D\in\mathbb{Z}\\ Q\in\mathcal{Q}_D}}Q_zQ(z,1)^{k-1}e^{-\frac{4\pi|Q(z,1)|^2v}{y^2}}e^{-2\pi iD\tau} \tag{2.8}$$

to find a theta kernel for the functions $\mathcal{F}_{1-k,D}$ and refound the relation between these functions and $f_{k,D}$ under ξ_{2-2k} via relations with the corresponding theta kernels. In particular, $\mathcal{F}_{1-k,D}$ was realized as the theta lift of a harmonic weak Maass form. Due to the theory built around theta lifts, this also allowed us to define a more general function which is a local Maass form with eigenvalue $4\lambda_s$ under Δ_{2-2k} for $s \in \mathbb{C}$, where

$$\lambda_s := \left(s - \frac{k}{2} - \frac{1}{4}\right)\left(1 - s - \frac{k}{2} - \frac{1}{4}\right)$$

is the eigenvalue of a corresponding half-integral weight weak Maass form under the theta lift. Here there are some technical restrictions on the allowable eigenvalues. To describe the full result, we next define a generalization of $f_{k,D}$ given by

$$f_{k,s,D}(z) := \sum_{Q \in \mathcal{Q}_D} Q(z,1)^{-k} \varphi_s\left(\frac{Dy^2}{\left|Q\left(z,1\right)\right|^2}\right) \tag{2.9}$$

of $f_{k,D}$. Here, for $0 < w \leq 1$ and $\mathrm{Re}(s) \geq \frac{k}{2} + \frac{1}{4}$, using the usual ${}_2F_1$ notation for Gauss's hypergeometric function, we define

$$\varphi_s(w) := \frac{\Gamma\left(s + \frac{k}{2} - \frac{1}{4}\right) D^{\frac{k}{2}+\frac{1}{4}}}{6\Gamma(2s)\left(4\pi\right)^{\frac{k}{2}-\frac{1}{4}}} w^{s-\frac{k}{2}-\frac{1}{4}} {}_2F_1\left(s + \frac{k}{2} - \frac{1}{4}, s - \frac{k}{2} - \frac{1}{4}; 2s; w\right),$$

which is easily seen to be a constant when $s = \frac{k}{2} + \frac{1}{4}$. Moreover, the functions $\mathcal{F}_{1-k,D}$ generalizes to

$$\mathcal{F}_{1-k,s,D}\left(z\right) := \sum_{Q \in \mathcal{Q}_D} \mathrm{sgn}\left(Q_z\right) Q(z,1)^{k-1} \varphi_s^*\left(\frac{Dy^2}{\left|Q(z,1)\right|^2}\right), \tag{2.10}$$

where, for $0 < w \leq 1$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geq \frac{k}{2} - \frac{3}{4}$, we define

$$\varphi_s^*(w) := \frac{\Gamma\left(s + \frac{k}{2} - \frac{1}{4}\right)\left(4\pi D\right)^{\frac{3}{4}-\frac{k}{2}}}{12\sqrt{\pi}\Gamma(2s)} w^{\frac{k}{2}-\frac{3}{4}+s} {}_2F_1\left(s - \frac{k}{2} + \frac{1}{4}, s + \frac{k}{2} - \frac{3}{4}; 2s; w\right).$$

Due to growth at ∞, the naive definition of the theta lift of a harmonic weak Maass form against Θ^* would diverge. Hence, we require a regularized Petersson inner product which was first introduced by Harvey–Moore [16] and Borcherds [2]. In particular, under this regularization, for a weak Maass form H with eigenvalue λ_s one may define the theta lift

$$\Phi^*_{1-k}(H)(z) := \left\langle H, \Theta^*\left(-\bar{z}, \cdot\right)\right\rangle^{\mathrm{reg}}.$$

Then the image of the weight $\frac{3}{2} - k$ Maass-Poincaré series (see [15]) $P_{\frac{3}{2}-k,D}$ in Kohnen's plus space for $\Gamma_0(4)$ under the theta lift is precisely $\mathcal{F}_{1-k,D}$. More generally, extending the definition of Φ_k to

$$\Phi_k(H)(z) := \left\langle f, \Theta\left(z, \cdot\right)\right\rangle^{\mathrm{reg}}$$

one obtains the following two theorems.

Theorem 2.3. *Suppose that $s \in \mathbb{C}$ satisfies $Re(s) \geq \frac{k}{2} + \frac{1}{4}$ and $D > 0$ is a discriminant. Then the following hold.*

1. *The function $f_{k,s,D}$ is a local Maass form of weight $2k$ and eigenvalue $4\lambda_s$ under Δ_{2k} with exceptional set E_D. Moreover,*
$$f_{k,\frac{k}{2}+\frac{1}{4},D} = \frac{2^{2k-3}}{3(2k-1)} (4\pi D)^{\frac{3}{4}-\frac{k}{2}} f_{k,D}, \tag{2.11}$$
which is a cusp form.

2. *The theta lift Φ_k maps weight $k + \frac{1}{2}$ weak Maass forms with eigenvalue λ_s under $\Delta_{k+\frac{1}{2}}$ to weight $2k$ local Maass forms with eigenvalue $4\lambda_s$ under Δ_{2k}. In particular, the image of the D-th Poincaré series under the theta lift Φ_k equals*
$$\Phi_k \left(P_{k+\frac{1}{2},s,D} \right) = f_{k,s,D}.$$

The corresponding theorem in negative weight is given below.

Theorem 2.4. *Suppose that k is even, $D > 0$ is a discriminant, and $s \in \mathbb{C}$ satisfies $Re(s) \geq \frac{k}{2} - \frac{3}{4}$. Then the following hold.*

1. *The function $\mathcal{F}_{1-k,s,D}$ is a local Maass form of weight $2-2k$ with eigenvalue $4\lambda_s$ under Δ_{2-2k} and exceptional set E_D.*

2. *The theta lift Φ^*_{1-k} maps weight $\frac{3}{2}-k$ weak Maass forms with eigenvalue λ_s under $\Delta_{\frac{3}{2}-k}$ to weight $2-2k$ local Maass forms with eigenvalue $4\lambda_s$ under Δ_{2-2k}. In particular, the image of $P_{\frac{3}{2}-k,s,D}$ under the theta lift is*
$$\Phi^*_{1-k} \left(P_{\frac{3}{2}-k,s,D} \right) = \mathcal{F}_{1-k,s,D}. \tag{2.12}$$

As with $f_{k,D}$ and $\mathcal{F}_{1-k,D}$, the functions $f_{k,s,D}$ and $\mathcal{F}_{1-k,s,D}$ are related through the ξ-operator.

Theorem 2.5. *Suppose that $k > 0$ is an even integer, D is a positive discriminant, and $s \in \mathbb{C}$ satisfies $Re(s) \geq \frac{k}{2} + \frac{1}{4}$.*

1. *For every $z \notin E_D$, we have that*
$$\xi_{2-2k} \left(\mathcal{F}_{1-k,s,D}(z) \right) = 2 \left(\overline{s} - \frac{3}{4} + \frac{k}{2} \right) f_{k,\overline{s},D}(z). \tag{2.13}$$

2. *For $z \notin E_D$, we have that*
$$\xi_{2k} \left(f_{k,s,D}(z) \right) = 2 \left(\overline{s} - \frac{k}{2} - \frac{1}{4} \right) \mathcal{F}_{1-k,\overline{s},D}(z). \tag{2.14}$$

The relation between these functions yields the following commutative diagram:

$$\begin{CD}
P_{\frac{3}{2}-k,s,D} @>{\Phi_{1-k}^{*}}>> \mathcal{F}_{1-k,s,D} \\
@V{\xi_{\frac{3}{2}-k}}VV @VV{\xi_{2-2k}}V \\
\left(\overline{s}-\frac{3}{4}+\frac{k}{2}\right) P_{k+\frac{1}{2},\overline{s},D} @>{2\Phi_k}>> 2\left(\overline{s}-\frac{3}{4}+\frac{k}{2}\right) f_{k,\overline{s},D} \\
@V{\xi_{k+\frac{1}{2}}}VV @VV{\xi_{2k}}V \\
-\lambda_s P_{\frac{3}{2}-k,s,D} @>{4\Phi_{1-k}^{*}}>> -4\lambda_s \mathcal{F}_{1-k,s,D}
\end{CD}$$

In the special case that $s = \frac{k}{2} + \frac{1}{4}$ (see Corollary 9 of [28] for the constant multiple of $\mathscr{S}_1$), the diagram becomes the following:

$$\begin{CD}
P_{\frac{3}{2}-k,D} @>{\Phi_{1-k}^{*}}>> \mathcal{F}_{1-k,D} \\
@V{\xi_{\frac{3}{2}-k}}VV @VV{\xi_{2-2k}}V \\
\left(k-\frac{1}{2}\right) P_{k+\frac{1}{2},D} @>{2\Phi_k}>{3^{-1}2^{-k}\mathscr{S}_1}> \frac{2^{2k-3}}{3}\left(4\pi D\right)^{\frac{3}{4}-\frac{k}{2}} f_{k,D}
\end{CD}$$

The same diagram was obtained by Hövel [17] in the case that $k = 1$.

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THE TRIANGULAR THEOREM OF EIGHT AND REPRESENTATION BY QUADRATIC POLYNOMIALS

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Abstract. We investigate here the representability of integers as sums of triangular numbers, where the n-th triangular number is given by $T_n = n(n+1)/2$. In particular, we show that $f(x_1, x_2, \ldots, x_k) = b_1 T_{x_1} + \cdots + b_k T_{x_k}$, for fixed positive integers $b_1, b_2, \ldots, b_k$, represents every nonnegative integer if and only if it represents 1, 2, 4, 5, and 8. Moreover, if 'cross-terms' are allowed in f, we show that no finite set of positive integers can play an analogous role, in turn showing that there is no overarching finiteness theorem which generalizes the statement from positive definite quadratic forms to totally positive quadratic polynomials.

1. Introduction

In 1638 Fermat claimed that every number is a sum of at most three triangular numbers, four square numbers, and in general k polygonal numbers of order k. The n-th polygonal number of order k is $\frac{(k-2)n^2-(k-4)n}{2}$, so the n-th triangular number is $T_n := \frac{n(n+1)}{2}$, where we include $T_0 = 0$ for simplicity. The claim for four squares was shown by Lagrange.

Theorem (Lagrange, 1770)**.** *Every positive integer is the sum of four squares.*

Gauss wrote "Eureka, $\triangle + \triangle + \triangle = n$" in his mathematical diary on July 10, 1796.

Theorem (Gauss, 1796)**.** *Every positive integer is the sum of three triangular numbers.*

The first proof of the full assertion of Fermat was given by Cauchy in 1813 [3]; cf. [12].

For a more complete history of related questions about sums of figurate numbers and some new results, see Duke's survey paper [8].

The current paper concerns questions of representability of integers by quadratic polynomials. If $f = f(x) = f(x_1, x_2, \ldots, x_k)$ is a rational polynomial in k variables, it *represents* the integer n if there exist integers n_i such that $n = f(n_1, n_2, \ldots, n_k)$, and it *oddly represents* the integer n if there exist odd integers n_i such that $f(n_1, n_2, \ldots, n_k) = n$. If f represents every element of a set $\mathcal{Z}$ of integers, it is said to *represent* $\mathcal{Z}$.

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If we let $S = S_x$ be the square polynomial x^2, and let $T = T_x$ denote the triangular polynomial $(x^2+x)/2$, the theorems of Lagrange and Gauss state that the positive integers are represented by $S_w + S_x + S_y + S_z$ and by $T_x + T_y + T_z$.

In 1917, Ramanujan extended the question about four squares to ask for which choices of quadruples $b = (b_1, b_2, b_3, b_4)$ of integers the form $b_1S_w+b_2S_x+b_3S_y+b_4S_z$ represents every positive integer; we shall refer to these as *universal diagonal forms*. He gave a list of 55 choices of b which he claimed to be the complete list of universal quarternary diagonal forms; 54 of them turned out to be universal, and this list is complete, as proven by Dickson [7].

Recently, Conway and Schneeberger proved in unpublished work a nice classification for universal positive definite quadratic forms whose corresponding matrices have integer entries. This answers the question of representability by positive definite homogeneous quadratic polynomials with *even* off-diagonal coefficients.

Theorem (Conway-Schneeberger). *A positive definite quadratic form $Q(x)=x^tAx$, where A is a positive symmetric matrix with integer coefficients, represents every positive integer if and only if it represents the integers* $1, 2, 3, 5, 6, 7, 10, 14,$ *and* 15.

Bhargava gave a simpler proof of the Conway-Schneeberger 15-Theorem in [1] and showed more generally that representability of any $\mathcal{Z}$ by such a form can always be checked on a finite subset $\mathcal{Y}$. In addition, he exhibited $\mathcal{Y}$ for $\mathcal{Z}$ consisting of all odd integers and for $\mathcal{Z}$ consisting of all primes.

More recently, Bhargava and Hanke [2] have shown the 290-Theorem, providing the necessary set (the largest element of which is 290) for universal forms when the corresponding matrix is half integral, that is, for totally positive integer quadratic forms.

In 1863, Liouville [11] proved the following generalization of Gauss's theorem, similar to Ramanujan's generalization of Lagrange's Four Squares Theorem.

Theorem (Liouville). *Let a, b, c be positive integers with $a \leq b \leq c$. Then every positive integer is represented by $aT_x + bT_y + cT_z$ if and only if (a, b, c) is one of the following:*

$$(1,1,1),\ (1,1,2),\ (1,1,4),\ (1,1,5),\ (1,2,2),\ (1,2,3),\ (1,2,4).$$

We will first prove a finiteness theorem similar to the results of the Conway-Schneeberger 15-Theorem or the Bhargava-Hanke 290-Theorem for sums of triangular numbers.

Theorem 1.1. *If $b_1, \ldots, b_k$ is a sequence of positive integers, then $\sum_{i=1}^k b_iT_{x_i}$ represents every nonnegative integer if and only if it represents* 1, 2, 4, 5, *and* 8.

Since $8T_x = (2x+1)^2-1$, clearly $\sum_{i=1}^k b_iT_{n_i} = n$ if and only if $\sum_{i=1}^k b_i(2n_i+1)^2 = 8n + \sum_{i=1}^k b_i$. Hence there is a close correspondence between representability by triangular polynomials and odd representability by diagonal quadratic forms.

Corollary 1.2. *If $b_1, \ldots, b_k$ is a sequence of positive integers with sum B, then $\sum_{i=1}^k b_ix_i^2$ oddly represents every integer of the form $8n + B$ with $n \geq 0$ if and only if it oddly represents* $8 + B$, $16 + B$, $32 + B$, $40 + B$, *and* $64 + B$.

It is not so difficult to establish Theorem 1.1 with the escalator techniques of Bhargava (and Liouville). We will prove a stronger statement in Section 2: if the integers $1, 2, 4, 5,$ and 8 are represented by the triangular form, then n is represented very many times unless $n + 1$ has high 3-divisibility.

We now turn to more general quadratic polynomials. Let f be a quadratic polynomial in $\mathbb{Q}[x_1, x_2, \ldots, x_k]$; then f is a *normalized totally positive* quadratic polynomial if the image of $\mathbb{Z}^k$ under f consists of nonnegative integers, while $f(x) = 0$ for some $x \in \mathbb{Z}^k$. Note that clearly $S_x = x^2$ is normalized totally positive, as is T_x: $T_0 = 0, T_1 = 1, T_2 = 3$ are the first of the increasing sequence of triangular numbers, and $T_{-m} = T_{m-1}$ for positive m.

It turns out that no finiteness theorem will hold in general for normalized totally positive quadratic polynomials, and moreover that checking no proper subset will suffice.

Proposition 1.3. *Let $\mathcal{Z}$ be a subset of the positive integers. For every proper subset $\mathcal{Y} \subsetneq \mathcal{Z}$ there exists a normalized totally positive quadratic polynomial that represents $\mathcal{Y}$ but does not represent $\mathcal{Z}$.*

Proposition 1.3 will follow directly from the corresponding result for *triangular sums with cross terms*. This class corresponds to integral quadratic forms with even off-diagonal terms, just as the ordinary triangular sums correspond to diagonal quadratic forms. We refer to Section 3 for a precise definition of this subclass of quadratic polynomials.

In Section 4 we construct a 'norm' m on this class that restores finite representability.

Theorem 1.4. *Fix an integer m and a subset $\mathcal{Z}$ of the positive integers. Then there is a finite subset $\mathcal{Y}_m \subset \mathcal{Z}$, depending only on m and $\mathcal{Z}$, such that every triangular sum t with cross terms satisfying $m(t) \leq m$ represents $\mathcal{Z}$ if and only if it represents $\mathcal{Y}_m$.*

Moreover, for $\mathcal{Z}$ equal to the positive integers, we find that $\max \mathcal{Y}_m \gg m^2$.

It may be of interest to investigate the growth of $\max \mathcal{Y}_m$; see Remark 4.3.

2. Theorem of eight

For background information on quadratic forms and genus theory, a good source is [9]. We prove Theorem 2.1, by using a standard argument to show that the theorem is equivalent to a statement about (diagonal) quadratic forms, and then prove the corresponding result for quadratic forms. We will only need some elementary results about quadratic forms and a theorem of Siegel to show the desired result. Theorem 1.1 and Corollary 1.2 follow immediately.

We will first introduce some useful notation and definitions. We abbreviate $t(x) = t(x_1, x_2, \ldots, x_k) := \sum b_i T_{x_i}$, and call it a *triangular sum*. For a vector b of length k we define the generating function

$$F(q) := F_b(q) := \sum_{x \in \mathbb{Z}^k} q^{t(x)} = \sum_{n=0}^{\infty} s_b(n) q^n,$$

where $s_b(n)$ is the number of solutions to $t(x) = n$. We will omit the subscript of $s_b(n)$ when it is clear from the context. We will furthermore use $r(n)$ to denote the number of representations of n by the corresponding (diagonal) quadratic form $\sum b_i x_i^2$ and $r_o(n)$ to denote the number of those representations with all x_i odd. For ease of notation, we will denote the triangular sum corresponding to b with $[b_1, b_2, \ldots, b_k]$ and the corresponding quadratic form by $(b_1, \ldots, b_k)$.

The Hurwitz class number for the imaginary quadratic order of discriminant $D < 0$ will play an important role in our analysis below. We recall the definition here. For a negative discriminant D, the Hurwitz class number $H(D)$ is the weighted number of equivalence classes of, not necessarily primitive, positive definite binary quadratic forms of discriminant D, where the weights are 1 except for classes of forms equivalent to a multiple of $\left(x^2+y^2\right)$, which are counted with weight $\frac{1}{2}$, and for classes of forms equivalent to a multiple of $\left(x^2+xy+y^2\right)$, which are counted with weight $\frac{1}{3}$. Every quadratic form of discriminant D is a multiple of a primitive form of discriminant $D'=D/f^2$, and the weights are reciprocal to $w(D')/2$, half the number of units in the unique order of discriminant D', or, accordingly, to half the number of representations of the integer 1 by the primitive form. The usual class numbers $h(D)$ are hence related to the Hurwitz class number by

$$H(D)=\sum_{f^2\mid D}\frac{h\left(\frac{D}{f^2}\right)}{\frac{1}{2}w\left(\frac{D}{f^2}\right)}.$$

For an integer n, we will set $a_n:=\frac{v_3(n+1)}{\log_3(n+1)}$, so that $3^{v_3(n+1)}=(n+1)^{a_n}$ gives the 3-part of $n+1$ as a power of $n+1$.

Theorem 2.1. *For $\epsilon>0$, there is an absolute constant c_ϵ such that if the triangular sum $t(x)$ represents $1,2,4,5$, and 8, then $t(x)$ represents every nonnegative integer n at least $\min\{c_\epsilon n^{\frac{1}{2}-\epsilon},16n^{1-a_n}\}$ times. In particular, if n is sufficiently large and $a_n<\frac{1}{2}$, then $t(x)$ represents n at least $c_\epsilon n^{\frac{1}{2}-\epsilon}$ times.*

Proof. We proceed with *escalator lattices* as in [1]. Without loss of generality we have $b_1\le b_2\le\cdots\le b_k$. Fixing $b=[b_1,\ldots,b_{k-1}]$, we will *escalate* to $[b_1,\ldots,b_k]$ by making all possible choices of $b_k\ge b_{k-1}$ for which it is possible to represent the next largest integer not already represented. We will then develop an *escalator tree* by forming an edge between b and $[b_1,\ldots,b_k]$, with $\emptyset$ as the root. If $\sum_i b_iT_{x_i}$ represents every integer, then b will be a leaf of our tree.

Since $s(1)>0$, it follows that $b_1=1$. We need $s(2)>0$, so $b_2=1$ or $b_2=2$. If $b_2=1$, then we need $s(5)>0$, so $1\le b_3\le 5$. For $b_3=3$, we need $s(8)>0$, so $3\le b_4\le 8$. Likewise, if $b_2=2$, then $2\le b_3\le 4$. Therefore, if $s(n)>0$ for every n, then we must have one of the above choices of b_i as a sublattice. By showing that each of these choices of b_i satisfies $s(n)>0$ for every n, we will see that this condition is both necessary and sufficient.

All of the cases other than $[1,1,3,k]$ with $3\le k\le 8$ are covered by Liouville's Theorem. However, to obtain the more precise version given in Theorem 2.1, we will use quadratic form genus theory.

One sees easily that

$$q^{\sum_{i=1}^k b_i}F(q^8)=\sum_x q^{\sum_{i=1}^k b_i(2x_i+1)^2},$$

so that $s(n)=r_o\left(8n-\sum_{i=1}^k b_i\right)$. For the forms $b=[1,1,1]$, $[1,1,4]$, $[1,1,5]$, $[1,2,2]$, and $[1,2,4]$, congruence conditions modulo 8 imply that

$$r_o\left(8n-\sum_{i=1}^k b_i\right)=r\left(8n-\sum_{i=1}^k b_i\right).$$

Moreover, for each of these choices of b, (b_1, b_2, b_3) is a genus 1 quadratic form. Therefore, extending the classification of Jones [9, Theorem 86] to primitive representations when the integer is not squarefree, $s_{[1,1,1]}(n) = 24H(-(8n+3))$, $s_{[1,1,4]}(n) = 4H(-4(8n+6))$, $s_{[1,2,2]}(n) = 4H(-4(8n+5))$, and $s_{[1,2,4]}(n) = 2H(-8(8n+7))$.

For $[1,1,5]$ we must be slightly more careful since 5 divides the discriminant. We will explain in some detail how to deal with this complication and then will henceforth ignore this difficulty when it arises. For $5 \nmid 8n+7$ we have $s_{[1,1,5]}(n) = 4H(-5(8n+7))$. Hence the only difficulty occurs with high divisibility by 5. For $p \neq 5$ the local densities are equal to those for bounded divisibility. Thus, entirely analogously to the result of Jones we have $s_{[1,1,5]}(n) = c_n H(-5(8n+7))$ for some constant $c_n > 0$ which only depends 5-adically on $8n+7$. We calculate the cases $v_5(8n+7) \leq 3$ by hand. Denote 5-primitive representations of m (i.e., $5 \nmid \gcd(x,y,z)$) by $r^*(m)$. Checking locally, for $5^2 \mid m := 8n+7$, we will obtain the result inductively by showing $\frac{r^*(25m)}{r^*(m)} = \frac{h(25m)/u(25m)}{h(m)/u(m)}$ and then summing to get $r(m) \geq 4H(-5m)$. But, since $5 \mid m$, we have $\frac{h(25m)/u(25m)}{h(m)/u(m)} = 5$ by the class number formula (see [5, Corollary 7.28, page 148]) so that this is a quick local check at the prime 5.

Our proofs for $[1,1,2]$, $[1,2,3]$, and $[1,1,3]$ will be essentially identical. For $[1,1,2]$, we note that if $x^2+y^2+2z^2 = 8n+4$ has a solution with x, y, and z not all odd, then taking each side modulo 8 leads us to the conclusion that x, y, and z must all be even. Therefore, the solutions without x, y, and z odd correspond to solutions of

$$4x^2+4y^2+8z^2 = 8n+4, \quad \text{that is, of} \quad x^2+y^2+2z^2 = 2n+1.$$

Using Siegel's theorem to compare the local density at 2, we see that the average of the number of representations over the genus is three times as large for $8n+4$ as $2n+1$. However, $(1,1,2)$ is again a genus 1 quadratic form, so $r(8n+4) = 3r(2n+1)$, and hence $s_{[1,1,2]}(n) = r_o(8n+4) = r(8n+4) - r(2n+1) = 2r(2n+1)$. Thus by Theorem 86 of Jones [9] we have $s_{[1,1,2]}(n) = 8H(-8(2n+1))$. Similar arguments show that

$$\begin{aligned} s_{[1,2,3]}(n) = r_{o,(1,2,3)}(8n+6) &= r_{(1,2,3)}(8n+6) - r_{(4,2,12)}(8n+6) \\ &= r_{(1,2,3)}(8n+6) - r_{(1,2,6)}(4n+3) = 2r_{(1,2,6)}(4n+3). \end{aligned}$$

Similar to the case $[1,1,5]$, we have $s_{[1,2,3]}(n) \geq 2H(-12(4n+3))$.

For $[1,1,3]$ we see analogously that

$$s_{[1,1,3]}(n) = r_{o,(1,1,3)}(8n+5) = r_{(1,1,3)}(8n+5) - r_{(1,1,12)}(8n+5) = r_{(1,1,12)}(8n+5),$$

and again $(1,1,12)$ is genus 1. We conclude in the case $3 \nmid (8n+5)$ that we have $s_{[1,1,3]}(n) = 4H(-3(8n+5))$, and we may henceforth assume that $3 \mid 8n+5$ (i.e. $n \equiv 2 \pmod 3$). Local conditions imply that $3^{2j+1}(3\ell+2)$ is not represented by $(1,1,3)$, so we have escalated to $[1,1,3,k]$ for k such that $3 \leq k \leq 8$. For $3 \nmid k$, by choosing $x_4 = 1$ we have $s_{[1,1,3,k]}(n) \geq 4H(-3(8(n-k)+5))$ since $3 \nmid 8(n-k)+5$. For $k = 3$ we have

$$s_{[1,1,3,3]}(n) = r_{(1,1,3,3)}(8(n+1)) + r_{(4,4,12,12)}(8(n+1)) - 2r_{(1,3,3,4)}(8(n+1)).$$

Denoting the usual d-th degeneracy V-operator by $V(d)$ and the usual U-operator by $U(d)$ (cf. p. 28 of [13]), one may write the difference of the θ-series $\sum_n r(8n)q^n$

for these quadratic forms as

$$\theta_{(1,1,3,3)}|U(8)+\theta_{(1,1,3,3)}|V(4)|U(8)-2\theta_{(1,3,3,4)}|U(8).$$

It is easy to conclude that the generating function $qF(z)=\sum_n s_{[1,1,3,3]}(n)q^{n+1}$, with $q=e^{2\pi iz}$, is a weight 2 modular form of level 48. Using Sturm's bound [15] and checking the first 16 coefficients reveals that $qF(z)=16\frac{\eta(2z)^4\eta(6z)^4}{\eta(z)^2\eta(3z)^2}$. The coefficients are multiplicative, so that if we have the factorization $n+1=2^e3^f\prod_{p>3}p^{e_p}$, then

$$s_{[1,1,3,3]}(n)=2^{e+4}\prod_{p>3}\frac{p^{e_p+1}-1}{p-1}\geq 16\frac{n+1}{3^f}=16(n+1)^{1-a_n}.$$

Finally, for $k=6$ we check $n<10$ by hand and then note that

$$s_{[1,3,6]}(n)=r_{(1,3,6)}(8n+10)-r_{(2,3,6)}(4n+5),$$

while both $(1,3,6)$ and $(2,3,6)$ are genus 1. Hence for $n\not\equiv 2 \pmod 3$ we have $s_{[1,3,6]}(n)\geq 2H(-4(4n+5))$. We then take the remaining variable $x_4=1$ to obtain for $n\equiv 2 \pmod 3$ that $s_{[1,1,3,6]}(n)\geq 2H(-4(4(n-1)+5))$, since $n-1\not\equiv 2 \pmod 3$.

Having seen that each of our choices of b is indeed a leaf to the tree, we conclude that representing the integers $1,2,4,5$, and 8 suffices. □

Remark 2.2. The constant c_ϵ in Theorem 2.1 is ineffective because it relies on Siegel's lower bound for the class number, but the bound of $c_\epsilon n^{\frac{1}{2}-\epsilon}$ may be replaced with the minimum of finitely many choices of a constant times a Hurwitz class number of a certain imaginary quadratic order whose discriminant is linear in n.

We have the following example. In this example, instead of considering $s_b(n)$, we normalize the number of representations by

$$s_b'(n):=\frac{s_b(n)}{2^k},$$

where k is the length of the sequence b. This normalization is made so that $T_{x_i}=T_{-x_i-1}$ appears exactly once and in particular implies that 0 is represented precisely once. Using this normalization and the explicit bound in terms of the Hurwitz class number, we obtain for instance that if $1,2,4,5$, and 8 are represented, then the integer 195727301431 is represented at least 270390 times and the integer 48291403767737750 is necessarily represented at least 90542761 times (here $a_n\approx 0.364$), while the integer $50031545098999706=3^{35}-1$ is only necessarily represented once. All of the bounds listed in these examples are sharp (i.e., there exists a triangular sum representing $1,2,4,5$, and 8 which represents 195727301431 precisely 270390 times).

3. Cross terms

Every quadratic polynomial f in k variables (over $\mathbb{Q}$) can be written uniquely as $f(x)=Q(x)+\Lambda(x)+C$, where $Q(x)$ is a quadratic form in k variables, $\Lambda(x)$ is a linear form, and C is a constant. We will only consider quadratic polynomials such that $f(x)\in\mathbb{Z}$ for every $x\in\mathbb{Z}^k$. The quadratic form $Q(x)$ is positive definite if and only if $f(x)$ is bounded from below. As in the introduction, $f(x_1,x_2,\ldots,x_k)$ is a normalized totally positive quadratic polynomial if f is quadratic, and the image of $\mathbb{Z}^k$ is contained in the nonnegative integers while it contains 0. Clearly, for every positive definite quadratic form $Q(x)$ and linear form $\Lambda(x)$ there is a unique $C\in\mathbb{Z}$ such that $f(x)=Q(x)+\Lambda(x)+C$ is normalized totally positive.

As noted before, $8T_x = (2x+1)^2 - 1 = X^2 - 1$ if we put $X = 2x+1$. The polynomial $X^2 - 1$ is normalized totally positive on the odd integers. With $Y = 2y+1$, we find $8B_{xy} = 4xy + 2x + 2y = XY - 1$, where $B_{xy} := \frac{1}{4}(2xy + x + y)$ is the polynomial in x, y satisfying $B_{xx} = T_x$. This way

$$8(aT_x + bT_y + cB_{xy}) = aX^2 + bY^2 + cXY - (a+b+c).$$

If C is the unique integer such that $aT_x + bT_y + cB_{xy} + C$ is normalized totally positive, then $aX^2 + bY^2 + cXY + (8C - a - b - c)$ will be the corresponding shifted quadratic form that is normalized totally positive on the odd integers.

In order to describe our construction, we will say for simplicity that two quadratic polynomials f_1 and f_2 are *(arithmetically) equivalent* if the number of solutions to $f_1(x) = n$ equals the number of solutions to $f_2(x) = n$ for every integer $n \geq 0$.

We will consider a positive definite integral quadratic form (in k variables) for which all cross terms in the matrix have *even* coefficients, so the cross terms of the quadratic form are 0 mod 4. This restriction is natural if one keeps in mind that we are interested in the integers *oddly* represented by forms.

If Q and $\widetilde{Q}$ are two equivalent quadratic forms such that the isomorphism preserves the condition that X_i is odd, then we shall refer to them as *equivalently odd* and denote the equivalence class of such forms as $[Q]_o$.

For any positive definite quadratic form with cross terms divisible by four, we write

$$Q = a_1 X_1^2 + \cdots + a_k X_k^2 + \sum_{i \neq j} 4c_{ij} X_i X_j.$$

We now define $f_Q = f_{[Q]_o}$ to be the unique normalized totally positive quadratic polynomial

$$f_Q := a_1 T_{x_1} + \cdots + a_k T_{x_k} + \sum_{i \neq j} 4c_{ij} B_{x_i x_j} + C.$$

We will refer to f_Q as a *triangular sum with cross terms*.

We will show that triangular sums with cross terms do not satisfy any finiteness theorem, and hence there is no overarching finiteness theorem for quadratic polynomials, as stated in Proposition 1.3. To do so, for every positive integer n we will construct a triangular sum with cross terms f_n which represents precisely every nonnegative integer other than n.

The following notation will be used. If f and g are polynomials in k and ℓ variables, we denote by $f \oplus g$ the sum of the two as a polynomial in $k + \ell$ variables (so f and g are assumed to share no variables).

Theorem 3.1. *Let $\mathcal{Z}$ be a subset of the positive integers. For every proper subset $\mathcal{Y} \subsetneq \mathcal{Z}$ there exists a triangular sum with cross terms representing $\mathcal{Y}$ but not representing $\mathcal{Z}$.*

Proof. Let a proper subset S_0 of a given subset S of the positive integers be given. Choose a positive integer $n \in S \backslash S_0$. We will proceed by explicit construction of the triangular sum with cross terms f_n which represents every integer other than n.

First note that if the smallest positive integer *not* represented by f is n, then, since the sum of three triangular numbers represents every nonnegative integer, we have that $f \oplus (n+1)(T_x \oplus T_y \oplus T_z)$ represents all $m \not\equiv n \pmod{n+1}$. But then we can choose $f_n := f \oplus (n+1)(T_x \oplus T_y \oplus T_z) \oplus (2n+1)T_w$. It is therefore equivalent to construct f for which n is the smallest positive integer not represented by f.

Consider the quadratic form

$$Q^{(N)}(X, Y) := NX^2 + NY^2 + 4XY$$

and denote the corresponding triangular sum with cross terms by $f^{(N)}$; then

$$f^{(N)}(x, y) = NT_x + NT_y + (2xy + x + y) + 1.$$

We first show that it is sufficient to determine that the generating function for $f^{(N)}$ is

$$2 + 2q + O(q^{N-12}). \tag{3.1}$$

Assuming equation (3.1), the generating function for $g_n := \bigoplus_{i=1}^n f^{(N)}$ is

$$2^n \left(1 + \binom{n}{1} q + \cdots + \binom{n}{n} q^n\right) + O(q^{N-12}).$$

If we choose $N > n + 13$, then the first integer not represented by g is $n + 1$. Therefore, we can take $f_n = g_{n-1}$; this also suffices for $n = 1$ (if we interpret the empty direct sum g_0 as 0).

We now show that the generating function satisfies (3.1). Note that $f^{(N)}(0, -1) = f^{(N)}(-1, 0) = 0$, while $f^{(N)}(0, 0) = f^{(N)}(-1, -1) = 1$. Now, without loss of generality, assume that $|x| \geq |y|$ and $x \notin \{0, -1\}$. Then,

$$|2xy + x + y| \leq 2|x|^2 + 2|x| = 4T_{|x|},$$

so that

$$f^{(N)}(x, y) \geq NT_x - 4T_{|x|} + NT_y.$$

When $x \leq -2$ it is easy to check that $4T_{|x|} \leq 12T_x$ so that

$$NT_x - 4T_{|x|} \geq (N - 12)T_x \geq N - 12$$

and when $x > 0$,

$$NT_x - 4T_{|x|} = (N - 4)T_x \geq N - 4,$$

since $T_x \geq 1$ for $x \notin \{0, -1\}$. Since $T_y \geq 0$, our assertion is verified. □

It is important here to note how the above counterexamples differ from the proof when we only have diagonal terms, since this observation will lead us to the proof of Theorem 1.4 when m_f is bounded.

Call a triangular sum with cross terms f_Q (and also any corresponding $\widetilde{f_Q}$) a *block* if the corresponding quadratic form Q has an irreducible matrix. We will build an escalator lattice by escalating (as a direct sum) by a block at each step. In Section 2, the breadth each time we escalated was finite, so that the overall tree was finite. In the above proof, however, there were infinitely many inequivalent blocks which represent 1, so that the breadth is infinite. What was expressed in the above proof was that the supremum of these depths went to infinity as we chose N increasing in terms of n in the proof.

For

$$f(x) = \sum_{i=1}^{k} b_i T_{x_i} + \sum_{1 \leq i < j \leq k} c_{ij}(2x_i x_j + x_i + x_j) + C$$

we will say that f has *(cross term) configuration* $c = (c_{ij})$. Since the matrix of f is irreducible and hence the corresponding adjacency matrix is connected, we can assume throughout (by a change of variables) that for each $j > 1$ there exists $i < j$ with $c_{ij} \neq 0$.

4. Bounded norm

We will now construct a natural norm on f_Q such that restricting this norm will again give a finiteness result. Let a positive definite quadratic form with even cross terms in the corresponding matrix,

$$Q(x) := \sum_{i=1}^{k} b_i x_i^2 + \sum_{i<j} 4c_{ij}x_ix_j, \tag{4.1}$$

be given. We define

$$\widetilde{f}(x) := \widetilde{f_Q}(x) := \sum_{i=1}^{k} b_i T_{x_i} + \sum_{i<j, c_{ij}\geq 0} c_{ij}(2x_ix_j + x_i + x_j) + \sum_{i<j, c_{ij}<0} c_{ij}(2x_ix_j + x_i + x_j + 1).$$

Remark 4.1. Note that the constant c_{ij} is added every time $c_{ij} < 0$; this may not seem canonical at first, but notice that if Q' is the equivalent quadratic form obtained by replacing x_1 with $-x_1$, then we find that this choice leads to $\widetilde{f_Q} = \widetilde{f_{Q'}}$.

We next define

$$\widetilde{m}_{\widetilde{f}} := -\min_{x\in\mathbb{Z}^k} \widetilde{f}(x),$$

which is added to obtain the unique (up to equivalence) normalized totally positive quadratic polynomial $f_Q = \widetilde{f_Q} + \widetilde{m}_{\widetilde{f}}$ corresponding to Q . Thus, we can define the norm

$$m_{f_Q} := m_{[Q]_o} := \min_{Q'\in[Q]_o} |\widetilde{m}_{\widetilde{f_{Q'}}}|.$$

In a sense, this norm measures the distance between f_Q and the closest $\widetilde{f_{Q'}}$ in the equivalence class, where the distance is merely given by the absolute value of the normalization factor required. If m_f is bounded, then we will again find that checking a finite subset will suffice. We may now state the following more precise version of Theorem 1.4.

Theorem 4.2. *Fix an integer m and a subset $\mathcal{Z}$ of the positive integers. Then there is a finite subset $\mathcal{Y}_m \subset \mathcal{Z}$ depending only on m and $\mathcal{Z}$ such that every triangular sum with cross terms f satisfying $m_f \leq m$ represents $\mathcal{Z}$ if and only if it represents $\mathcal{Y}_m$.*

Moreover, for $\mathcal{Z}$ equal to the positive integers, we find $\max \mathcal{Y}_m \gg m^2$.

Remark 4.3. It may be of interest to investigate the growth of $\max \mathcal{Y}_m$ in terms of m in the case where $\mathcal{Z}$ consists of all positive integers. The $m = 0$ case is precisely Theorem 1.1. Following the bounds given in the proof of Theorem 1.4, computational evidence suggests that $\mathcal{Y}_1(\mathbb{Z}_{>0})$ equals

$$\{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 16, 17, 19, 20, 23, 24, 25, 26, 29, 32, 33, 34, 35, 38, 41, 46, 47, 48, 50, 53, 54, 58, 62, 63, 75, 86, 96, 101, 102, 113, 117, 129, 162, 195, 204, 233\}.$$

A proof of the above identity using the techniques of Bhargava and Hanke [2] developed in the proof of the 290-Theorem may require a careful analysis of a possible Siegel zero. To exhibit this difficulty, consider the sum $g(x, y, z) = T_x + 2T_y + 6T_z$. In the construction of $\mathcal{Y}_1(\mathbb{Z}_{>0})$ the computations imply that there are infinitely many Q with $m_{f_Q} = 1$ for which $g \oplus f_Q$ represents every positive integer.

Hence we cannot merely check each case individually and must know information about the integers represented by g independently.

Although it seems that g represents all odd integers, a proof of this appears to be beyond current techniques due to ineffective lower bounds for the class number (see [10]). However, since a possible Siegel zero for $L(\chi_d, s)$ would give a lower bound for the class number when $d' \neq d$ (both fundamental), one may be able to show that g represents at least one of n or $n-1$ for every positive integer n, which would suffice for showing the above identity.

We will first give an overview of the proof; details can be found in the next section.

Fix a positive integer m. As in the above remark, we will escalate with blocks. We will first show that when $m_f \leq m$, the number of blocks that are not dimension 1 in any branch of the escalator tree is bounded and that there are only finitely many choices for the configuration of each block. We will then proceed by defining $N(M_1, M_2, \ldots, M_k, c)$ to be the smallest integer not represented by the totally positive quadratic polynomial corresponding to

$$\widetilde{f}(x) := \sum_{i=1}^{k} M_i T_{x_i} + \sum_{i<j, c_{ij}\geq 0} c_{ij}(2x_ix_j + x_i + x_j) + \sum_{i<j, c_{ij}<0} c_{ij}(2x_ix_j + x_i + x_j + 1).$$

Our claim is then equivalent to showing that in the escalator tree,

$$\sup_{M_1,\ldots,M_k,c} N(M_1, M_2, \ldots, M_k, c)$$

is finite. To do so, we will effectively show that with the configurations of blocks of dimension greater than one fixed, the supremum with M_i sufficiently large is finite and independent of the choice of M_i, and then fix $M_1 \leq m_1$, and again show that the resulting supremum is independent of $M_2, \ldots, M_k$, and so forth. Since there are only finitely many such choices of c, the result comes from taking the maximum of each of these suprema.

5. Proof

To prove Theorem 4.2, and hence Theorem 1.4, we begin with a lemma that will show that there are only finitely many choices of the cross term configuration.

Lemma 5.1. *If $m_f \leq m$, then there are only finitely many choices of the cross term configurations c_{ij} of all blocks of dimension greater than one, up to equivalent forms.*

Proof. First note that $m_{f\oplus g} = m_f + m_g$, so that we can only have at most m blocks f with $m_f > 0$, while we will see that $m_f > 0$ unless f is one dimensional (and hence the block is a constant times T_x). It therefore suffices to show that each block f of dimension greater than one has $m_f > 0$ and those with the restriction $m_f \leq m$ have bounded dimension and bounded coefficients in the configuration. Fix the configuration c of a block $\widetilde{f}$ with dimension k such that $\widetilde{m}_{\widetilde{f}} = m_f$, namely a minimal element. We will recursively show a particular choice of x_i such that

$$\widetilde{f}(x) \leq -\max\{\max_{i,j} |c_{ij}|, k-1\},$$

so that the max of the c_{ij} is bounded by m, and the dimension is bounded by $m+1$.

First set $x_1 = 0$. Since $\widetilde{f}$ is a block, we know at step j that there is some $i < j$ such that $c_{ij} \neq 0$. Choose $i < j$ such that $|c_{ij}|$ is maximal. If $x_i = 0$, then we set $x_j = -1$ if $c_{ij} > 0$ and $x_j = 0$ otherwise. If $x_i = -1$, then we set $x_j = 0$ if $c_{ij} > 0$ and $x_j = -1$ otherwise.

Since all of our choices of x_i are 0 or -1 and $T_{-1} = T_0 = 0$, the integer represented is independent of the diagonal terms M_i. Now we note that for $x_i, x_j \in \{0, -1\}$ we have $2x_ix_j + x_i + x_j = 0$ if $x_i = x_j$ and $2x_ix_j + x_i + x_j = -1$ otherwise. Therefore, if $x_i = x_j$, then from our definition of $\widetilde{f}$, the cross term corresponding to c_{ij} adds 0 if $c_{ij} \geq 0$ and adds $-|c_{ij}|$ otherwise. If $x_i = 0$ and $x_j = -1$, then the cross term adds $-|c_{ij}|$ if $c_{ij} \geq 0$ and adds 0 otherwise. Therefore by our construction above, we know that for $|c_{ij}|$ maximal, we have added $-|c_{ij}|$ to our sum, and we never add a positive integer, so the sum is at most $-|c_{ij}|$. Moreover, since the block is connected, we have added at most -1 at each inductive step, so that the sum is at most $-(k-1)$. □

For simplicity, in our escalator tree, we will "push" up all of the blocks to the top of the tree which are not dimension 1. To do so, we will first build the tree with all possible choices of blocks which are not dimension 1 and then escalate with only dimension 1 blocks from each of the nodes of the tree, including the root (the empty set). Thus, every possible form will show up in our representation. This tree (without the blocks of dimension 1) is depth at most m in the number of blocks, but is of infinite breadth. Henceforth, we can consider the configuration c to be fixed, and take the maximum over all choices of c.

We will now see that the subtree from each fixed node is of finite depth. Consider the corresponding quadratic form Q. First note that the generating function for Q when all x_i are odd is the generating function for Q minus the generating function with some x_i even, and the others arbitrary, which is simply the generating function for another quadratic form without any restrictions, taking $x_i \to 2x_i$. Thus, we have the difference of θ-series for finitely many quadratic forms, and hence the Fourier expansion is a modular form. Now we simply note that any modular form can be decomposed into an Eisenstein series and a cusp form (cf. [13]). Using the bounds of Tartakowsky [16] and Deligne [6], as long as the Eisenstein series is nonzero, the growth of the coefficients of the Eisenstein series can be shown to grow more quickly than the coefficients of the cusp form whenever the dimension is greater than or equal to 5, other than finitely many congruence classes for which the coefficients of both the Eisenstein series and the cusp form are zero.

Therefore, as long as the Eisenstein series is nonzero, there are only finitely many congruence classes and finitely many "sporadic" integers which are not represented by the quadratic form. Thus, after dimension 5, there are only finitely many congruence classes and finitely many sporadic integers not represented by the form f. If at any step of the escalation, any of the integers in these congruence classes is represented, then we have fewer congruence classes, and only finitely many more sporadic integers which are not represented, so that the resulting depth is bounded. For the dimension 1 blocks, it is clear that the breadth of each escalation is finite, so there are only finitely many escalators coming from this node. Therefore, it suffices to show that the Eisenstein series is nonzero.

Again using Siegel's theorem [14], the coefficients of the Eisenstein series are simply a linear combination of the values given by the local densities of the quadratic forms from the above linear combination of θ-series. At every prime other than $p = 2$, the local densities of the quadratic forms, of which we are taking the difference of θ-series, are equal, so we only need to show that the difference of the local densities at $p = 2$ is positive. However, the difference of the number of local representations at a fixed 2-power must be positive, since the integer is locally represented with x_i odd, except possibly for finitely many congruence classes if a high 2-power divides the discriminant.

Therefore, we can define $\widetilde{N}(M_1, \ldots, M_k, c)$ to be the maximum of $N(M_1, \ldots, M_k, M_{k+1}, \ldots, M_l, c)$, where M_{k+1} to M_l are the dimension 1 blocks coming from the (finite) subtree of this node. We will show that $\widetilde{N}(M_1, \ldots, M_k, c)$ is independent of the choice of M_i whenever M_i is sufficiently large by showing that the resulting subtrees are identical. We need the following lemma to obtain this goal. We will need some notation before we proceed.

For a set T, define $q^T := \sum\limits_{t \in T} q^t$, a formal power series in q. For fixed sets $S, T \subseteq \mathbb{N}$, we will say that a form $f(x) := \sum b_i T_{x_i}$ *represents* S/T if for every $s \in S$ the coefficient of q^s in $q^T g(q)$ is positive, where $g(q)$ is the generating function for $f(x)$ given by $g(q) := \sum_{x \in \mathbb{Z}^k} q^{f(x)}$.

Lemma 5.2. *Let a (diagonal) triangular form f be given. Fix $S, T_1, T_2 \subseteq \mathbb{N}$ and $M \in \mathbb{N}$ such that $\min_{n \in T_2} \geq M$. Define $T := T_1 \cup T_2$. Then there exists a bound $M_{T_1,S}$ and a finite subset $S_0 \subseteq S$, depending only on T_1 and S such that if $M > M_{T_1,S}$, then f represents S/T if and only if f represents S_0/T_1.*

Proof. We will escalate as in [1] with a slight deviation. At each escalation node, there is a least element $s \in S$ such that S/T_1 is not represented by the form f corresponding to this node. As in [1], we shall refer to s as the *truant* of f. To represent $\{s\}/T_1$, we must have some $t_1 \in T_1$ such that $s - t_1$ is represented by $f + bT_x$. Therefore, for each $t_1 < s$ we escalate with finitely many choices of b, and there are only finitely many choices of t_1. Thus, the breadth at each escalation is finite, and our argument above using modular forms shows that the depth is also finite, so there are only finitely many choices of $s \in S$ which are truants in the escalation tree. Take S_0 to be the set of truants in the escalation tree and define $M_{T_1,S} := \max s \in S_0 s + 1$. The argument above shows that representing S/T_1 is equivalent to representing S_0/T_1. When following the above process with T instead of T_1 whenever $M > M_{T_1,S}$, we will have the same subtree and the same truants at each step, so that representing S/T is equivalent to representing S/T_1, and hence representing S/T is equivalent to representing S_0/T_1. □

Remark 5.3. It is of interest to note that if we replace "(diagonal) triangular form" with "quadratic form" (without the odd condition), the proof follows verbatim, since the breadth is also finite, so that this can be considered a generalization of Bhargava's result that there is always a finite subset S_0 of S such that the quadratic form represents S if and only if it represents S_0, since this is obtained by taking $T_1 = T = \{0\}$.

Now consider $X_j := \{x : x_i \text{ arbitrary for } i \leq j,\ x_i \in \{0, -1\} \text{ otherwise}\}$ and define $T_{1,j} := \{f(x) : x \in X_j\}$ and $T_{2,j} := \{f(x) : x \notin X_j\}$. We will use Lemma 5.2 with $T_1 = T_{1,j}$ and $T_2 = T_{2,j}$ for each $0 \leq j \leq k$. To use the lemma effectively, we will show the following lemma.

Lemma 5.4. *There exist bounds $M_{X_j}^{(i)}$ depending only on $M_1, \ldots, M_j, c$ such that if $M_i \geq M_{X_j}^{(i)}$ for every $i > j$, then the smallest element of $T_{2,j}$ is greater than $M_{T_1,\mathbb{N}}$, where $M_{T_1,\mathbb{N}}$ is as defined in Lemma 5.2.*

Proof. We will proceed by induction. For $j = 0$, we will take $M_{X_0}^{(i)} = M_{T_{1,0},\mathbb{N}} + 6\sum_j |c_{ij}|$. Noting that for $|x_j| < |x_i|$ we have $\left|2\left(x_i - \frac{x_j}{2}\right)x_j\right| \leq x_i^2$, we get the inequality

$$c_{ij}(2x_ix_j + x_i + x_j) \geq -|c_{ij}|(2T_{|x_i|} + 2T_{|x_j|}).$$

The case $j = 0$ then follows from the fact that for $x_i \notin \{0, -1\}$ we have $T_{|x_i|} \leq 3T_{x_i}$.

We now continue by induction on j. For the corresponding quadratic form, we note that plugging in $x_1 = \frac{-\sum_{j>1} c_{1j}x_j}{2M_1}$ gives the minimal value over the reals. The quadratic form Q' obtained by specializing this value of x_1 has rational coefficients with denominator dividing $2M_1$. We therefore can consider $\widetilde{Q} := 4M_1 \cdot Q'$, which is a quadratic form of the desired type. Thus, we can use the inductive step for $\widetilde{Q}$. But this gives a bound which minimizes $\widetilde{Q}$, and hence Q', but an arbitrary choice of x_1 must give a value greater than or equal to this, so the result follows. □

Now, by our choice of X_j, $T_{1,j}$ is independent of M_i for $i > j$, since $T_{x_i} = 0$. Thus, fix c and take $M_i \geq M_{X_0}^{(i)}$. Then the corresponding subtrees are independent of the choice of M_i, so that $\sup \widetilde{N}(M_1, \ldots, M_k, c)$ is the unique largest truant in the subtree (effectively we may replace $M_i = \infty$). We may now fix $M_1 \leq M_{X_0}^{(1)}$, since there are only finitely many such choices. With this M_1 fixed, we define $T_{1,1}$ as above, and again find bounds for the other M_i. Continuing recursively gives the desired result, since we know that $k \leq m$, so there are only finitely many suprema that we take.

To show that $\max \mathcal{Y}_m(\mathbb{Z}_{>0}) \gg m^2$, we consider again the construction of our counterexamples. Consider $f(x,y) := \bigoplus_{i=1}^m f^{(N)} \oplus T_y$. Since $T_r = \sum_{n=1}^r n$, for N sufficiently large the smallest integer not represented by f is clearly $T_{m+1} - 1 \gg m^2$.

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Representing Sets with Sums of Triangular Numbers

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We investigate here sums of triangular numbers $f(x) := \sum_i b_i T_{x_i}$ where T_n is the nth triangular number. We show that for a set of positive integers S, there is a finite subset S_0 such that f represents S if and only if f represents S_0. However, computationally determining S_0 is ineffective for many choices of S. We give an explicit and efficient algorithm to determine the set S_0 under certain generalized Riemann hypotheses, and implement the algorithm to determine S_0 when S is the set of all odd integers.

1 Introduction

In 1638 Fermat wrote that every number is a sum of at most three triangular numbers, four square numbers, and in general n polygonal numbers of order n. Here the triangular numbers are $T_x := \frac{x(x+1)}{2}$, where we include $x = 0$ for simplicity. The claim for four squares was shown by Lagrange in 1772, while Gauss famously wrote "Eureka, $\triangle + \triangle + \triangle = n$" in his mathematical diary on July 10, 1796.

Theorem (Gauss, 1796). Every positive integer is the sum of three triangular numbers. □

The first proof of the full assertion of Fermat was given by Cauchy in 1813.

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In 1917, Ramanujan extended the question about four squares to consider which choices of $b = (b_1, b_2, b_3, b_4)$ satisfy $b_1x_1^2 + b_2x_2^2 + b_3x_3^2 + b_4x_4^2$, representing every positive integer. We shall refer to such forms as *universal diagonal forms*. He gives a list of 55 possible choices of b which he then claims are the complete list of universal quarternary diagonal forms (54 forms actually turned out to be universal).

In 1862 Liouville similarly proved the following generalization of Gauss' theorem.

Theorem. Let a, b, c be positive integers with $a \le b \le c$. Then every $n \in \mathbb{N}$ can be written as $aT_x + bT_y + cT_z$ if and only if (a, b, c) is one of the following:

$$(1,1,1),\ (1,1,2),\ (1,1,4),\ (1,1,5),\ (1,2,2),\ (1,2,3),\ (1,2,4). \qquad \square$$

In fact, the following simple condition determines whether a fixed set of positive integers $(b_1, \ldots, b_k)$ gives rise to a sum of triangular numbers $\sum_{i=1}^{k} b_k T_{x_k}$ which represents every integer as shown by Bosma and the author in [4].

Theorem 1.1. Fix the sequence $b_1 \le \cdots \le b_k \in \mathbb{N}$. Then

1. the sum of triangular numbers
$$f(x) := f_b(x) := \sum_{i=1}^{k} b_i T_{x_i}$$
represents every positive integer if and only if f_b represents the integers 1, 2, 4, 5, and 8.
2. The corresponding diagonal quadratic form $Q(x) = \sum_{i=1}^{k} b_i x_i^2$ with x_i all odd represents every integer of the form $8n + \sum_{i=1}^{k} b_i$ with $n \ge 0$ if and only if it represents $8 + \sum_{i=1}^{k} b_i$, $16 + \sum_{i=1}^{k} b_i$, $32 + \sum_{i=1}^{k} b_i$, $40 + \sum_{i=1}^{k} b_i$, and $64 + \sum_{i=1}^{k} b_i$. $\square$

Recently, Conway and Schneeberger proved a very nice similar condition for positive definite quadratic forms whose corresponding matrix has integer entries, but without publishing their results.

Theorem (Conway–Schneeberger). A positive definite quadratic form $Q(x) = x^t Ax$, where A is a positive symmetric matrix with integer coefficients, represents every positive integer if and only if it represents the integers $1, 2, 3, 5, 6, 7, 10, 14$, and 15. $\square$

Bhargava gave an elegant simpler proof of the Conway–Schneeberger 15 theorem in [2], in addition to showing more generally that for any set $S \subseteq \mathbb{N}$, it is always sufficient to check whether Q represents a finite subset S_0, and showed the set S_0 for the two sets $S = \{2n+1 : n \in \mathbb{Z}^+\}$ and $S = \{p \text{ prime}\}$.

In this paper, we will consider a similar generalization of Theorem 1.1.

Theorem 1.2. Let a set $S \subseteq \mathbb{N}$ be given. Then there is a finite subset S_0 of S such that $f(x)$ represents S if and only if f represents S_0. □

A simple computer calculation leads us to conjecture a set S_0 when S is the set of all odd integers, for example.

Conjecture 1.3. A sum of triangular numbers f represents all odd integers if and only if it represents the integers

$$1, 5, 7, 9, 11, 13, 17, 19, 25, 29, 35, 49, 89.$$ □

Unlike in Bhargava's theorem, however, current techniques are insufficient for computationally determining a suitable S_0 for most choices of S, due to ineffective bounds for the class numbers of imaginary quadratic fields. We shall briefly explain this complication. Let $f(x) = b_1 T_{x_1} + b_2 T_{x_2} + b_3 T_{x_3}$ be given such that f represents all of the integers in S, but the corresponding (diagonal) quadratic form is not (spinor) genus 1. Then the corresponding weight 3/2 modular form with x_i all odd can be written as an Eisenstein series plus a cusp form. Siegel has shown that the Fourier coefficients (with bounded divisibility at the anisotropic primes) of the Eisenstein series grow like the class number [21]. Siegel has also shown that the class number grows faster than $n^{\frac{1}{2}-\epsilon}$ [20], but the bound was ineffective because Siegel showed the result by first assuming a certain Riemann hypothesis and showing the result, and then assuming that the Riemann hypothesis was false, and getting a different constant, depending on the location of possible zeros. The best known effective bound is given by Oesterle [14], but is only $O(\log n)$. After decomposing the cusp form into $g_1 + g_2$, where g_1 is in the space of lifts of one-dimensional theta series and the Shimura lift of g_2 is cuspidal, we note that the coefficients of g_2 grow slower than $n^{\frac{1}{2}-\frac{1}{28}+\epsilon}$ by the work of Duke [7], and g_1 is supported at finitely many square classes with the same growth as the Eisenstein series. Since the coefficients with bounded divisibility at the anisotropic primes of the Eisenstein series grow faster than the coefficients of g_2, every sufficiently large n with bounded divisibility by the anisotropic primes and outside of the support of the coefficients of g_1 must be represented. However, we do not know the implied constant from Siegel's ineffective bound, so we cannot effectively determine when n is sufficiently large.

Assuming the generalized Riemann hypothesis (GRH) for Dirichlet L-functions and using Duke's (effective) bound of $O(n^{\frac{13}{28}+\epsilon})$ [7], we would have an algorithm to determine whether f represents S or not. However, although Duke's result is effective, the author is unaware of any paper where the constant is explicitly computed. Even assuming that the implied constant was 1, the bound obtained is entirely infeasible with current computer technology. Using an idea of Ono and Soundararajan [16], and a generalization by the author [13], we will be able to determine an algorithm to determine the set S_0 under the additional generalized Riemann hypothesis for L-functions of weight 2 newforms. For notational ease, we will refer to an integer which is locally represented at every prime as an eligible integer.

Theorem (Ono–Soundararajan [16]). Assume GRH for Dirichlet L-functions and GRH for L-functions of weight 2 newforms. Then the eligible integers not represented by Ramanujan's ternary quadratic form $x^2 + y^2 + 10z^2$ are precisely

$$3, 7, 21, 31, 33, 43, 67, 79, 87, 133, 217, 219, 223, 253, 307, 391, 679, 2719. \qquad \square$$

Using the ideas in [16], [13], and [12], we obtain the following result.

Theorem 1.4. Assume GRH for Dirichlet L-functions and GRH for L-functions of weight 2 newforms. Let a set of positive integers S of nonzero density be given. Then there exists an explicit algorithm to determine the (unique) smallest (finite) set S_0 such that a sum of triangular numbers f represents S if and only if f represents S_0. □

Remark 1.5. It is important to note that this algorithm is computationally feasible for sets S which contain many small integers. Since many interesting sets (such as odd integers and primes) are not very sparse in the small integers, the algorithm will be practical in many cases. □

We also implement the algorithm to prove Conjecture 1.3 under these GRH assumptions.

Theorem 1.6. Assume GRH for Dirichlet L-functions and GRH for L-functions of weight 2 newforms. Then a sum of triangular numbers f represents all odd integers if and only if it represents the integers

$$1, 5, 7, 9, 11, 13, 17, 19, 25, 29, 35, 49, 89.$$

Moreover, this is the smallest such set with this property. □

Remark 1.7. We only need to assume GRH specifically for weight 2 newforms that occur in the decomposition of any f for which, under the GRH assumption we can show, will generate all integers in S. However, to make our algorithm more efficient, we will make the GRH assumption in general so that we do not need to check many coefficients of a large number of weight 2 modular forms (for more information on modular forms, please see [15]). Thus, we note here that for determining the set S_0 when S is all odd integers, we could have avoided the GRH assumptions except in three cases at the added disadvantage of more computation. □

We conclude with a curious example in which we obtain an unconditional result.

Example 1.8. Consider the set

$$S := \{n \in \mathbb{N} : 2x^2 + 3y^2 + 4z^2 = 8n + 9 \text{ has a solution}\}.$$

Current techniques appear insufficient to determine the set S explicitly, while GRH predicts $S = \widetilde{S} := \mathbb{N} \setminus \{1, 8, 31\}$. Following the algorithm given above, one obtains (unconditionally)

$$S_0 = \{2, 3, 4, 5, 10, 16, 17, 19, 89\},$$

but we know of no algorithm to compute $\widetilde{S}_0$ without GRH. This occurs because the only difficulty coming from ternary quadratic forms in the algorithm to determine S_0 and $\widetilde{S}_0$ occurs for the form $2T_x + 3T_y + 4T_z$, while this form trivially represents S. Further details are left to the reader. □

2 Existence of S_0

We will begin by showing that fixing a subset S of the positive integers and then, indeed, checking a finite subset S_0 will suffice.

Proof of Theorem 1.2. We will follow the basic argument of escalator lattices in Bhargava's argument [2] for quadratic forms. Without loss of generality, we will denote the triangular form $f = f_b$ simply by the sequence of coefficients, $b = [b_1, \ldots, b_k]$

with $b_1 \le b_2 \le \cdots \le b_k$. Note that we must represent the smallest integer $s_\emptyset \in S$, so it follows easily that $b_1 \le s_\emptyset$. For each choice of b_1, we find the smallest integer $s_{[b_1]} \in S$ which is not represented by $[b_1]$, and conclude that $b_2 \le s_{[b_1]}$. We recursively continue to build a tree of possible choices of b_k, depending on the previous choices of $b_1, \ldots, b_{k-1}$. Note that we never need to choose the same integer b_i more than three times in one branch, since this precisely represents every integer congruent to zero modulo b_i by Gauss' theorem. Whenever b represents all odd integers, we will say that b is a leaf of the tree, since any arbitrary choice b', containing b as a subsequence, will automatically represent every integer in S. For $b \in T$ not a leaf, we will denote by s_b the smallest $s \in S$ not represented by b, and we will call s_b the *truant* of b. Thus, taking for our tree T

$$S_0 := \{s_b : b \in T\},$$

it follows easily that a triangular form f represents S if and only if it represents S_0 by our construction of T. Moreover, such a choice of S_0 is smallest possible and unique, since for the form b, s_b is the smallest $s \in S$ not represented by b, noting that

$$[b_1, \ldots, b_k, s_b+1, s_b+1, s_b+1, s_b+2, s_b+2, s_b+2, \ldots, (s_b+1)(s_b+2)-1]$$

represents exactly every integer $s \in S$ other than s_b, using Gauss' theorem that every integer is the sum of three triangular numbers.

It, therefore, remains to show that S_0 is finite. Since at each step there are only finitely many choices for b_k, the breadth at each node of the tree is finite, so it suffices to show that the supremum of the depth is finite. To do so, we will consider all nodes at depth 5. Since the breadth is finite, there are only finitely many such nodes, and only finitely many leaves of depth less than or equal to 5. Therefore, it suffices to fix one such node and show that the depth of the resulting subtree is finite.

Let f be a triangular form of dimension at least 5, and consider Q_{odd} to be the corresponding quadratic form with x_i odd. Then the nth coefficient of f is equal to the $8n + \sum_{i=1}^{k} b_k$th coefficient of Q_{odd}. Therefore, the coefficients $a_f(n)$ are precisely the coefficients of the modular form, corresponding to Q_{odd}. Hence, we may decompose these coefficients into coefficients of an Eisenstein series plus coefficients of a cusp form. Since the dimension of f is at least 5, the coefficients of the Eisenstein series grow faster than the coefficients of the cusp form as long as the Eisenstein series is nonzero. We note that there are finitely many congruence classes where the coefficients of both are zero, namely those integers not locally represented by the quadratic form. Hence, as long as the Eisenstein series is nonzero, there are only finitely many congruence classes and finitely many "sporadic" integers not represented by f, since the coefficients of the

Eisenstein series are always positive. It is simple to show that the Eisenstein series must be nonzero, however, since by Siegel's local density formula [21], this Eisenstein series is given by the difference of the local densities with arbitrary x_i and the local densities with x_i even for some i. However, a quick check show that when $p \neq 2$ the local densities are the same, and, since the integer $8n + \sum_{i=1}^{k} b_k$ is locally represented modulo 8 with x_i all odd (namely $x_i = 1$), the local density for x_i arbitrary must be greater than when x_i is even except at finitely many congruence classes.

Let one of the nodes of our tree, $f \in T$ be given of exactly dimension 5. Then there are only finitely many congruence classes and finitely many "sporadic" integers not represented by f, and hence at each step of our escalation, our next choice of $s_b \in S$ must be one such integer. We may only escalate finitely many times for the "sporadic" integers, and each time we escalate to include an element of one of the congruence classes, the congruence class is replaced with finitely many new "sporadic" integers at the next step. Therefore, since there are only finitely many congruence classes, we can only add finitely many new "sporadic" integers overall, and hence the subtree of f is of finite depth. ■

3 Determining S_0

We will describe the algorithm to determine the set S_0. For complete details of how to compute the bounds obtained given GRH, see [12] and [13].

Proof of Theorem 1.4. From the proof in Section 2, it is clear that S_0 is uniquely determined by the tree T, so this algorithm is equivalent to determining the tree T, since it is a simple check to determine the smallest $s \in S$ which is not represented by a fixed form f. Constructing the tree as in the proof is also quite simple, so that the only remaining obstacle is determining whether a node of the tree is a leaf. Our task is thus equivalent to determining (effectively) which integers are represented by Q_{odd}. If the dimension (and hence the depth in the tree) of f is at least 5, then, using the trivial bounds for the coefficients of the Eisenstein series and the cusp forms, we may effectively determine a bound beyond which every integer locally represented is globally represented as in Tartakowsky's work [22], and local conditions are a simple check at the primes dividing the discriminant. For depth 4, the wonderful optimal bound for cusp forms of Deligne [6] and calculation of the anisotropic primes allow us to again effectively determine the set of integers represented by Q_{odd} (see Hanke [10]). For forms of depth 2 or less, we note that Q_{odd} only represents a set of density zero, so that the form cannot be a leaf.

It remains to determine whether a form of dimension 3 is a leaf or not. However, additional complications arise for ternary quadratic forms. First, note that the inclusion/exclusion of theta series

$$\theta_{Q_{\text{odd}}} := \theta_{Q(x,y,z)} - \theta_{Q(2x,y,z)} - \theta_{Q(x,2y,z)} - \theta_{Q(x,y,2z)} + \theta_{Q(2x,2y,z)} + \theta_{Q(2x,y,2z)} + \theta_{Q(x,2y,2z)} - \theta_{Q(2x,2y,2z)}$$

has nth coefficient nonzero if and only if Q_{odd} represents n, where Q is the quadratic form without the odd restriction. We note that $\theta := \theta_{Q_{\text{odd}}}$ decomposes as follows:

$$\theta = (\theta - \theta_{\text{spin}}) + (\theta_{\text{spin}} - \theta_{\text{Gen}}) + (\theta_{\text{Gen}}),$$

where θ_{spin} is the inclusion/exclusion of the (Siegel) weighted average over the spinor genus and θ_{Gen} is the inclusion/exclusion of the weighted average over the genus. For a quadratic form Q, it is well known (see [8]) that $\theta_Q - \theta_{\text{spin}(Q)}$ is in the orthogonal complement of U, $\theta_{\text{spin}(Q)} - \theta_{\text{Gen}(Q)} \in U$, and $\theta_{\text{Gen}(Q)}$ is the Eisenstein series given by the local densities in [21].

Firstly, as noted in the introduction, the coefficients of the Eisenstein series grow like the class number, so that we have made the assumption of GRH for Dirichlet L-functions. Complications that arise are referred to as *anisotropic primes*, that is, primes p for which the nth coefficient of the Eisenstein series does not grow when n grows with high divisibility by p. Luckily, a local condition allows one to check only finitely many primes (those dividing $2D$, where D is the discriminant of Q) to determine these anisotropic primes. Additionally, the genus of the quadratic form may be broken into what are referred to as *spinor genera*. In each spinor genus, there are finitely many eligible integers t, called *spinor exceptions*, for which a certain subset of the square class $t\mathbb{Z}^2$ is not represented by any form in the spinor genus. This comes from the fact that the form splits naturally into three parts, namely an Eisenstein series, a cusp form in the space spanned by lifts of one-dimensional theta series (we will denote this space by U), and a cusp form in the orthogonal complement of U. The forms in the space spanned by the lifts of one-dimensional theta series have coefficients which grow like $n^{\frac{1}{2}}$ in square classes $t\mathbb{Z}^2$, and hence with the same growth as the coefficients of the Eisenstein series. However, the coefficients of each lift are nonzero exactly at t times a square, and t is restricted by the level of the corresponding modular form. Schulze-Pillot has given an explicit algorithm to determine the full set of spinor exceptions for a given quadratic form [17], so this problem will be resolved by using Schulze-Pillot's algorithm. If the t-th coefficient of θ_{spin} is nonzero, then the coefficients of θ_{spin} grow like the class number in this square class. Otherwise, by investigating the spinor norm mapping, Schulze-Pillot determines explicitly the integers m such that the tm^2th coefficient of θ_{spin} is zero and those for which the coefficient is equal to a positive constant times the Eisenstein series (see Schulze-Pillot [18]), for which we can reduce the problem to the case where t is not a spinor exception for the genus.

Since we can determine the finitely many spinor exceptions, we may then ignore the part of the decomposition resulting from a cusp form in U, and it remains to give (effectively) a bound N_θ such that the nth coefficient of the Eisenstein series is larger than the nth coefficient of the cusp form in $U^\perp$ whenever $n > N_\theta$.

To do so, we must have effective bounds for the coefficients of both the Eisenstein series and the cusp form. The nth coefficient of the Eisenstein series is

$$C_\theta \sum_{d^2|n} a_{D,n/d^2} \frac{h(-4D\frac{n}{d^2})}{u(-4D\frac{n}{d^2})},$$

where C_θ may be determined by the local densities in [21], $h(-m)$ is the class number of the imaginary quadratic field $K_m := \mathbb{Q}(\sqrt{-m})$, $u(m) = \#O_{K_m}^*$, D is the discriminant divided by the square of the gcd of 2×2 minors and $a_{D,n/d^2}$ is a constant depending only on $\gcd(n/d^2, 4D^2)$ and the Kronecker symbol $(\frac{4D}{n/d^2})$ [Theorem 86 in 11].

For square-free integers, we will use Dirichlet's class number formula (see [5]) to rewrite the class number with the special value at $s = 1$ of the Dirichlet character χ_{-N},

$$h(-N) = \frac{\sqrt{n}L(\chi_{-N}, 1)}{\pi}.$$

For square-free integers, a celebrated result of Waldspurger [23] allows us to rewrite the square of the absolute value of the coefficient of a Hecke eigenform as the central value of a twist of the (integral weight 2) Shimura lift [19] of the Hecke eigenform. Hence, we decompose the cusp form into Hecke eigenforms and then use Schwarz's inequality, giving

$$|a_{\theta_{\text{spin}}-\theta_{\text{Gen}}}(n)| \le \sqrt{\sum_{i=1}^{t} c_i L(G_i, \chi\chi_{-n}, 1)},$$

where c_i is some explicitly computable constant given by a fixed $n_0 \equiv n \pmod{4D^2}$, G_i is the Shimura lift, χ is the Nebentypus (see [15]) of the weight 3/2 cusp form, and $L(G_i, \psi, 1)$ is the central value of G_i twisted by ψ.

Hence, rearranging, if n is square free such that the coefficient $a_\theta(n)$ is zero (and n is not a spinor exception), then

$$c_\theta n^{\frac{1}{2}} \le \sum_{i=1}^{t} c_i \frac{L(G_i, \chi\chi_{-n}, 1)}{L(\chi_{-4Dn}, 1)^2},$$

where c_θ is an explicitly computable constant. Thus, it only remains to bound $\frac{L(G_i,\psi_1,1)}{L(\psi_2,1)^2} \ll_\delta n^\delta$ effectively and with the implied constant given explicitly for some $\delta < \frac{1}{2}$. Defining for q, the conductor of G_i twisted by ψ_1

$$F_i(s) := \left(\frac{\sqrt{q}}{2\pi}\right)^{s-1} \frac{L(G_i,\psi_1,s)\Gamma(s)}{L(\psi_2,s)L(\psi_2,2-s)}, \tag{3.1}$$

Ono and Soundararajan [16] have shown under GRH how to obtain an explicit bound for $\frac{L(G_i,\psi_1,1)}{L(\psi_2,1)^2} = F_i(1)$. We shall briefly explain the details necessary to obtain their bound.

We first choose $1 < \sigma < \frac{3}{2}$ and $\boldsymbol{X} > 0$ (Ono and Soundararajan chose $\sigma = 7/6$ and gave bounds depending on $\boldsymbol{X}$ in piecewise intervals, while the author [13] gave a formula in terms of arbitrary σ and $\boldsymbol{X}$). Using the Phragmén–Lindelöf principle along with the functional equation $F_i(s) = F_i(1-s)$, one obtains

$$F_i(1) \leq \max_{t\in\mathbb{R}} |F_i(\sigma+it)|.$$

Therefore, it suffices to bound $L(G_i,\psi_1,s)$, $L(\psi_2,s)$, and $L(\psi_2,2-s)$ with $\mathrm{Re}(s)=\sigma$. For L-series $L(s)$ and $c>0$ chosen such that $s+c$ is in the region of absolute convergence, consider the integral

$$\int_{c-i\infty}^{c+i\infty} \frac{L'}{L}(s+w)\Gamma(w)\boldsymbol{X}^w\,dw.$$

On the one hand, if $L(s) = \sum_{n=1}^\infty \frac{a(n)}{n^s}$ in the region of convergence, then this can be computed as the sum

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)a(n)}{n^s} e^{-n/\boldsymbol{X}}, \tag{3.2}$$

using the fact that

$$\int_{c-i\infty}^{c+i\infty} \Gamma(w)\left(\frac{\boldsymbol{X}}{n}\right)^w dw = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\left(\frac{n}{\boldsymbol{X}}\right)^m = e^{-n/\boldsymbol{X}},$$

which follows by shifting the integral $\mathrm{Re}(w) \to -\infty$ and counting the residues at negative integers. On the other hand, we can shift the original integral to the left and count the contribution from residues at each of the poles. The contribution from $w=0$ gives $\frac{L'}{L}(w)$. The assumption of GRH allows us to determine the real part of the zeros of $L(s+w)$ and hence the poles of $\frac{L'}{L}(s+w)$. Collecting all of the residues and rewriting, we obtain a formula for $\frac{L'}{L}(s)$, which we then integrate to obtain $\log|L(s)|$ [16, Lemmas 1 and 2]. Most of the resulting terms can then be bounded in terms of $\frac{\Gamma'}{\Gamma}$ factors by a careful and technical analytic analysis, but the terms coming from the zeros of $L(s+w)$ pose a more formidable challenge. For these, we take advantage of the fact that we have a sum over all of the zeros of L and may hence use Hadamard's factorization (see [16, Lemma 3]) to rewrite this sum in terms of $\frac{L'}{L}(s)$ and $\frac{\Gamma'}{\Gamma}$ factors. Rearranging gives the desired bound for $\log|L(s)|$. The bounds we will use here are given in Theorems 6.1 and 7.1 of [13]. The statements of these bounds are rather long and will be omitted here, but we note that after fixing σ the remaining constants are calculated in a straightforward manner. We would like to emphasize one technical detail here which aided Ono and Soundararajan in obtaining a better bound for Ramanujan's ternary quadratic form and will aid us in many cases. When a prime p divides the conductor of both ψ_1 and ψ_2 then the coefficient of n^s in formula (3.2) will vanish when $p \mid n$, since $\psi_i(n)=0$ implies $a(n)=0$. Since powers of small primes contribute significantly to (3.2), this trick for $p=2$ and $p=5$ allowed Ono and Soundararajan to obtain a bound many orders of magnitude better [16, p. 451].

For n not square free, the Hecke operators may be used to show an explicit bound for the square part beyond which integers must be represented, away from the spinor exceptions. For more details, please see [13, Section 4]. Therefore, we can conclude that the set of square-free integers not represented by this form may be exactly determined by checking up to the bound obtained, and hence we may (effectively and efficiently) determine whether this form is a leaf. ■

Remark 3.1. In practice, whenever a leaf exists at depth 3, we will determine in general the set of integers not represented by Q_{odd} for the nodes at depth 3 (not just the leaves) and then note which of these integers remain at each step of the escalation, instead of using the arguments of Tartakowsky [22] at depth at least 5 or the bounds of Deligne [6] at depth 4. □

We now implement the above algorithm to show that the set S_0 given in Conjecture 1.3, when S is the set of all odd integers, is the correct smallest such set under GRH.

Proof of Theorem 1.6. We will proceed by considering each node at depth 3 and determining the corresponding subtree under these GRH assumptions. For notational ease, we will refer to the form

$$f := \sum_{i=1}^{r} b_i T_{x_i}$$

by $[b_1,\dots,b_r]$, and the corresponding quadratic form by $(b_1,\dots,b_k)$.

The forms $[1,1,1]$, $[1,1,2]$, $[1,1,4]$, $[1,1,5]$, $[1,2,2]$, $[1,2,3]$, and $[1,2,4]$ represent every natural number by Liouville's theorem. Since $[1,1,3]$ is a genus-1 form, the integers not represented by $[1,1,3]$ are precisely the integers n such that

$$8n+5=3^{2r+1}(3\ell+2).$$

Therefore, we are missing the integer 17, and we must escalate to $[1,1,3,k]$ for some $k\leq 17$. If n is not represented by $[1,1,3,k]$, then it is not represented by $[1,1,3]$ either, so $8n+5=3^{2r+1}(3\ell+2)$, so that $n\equiv 5 \pmod 9$ or $n\equiv 8 \pmod 9$, depending on whether $r>0$ or $r=0$, respectively. But then, taking $x_4=1$, it follows for $n>k$ that $n-k\equiv 5 \pmod 9$ or $n-k\equiv 8 \pmod 9$, and hence it follows that $3\mid k$. If $r>0$ then it follows that $k\equiv 0,6 \pmod 9$, and if $r=0$ it follows that $k\equiv 0,3 \pmod 9$.

For the case $[1,1,3,3]$ we then note that the form $Q=(1,1,3,3)$ is genus 1. Our inclusion/exclusion of theta series gives that n is represented if and only if the $2n+2$-th coefficient of θ_Q is positive, and it follows that every n is represented because the local conditions are always satisfied.

For the cases $[1,1,3,k]$ with $k=6$ or $k=15$, we have $r>0$, so that we only need to consider $n\equiv 5 \pmod 9$, or in other words, $8n+5=3^{2r+1}(3\ell+2)$ with $r>0$. We check the cases $n\leq 3k$ by hand and for $n>3k$, the choice $x_4=2$ shows that $8(n-3k)+5=3^{2r'+1}(3\ell'+2)$, with $r'>0$ by congruence conditions modulo 9. Taking the difference and denoting $R=\min\{r,r'\}$, we have

$$\begin{aligned}24k&=8n+5-(8(n-3k)+5)=3^{2r+1}(3\ell+2)-3^{2r'+1}(3\ell'+2)\\&=3^{2R+1}(3^{2(r-R)+1}(3\ell+2)-3^{2(r'-R)+1}(3\ell'+2)).\end{aligned}$$

But $v_3(24k)=2$ and 3^{2R+1} divides the right-hand side, giving a contradiction since $R>0$.

In the case $[1,1,3,12]$ we note that we have the truant 89 and $r=0$ from above. In this case we take $x_4=1$ to obtain $n'=8n+5=3^{2r'+1}(3\ell'+2)+96$ with $r'>0$. We then have either $r'=1$ and $n'\equiv 69 \pmod{81}$ or $r'>1$ and $n'\equiv 15 \pmod{81}$. Assume $r'>1$ and set $x_4=4$ to obtain $n'=3^{2r''+1}(3\ell''+2)+960$. Taking the difference, we get

$$54\equiv 864=3^{2r'+1}(3\ell'+2)-3^{2r''+1}(3\ell''+2)\equiv -3^{2r''+1}(3\ell''+2) \pmod{81}.$$

It follows immediately that $r''=0$ because otherwise the right-hand side would be zero. But now we have $n'=3^3(3\ell''+2)+960\equiv 42 \pmod{81}$, which contradicts the fact that $n'\equiv 15 \pmod{81}$. Hence, we have only the case $n'\equiv 69 \pmod{81}$ remaining. We now escalate to $[1,1,3,12,k]$ for $9\leq k\leq 89$ (although we have restricted our coefficients to be monotone, we include the case $k=9$ for usage below). Since $[1,1,3,12]$ represents every integer not congruent to 69 modulo 81, we are done when $k\neq 81$, and for $k=81$ the truant 89 remains. Furthermore, if we escalate further with $k=81$ we will never obtain 89 since $[k,k,k]$ represents precisely $k\mathbb{N}$ by Gauss' theorem. Furthermore, if we ever choose $k = 81$ and then escalate to another integer (such as $[1, 1, 3, 12, 81, k]$), we do not obtain a new truant because $[1, 1, 3, 12, k]$ has no truants. Whenever this situation occurs, henceforth, we shall say that we are "stuck" at $k = 81$. Following the above, for $[1, 1, 3, 9]$ we are stuck at $k = 9$. This concludes the subtree of $[1, 1, 3]$, and since $[1, 1]$ does not represent 5, we have included the entire subtree of $[1, 1]$.

Our arguments for $[1, 2, 5]$, $[1, 2, 10]$, $[1, 3, 4]$, and $[1, 4, 6]$ will all be identical and similar to the cases above, so we will combine them together. We will demonstrate the argument for $[1, 2, 5]$ and leave the other cases to the reader. First, we note that the number of representations of n by $[1, 2, 5]$ is the same as the number of representations of $8n + 8$ by $(1, 2, 5)$ minus the number of representations of $8n + 8$ by $(4, 8, 20)$, since if any are even, then all must be even, taking everything modulo 8. For simplicity, we will denote $t_{[b]}(n)$ to be the number of times that n is represented by the sum of triangular numbers corresponding to b, $r_{(b)}(n)$ for the number of times the quadratic form represents n, and $r^o_{(b)}(n)$ for the number of times the quadratic form with all x_i odd represents n. So the above is simply

$$t_{[1,2,5]}(n) = r^o_{(1,2,5)}(8n + 8) = r_{(1,2,5)}(8n + 8) - r_{(4,8,20)}(8n + 8).$$

Now, $r_{(4,8,20)}(8n + 8) = r_{(1,2,5)}(2n + 2)$. However, $(1, 2, 5)$ is genus 1, so $r_{(1,2,5)}(m)$ is given precisely by the local density [21]. Checking the local densities, we see that $r_{(1,2,5)}(8n + 8) = 2r_{(1,2,5)}(2n + 2)$, since the local densities are clearly equal when $p \neq 2$ (taking the isomorphisms $x_i \mapsto 2^{-1}x_i$), and for $p = 2$ a simple computation shows exactly twice as many solutions modulo the same 2 power. Therefore,

$$t_{[1,2,5]}(n) = r_{(1,2,5)}(2n + 2).$$

Again, noting that $(1, 2, 5)$ is genus 1, we know that n is represented globally if and only if it is represented locally. The integers not represented locally by $(1, 2, 5)$ are integers of the form $5^{2r+1}(5n + m)$ where m is a nonsquare modulo 5. Therefore, it follows that if n is not represented by $[1, 2, 5]$, then $5 \mid (n + 1)$. So $[1, 2, 5, k]$ must be a leaf if $5 \nmid k$. Since the truant for $[1, 2, 5]$ is 19, we only need to check $k = 5$, $k = 10$, and $k = 15$. We are stuck at $k = 15$, so we only need to show the cases $k = 5$ and $k = 10$.

Let m be smallest such that $[1, 2, 5, 5]$ does not represent m. Then $5 \mid m + 1$ from above and $\frac{2m+2}{5^{2r+1}}$ is not a square modulo 5, taking $x_4 = 0$. If $r \neq 0$, then take $x_4 = 2$, and note that $5^{2r+1}(5n + \not\square) - 2 \times 3 \times 5 \equiv 20 \pmod{25}$, so that $m - 15$ is represented by $[1, 2, 5]$, and hence m is represented by $[1, 2, 5, 5]$ as long as $m \geq 15$, which is as desired. Thus, we only need to consider $r = 0$. The nonsquares modulo 5 are 2 and 3. As above, taking $x_4 = 2$ when $\frac{2m+2}{5} \equiv 2 \pmod{5}$ and $x_4 = 1$ when $\frac{2m+2}{5} \equiv 3 \pmod{5}$ gives the desired result.

For $k = 10$, we again conclude that $r \neq 0$ by taking $x_4 = 1$. Furthermore, if $\frac{2m+2}{5} \equiv 3 \pmod 5$, then $x_4 = 1$ again gives us the desired conclusion. Hence, only the case $\frac{2m+2}{5} \equiv 2 \pmod 5$ remains. For $[1, 2, 5, 10, k]$, with $5 \mid k$, we may again conclude, by taking $x_5 = 0$ and $x_5 = 1$, that $25 \mid k$ since $2(m + 2) = 5(5n + 2)$ and $2(m + 2 - k) = 5(5n' + 2)$, so $2k = 25(n - n')$. Since 29 is the truant of $[1, 2, 5, 10]$, the result follows when $k \neq 25$. But we are stuck at $k = 25$, so we have the desired result.

We will leave out the analogous proofs for $[1, 2, 10]$, $[1, 3, 4]$, and $[1, 4, 6]$, but list the truants from their subtrees for completeness. The truants coming from $[1, 2, 10]$ are 29 and 49 (from $[1, 2, 10, 20]$). The only truant from the subtree of $[1, 3, 4]$ is 11. The truants from the $[1, 4, 6]$ subtree are 17, 29 (from $[1, 4, 6, 12]$), and 35 (from $[1, 4, 6, 6]$).

We will now show the subtree for $[1, 4, 4]$. While $(1, 4, 4)$ is not genus 1, Benham, Earnest, Hsia, and Hung [1] have shown that it is spinor genus 1. Moreover, they have shown that $(1, 4, 16)$ is spinor genus 1 by showing that the other member of its genus, namely $4x^2 + 4y^2 + 5z^2 + 4xz$, is spinor genus 1. Therefore, the difference $r_{(1,4,4)}(8n + 9) - r_{(1,4,16)}(8n + 9)$ can be decomposed into coefficients of the Siegel averaging of the genus, and a cusp form in U which has nonzero coefficients only at finitely many square classes. Using Schulze-Pillot's classification [17] or the generalization of Earnest, Hsia, and Hung [9] to determine all t such that the square class $t\mathbb{Z}^2$ has nonzero coefficients for the resulting cusp forms in U with these two quadratic forms, we conclude that only $t = 1$ occurs. Therefore, it follows that if m is not represented by $[1, 4, 4]$, then $8m + 9$ must be a square. The first truant is $m = 35$, so we consider $[1, 4, 4, k]$ for $k \leq 35$. Let m not represented by $[1, 4, 4, k]$ be given. Then $m - k$ is also not represented by $[1, 4, 4]$, so that $8m + 9$ is a square, say s^2 and $8m + 9 - 8k$ is a square, say t^2. Therefore, $s^2 - t^2 = 8k$. But the difference between s^2 and t^2 must be at least the difference between s^2 and $(s - 1)^2$, which is $2s - 1$. Therefore, $2s - 1 \leq 8k$. This restricts the possible choices for s to a (small) finite set, and hence the possible choices for m. Checking each such choice of s for each k allows us to determine the integers not represented by $[1, 4, 4, k]$. We conclude that we are done for every integer other than $k = 15$ and $k = 33$. For $k = 15$, we represent exactly every integer other than 2 and 35, so $[1, 4, 4, 15, k']$ will represent every integer except when $k' = 33$. For $k = 33$ or $k' = 33$, we are stuck at 35, and hence we are done with the $[1, 4, 4]$ subtree.

We will begin to use our GRH assumptions now. We will indicate clearly when we make these assumptions. The eight cases $[1, 2, 6]$, $[1, 2, 8]$, $[1, 2, 9]$, $[1, 2, 11]$, $[1, 4, 5]$, $[1, 4, 8]$, $[1, 4, 9]$, and $[1, 5, 6]$ will all follow analogous arguments. The cases $[1, 2, 6]$, $[1, 2, 9]$, and $[1, 4, 5]$ are the three cases where the GRH assumptions were seemingly unavoidable. In each case, we will be able to decompose the theta series for the corresponding quadratic

Table 1 Equivalent quadratic forms

Triangular form	represents	n	$\Longleftrightarrow$	Quadratic form	represents	m
$[1,2,6]$		n		$(2,4,7,0,0,4)$		$8n+9$
$[1,2,8]$		n		$(2,4,9,0,0,4)$		$8n+11$
$[1,2,9]$		n		$(2,3,4,2,0,2)$		$2n+3$
$[1,2,11]$		n		$(1,6,8,0,0,4)$		$4n+7$
$[1,4,5]$		n		$(1,4,5,0,0,0)$		$8n+10$
$[1,4,8]$		n		$(4,4,9,0,4,0)$		$8n+13$
$[1,4,9]$		n		$(1,4,9,0,0,0)$		$8n+14$
$[1,5,6]$		n		$(3,3,4,0,2,2)$		$2n+3$

forms with odd conditions into the Eisenstein series plus a Hecke eigenform. The Hecke eigenform in each case is in the complement of the space spanned by lifts of one - dimensional theta series. Because of the fact that these are all genus 2 quadratic forms, we are able to first obtain the following proposition.

Proposition 3.2. Each triangular form $[b_1, b_2, b_3]$ in Table 1 represents n if and only if the corresponding quadratic form

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz$$

represents m. Here we do not have the "odd" condition for the quadratic forms. □

Proof. Each of the above assertions follows the same simple argument, which we will demonstrate explicitly for the form $[1,2,11]$.

First, note that if not all of x, y, and z are odd, then $x^2 + 2y^2 + 11z^2 = 8n + 14$ has a solution modulo 8 only if x and z are both even. Therefore, if $t_b(n)$ is the number of solutions of the triangular form represented by b, and $r_Q(m)$ is the number of solutions to $Q(x) = m$, then

$$\begin{aligned} t_{[1,2,11]}(n) &= r_{(1,2,11,0,0,0)}(8n+14) - r_{(4,2,44,0,0,0)}(8n+14) \\ &= r_{(1,2,11,0,0,0)}(8n+14) - r_{(1,2,22,0,0,0)}(4n+7). \end{aligned}$$

Now we note that $(1,2,11,0,0,0)$ is a genus-2 quadratic form. The other representative of the genus is $(2,3,4,0,0,2)$. If $2x^2 + 3y^2 + 4z^2 + 2yz = 8n + 14$ has a solution, then it is clear that y must be even. Therefore,

$$r_{(2,3,4,0,0,2)}(8n+14) = r_{(2,12,4,0,0,4)}(8n+14) = r_{(1,2,6,0,0,2)}(4n+7).$$

But $(1,2,6,0,0,2)$ is a genus-1 quadratic form, so it follows that $r_{(1,2,6,0,0,2)}(4n+7)$ is merely the value given by Siegel's local density formula as in [21]. Using this observation and the fact that the local densities are equal for $(1,2,6,0,0,2)$ and $(2,3,4,0,0,2)$, it follows that $r_{(1,2,6,0,0,2)}(4n+7) = r_{(2,3,4,0,0,2)}(8n+14) = r_{(1,2,11,0,0,0)}(8n+14)$.

The form $(1,2,22,0,0,0)$ is again genus 2, and the other representative of the genus is $(1,6,8,0,0,4)$. Therefore, the theta series for $Q=(1,2,22,0,0,0)$,

$$\theta_Q(z) := \sum_x q^{Q(x)},$$

satisfies $\theta_Q = E_Q + g$, where E_Q is the Eisenstein series obtained by taking the local densities and g is a cusp form which is a Hecke eigenform. Siegel's formula shows for $Q' := (1,6,8,0,0,4)$ that

$$E_Q = \frac{\frac{\theta_Q}{4} + \frac{\theta_{Q'}}{2}}{\frac{3}{4}}.$$

Therefore, $\theta_{Q'} = E_Q - \frac{1}{2}g$. By observing that the local densities for $(1,2,11,0,0,0)$ and $(4,2,44,0,0,0)$ are the same other than at $p=2$, one can easily see by explicitly computing the local density at $p=2$ that $a_{E_{(1,2,11,0,0,0)}}(8n+14) = 3a_{E_{1,2,22,0,0,0}}(4n+7)$. Therefore, we have shown that

$$\begin{aligned} r_{(1,2,11,0,0,0)}(8n+14) - r_{(1,2,22,0,0,0)}(4n+7) &= 3a_{E_{1,2,22,0,0,0}}(4n+7) - (a_{E_{1,2,22,0,0,0}}(4n+7) + g) \\ &= 2a_{E_{1,2,22,0,0,0}}(4n+7) - g = 2r_{(1,6,8,0,0,4)}(4n+7). \end{aligned}$$

This is precisely what we wanted to show. The other cases follow analogously. ∎

We now proceed by determining which integers in these arithmetic progressions are represented by these quadratic forms. Since each of these forms is genus 2, as well as spinor genus 2, we know that $\theta = E + g$, where g is a Hecke eigenform in the complement of the space spanned by lifts of one -dimensional theta series. Thus, we will employ the argument given in [13] in the case where there is precisely one Hecke eigenform.

We will begin by constructing an algorithm that given a nonzero Hecke eigenform g, a constant c_E depending only on the local densities, such that the Eisenstein series has coefficients $a_E(N) = c_E h(-mN)$ for some fixed integer m and N square free, where $h(-D)$ is the class number, the modulus q such that the corresponding twist from Waldspurger's theorem [23] of the Shimura lift G of g is of modulus qN^2, the integer m given above in the class number, an integer D_0 such that $a_g(D_0)$ is nonzero, $a_g(D_0)$, and the character χ that we are twisting by and returning a bound D beyond which the coefficients of the theta series are nonzero in the congruence class corresponding to D_0.

We do so by fixing $\boldsymbol{X} := 455$, $\sigma := 1.1573$, and $\sigma_2 := 1.3465$ and calculating the bounds given in Theorems 6.1 and 7.1 of [13] for $L(\chi_{mN}, \sigma + it)$ and $L(G, \chi, \sigma + it)$, where G is the Shimura lift of g. The choices of $\boldsymbol{X}$, σ, and σ_2, although arbitrary, were chosen by doing a binary search for local minima, first in terms of σ (due to a heavy dependence on σ in the power of n), then $\boldsymbol{X}$, and finally σ_2. One reason why it is difficult to choose these constants is that choosing $\sigma \to 1$ gives the optimal bound asymptotically in terms of the power of the conductor, but the implied constant explodes. Therefore, it is desirable to have flexibility to choose σ larger when q is small and to choose σ smaller when q is large (here q is the level of a twist of G). The bounds from [13] rely heavily on bounds for the logarithmic derivative of Γ and are rather technical, but the only serious difficulties occur in the terms contributed by the zeros of $L(s)$, as described briefly in the paragraph following the definition (3.1) of F_i. Most of the constants are then straightforward to calculate in terms of special values of $\Gamma(s)$ and $\zeta(s)$. Although the bounds exhibit polynomial growth in the variable t, one can use the functional equation of the Γ-function to essentially gain an extra factor of t^a in the denominator at the cost of shifting the real part to the right by a, and hence deal with this technical difficulty by using the $\Gamma(s)$ factor from the definition (3.1) of F_i. The constants $\alpha(\boldsymbol{X})$, $\gamma(\boldsymbol{X})$, and $\delta(\boldsymbol{X})$ are given in terms of the maximum over $0 \le y \in \mathbb{R}$ of certain functions $f(y) = g(y)h(y)$ with $g(y)$, defined in terms of a certain factor $\Gamma(x + iy)$, monotone decreasing exponentially with y and $h(y)$ monotone increasing polynomially. Hence, in the interval $y \in [y_0, y_1]$ we have $f(y) \le g(y_0)h(y_1)$. Breaking up $[0, N]$ into small intervals and using this trick repeatedly, we obtain a bound for the maximum in $[0, N]$. For the interval $[N, \infty)$ we use the functional equation of Γ to divide by powers of y and, hence, cancel the growth coming from $h(y)$. Bounding α, β, γ, and δ in this manner in Theorem 8.1 of [13], we see with the help of a computer using MAGMA [3] that

$$\alpha(\boldsymbol{X}) \le 0.089028567932572,\ \beta(\boldsymbol{X}) \ge 0.0886630818642167,$$
$$\gamma(\boldsymbol{X}) \le 0.249235264918139,\ \delta(\boldsymbol{X}) \le 0.0963544917482776.$$

The main technical difficulty which remains is to bound the terms arising from the sums obtained by (3.2). Based on the uniform shape of these sums, we are able to combine them as a single sum, allowing us to take advantage of cancellation between terms which would otherwise be lost when pulling the absolute value inside. We then explicitly compute the

Table 2 Sufficient bounds under GRH

Triangular form	Bound D_0 for quadratic form	Bound D_0' for odd squares
[1, 2, 6]	1.23×10^9	1.23×10^9
[1, 2, 8]	6.0×10^8	6.0×10^8
[1, 2, 9]	1.68×10^6	6.72×10^6
[1, 2, 11]	8.0×10^4	1.6×10^5
[1, 4, 5]	2.6×10^5	2.6×10^5
[1, 4, 8]	5.7×10^8	5.7×10^8
[1, 4, 9]	1.1×10^{12}	1.1×10^{12}
[1, 5, 6]	4.55×10^9	1.82×10^{10}

sum for small n, while for large n we bound against a constant times

$$\sum_{n=N+1}^{\infty} \frac{\Lambda(n)}{n^a \log(n)} e^{-n/X}.$$

This sum is then rewritten as an integral, and integration by parts gives us a bound in terms of $\psi(x) = \sum_{n \leq x} \Lambda(n)$ and an incomplete Γ-function (see [16, Lemma 5] or [13, Lemma 8.5]).

Notice that the bound obtained by this algorithm is valid only in the congruence class congruent to D_0 modulo 8 times the square of the determinant of the corresponding quadratic form, and that for each congruence class the choice of m and χ may vary. We, therefore, will run the algorithm for a choice of D_0 in each congruence class which satisfies the congruence modulo a 2 power given above, and we will merely state the largest such bound obtained in Table 2.

We then check up to the bound D_0' for odd squares with a computer (using the fact that it is diagonal to our advantage by splitting off one dimension, as in [12]) and list the integers not represented by each triangular form. We will list the triangular form, then the congruence classes not represented locally by the form, and finally the finite list of "sporadic" integers not represented globally but represented locally by the form. Note that we have a leaf (and hence GRH seems unavoidable with current techniques) if and only if there are no congruence classes and no sporadic odd integers, namely the cases [1, 2, 6], [1, 2, 9], and [1, 4, 5].

Given Table 3, it is easy to see that the only truants which arise from all of the subtrees other than [1, 4, 9] are 13, 17, 19, and 25. The first odd integer not represented by [1, 4, 9] is 11, so we only need to consider $[1, 4, 9, k]$ for $9 \leq k \leq 11$. The choice $k = 9$ is stuck at 11, and $k = 10$ or $k = 11$ clearly represent every integer other than 2 and 8, by taking $x_4 = 0$ or $x_4 = 1$.

Table 3 Exceptional integers not represented

Triangular form	Congruence classes	Sporadic integers
[1, 2, 6]	∅	∅
[1, 2, 8]	∅	{4, 19, 112}
[1, 2, 9]	∅	∅
[1, 2, 11]	∅	{4, 25}
[1, 4, 5]	∅	{2, 26, 38}
[1, 4, 8]	∅	{2, 16, 17}
[1, 4, 9]	$m \equiv 2 \mod 9$ and $m \equiv 8 \mod 9$	∅
[1, 5, 6]	∅	{2, 4, 13, 35}

The case $[1, 2, 7, k]$ with $8 \leq k \leq 11$ follows from the above cases and $k = 7$ is stuck at 11 so we get no new truants. The case [1, 4, 7] analogously gives no new truants, and [1, 3, 3] and [1, 5, 5] will follow similarly after we show the argument for [1, 3, 5] and [1, 5, 7]. Thus, it only remains to show the subtrees for [1, 3, 5] and [1, 5, 7].

For [1, 3, 5], we will follow a similar argument as above, but we must be slightly more careful. We note that

$$t_{[1,3,5]}(n) = r_{(1,3,5,0,0,0)}(8n+9) - r_{(1,5,12,0,0,0)}(8n+9).$$

The theta series for (1, 3, 5, 0, 0, 0) decomposes as $E + g_1$, while the theta series for (1, 5, 12, 0, 0, 0) decomposes as $\frac{1}{2}E + g_2$. In this case, the Shimura lift of $g_1 - g_2$ is $\frac{1}{2}(-G_{15} + G_{15}|V(2) - 4G_{15}|V(4))$, where G_{15} is the (unique) newform of weight 2 and level 15 and $V(d)$ is the dth V-operator (see [15, p. 28]). Therefore, since the Hecke operators commute with the Shimura lift, it follows that $g_1 - g_2$ is a Hecke eigenform, so we may use the above algorithm with $g = g_1 - g_2$.

When 5 does not divide $8n + 9$, our algorithm gives the bound $D_0' = 9.4 \times 10^8$. Checking up to this bound, we find no sporadic integers outside of $n \equiv 2 \mod 5$. Therefore, all integers other than a subset of $n \equiv 2 \mod 5$ are represented by [1, 3, 5]. However, since the truant of [1, 3, 5] is 7, we only need to consider $[1, 3, 5, k]$ for $5 \leq k \leq 7$, and 5 is stuck on 7. Hence, using $x_4 = 0$ or $x_4 = 1$, we represent every integer other than 2. The reason for separating the case of $n \equiv 2 \mod 5$ is that the bound obtained by the algorithm was not computationally feasible and was not needed to obtain the desired result.

In the case of [1, 5, 7],

$$t_{[1,5,7]}(n) = r_{(1,5,7,0,0,0)}(8n+13) - r_{(1,5,28,0,0,0)}(8n+13),$$

and both $(1,5,7,0,0,0)$ and $(1,5,28,0,0,0)$ are genus 3 and spinor genus 3. Therefore, the theta series for $(1,5,7,0,0,0)$ decomposes into $E+g_1+g_2$ and the theta series for $(1,5,28,0,0,0)$ decomposes into $\frac{1}{2}E+g_3+g_4$, where g_1, g_2, g_3, and g_4 are Hecke eigenforms in the complement of the space spanned by lifts of one-dimensional theta series. Moreover, computation shows that the Shimura lift of g_1-g_3 is $\frac{1+\sqrt{17}}{2\sqrt{17}}(G_{35}+c_1G_{35}|V(2)+c_2G_{35}|V(4))$, where G_{35} is the newform of weight 2 and level 35 whose second coefficient is $\frac{-1+\sqrt{17}}{2}$. The constants c_1 and c_2 are irrelevant, since we will twist the Hecke eigenform by χ_4, killing all of the coefficients divisible by 2, while the nth coefficient of $G|V(d)$ is zero unless $d\mid n$. Moreover, the Shimura lift of g_2-g_4 is $(1-\frac{1+\sqrt{17}}{2\sqrt{17}})(\sigma G_{35}+\sigma(c_1)\sigma G_{35}|V(2)+\sigma(c_2)\sigma G_{35}|V(4))$. Here σ is the Galois map sending $\sqrt{17}$ to $-\sqrt{17}$.

We could write a separate algorithm from the one above for two (or any arbitrary number of) eigenforms, but for simplicity we will simply rewrite the above sum as

$$(\alpha E+(g_1-g_3))+\left(\left(\frac{1}{2}-\alpha\right)E+(g_2-g_4)\right)$$

for some appropriate choice of α, and then bound each half separately as above, taking the maximum of the two bounds, since beyond the corresponding maximum bound D_0, $a_{\alpha E+(g_1-g_3)}>0$, and $a_{(\frac{1}{2}-\alpha)E+(g_2-g_4)}>0$. The optimal choice of α would give equal D_0 bounds for each.

We choose $\alpha=\frac{17}{36}$ here, so that $\frac{1}{2}-\alpha=\frac{1}{36}$. Doing so, whenever 5 does not divide $8n+13$, we get a bound for the part corresponding to g_1-g_3 of $D_0'=4.53\times10^8$ and a bound of $\tilde{D_0}'=1.65\times10^8$ for the part corresponding to g_2-g_4. The only sporadic integer not congruent to 4 mod 5 is 2. Therefore, we are again done, since $[1,5,7,k]$ represents every odd when $8\le k\le 9$ since $n\equiv 4 \mod 5$ is represented by $x_4=1$, and $k=7$ is stuck at 9. ■

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ON TWO CONJECTURES ABOUT MIXED SUMS OF SQUARES AND TRIANGULAR NUMBERS

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Abstract

We investigate mixed sums of triangular numbers and squares. We resolve a conjecture of Z.-W. Sun about representability of sums of this type by proving 6 of the 10 parts and giving counterexamples to the 4 other parts. We also show that the generalized Riemann hypothesis implies another conjecture of Z.-W. Sun about which explicit natural numbers may be represented.

Key Words: triangular numbers, odd Squares, sums of squares, quadratic forms, half-integral weight modular forms
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1. Introduction

Sums of squares and sums of triangular numbers have been studied extensively, going as far back as Fermat. Fermat asserted that every natural number was the sum of three triangular numbers, four squares, five pentagonal numbers, etc. Lagrange showed this claim for sums of four squares in 1770, while Gauss showed the result for three triangular numbers in 1796. The full assertion was later shown by Cauchy in 1813.

More recently, Sun [18] has considered mixed hybrid sums involving both triangular numbers and squares. That is, Sun has considered sums of the type

$$f_{a,b}(x,y) := a_1x_1^2 + \cdots + a_{m_1}x_{m_1}^2 + b_1T_{y_1} + \cdots + b_{m_2}T_{y_{m_2}},$$

where a_i and b_i are natural numbers and $T_n = n(n+1)/2$ is the n-th triangular number.

In [18], Sun investigates which sums with three terms represent every integer, so called *universal forms*, reducing the possible candidates to a short list which he then conjectured to be universal. Guo, Pan, and Sun [5] showed that at least all but one of these were indeed universal, while Sun and Oh [10] showed that every natural number could be written as a square plus an odd square plus a triangular number to complete the classification.

Theorem 1.1 (Sun and Oh, see [10]). *Every natural number has the form*

$$x^2 + 8T_y + T_z.$$

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Sun then conjectured the following (cf. [19]).

Conjecture 1.2 (Sun [19]). *Let m and n be any nonnegative integers. Then every sufficiently large natural number can be written in any of the following forms:*

$$2^m x^2 + 2^n y^2 + T_z \tag{1.1}$$
$$2^m x^2 + 2^n T_y + T_z \tag{1.2}$$
$$2^m T_x + 2^n T_y + T_z \tag{1.3}$$
$$x^2 + 2^n \cdot 3y^2 + T_z \tag{1.4}$$
$$x^2 + 2^n \cdot 3T_y + T_z \tag{1.5}$$
$$2^n \cdot 3x^2 + 2T_y + T_z \tag{1.6}$$
$$2^n \cdot 3T_x + 2T_y + T_z \tag{1.7}$$
$$2^n \cdot 5T_x + T_y + T_z \tag{1.8}$$
$$2T_x + 3T_y + 4T_z \tag{1.9}$$
$$2x^2 + 3y^2 + 2T_z. \tag{1.10}$$

We will see first that this conjecture does not hold in general. For formulas (1.2), (1.3), (1.7), and (1.8) we obtain explicit counterexamples to the conjecture.

Theorem 1.3.

$$x^2 + 16T_y + T_z$$

does not represent any natural number of the form $(p^2 - 17)/8$, where p is any prime congruent to 1 or 3 modulo 8, and is hence a counterexample to (1.2).

$$4T_x + 4T_y + T_z$$

and

$$8T_x + T_y + T_z$$

represent precisely the natural numbers not of the form $(a^2 - 9)/8$ and $(a^2 - 5)/4$, respectively, where a is any integer all of whose prime factors are congruent to 1 modulo 4. Hence both are counterexamples to (1.3).

$$192T_x + 2T_y + T_z$$

does not represent any natural number of the form $(3p^2 - 195)/8$ with p a prime congruent to 5 or 7 modulo 8, and hence it is a counterexample to (1.7).

$$160T_x + T_y + T_z$$

does not represent any natural number of the form $(5p^2 - 162)/8$ with p is a prime congruent to 5 or 7 modulo 8, and hence it is a counterexample to (1.8).

Remark 1.4. After submission, Sun has pointed out that the cases $4T_x + 4T_y + T_z$ and $8T_x + T_y + T_z$ are in fact implied in Oh and Sun's paper [10], as follows.

From Oh and Sun [10, Theorem 1.1(ii)],

$$\{n \in \mathbb{Z}^+ : n \neq (2x+1)^2 + T_y + T_z, x, y, z \in \mathbb{Z}\}$$
$$= \{2T_m : m > 0, \text{ and all prime divisors of } 2m+1 \text{ are congruent to } 1 \pmod 4\}.$$

It follows that a nonnegative integer n cannot be represented by $8T_x + T_y + T_z$ if and only if n has the form $2T_m - 1$ where $2m + 1$ has no prime divisors congruent to 3 modulo 4.

By Sun [18, Theorem 1(iii)], and Oh-Sun [10, Theorem 2.1(ii)]

$$\{n \in \mathbb{Z}^+ : n \neq (2x+1)^2 + (2y)^2 + T_z, x, y, z \in \mathbb{Z}\}$$
$$= \{T_m : m > 0, \text{ and all prime divisors of } 2m+1 \text{ are congruent to } 1 \pmod 4\}.$$

Observe that

$$\begin{aligned} n &= 4T_x + 4T_y + T_z \text{ for some } x, y, z \in \mathbb{Z} \\ \Longleftrightarrow 2n+2 &= (2x+1)^2 + (2y+1)^2 + 2T_z \text{ for some } x, y, z \in \mathbb{Z} \\ \Longleftrightarrow n+1 &= (x+y+1)^2 + (x-y)^2 + T_z \text{ for some } x, y, z \in \mathbb{Z} \\ \Longleftrightarrow n+1 &= (2u+1)^2 + (2v)^2 + T_z \text{ for some } u, v, z \in \mathbb{Z}. \end{aligned}$$

Thus, an integer n cannot be represented by $4T_x + 4T_y + T_z$ if and only if n has the form $T_m - 1$ with $2m + 1$ having no prime divisors congruent to 3 modulo 4. Note also that the "if" part is equivalent to Oh and Sun [10, Corollary 1.1(ii)].

Although such counterexamples to the conjecture exist, the nature of such counterexamples is tractable, and we will prove a revised version of the conjecture, resolving the conjecture conclusively in each case.

Theorem 1.5. *Let m and n be any nonnegative integers. Then for a sufficiently large natural number r, depending on n and m, the following equations hold*

1.
$$2^m x^2 + 2^n y^2 + T_z = r.$$

2.
$$2^m x^2 + 2^n T_y + T_z = r$$

whenever $8r + 2^n + 1$ is not a square. This condition is empty when $n < 3$.

3.
$$2^m T_x + 2^n T_y + T_z = r$$

whenever $8r+2^n+2^m+1$ is not a square, or when $n = 0$ (or, symmetrically, $m = 0$) and $8r + 2^m + 2$ ($8r + 2^n + 2$, respectively) is twice a square.

4.
$$x^2 + 2^n \cdot 3y^2 + T_z = r$$

5.
$$x^2 + 2^n \cdot 3T_y + T_z = r$$

6.

$$2^n \cdot 3x^2 + 2T_y + T_z = r$$

7.

$$2^n \cdot 3T_x + 2T_y + T_z = r$$

whenever $8r + 3 \cdot 2^n + 3$ *is not* 3 *times a square.*

8.

$$2^n \cdot 5T_x + T_y + T_z = r$$

whenever $8r + 5 \cdot 2^n + 2$ *is not 10 times a square.*

9.

$$2T_x + 3T_y + 4T_z = r.$$

10.

$$2x^2 + 3y^2 + 2T_z = r.$$

Remark 1.6. This result is best possible in the sense that Theorem 1.3 gives forms which do not represent infinitely many natural numbers r such that $8r + c$ are in each of the exceptional square classes $t\mathbb{Z}^2$ listed in Theorem 1.5. In particular, $x^2 + 16T_y + T_z$ does not represent r if $8r + 17 = p^2$, whenever p is any prime congruent to 1 or 3 modulo 8, $4T_x + 4T_y + T_z$ and $8T_x + T_y + T_z$ do not represent r if $8r + 9 = a^2$ or $8r + 10 = 2a^2$, respectively, whenever all prime divisors of a are congruent to 1 modulo 4, $192T_x + 2T_y + T_z$ and $160T_x + T_y + T_z$ do not represent r if $8r + 195 = 3p^2$ or $8r + 162 = 5p^2$, respectively, whenever p is a prime congruent to 5 or 7 modulo 8. It is important to note that for any fixed m, n our method will be sufficient to determine whether every sufficiently large integer is represented, or if there are infinitely many natural numbers of the form $8r + c = t\mathbb{Z}^2$ which are not represented.

Sun also makes several concrete observations based on computational evidence for many of these forms to determine what is "sufficiently large." However, our proof relies on a lower bound for the class numbers, and is hence ineffective, so that we cannot determine an explicit bound on r. Under the assumption of the Generalized Riemann Hypothesis (GRH), we can verify that the list given by Sun is complete. Sun also makes the following explicit conjecture (Conjecture 3 in [18]).

Conjecture 1.7. *Every natural number can be written in the form* $x^2 + 2y^2 + 3T_z$ *except* $r = 23$*, in the form* $x^2 + 5y^2 + 2T_z$ *except* $r = 19$*, in the form* $x^2 + 6y^2 + T_z$ *except* $r = 47$*, and in the form* $2x^2 + 4y^2 + T_z$ *except* $r = 20$.

Although our methods are not sufficient to completely resolve this conjecture, due to the ineffective nature of our bounds, we are able to obtain a partial result and a conditional proof of Conjecture 1.7, with the help of a computer, using the method of Ono and Soundararajan [12] which was used to (conditionally) determine the integers represented by $x^2+y^2+10z^2$.

Theorem 1.8. *Every sufficiently large natural number may be written in each of the forms given in Conjecture 1.7.*

Moreover, assuming GRH for Dirichlet L-functions and GRH for the L-functions of weight 2 new forms, Conjecture 1.7 holds.

Table 1. Equivalent Quadratic Forms

Mixed Sum f	Exceptional r	Quadratic Form Q	Congruence	Exceptional r'
$x^2 + 2y^2 + 3T_z$	$\{23\}$	$2x^2 + 3y^2 + 4z^2$	$3 \pmod 8$	$\{187\}$
$x^2 + 5y^2 + 2T_z$	$\{19\}$	$x^2 + y^2 + 20z^2$	$1 \pmod 4$	$\{77\}$
$x^2 + 6y^2 + T_z$	$\{47\}$	$x^2 + 2y^2 + 12z^2$	$1 \pmod 8$	$\{377\}$
$2x^2 + 4y^2 + T_z$	$\{20\}$	$x^2 + 4y^2 + 32z^2$	$1 \pmod 8$	$\{161\}$

Remark 1.9. In light of Theorem 1.8, there is an elliptic curve E for each form such that any counterexample r to Conjecture 1.7 will give a (specific) discriminant D_r such that $L(\chi_{D_r}, s)$ has a Siegel zero or a (specific) discriminant D_r' such that the L-series of the D_r'-th quadratic twist of E contains a Siegel zero. Here D_r and D_r' vary linearly in r as a constant times $8r + c$ with c the constant in front of the term T_z

In our proof of Theorem 1.8, we also show that Conjecture 1.7 leads to the following equivalent statements.

Proposition 1.10. *In Table 1, the mixed sum f represents precisely every natural number other than the exceptional set of r if and only if the quadratic form Q represents every natural number in the given congruence class other than the exceptional set of r'.*

Our methods will be based upon the theory of (ternary) quadratic forms and half-integral weight modular forms. A good reference for quadratic forms is [6] and the survey paper of Schulze-Pillot [13], while a good reference for modular forms is [11].

2. Representations by Sufficiently Large Integers

In this section we will show our main results, Theorems 1.5 and 1.3. We will first show the main result and then show how the counterexamples arise naturally from our proof.

Proof of Theorem 1.5. Consider one of the forms of the conjecture written as

$$f_{a,b}(x, y) = a_1 x_1^2 + \cdots + a_{m_1} x_{m_1}^2 + b_1 T_{y_1} + \cdots + b_{m_2} T_{y_{m_2}}.$$

We first note that (extending the definition of triangular number to $T_{-x} = -x(-x + 1)/2$ for symmetry), $f_{a,b}(x, y) = r$ if and only if

$$Q_{a,b}(x, y) := 8a_1 x_1^2 + \cdots + 8a_{m_1} x_{m_1}^2 + b_1(2y_1 + 1)^2 + \cdots + b_{m_2}(2y_{m_2} + 1)^2 = 8r + \sum_{i=1}^{m_2} b_i. \tag{2.1}$$

This is obtained simply by multiplying both sides of the equation by 8 and then adding $\sum_{i=1}^{m_2} b_i$ to both sides. Therefore, we will consider sums of the type $Q_{a,b}(x, y)$. Note that in each of our cases $Q = Q_{a,b}$ is a (ternary) quadratic form (which we shall denote Q' for the quadratic form) with the added condition that the b_i terms must be odd.

Consider the associated theta-series

$$\theta_Q := \sum_{x,y,z \in \mathbb{Z}} q^{Q(x,y,z)} = \sum_{r \in \mathbb{N}} a_Q(r) q^r,$$

where $a_Q(r)$ is the number of representations of r by Q. We may write, using inclusion/exclusion,

$$\theta_{Q(x_1,x_2,y_1)} = \theta_{Q'(x_1,x_2,y_1)} - \theta_{Q'(x_1,x_2,2y_1)}, \tag{2.2}$$

$$\theta_{Q(x_1,y_1,y_2)} = \theta_{Q'(x_1,y_1,y_2)} - \theta_{Q'(x_1,2y_1,y_2)} - \theta_{Q'(x_1,y_1,2y_2)} + \theta_{Q'(x_1,2y_1,2y_2)}, \tag{2.3}$$

and

$$\begin{aligned}\theta_{Q(y_1,y_2,y_3)} &= \theta_{Q'(y_1,y_2,y_3)} - \theta_{Q'(2y_1,y_2,y_3)} - \theta_{Q'(y_1,2y_2,y_3)} - \theta_{Q'(y_1,y_2,2y_3)} \\ &\quad + \theta_{Q'(2y_1,2y_2,y_3)} + \theta_{Q'(2y_1,y_2,2y_3)} + \theta_{Q'(y_1,2y_2,2y_3)} - \theta_{Q'(2y_1,2y_2,2y_3)}.\end{aligned} \tag{2.4}$$

Thus, θ_Q is a sum of finitely many modular forms (the theta-series of the above quadratic forms), and is thus itself a modular form.

The theta-series of a ternary quadratic form decomposes as follows:

$$\theta_{Q'} = (\theta_{Q'} - \theta_{\text{Spn}(Q')}) + (\theta_{\text{Spn}(Q')} - \theta_{\text{Gen}(Q')}) + \theta_{\text{Gen}(Q')},$$

where $\theta_{\text{Gen}(Q')}$ denotes the weighted average over the genus and $\theta_{\text{Spn}(Q')}$ denotes the weighted average over the spinor genus. Moreover, $(\theta_{Q'} - \theta_{\text{Spn}(Q')})$ is a cuspidal modular form whose Shimura lift is also cuspidal, $(\theta_{\text{Spn}(Q')} - \theta_{\text{Gen}(Q')})$ is a cuspidal modular form in the space of lifts of one dimensional theta-series, where only $t\mathbb{Z}^2$ coefficients are supported (all others are equal to zero) for finitely many squarefree integers t dividing the discriminant (cf. Schulze-Pillot [13], p. 7-9). We will call ta^2 a (primitive) *spinor exception* for Q' if ta^2 is not (primitively) represented by the spinor genus of Q', and we will call $t\mathbb{Z}^2$ a *spinor exceptional class* for Q' if t is not represented by one of the spinor genera in the genus of Q'. The r-th coefficient of the weighted average of the genus grows like a certain class number (see Jones [6]) when r has bounded divisibility by the *anisotropic primes* (primes p dividing twice the discriminant in which the number of representations does not grow locally), and hence the r-th coefficient grows like

$$a_{\theta_{\text{Gen}(Q')}}(r) \gg r^{1/2-\epsilon},$$

whenever r is locally represented, by Siegel's (ineffective) lower bound for the class numbers [16]. Since the Shimura lift of $(\theta_{Q'} - \theta_{\text{Spn}(Q')})$ is cuspidal, Duke's bound [2] gives

$$a_{\theta_{Q'}-\theta_{\text{Spn}(Q')}}(r) \ll r^{3/7+\epsilon},$$

as observed by Duke and Schulze-Pillot [3]. Therefore, outside of the coefficients which are supported by $(\theta_{\text{Spn}(Q')} - \theta_{\text{Gen}(Q')})$ or $(\theta_{\text{Spn}(Q'(x,y,2z))} - \theta_{\text{Gen}(Q'(x,y,2z))})$, Equation (2.2) gives in that case

$$a_Q(r) = a_{\theta_{\text{Gen}(Q'(x,y,z))}}(r) - a_{\theta_{\text{Gen}(Q'(x,y,2z))}}(r) + O(r^{3/7+\epsilon}).$$

We now investigate the difference

$$a_{\theta_{\text{Gen}(Q'(x,y,z))}}(r) - a_{\theta_{\text{Gen}(Q'(x,y,2z))}}(r).$$

Using Siegel's averaging formula, the coefficients of these forms are given by the product of the limit of the number of solutions modulo a prime power p^m divided by p^m. Since these forms are equivalent p-adically for all primes $p \neq 2$, it follows that $a_{\theta_{\text{Gen}(Q'(x,y,2z))}}(r) = c_r a_{\theta_{\text{Gen}(Q'(x,y,z))}}(r)$, where c_r is a constant which only depends on r modulo a fixed power of 2. Clearly $c_r \leq 1$, since the number of representations of r with z arbitrary is less than or equal to the number of representations with z even. We then note that modulo a fixed power of 2 this difference of local densities is equal to the number of solutions with z odd. Moreover, Siegel's averaging theorem [17] shows that r is represented by one of the forms in the genus if and only if it is locally represented at all of the primes. Thus, $c_r = 1$ if and only if r is not locally represented by Q' with z odd.

Taking c'_r to be the weighted sum of the above c_r from the inclusion/exclusion, the same argument as above shows that Equations (2.3) and (2.4) also give

$$a_Q(r) = (1 - c'_r) a_{\theta_{\text{Gen}(Q'(x,y,z))}}(r) + O(r^{3/7+\epsilon})$$

for coefficients not supported by $\theta_{\text{Spn}(Q'')} - \theta_{\text{Gen}(Q'')}$ for any Q'' occuring in the inclusion/exclusion, where $1 - c'_r = 0$ if and only if r is not locally represented.

Thus, any sufficiently large integer which is locally represented by Q and has bounded divisibility by the anisotropic primes, other than (possibly) spinor exceptional square classes $t\mathbb{Z}^2$, with t a squarefree divisor of twice the discriminant of Q', are represented globally by Q.

We will now proceed to show that each of the forms (1.1), (1.2), (1.3), (1.4), (1.5), (1.6), (1.7), (1.8), (1.9), and (1.10) give a form $Q_{a,b}$ which locally represents every integer of the form

$$8r + \sum_{i=1}^{m_2} b_i$$

from Equation (2.1). We will then show which possible exceptional square classes may occur in each case.

We first note that the anisotropic primes must divide twice the discriminant and hence in each case these can only be 2, 3, or 5. The prime 2 can be ignored, since the congruence conditions modulo 8 of the integers we would like to represent by $Q_{a,b}$ in each of our examples automatically implies bounded divisibility at 2. For those cases where 3 and 5 occur in the discriminant we use the fact that 2 is invertible p-adically so that we only need to check the local conditions at 3 and 5 for $n = 0$ or $n = 1$ and $m = 0$ or $m = 1$, verifying in each case that 3 and 5 are not anisotropic. Therefore we only need to check p-adically at each prime for existence of a solution.

It is well known that a solution exists for primes p relatively prime to the discriminant, so we only need to consider primes which divide the discriminant. For (1.1), (1.2), and (1.3), the discriminant is a power of 2, so we only need to consider solutions modulo a sufficiently large power of 2. Checking a fixed power of 2 and applying Hensel's lemma shows that the 2-adic conditions are indeed satisfied in each case.

Therefore, other than spinor exceptional square classes, we have the desired result for these three types of forms. Since the discriminants of each of these forms are a power of 2, the only possible spinor exceptional classes are $\mathbb{Z}^2$ and $2\mathbb{Z}^2$, so $t = 1$ or $t = 2$.

For forms of type (1.1) we note that $8r + 1$ is never 2 times a square and for a square $(2a+1)^2 = 8r + 1$ we have the explicit solution $x = y = 0$ and $z = a$, so we obtain the desired statement.

For forms of type (1.2), $8r + 2^n + 1$ is even if and only if $n = 0$ (and in that case 2 $\pmod 8$). But in this case we have the solution $x = 0$ and $y = z = a$ for $2(2a + 1)^2$, so $t = 2$ does not appear in our analysis. When $n < 3$, $8r + 2^n + 1$ is not 1 or 0 mod 8, so squares do not appear in our analysis when $n < 3$. The case $t = 1$ gives the condition of the theorem.

For forms of type (1.3) $8r + 2^n + 2^m + 1$ is even if and only if $m > 0$ and $n = 0$ (up to symmetry), so we only need to consider twice a square when $n = 0$. The case $t = 1$ gives the other condition of the theorem.

Forms of type (1.4), (1.5), (1.6), (1.7), (1.9) and (1.10) give Q' with discriminant a power of two times 3. Therefore, for these forms we need to check the local condition at 2 and at 3. The 2-adic argument for forms of type (1.4), (1.5), (1.6), (1.7) follow exactly as above for the previous 3 types of forms, using Hensel's Lemma. For the 3-adic argument we only need to show that there is a solution modulo 9 and then use Hensel's Lemma. Since 2 is invertible modulo a 3 power, we only need to consider the cases $n = 0$ or $n = 1$. A simple check shows that the local conditions are satisfied in this case. The local conditions for the forms (1.9) and (1.10) follow directly by direct calculation.

Therefore, the result holds outside of the spinor exceptional square classes for these forms. For (1.9) and (1.10) the genus only has one spinor genus so there are no spinor exceptional square classes, and these follow immediately. For all others, the only possible spinor exceptional classes are $\mathbb{Z}^2$, $2\mathbb{Z}^2$, $3\mathbb{Z}^2$, and $6\mathbb{Z}^2$. For (1.4), $8r + 1 = ta^2$ only has a solution modulo 8 if $t_1 = 1$, but in this case we have the solution $x = y = 0$ and $z = a$. For forms of type (1.5) we have

$$8r + 3 \cdot 2^n + 1 \equiv \begin{cases} 1 & \text{if } n \geq 3 \\ 5 & \text{if } n = 2 \\ 7 & \text{if } n = 1 \\ 4 & \text{if } n = 0 \end{cases} \pmod 8$$

Hence we only need to consider the spinor exceptional class with $t = 1$ for $n \geq 3$ and $t = 2$ for $n = 1$. A quick check shows that $2\mathbb{Z}^2$ is not a spinor exceptional class for $n = 1$ since the genus equals the spinor genus in this case. Therefore only the case $t = 1$ is possible. However, Schulze Pillot [14] gives necessary and sufficient conditions p-adically for t to be a spinor exception. We will only need here the necessary condition 3-adically (which is due to Kneser [9]). Earnest, Hsia, and Hung have given an easy determination of when these conditions are satisfied [4]. They show that the necessary condition implies that if p is ramified in $\mathbb{Q}(\sqrt{-td})$ then

$$L_p \cong b_1 x^2 + b_2 3^r y^2 + b_3 3^s z^2$$

with b_i being p-adic units and $0 < r < s$. However, we have 3 ramified in $\mathbb{Q}(\sqrt{-3 \cdot 2^{n+3} t})$ whenever 3 does not divide t, and $r = 0$ in our case, so it follows that 1 cannot be a spinor exception for $8x^2 + 2^n 3y^2 + z^2$ for any n. But our sum (2.3) only contains quadratic forms of this type, and the result follows.

For (1.6), the congruence $8r + 3 \equiv 3 \pmod 8$ implies that only $t = 3$ may occur, but $x = 0$, $y = z = a$ gives a solution to $3a^2$. For (1.7) the congruence condition modulo 8 implies that only $t = 6$ is possible for $n = 0$, $t = 1$ for $n = 1$, and $t = 3$ is possible for

$n \geq 3$. A quick check for $n = 0$ and $n = 1$ show that these spinor exceptions do not occur, and we are left with the remaining condition.

Finally we show the result for forms of type (1.8). In this case, the discriminant is a power of 2 times 5, and the local conditions are shown as above. The only possible spinor exceptional classes are those with $t = 1$, $t = 2$, $t = 5$, and $t = 10$. We again look at the congruence conditions modulo 8 to determine that the only possible spinor exceptions equal to $8r + 5 \cdot 2^n + 2$ are twice a square or 10 times a square when $n \geq 3$. As in the case of (1.5) we then argue 5-adically to show that 5 must be a divisor of t, so the case $t = 2$ cannot occur.

□

We will now show that our counterexamples to the original conjecture are of the exceptional type arising from spinor exceptional square classes in the associated quadratic form, as evidenced in the above proof.

Proof of Theorem 1.3. In light of Theorem 1.5, for each of the counterexamples we will show that the associated form Q does not represent ta^2 for infinitely many integers a, with $t\mathbb{Z}^2$ the possible spinor exceptional square class which occurs as a condition in the given statement.

We will first show the case for $4T_x + 4T_y + T_z$. The associated form $Q'(x, y, z) := 4x^2 + 4y^2 + z^2$ is genus 1. Local conditions (modulo 8) show that the difference of sums to obtain $8r+9$ is $Q(x, y, z) = Q'(x, y, z) - Q'(x, 2y, z)$, since otherwise the local conditions are not satisfied. However, $Q''(x, y, z) := 4x^2 + 16y^2 + z^2$ is spinor genus 1. Therefore, $\theta_{Q'} - \theta_{\text{Spn}(Q')} = 0$, $\theta_{\text{Gen}(Q')} - \theta_{\text{Spn}(Q')} = 0$, and $\theta_{Q''} - \theta_{\text{Spn}(Q'')} = 0$, so that, calculating the constant in front of $\theta_{\text{Gen}(Q')}$ exactly,

$$\theta_Q = c\theta_{\text{Gen}(Q')} - (\theta_{\text{Spn}(Q'')} - \theta_{\text{Gen}(Q'')}),$$

so that $a_Q(8r + 9) = 2a_{Q'''}(8r + 9)$, where $Q''' = 4x^2 + 4y^2 + 5z^2 + 4xz$ is the (unique) representative of the other spinor genus in the genus of Q''. Therefore, r will be represented if and only if $8r + 9$ is represented by Q''', which is spinor genus 1 (and satisfies local conditions), and hence represents every integer of this type except for the spinor exceptions. However 1 is a spinor exception for Q''', so it follows from the work of Schulze-Pillot [14] that if p is a prime which splits in $K = \mathbb{Q}(\sqrt{-16}) = \mathbb{Q}(i)$ then Q''' will not represent p^2, and hence neither will Q. To determine completely the integers not represented by Q''', one may then follow Schulze-Pillot [14] to see that the integers r not represented are precisely those for which $8r + 9$ is a square which has divisors that all split in $\mathbb{Q}(i)$, which occurs if and only if every prime divisor of $8r + 9$ is congruent to 1 modulo 4.

For $8T_x+T_y+T_z$, we similarly have that $8x^2+y^2+z^2$ is genus 1 and that $32x^2+y^2+z^2$ is spinor regular with $2\mathbb{Z}^2$ a spinor exceptional square class. Again $K = \mathbb{Q}(\sqrt{-8 \cdot 2}) = \mathbb{Q}(i)$. Therefore, the integers not represented by $8T_x + T_y + T_z$ are precisely those r for which $8r + 10$ is twice a square a where all of the divisors of a are congruent to 1 modulo 4. We include this case as a second counterexample to (1.3) to show that both conditions which we have in the theorem are necessary.

For $x^2 + 16T_y + T_z$, the inclusion/exclusion sum gives $Q'(x, y, z) - Q'(x, 2y, z)$ with $Q' = 8x^2 + 16y^2 + z^2$, a genus 1 form. Since $\mathbb{Z}^2$ is a spinor exceptional square class for

$32x^2 + 16y^2 + z^2$ we argue as above to obtain $8r + 17$ is not represented by Q if and only if $8r + 17$ is a square with all divisors split in $\mathbb{Q}(\sqrt{-2})$.

For the other cases, $t\mathbb{Z}^2$ will be a spinor exceptional square class for both $Q'(x, y, z)$ and $Q'(2x, y, z)$ which is represented by both $Q'(x, y, z)$ and $Q'(2x, y, z)$. Therefore, for any prime p which is inert in $K := \mathbb{Q}(\sqrt{-td})$ (Here d is the discriminant of the form, which is the same up to a square for both quadratic forms), then tp^2 will not be represented primitively by the spinor genus of $Q'(x, y, z)$ or $Q'(2x, y, z)$, and hence not by these forms (see [15], page 352).

For $192T_x + 2T_y + T_z$ we have the spinor exceptional class from $t = 3$. Clearly, $Q(x, y, z)$ does not represent $t = 3$, because the odd condition dictates that the smallest integer represented by Q is $192 + 2 + 1 = 195$. Therefore, the number of representations of $t = 3$ by $Q'(x, y, z)$ and $Q'(2x, y, z)$ are equal. Fix an arbitrary prime p inert in $K = \mathbb{Q}(\sqrt{-2})$. Since there are no primitive representations of tp^2 by $Q'(x, y, z)$ and $Q'(2x, y, z)$, it follows that the number of representations of tp^2 by $Q'(x, y, z)$ equals the number of representation of tp^2 by $Q'(2x, y, z)$. For $160T_x + T_y + T_z$, we have $t = 10$ and $K = \mathbb{Q}(i)$, and the argument follows as above. □

3. GRH and Mixed Sums

In [8], the author considers sums of the type $f_{a,b}$ where $a = 0$. Using the decomposition given in Equation (2.4), an algorithm is shown to determine, conditional upon GRH, which integers are represented by $f_{0,b}$. This is based upon an algorithm described by Ono and Soundararajan [12] to determine the integers represented by the particular form $x^2 + y^2 + 10z^2$, as generalized to more general forms by the author in [7]. We will briefly explain the theory behind the algorithm and then use the algorithm to conclude Theorem 1.8.

We start by decomposing the associated quadratic form Q' as described in the previous section, namely

$$\theta_{Q'} = (\theta_{Q'} - \theta_{\mathrm{Spn}(Q')}) + (\theta_{\mathrm{Spn}(Q')} - \theta_{\mathrm{Gen}(Q')}) + \theta_{\mathrm{Gen}(Q')}.$$

For $(\theta_{\mathrm{Spn}(Q')} - \theta_{\mathrm{Gen}(Q')})$, we use the results of Schulze-Pillot [14] to determine all (primitive) spinor exceptions. Outside of these finitely many square classes, we have a cuspidal weight $3/2$ modular form $g := (\theta_{Q'} - \theta_{\mathrm{Spn}(Q')})$ whose Shimura lift is cuspidal plus $E_{Q'} := \theta_{\mathrm{Gen}(Q')}$. Since the Shimura lift of g is cuspidal and the Hecke operators commute with the Shimura lift, we may further decompose g into

$$g = \sum_{i=1}^{m} b_i g_i,$$

where g_i are a fixed set of weight $3/2$ eigenforms which each lift to weight 2 normalized eigenforms G_i ($a_{G_i}(1) = 1$) under our choice of Shimura lift. One then uses the following result of Waldspurger [20].

Theorem (Waldspurger)**.** *Let a weight $3/2$ Hecke eigenform g_i of level N with Nebentypus χ whose Shimura lift is G_i. Then if $r_1/r_2 \in \mathbb{Q}_p^{x^2}$ for every $p \mid N$,*

$$a_{g_i}^2(r_1)L(G_i, \left(\frac{-1}{\cdot}\right)\chi^{-1}\chi_{r_2}, 1)\chi\left(\frac{r_2}{r_1}\right) r_2^{1/2} = a_{g_i}^2(r_2)L(G_i, \left(\frac{-1}{\cdot}\right)\chi^{-1}\chi_{r_1}, 1)r_1^{1/2},$$

where $L(G_i, \chi', s)$ is the L-series of G_i twisted by the character χ'.

We then find a representative r_2 modulo squares in $\mathbb{Q}_p$ so that the coefficient $a_{g_i}(r_2) \neq 0$ (if one exists). If we define

$$c_i := \frac{a_{g_i}^2(r_2)}{r_2^{1/2} L(G_i, \left(\frac{-1}{\cdot}\right) \chi^{-1}\chi_{r_2}, 1)}$$

then for each r_1 equivalent to r_2 modulo squares, we have

$$a_{g_i}^2(r_1) = \frac{c_i}{\chi(r_1/r_2)} L(G_i, \chi'_{r_1}, 1) r_1^{1/2}.$$

Now we note that to obtain the theta series for Q we are taking the sums and differences of finitely many of these theta series for quadratic forms Q'', and in each case $E_{Q''} = c_{Q''} E_{Q'}$, where $c_{Q''}$ is some constant which only depends modulo squares 2-adically. But, as shown above, $\sum_{Q''} c_{Q''} > 0$ whenever the integer is represented locally with the appropriate odd conditions. Thus, taking the sum of each of these from the appropriate Equation (2.2), (2.3), or (2.4) we have, for integers represented locally,

$$a_Q(r) = c a_E(r) + \sum_{i=1}^{m} b_i \sqrt{\frac{c_i}{\chi(r/r_2)} L(G_i, \chi'_r, 1) r^{1/2}}.$$

For r square free, the coefficients $a_E(r)$ are certain class numbers, so Dirichlet's class number formula (cf. [1]) allows us to write

$$a_E(r) = c' L(\chi''_r, 1) r^{1/2},$$

where $L(\chi''_r, s)$ is the L-series of the appropriate character χ''_r. We then simply note that r is not represented by Q if and only if $a_Q(r) = 0$, and then rearrange and divide by $L(\chi'', 1)$, bounding the ratios $L(G_i, \chi'_r, 1)/L(\chi'', 1)^2$ using the bounds in [7].

We now use this algorithm to solve the conjecture assuming GRH.

Proof of Theorem 1.8. We first note that in each of these cases we have $Q(x, y, z) = Q'(x, y, z) - Q'(x, y, 2z)$ with $Q'(x, y, z) = 8ax^2 + 8by^2 + cT_z$ for some a, b, with $c = 1, 2$ or 3. Thus there are no solutions to $8r + c = Q'(x, y, 2z) \pmod 8$, so $Q(x, y, z) = Q'(x, y, z)$. We check the local conditions for Q' and note that Q' does not have c as a spinor exception (the only one possible because of the congruence conditions modulo 8) in each case (one can merely check trivially that it represents c). Therefore, every sufficiently large integer is represented by the form.

We now proceed to show Proposition 1.10 and then use our algorithm given above to determine the integers represented in each case. In each case, the resulting form is genus two and thus will decompose as $E + g$, with E the weighted average among the genus and g a Hecke eigenform (hence, since g has rational coefficients, its lift G will be the L-series of an elliptic curve). We then in each case use an argument similar to that given in [7] to determine (unconditionally) that all of the non squarefree integers must be represented by the form.

For $r = x^2+5y^2+2T_z$, we get $Q'(x,y,z) = 8x^2+40y^2+2z^2$. But $Q'(x,y,z) = 8r+2$ if and only if

$$4x^2 + 20y^2 + z^2 = 4r + 1.$$

However, solutions to $x_2^2 + 20y_2^2 + z^2 = 8r + 1$ can only exist if x is even (up to symmetry of x and z), so we have half the number of solutions to

$$x^2 + 20y^2 + z^2 = 4r + 1,$$

giving the assertion of Proposition 1.10. We then use our algorithm to show the integers (not) represented by $x^2+20y^2+z^2$ and check which are 1 modulo 4. This form is genus 2, so the theta series decomposes as $E+g$ with g a Hecke eigenform. Using the algorithm in [8], all squarefree integers 1 mod 4 greater than 12288 (we get three different bounds depending on the value of $\left(\frac{-20}{4r+1}\right)$, and take the largest one) are represented by $x^2 + 20y^2 + z^2$. A quick computer check then verifies that the only squarefree integer smaller than 10^8 which is 1 modulo 4 and not represented by $x^2 + 20y^2 + z^2$ is 77.

For $x^2 + 2y^2 + 3T_z$, we need to find solutions to $8r+3 = 8x^2+16y^2+3z^2$. However, any solution to $2x_2^2 + 4y_2^2 + 3z^2 = 8r + 3$ must have x and y even, so this is equivalent to $2x^2 + 4y^2 + 3z^2 = 8r + 3$ and we get the assertion of Proposition 1.10. The form is genus 2, so the theta series decomposes as $E + g$ with g a Hecke eigenform. We then use the algorithm in [8] to show that every squarefree integer which is 3 modulo 8, relatively prime to 3 and greater than 1.89×10^9 is represented by $2x^2 + 4y^2 + 3z^2$, while those which are not relatively prime to 3 and greater than 21291 are represented. We then do a quick check by computer to verify that every natural number less than 5×10^{10}, other than 187, which is congruent to 3 modulo 8 is represented by $2x^2 + 4y^2 + 3z^2$.

Next we consider $x^2 + 6y^2 + T_z$. In this case we have solutions to $8r + 1 = 8x^2 + 48y^2 + z^2$. But the number of solutions to $2x^2 + 12y^2 + z^2 = 8r + 1$ equals the number of solutions to $8r+1 = 8x^2+48y^2+z^2$, so r is represented if and only if $8r+1$ is represented by $2x^2 + 12y^2 + z^2$, verifying the statement in Proposition 1.10. As above, our algorithm shows the result for every natural number less than 1.6×10^8. We then check by computer to verify that every natural number less than 2×10^9, other than 377, which is congruent to 1 modulo 8 is represented by $2x^2 + 12y^2 + z^2$.

Finally, for $2x^2 + 4y^2 + T_z$, we need to find solutions to $8r + 1 = 16x^2 + 32y^2 + z^2$. Similarly to above, the number of solutions to $8r+1 = 4x^2+32y^2+z^2$ equals the number of solutions of $8r+1 = 16x^2+32y^2+z^2$, verifying the statement in Proposition 1.10. Our algorithm shows the result for every natural number greater than 5.2×10^8. We then check by computer to verify that every natural number less than 5×10^{10}, other than 161, which is congruent to 1 modulo 8 is represented by $4x^2 + 32y^2 + z^2$. □

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ON ALMOST UNIVERSAL MIXED SUMS OF SQUARES AND TRIANGULAR NUMBERS

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ABSTRACT. In 1997 K. Ono and K. Soundararajan [Invent. Math. 130(1997)] proved that under the generalized Riemann hypothesis any positive odd integer greater than 2719 can be represented by the famous Ramanujan form $x^2 + y^2 + 10z^2$; equivalently the form $2x^2 + 5y^2 + 4T_z$ represents all integers greater than 1359, where T_z denotes the triangular number $z(z+1)/2$. Given positive integers a, b, c we employ modular forms and the theory of quadratic forms to determine completely when the general form $ax^2 + by^2 + cT_z$ represents sufficiently large integers and to establish similar results for the forms $ax^2 + bT_y + cT_z$ and $aT_x + bT_y + cT_z$. Here are some consequences of our main theorems: (i) All sufficiently large odd numbers have the form $2ax^2 + y^2 + z^2$ if and only if all prime divisors of a are congruent to 1 modulo 4. (ii) The form $ax^2 + y^2 + T_z$ is almost universal (i.e., it represents sufficiently large integers) if and only if each odd prime divisor of a is congruent to 1 or 3 modulo 8. (iii) $ax^2 + T_y + T_z$ is almost universal if and only if all odd prime divisors of a are congruent to 1 modulo 4. (iv) When $v_2(a) \neq 3$, the form $aT_x + T_y + T_z$ is almost universal if and only if all odd prime divisors of a are congruent to 1 modulo 4 and $v_2(a) \neq 5, 7, \ldots$, where $v_2(a)$ is the 2-adic order of a.

1. INTRODUCTION AND THE MAIN RESULTS

A classical theorem of Lagrange states that any $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ can be written as a sum of four squares (of integers). In 1916 S. Ramanujan [22] found all the finitely many vectors (a, b, c, d) with $a, b, c, d \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ such that the form $ax^2 + by^2 + cz^2 + dw^2$ (with $x, y, z, w \in \mathbb{Z}$) represents all natural numbers. Ramanujan also asked for determining those vectors $(a, b, c, d) \in (\mathbb{Z}^+)^4$ such that the form $ax^2 + by^2 + cz^2 + dw^2$ represents all sufficiently large integers; this problem was essentially solved by H. D. Kloosterman [13] with help from the useful Kloosterman sum, and this work represents a major breakthrough in the field of quadratic forms.

What about ternary quadratic forms? A well-known theorem of Gauss and Legendre states that $n \in \mathbb{N}$ is a sum of three squares if and only if it is not of the form $4^k(8l + 7)$ with $k, l \in \mathbb{N}$. In general, it is known that for any $a, b, c \in \mathbb{Z}^+$ the

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subset $\{ax^2+by^2+cz^2:\ x,y,z\in\mathbb{Z}\}$ of $\mathbb{N}$ cannot have asymptotic density 1 because there is always a congruence class modulo a power of some prime p dividing $2abc$ which is not even locally represented by the form $ax^2+by^2+cz^2$.

For $x\in\mathbb{Z}$ let T_x denote the triangular number $x(x+1)/2$. Clearly $T_n=T_{-n-1}$ for all $n\in\mathbb{N}$. A famous assertion of Fermat states that each $n\in\mathbb{N}$ can be expressed as a sum of three triangular numbers, equivalently $8n+3$ is a sum of three (odd) squares; this follows immediately from the Gauss-Legendre theorem. Here is another consequence of the Gauss-Legendre theorem observed by Euler: Each natural number can be written in the form $x^2+y^2+T_z$ with $x,y,z\in\mathbb{Z}$. Recently, B. K. Oh and the second author [16] showed that for any $n\in\mathbb{Z}^+$ there are $x,y,z\in\mathbb{Z}$ such that $n=x^2+(2y+1)^2+T_z$, i.e., $n-1=x^2+8T_y+T_z$.

In view of the above, it is natural to study mixed sums of squares and triangular numbers of the following three types:

$$ax^2+by^2+cT_z,\ ax^2+bT_y+cT_z,\ aT_x+bT_y+cT_z,$$

where $a,b,c\in\mathbb{Z}^+=\{1,2,3,\ldots\}$. Let $f(x,y,z)$ be any of the three forms, and define the exceptional set

$$E(f):=\{n\in\mathbb{N}:f(x,y,z)=n\text{ has no integral solution}\}.$$

If $E(f)=\emptyset$, then f is said to be *universal*; if $E(f)$ is finite, then we call f *almost universal*. When the set $E(f)$ has asymptotic density zero, i.e.,

$$\lim_{N\to+\infty}\frac{|\{1\le n\le N:\ f(x,y,z)=n\text{ for some }x,y,z\in\mathbb{Z}\}|}{N}=1,$$

we say that f is *asymptotically universal*. In the case $\gcd(a,b,c)>1$, obviously f is neither almost universal nor asymptotically universal.

In 1862 J. Liouville (cf. [4, p. 23]) proved the following result: For positive integers $a\le b\le c$, the form $aT_x+bT_y+cT_z$ is universal if and only if (a,b,c) is among the following vectors:

$$(1,1,1),\ (1,1,2),\ \ (1,1,4),\ (1,1,5),\ (1,2,2),\ (1,2,3),\ (1,2,4).$$

Recently the second author [27] initiated the determination of all universal forms of the type $ax^2+by^2+cT_z$ or $ax^2+bT_y+cT_z$, and the project was finally completed by combining the results in [27], [9] and [16]. Namely, for $a,b,c\in\mathbb{Z}^+$ with $a\le b$, the form $ax^2+by^2+cT_z$ is universal if and only if (a,b,c) is among the following vectors:

$$(1,1,1),\ (1,1,2),\ (1,2,1),\ (1,2,2),\ (1,2,4),$$
$$(1,3,1),\ (1,4,1),\ (1,4,2),\ (1,8,1),\ (2,2,1).$$

Also, for $a,b,c\in\mathbb{Z}^+$ with $b\ge c$, the form $ax^2+bT_y+cT_z$ is universal if and only if (a,b,c) is among the following vectors:

$$(1,1,1),\ (1,2,1),\ (1,2,2),\ (1,3,1),\ (1,4,1),\ (1,4,2),\ (1,5,2),$$
$$(1,6,1),\ (1,8,1),\ (2,1,1),\ (2,2,1),\ (2,4,1),\ (3,2,1),\ (4,1,1),\ (4,2,1).$$

In 1916 Ramanujan (cf. [22] and [19]) conjectured that those positive even integers not represented by $x^2+y^2+10z^2$ are exactly those of the form $4^k(16l+6)$ with $k,l\in\mathbb{N}$ and that those positive odd integers not represented by $x^2+y^2+10z^2$ are as follows:

$$3,\ 7,\ 21,\ 31,\ 33,\ 43,\ 67,\ 79,\ 87,\ 133,\ 217,\ 219,\ 223,\ 253,\ 307,\ 391,\ldots.$$

In 1927 L. E. Dickson [3] proved Ramanujan's conjecture about even numbers by a simple argument. However, Ramanujan's conjecture about odd numbers is very difficult. For $n \in \mathbb{N}$, clearly

$$\begin{aligned} &2n+1 = x^2+y^2+10z^2 \text{ for some } x,y,z \in \mathbb{Z} \\ \iff &2n+1 = (2x)^2+10y^2+(2z+1)^2 \text{ for some } x,y,z \in \mathbb{Z} \\ \iff &n = 2x^2+5y^2+4T_z \text{ for some } x,y,z \in \mathbb{Z}. \end{aligned}$$

Only in 1990 were W. Duke and R. Schulze-Pillot [6] able to show that sufficiently large odd integers can be written in the form $x^2+y^2+10z^2$, or equivalently that the form $2x^2+5y^2+4T_z$ is almost universal. In 1997 K. Ono and K. Soundararajan [20] showed further that the generalized Riemann hypothesis implies that the only positive odd integers not in the form $x^2+y^2+10z^2$ are those listed by Ramanujan together with 679 and 2719; in other words $E(2x^2+5y^2+4T_z)$ consists of the following numbers:

1, 3, 10, 15, 16, 21, 33, 39, 43, 66, 108, 109, 111, 126, 153, 195, 339, 1359.

Motivated by his conjecture on sums of primes and triangular numbers (cf. [29, Conjecture 1.1]), the second author [28] conjectured that for any $k,l \in \mathbb{N}$ the form $2^kx^2+2^ly^2+T_z$ is almost universal. The first author [12] showed that all of those forms conjectured to be almost universal in [28] are asymptotically universal and that many of them are almost universal.

In this paper we aim at determining all asymptotically universal forms and almost universal forms of the three types via modular forms and the theory of quadratic forms.

For convenience we introduce some basic notation. We may write a positive integer a in the form $2^{v_2(a)}a'$ with $v_2(a) \in \mathbb{N}$ and a' odd; $v_2(a)$ is called the 2-adic order of a (equivalently, $2^{v_2(a)}\|a$), while a' is said to be the odd part of a. For $a \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$, by $a\ R\ m$ we mean that a is quadratic residue modulo m, i.e., a is relatively prime to m and the equation $x^2 \equiv a \pmod{m}$ is solvable over $\mathbb{Z}$. For an integer a and a positive odd integer m, it is well known that $a\ R\ m$ if and only if the Legendre symbol $(\frac{a}{p})$ equals 1 for every prime divisor p of m.

Now we state our results on asymptotically universal forms.

Theorem 1.1. *Fix $a,b,c \in \mathbb{Z}^+$ with $\gcd(a,b,c)=1$. Then the form*

$$f(x,y,z) := ax^2+by^2+cT_z$$

is asymptotically universal if and only if we have the following:

(1) $-2bc\ R\ a'$, $-2ac\ R\ b'$, *and* $-ab\ R\ c'$.
(2) *Either* $4 \nmid c$, *or both* $4\|c$ *and* $2\|ab$.

Theorem 1.2. *Fix $a,b,c \in \mathbb{Z}^+$ with $\gcd(a,b,c)=1$. Then the form*

$$f(x,y,z) := ax^2+bT_y+cT_z$$

is asymptotically universal if and only if we have the following:

(1) $-bc\ R\ a'$, $-2ac\ R\ b'$, *and* $-2ab\ R\ c'$.
(2) *Either* $4 \nmid b$ *or* $4 \nmid c$.

Theorem 1.3. *Fix $a,b,c \in \mathbb{Z}^+$ with $\gcd(a,b,c)=1$. Then the form*

$$f(x,y,z) := aT_x+bT_y+cT_z$$

is asymptotically universal if and only if

$$-bc\ R\ a',\ -ac\ R\ b',\ and\ -ab\ R\ c'.$$

Remark 1.1. For $a, b, c \in \mathbb{Z}^+$, if the form $ax^2 + by^2 + cT_z$ or $ax^2 + bT_y + cT_z$ or $aT_x + bT_y + cT_z$ is asymptotically universal, then a', b', c' must be pairwise coprime by Theorems 1.1-1.3.

The law of quadratic reciprocity gives restrictions under which the relations in the above theorems cannot occur.

Corollary 1.4. *Fix $a, b, c \in \mathbb{Z}^+$ and consider*

$$(1)\ ax^2 + by^2 + cT_z, \qquad (2)\ aT_x + bT_y + cz^2, \qquad (3)\ aT_x + bT_y + cT_z,$$
$$(4)\ ax^2 + bT_y + cz^2, \qquad (5)\ ax^2 + bT_y + cT_z.$$

(i) *If $a' \equiv b' \equiv -c' \pmod 8$, then none of (1)-(5) is asymptotically universal.*
(ii) *If*

$$\begin{cases} a' \equiv b' \equiv c' + 4 \pmod 8, \\ v_2(a) \not\equiv v_2(b) \pmod 2 \end{cases} \quad or \quad \begin{cases} \pm a' \equiv -b' \equiv c' + 4 \pmod 8, \\ v_2(a) \equiv v_2(b) \pmod 2, \end{cases}$$

then none of (1)-(3) is asymptotically universal.
(iii) *If*

$$\begin{cases} a' \equiv b' \equiv c' + 4 \pmod 8, \\ v_2(a) \equiv v_2(b) \pmod 2 \end{cases} \quad or \quad \begin{cases} \pm a' \equiv -b' \equiv c' + 4 \pmod 8, \\ v_2(a) \not\equiv v_2(b) \pmod 2, \end{cases}$$

then neither (4) nor (5) is asymptotically universal.

Corollary 1.5. *Let $a, b, c \in \mathbb{Z}^+$ with $v_2(b) \equiv v_2(c) \pmod 2$. Assume that $a' \equiv b' \equiv c' \pmod 4$ fails. Then, either none of the forms $ax^2+by^2+2cT_z$ and $ax^2+cy^2+2bT_z$ is asymptotically universal or none of the forms $ax^2+2cy^2+bT_z$ and $ax^2+2by^2+cT_z$ is asymptotically universal.*

Any $n \in \mathbb{Z}^+$ can be uniquely written in the form a^2q with $a, q \in \mathbb{Z}^+$ and q squarefree, and we use $\mathcal{SF}(n)$ to denote $q = \prod_{p|n,\, 2\nmid v_p(n)} p$, the squarefree part of n.

Now we turn to almost universal forms.

Theorem 1.6. *Let $a, b, c \in \mathbb{Z}^+$ with $\gcd(a, b, c) = 1$ and $v_2(a) \geq v_2(b)$. Suppose that both (1) and (2) in Theorem 1.1 hold. Then there are infinitely many positive integers not represented by the form*

$$f(x, y, z) := ax^2 + by^2 + cT_z$$

(i.e., f is not almost universal) if and only if we have the following:

(1) $2|a$, $4 \nmid c$, $a' \equiv b' \pmod{2^{3-v_2(c)}}$, *and*

$$\begin{cases} 4 \nmid b \Rightarrow v_2(a) \equiv c \pmod 2, \\ 2 \nmid bc \Rightarrow 8 \mid a \ \&\ 8 \mid (b - c). \end{cases}$$

(2) *All prime divisors of $\mathcal{SF}(a'b'c')$ are congruent to 1 modulo 4 if $v_2(a) \equiv v_2(b) \pmod 2$, and congruent to 1 or 3 modulo 8 otherwise.*
(3) $2^{3-v_2(c)}(ax^2 + by^2) + c'z^2 = \mathcal{SF}(a'b'c')$ *has no integral solutions.*

Remark 1.2. When $ax^2 + by^2 + cT_z$ (with $a, b, c \in \mathbb{Z}^+$) is asymptotically universal it is not necessary that (2) in Theorem 1.6 holds. For example, $6x^2 + y^2 + 10T_z$ is asymptotically universal, but we don't have (2) in Theorem 1.6 with $a = 6$, $b = 1$ and $c = 10$.

Example 1.1. Consider those forms $ax^2+by^2+cT_z$ with $a, b, c \in \mathbb{Z}^+$ and $a+b+c \le 10$. By Theorem 1.6, we find that those asymptotically universal ones are all almost universal. Below is a complete list of those forms $ax^2 + by^2 + cT_z$ with $a, b, c \in \mathbb{Z}^+$ and $a + b + c \le 10$ which are almost universal but not universal:

$$\begin{aligned}
&x^2 + 2y^2 + 3T_z,\ 2x^2 + 4y^2 + T_z,\ x^2 + 5y^2 + 2T_z,\ x^2 + 6y^2 + T_z,\\
&x^2 + y^2 + 5T_z,\ 2x^2 + 3y^2 + 2T_z,\ \ 2x^2 + 5y^2 + T_z,\ 3x^2 + 4y^2 + T_z,\\
&x^2 + 2y^2 + 6T_z,\ x^2 + 5y^2 + 3T_z,\ 2x^2 + 2y^2 + 5T_z,\ 2x^2 + 4y^2 + 3T_z,\\
&4x^2 + 4y^2 + T_z,\ x^2 + 4y^2 + 5T_z,\ 2x^2 + 3y^2 + 5T_z.
\end{aligned}$$

For the four forms in the first row, the second author [27] conjectured that

$$\begin{aligned}
&E(x^2 + 2y^2 + 3T_z) = \{23\},\ E(2x^2 + 4y^2 + T_z) = \{20\},\\
&E(x^2 + 5y^2 + 2T_z) = \{19\},\ E(x^2 + 6y^2 + T_z) = \{47\},
\end{aligned}$$

which was confirmed by the first author [12] under the generalized Riemann hypothesis. For the form $4x^2 + 4y^2 + T_z$ the second author [29] conjectured that $E(4x^2 + 4y^2 + T_z)$ consists of the following 19 numbers:

$$\begin{aligned}
&2,\ 12,\ 13,\ 24,\ 27,\ 34,\ 54,\ 84,\ 112,\ 133,\\
&162,\ 234,\ 237,\ 279,\ 342,\ 399,\ 652,\ 834,\ 864.
\end{aligned}$$

For the ten remaining forms on the above list, our computation via computer suggests the following information:

$$\begin{aligned}
&E(x^2 + y^2 + 5T_z) = \{3,\ 11,\ 12,\ 27,\ 129,\ 138,\ 273\},\\
&E(2x^2 + 3y^2 + 2T_z) = \{1,\ 19,\ 43,\ 94\},\ E(2x^2 + 5y^2 + T_z) = \{4,\ 27\},\\
&E(3x^2 + 4y^2 + T_z) = \{2,\ 11,\ 23,\ 50,\ 116,\ 135,\ 138\},\\
&E(x^2 + 2y^2 + 6T_z) = \{5,\ 13,\ 46,\ 161\},\\
&E(x^2 + 5y^2 + 3T_z) = \{2,\ 11,\ 26,\ 37,\ 40,\ 53,\ 62,\ 142,\ 220,\ 425,\ 692\},\\
&\max E(2x^2 + 2y^2 + 5T_z) = 2748,\ \max E(2x^2 + 4y^2 + 3T_z) = 3185,\\
&\max E(x^2 + 4y^2 + 5T_z) = 2352,\ \max E(2x^2 + 3y^2 + 5T_z) = 933.
\end{aligned}$$

Under the generalized Riemann hypothesis, the argument of Ono and Soundararajan [20] would allow one to use Waldspurger's theorem [30] (or a Kohnen-Zagier variant [15] when the corresponding modular form is in Kohnen's plus space) to determine effectively a computationally feasible bound beyond which every integer is represented and hence verify that the above lists (and all lists contained herein) are indeed complete. This is done by carefully comparing the growth of the class numbers of imaginary quadratic fields with the growth of coefficients of a particular cusp form.

Recall that $\{x^2 + 2T_y : \ x, y \in \mathbb{Z}\} = \{T_x + T_y : \ x, y \in \mathbb{Z}\}$ as observed by Euler. (See, e.g., (3.6.3) of [1, p.71], and [27, Lemma 1].) Thus we say that $x^2 + 2T_y$ is equivalent to $T_x + T_y$ and denote this by $x^2 + 2T_y \sim T_x + T_y$.

Corollary 1.7. *Let $a, b, c \in \mathbb{Z}^+$ with c odd. Then,*

$$\begin{aligned} &\text{all sufficiently large odd integers have the form } 2ax^2 + 2by^2 + cz^2 \\ \iff\ & ax^2 + by^2 + 4cT_z \text{ is almost universal} \\ \iff\ & 2\|ab,\ -ab\ R\ c,\ -2ac\ R\ b' \text{ and } -2bc\ R\ a'. \end{aligned}$$

In particular,

$$\begin{aligned} &\text{all sufficiently large odd numbers are represented by } 2ax^2 + c(y^2 + z^2) \\ \iff\ & ax^2 + 2cy^2 + 4cT_z \sim ax^2 + 2c(T_y + T_z) \text{ is almost universal} \\ \iff\ & c = 1, \text{ and all prime divisors of } a \text{ are congruent to } 1 \bmod 4. \end{aligned}$$

Remark 1.3. In 2005 L. Panaitopol [21] showed that for $a, b, c \in \mathbb{Z}^+$ with $a \le b \le c$ all positive odd integers can be written in the form $ax^2 + by^2 + cz^2$ with $x, y, z \in \mathbb{Z}$, if and only if the vector (a, b, c) is $(1, 1, 2)$ or $(1, 2, 3)$ or $(1, 2, 4)$. For $n \in \mathbb{N}$, clearly $2n + 1 = x^2 + 2y^2 + 3z^2$ for some $x, y, z \in \mathbb{Z}$ if and only if there are $x, y, z \in \mathbb{Z}$ such that $2n + 1 = (8T_x + 1) + 2y^2 + 3(2z)^2$ (i.e., $n = 4T_x + y^2 + 6z^2$) or $2n + 1 = (2x)^2 + 2y^2 + 3(8T_z + 1)$ (i.e., $n - 1 = 2x^2 + y^2 + 12T_z$). By Corollary 1.7, the forms $6x^2 + y^2 + 4T_z$ and $2x^2 + y^2 + 12T_z$ are almost universal. Our computation suggests that

$$E(6x^2 + y^2 + 4T_z) = \{2,\ 3,\ 17,\ 23,\ 38,\ 51,\ 86,\ 188\}$$

and

$$E(2x^2 + y^2 + 12T_z) = \{5,\ 7,\ 10,\ 26,\ 35,\ 65,\ 92,\ 127,\ 322\}.$$

Corollary 1.8. *Let $a, b \in \mathbb{Z}^+$ with b odd. If $\mathcal{SF}(a')$ or $\mathcal{SF}(b)$ has a prime divisor $p \equiv 3 \pmod 4$ (which happens when a' or b is congruent to $3 \bmod 4$), then*

$$\begin{aligned} & ax^2 + by^2 + 2T_z \text{ is almost universal} \\ \iff\ & ax^2 + y^2 + 2bT_z \text{ is almost universal} \\ \iff\ & -a\ R\ b \text{ and } -b\ R\ a' \end{aligned}$$

and

$$\begin{aligned} & ax^2 + 2y^2 + bT_z \text{ is almost universal} \\ \iff\ & ax^2 + 2by^2 + T_z \text{ is almost universal} \\ \iff\ & -2a\ R\ b \text{ and } -b\ R\ a'. \end{aligned}$$

Corollary 1.9. *Let a be any positive integer.*

(i) *The form $ax^2 + y^2 + T_z$ is almost universal if and only if $-2\ R\ a'$ (i.e., every odd prime divisor of a is congruent to 1 or 3 modulo 8). Also, $ax^2 + 2y^2 + 2T_z$ is almost universal if and only if each prime divisor of a is congruent to 1 or 3 modulo 8.*

(ii)

$$\begin{aligned} & ax^2 + 2y^2 + T_z \text{ is almost universal} \\ \iff\ & ax^2 + y^2 + 2T_z \sim ax^2 + T_y + T_z \text{ is almost universal} \\ \iff\ & -1\ R\ a', \text{ i.e., every odd prime divisor of } a \text{ is congruent to } 1 \bmod 4. \end{aligned}$$

Also,

$$ax^2 + 2y^2 + 4T_z \sim ax^2 + 2T_y + 2T_z \text{ is almost universal}$$
$$\iff \text{all prime divisors of } a \text{ are congruent to } 1 \bmod 4$$

and

$$ax^2 + 4y^2 + 2T_z \text{ is almost universal}$$
$$\iff a \equiv 1 \pmod 8 \text{ and each prime divisor of } a \text{ is congruent to } 1 \bmod 4.$$

Example 1.2. By Corollary 1.9, the form $5x^2 + 4y^2 + 2T_z$ is not almost universal, although it is asymptotically universal. Also, our computation suggests the following information:

$$E(11x^2 + y^2 + T_z) = \{8, 34, 348\} \quad \text{and} \quad E(12x^2 + y^2 + T_z) = \{8, 20, 146, 275\}.$$

Corollary 1.10. *Let $a \in \mathbb{Z}^+$. Then*

$$ax^2 + 3y^2 + T_z \text{ is almost universal (or asymptotically universal)}$$
$$\iff a \equiv 1 \pmod 3, \text{ and } \lfloor p/12 \rfloor \text{ is even for any odd prime divisor } p \text{ of } a$$

and

$$ax^2 + y^2 + 3T_z \text{ is almost universal (or asymptotically universal)}$$
$$\iff a \equiv 2 \pmod 3, \text{ and } \lfloor p/12 \rfloor \text{ is even for any odd prime divisor } p \text{ of } a.$$

Also,

$$ax^2 + 2y^2 + 6T_z \text{ is almost universal (or asymptotically universal)}$$
$$\iff a \equiv 1 \pmod 6, \text{ and } \lfloor p/12 \rfloor \text{ is even for any prime divisor } p \text{ of } a$$

and

$$ax^2 + 6y^2 + 2T_z \text{ is almost universal (or asymptotically universal)}$$
$$\iff a \equiv 5 \pmod 6, \text{ and } \lfloor p/12 \rfloor \text{ is even for any prime divisor } p \text{ of } a.$$

Corollary 1.11. *Let m be a positive integer.*

(i) *$x^2 + y^2 + mT_z$ is almost universal if and only if $4 \nmid m$ and all odd prime divisors of m are congruent to $1 \bmod 4$. Also, $2x^2 + y^2 + mT_z$ is almost universal if and only if $8 \nmid m$ and each odd prime divisor of m is congruent to 1 or $3 \bmod 8$.*

(ii) *Let $k \in \mathbb{Z}^+$. Then the form $2^{2k}x^2 + y^2 + mT_z$ is almost universal if and only if $4 \nmid m$, $-1\ R\ m'$ and*

$$2 \| m \implies m \text{ is squarefree.}$$

Also, the form $2^{2k+1}x^2 + y^2 + mT_z$ is almost universal if and only if $4 \nmid m$, $-2Rm'$ and

$$m \equiv 1 \pmod 8 \implies m \text{ is squarefree.}$$

(iii) *Let $k, l \in \mathbb{Z}^+$ with $k \geq l$. Then $2^k x^2 + 2^l y^2 + mT_z$ is asymptotically universal if and only if for each prime divisor p of m we have*

$$\begin{cases} p \equiv 1 \pmod 4 & \text{if } k \equiv l \pmod 2, \\ p \equiv 1 \text{ or } 3 \pmod 8 & \text{otherwise.} \end{cases}$$

When $2^k x^2 + 2^l y^2 + mT_z$ is asymptotically universal, it is almost universal if and only if m is squarefree, or both $2 \mid k$ and $l = 1$.

Example 1.3. By Corollary 1.11, the forms $4x^2 + y^2 + 50T_z$, $8x^2 + y^2 + 9T_z$ and $2x^2 + 2y^2 + 25T_z$ are not almost universal, although they are asymptotically universal. We also have the following observations via computation:

$$\max E(x^2 + y^2 + 10T_z) = 546,\ \max E(2x^2 + y^2 + 11T_z) = 985,$$
$$\max E(4x^2 + y^2 + 10T_z) = 5496,\ \max E(4x^2 + 2y^2 + 9T_z) = 9555,$$
$$\max E(2x^2 + 2y^2 + 13T_z) = 22176,\ \max E(8x^2 + y^2 + 3T_z) = 499.$$

Corollary 1.12. *Let $a \in \mathbb{Z}^+$ be even.*

(i) *Suppose that $v_2(a)$ is even and each odd prime divisor of a is congruent to 1 modulo 3. Then $ax^2 + 216y^2 + T_z$ is asymptotically universal. Moreover, this form is not almost universal if and only if every prime divisor of $\mathcal{SF}(a')$ is congruent to 1 or 19 modulo 24, and the number of prime divisors congruent to 19 modulo 24 is odd.*

(ii) *Assume that $v_2(a)$ is odd, $a' \equiv \pm 1 \pmod{10}$ and $2 \mid \lfloor p/10 \rfloor$ for every prime divisor p of a'. Then $ax^2 + 250y^2 + T_z$ is asymptotically universal. Moreover, this form is not almost universal if and only if $a' \equiv 21, 29 \pmod{40}$ and every prime divisor of $\mathcal{SF}(a')$ is congruent to 1 or 9 modulo 20.*

Remark 1.4. Corollary 1.12 implies that the forms $76x^2 + 216y^2 + T_z$ and $58x^2 + 250y^2 + T_z$ are asymptotically universal but not almost universal.

For the form $ax^2 + bT_y + cT_z$, we obtain the following result.

Theorem 1.13. *Let $a, b, c \in \mathbb{Z}^+$ with $\gcd(a, b, c) = 1$ and $v_2(b) \geq v_2(c)$. Consider the form*

$$f(x, y, z) := ax^2 + bT_y + cT_z$$

and assume that both (1) and (2) in Theorem 1.2 hold.

(i) *When $v_2(b) \notin \{3, 4\}$, f is not almost universal if and only if we have the following:*

(1) *$4 \nmid b + c$ and $\mathcal{SF}(a'b'c') \equiv (b + c)' \pmod{2^{3-v}}$, where $v := v_2(b + c) < 2$.*
(2) *All prime divisors of $\mathcal{SF}(a'b'c')$ are congruent to 1 or 3 modulo 8 if $\mathcal{SF}(abc) \equiv b + c \pmod 2$ and are congruent to 1 modulo 4 otherwise.*
(3) *$8ax^2 + by^2 + cz^2 = 2^v\mathcal{SF}(a'b'c')$ has no integral solutions with y and z odd.*
(4)

$$\begin{cases} v_2(b) \leq 1 \Rightarrow v_2(a) - v_2(b) \in \{2, 4, 6, \ldots\}, \\ v_2(b) = 2 \Rightarrow v_2(a) \in \{1, 3, 5, \ldots\}, \\ v_2(b) \in \{5, 7, \ldots\} \Rightarrow (4 \mid a \text{ or } 2 \mid c), \\ v_2(b) \in \{6, 8, \ldots\} \Rightarrow (2 \mid a \text{ or } a \equiv c \pmod 8). \end{cases}$$

(ii) *In the case $v_2(b) \in \{3, 4\}$, if f is not almost universal, then $(1) - (3)$ above hold and also*

$$\begin{cases} v_2(b) = 3 \Rightarrow (4 \mid a \text{ or } 2 \mid c), \\ v_2(b) = 4 \Rightarrow (2 \mid a \text{ or } a \equiv c \pmod 8). \end{cases}$$

Moreover, provided $(1) - (3)$ in part (i) and the condition $2 \nmid v_2(a)$, f is not almost universal if $v_2(b) = 4$, or $v_2(a) \geq v_2(b) = 3$ and $b' \equiv c' \pmod 8$.

Example 1.4. Consider those forms $ax^2 + bT_y + cT_z$ with $a, b, c \in \mathbb{Z}^+$ and $a + b + c \leq 10$. By Theorem 1.13, we find that those asymptotically universal ones are almost universal. Below is a complete list of those forms $ax^2+bT_y+cT_z$ with $a,b,c\in\mathbb{Z}^+$ and $a+b+c\le 10$ which are almost universal but not universal:

$$5x^2+T_y+T_z\sim x^2+5y^2+2T_z,\ 5x^2+2T_y+2T_z\sim 2x^2+5y^2+4T_z,$$
$$8x^2+T_y+T_z\sim x^2+8y^2+2T_z,\ 2x^2+3T_y+2T_z,\ x^2+4T_y+3T_z,$$
$$2x^2+5T_y+T_z,\ 4x^2+3T_y+T_z,\ 3x^2+5T_y+T_z,\ 3x^2+4T_y+2T_z,$$
$$4x^2+4T_y+T_z,\ 6x^2+2T_y+T_z,\ 5x^2+3T_y+2T_z,\ 5x^2+4T_y+T_z.$$

For the above forms from the second line, our computation via computer suggests the following information:

$$E(8x^2+T_y+T_z)=E(x^2+8y^2+2T_z)=\{5,\ 40,\ 217\},$$
$$E(2x^2+3T_y+2T_z)=\{1,\ 16\},\ E(x^2+4T_y+3T_z)=\{2,\ 6,\ 80\},$$
$$E(2x^2+5T_y+T_z)=\{4\},\ E(4x^2+3T_y+T_z)=\{2,\ 11,\ 27,\ 38,\ 86,\ 93,\ 188,\ 323\},$$
$$E(3x^2+5T_y+T_z)=\{2,\ 7\},\ E(3x^2+4T_y+2T_z)=\{1,\ 8,\ 11,\ 25\},$$
$$E(4x^2+4T_y+T_z)=\{2,\ 108\},\ E(6x^2+2T_y+T_z)=\{4\},$$
$$E(5x^2+3T_y+2T_z)=\{1,4,13,19,27,46,73,97,111,123,151,168\},$$
$$E(5x^2+4T_y+T_z)=\{2,\ 16,\ 31\}.$$

In Corollary 1.9 we determined when $ax^2+T_x+T_y$ or $ax^2+2T_x+2T_y$ is almost universal. The following corollary deals with two other similar forms.

Corollary 1.14. *Let a be a positive integer. Then $ax^2+2T_y+T_z$ is almost universal if and only if all odd prime divisors of a are congruent to 1 or 3 $\mathrm{mod}\ 8$. Also, $ax^2+4T_y+T_z$ is almost universal if and only if all odd prime divisors of a are congruent to 1 $\mathrm{mod}\ 4$.*

Example 1.5. By means of computation, we believe that

$$E(9x^2+2T_y+T_z)=\{4\}\quad\text{and}\quad E(11x^2+2T_y+T_z)=\{4,\ 25,\ 94,\ 123\}.$$

Corollary 1.15. *Let m be any positive integer.*

(i) *If all odd prime divisors of m are congruent to 1 or 3 $\mathrm{mod}\ 8$, and $m'\equiv 3 \pmod 8$ or $v_2(m)\ne 4,6,\ldots$, then $x^2+T_y+mT_z$ is almost universal. The converse also holds when $v_2(m)\ne 4$.*

(ii) *For $k\in\mathbb{Z}^+\setminus\{3,4\}$, the form $2^k(x^2+T_y)+mT_z$ is almost universal if and only if $k\in\{1,2\}$ and all prime divisors of m are congruent to 1 or 3 $\mathrm{mod}\ 8$. When $m\equiv 1 \pmod 8$, the form $8(x^2+T_y)+mT_z$ is not almost universal.*

Example 1.6. By Corollary 1.15, the form $8x^2+8T_y+T_z$ is not almost universal, although it is asymptotically universal. We also have the following guess based on our computation:

$$E(x^2+T_y+9T_z)=\{8,\ 47\},\ E(x^2+T_y+11T_z)=\{8\},\ E(x^2+T_y+12T_z)=\{8,\ 20\}.$$

Corollary 1.16. *Let m be any positive integer.*

(i) *When $v_2(m)\ne 3$, the form $x^2+2T_y+mT_z\sim T_x+T_y+mT_z$ is almost universal if and only if all odd prime divisors of m are congruent to 1 $\mathrm{mod}\ 4$ and $v_2(m)\ne 5,7,\ldots$.*

(ii) *For $k\in\mathbb{Z}^+\setminus\{2\}$, the form $2^k(x^2+2T_y)+mT_z\sim 2^k(T_x+T_y)+mT_z$ is almost universal if and only if $k=1$ and all prime divisors of m are congruent to 1 $\mathrm{mod}\ 4$.*

Remark 1.5. By Corollary 1.16(ii), $8x^2 + 16T_y + T_z \sim 8(T_x + T_y) + T_z$ is not almost universal, although it is asymptotically universal. In [29] the second author conjectured that any integer $n > 1029$ is either a triangular number or a sum of two odd squares and a triangular number (i.e., $n = (8T_x + 1) + (8T_y + 1) + T_z$ for some $x, y, z \in \mathbb{Z}$); in other words,

$$E(8T_x + 8T_y + T_z) \cap [1028, +\infty) \subseteq \{T_m - 2 : \ m \in \mathbb{Z}^+\}.$$

Recently, Oh and the second author [16] showed that $T_m - 2 \in E(8T_x + 8T_y + T_z)$ (i.e., T_m is not a sum of two odd squares and a triangular number) if and only if $2m + 1$ is a prime congruent to 3 mod 4.

Example 1.7. Via computation we make the following observation:

$$\begin{aligned} E(x^2 + 2T_y + 10T_z) &= E(T_x + T_y + 10T_z) = \{5, 8\}, \\ E(x^2 + 2T_y + 13T_z) &= E(T_x + T_y + 13T_z) = \{5, 8, 32, 53\}. \end{aligned}$$

Theorem 1.17. *Let* $a, b, c \in \mathbb{Z}^+$ *with* $v_2(a) \ge v_2(b) \ge v_2(c) = 0$. *Assume that* $-bc \ R \ a'$, $-ac \ R \ b'$ *and* $-ab \ R \ c'$. *Consider the form*

$$f(x, y, z) := aT_x + bT_y + cT_z.$$

(i) *If* f *is not almost universal, then we have the following:*

(1) $4 \nmid a + b + c$ *and* $\mathcal{SF}(a'b'c') \equiv (a + b + c)' \pmod{2^{3-v}}$, *where* $v = v_2(a + b + c) < 2$.

(2) *All prime divisors of* $\mathcal{SF}(a'b'c')$ *are congruent to* 1 *modulo* 4 *if* $\mathcal{SF}(abc) \equiv a + b + c \pmod 2$ *and are congruent to* 1 *or* 3 *modulo* 8 *otherwise.*

(3) $ax^2 + by^2 + cz^2 = 2^v \mathcal{SF}(a'b'c')$ *has no integral solution with* x, y, z *all odd.*

(4)

$$\begin{cases} v_2(b) \le 1 \Rightarrow v_2(a) - v_2(b) \in \{3, 5, 7, \ldots\}, \\ v_2(b) = 2 \Rightarrow v_2(a) \in \{2, 4, 6, \ldots\}. \end{cases}$$

(ii) f *is not almost universal under* (1)-(3) *in part* (i), *and the following condition stronger than* (4):

$$\begin{cases} v_2(b) \le 1 \Rightarrow v_2(a) - v_2(b) \in \{5, 7, \ldots\}, \\ v_2(b) \in \{2, 4\} \Rightarrow v_2(a) \in \{4, 6, \ldots\}, \\ v_2(b) = 3 \Rightarrow (v_2(a) \in \{6, 8, \ldots\} \ \& \ b' \equiv c \pmod 8). \end{cases}$$

Example 1.8. Consider those forms $aT_x + bT_y + cT_z$ with $a, b, c \in \mathbb{Z}^+$ and $a + b + c \le 10$. By Theorem 1.3, we find the following complete list of those asymptotically universal forms which are not universal:

$$\begin{aligned} &T_x + 4T_y + 4T_z \sim 4x^2 + 8T_y + T_z, \\ &T_x + T_y + 8T_z \sim x^2 + 8T_y + 2T_z, \\ &2T_x + 2T_y + 5T_z \sim 2x^2 + 4T_y + 5T_z, \\ &T_x + 2T_y + 6T_z, \ 2T_x + 3T_y + 4T_z, \ T_x + 4T_y + 5T_z. \end{aligned}$$

By Theorem 1.17, the last four forms are in fact almost universal; our computation via computer suggests the following information:

$$E(2T_x + 2T_y + 5T_z) = E(2x^2 + 4T_y + 5T_z) = \{1, 3, 10, 16, 28, 43, 46, 85, 169, 175, 211, 223\}$$

and

$$E(T_x+2T_y+6T_z)=\{4,\ 50\}, E(2T_x+3T_y+4T_z)=\{1,\ 8,\ 31\},\ E(T_x+4T_y+5T_z)=\{2\}.$$

As for the first two forms $T_x + 4T_y + 4T_z$ and $T_x + T_y + 8T_z$, neither Theorem 1.17 nor Theorem 1.13 tells us whether or not they are almost universal. However, with some special arguments, the first author [12] was able to show that they are not almost universal, although they are asymptotically universal. By Theorem 1.1(ii) of an earlier paper [16], $E(T_x + T_y + 8T_z)$ actually consists of those $2T_m - 1$ ($m \in \mathbb{Z}^+$) with $2m + 1$ having no prime divisors congruent to 3 mod 4. Similarly, by [27, Theorem 1(iii)] and [16, Theorem 2.1(ii)], $E(T_x + 4T_y + 4T_z)$ consists of those $T_m - 1$ ($m \in \mathbb{Z}^+$) with $2m + 1$ having no prime divisors congruent to 3 mod 4.

Corollary 1.18. *Let $a \in \mathbb{Z}^+$. Then the form $aT_x + 2T_y + T_z$ is almost universal if each odd prime divisor of a is congruent to* 1 *or* 3 mod 8*, and either $a' \equiv 1 \pmod 8$ or $v_2(a) \neq 4, 6, \ldots$. We also have the converse when $v_2(a) \neq 4$.*

Remark 1.6. In [12] the first author was able to show that the special form $48T_x + 2T_y + T_z$ is not almost universal (though it is asymptotically universal by Theorem 1.3).

Example 1.9. Our computation leads us to make the following observation:

$$E(9T_x + 2T_y + T_z) = \{4,\ 46\},\ E(11T_x + 2T_y + T_z) = \{4,\ 25\},$$
$$E(22T_x + 2T_y + T_z) = \{4,\ 11,\ 14,\ 19,\ 46,\ 54\}.$$

Our following conjecture is a supplement to Theorems 1.13 and 1.17; its solution might involve a further investigation of the spinor norm mapping or alternation of certain coefficients of cusp forms.

Conjecture 1.19. *Let a, b, c be positive integers.*

(i) *In the case $v_2(b) \geq v_2(c)$ and $v_2(b) \in \{3, 4\}$, if $(1) - (3)$ in Theorem* 1.13 *hold and also*

$$\begin{cases} v_2(b) = 3 \Rightarrow (4 \mid a \ \text{or}\ 2 \mid c), \\ v_2(b) = 4 \Rightarrow (2 \mid a \ \text{or}\ a \equiv c \pmod 8), \end{cases}$$

then the form $ax^2 + bT_y + cT_z$ is not almost universal.

(ii) *In the case $v_2(a) \geq v_2(b) \geq v_2(c) = 0$, if $(1) - (3)$ in Theorem* 1.17 *hold and*

$$v_2(a) = v_2(b) = 2 \quad \text{or} \quad v_2(a) = v_2(b) + 3 \in \{3, 4\} \quad \text{or} \quad v_2(b) \in \{3, 4\},$$

then the form $aT_x + bT_y + cT_z$ is not almost universal.

In the next section we are going to introduce some further notation and give an overview of our method. In Section 3 we will deal with asymptotically universal forms and prove Theorems 1.1-1.3 and Corollaries 1.4-1.5. Section 4 is devoted to the proofs of the remaining theorems and corollaries concerning almost universal forms.

2. Notation and brief overview

Our arguments will involve the theory of modular forms and spinor exceptional square classes for quadratic forms. A good introduction to modular forms may be found in Ono's book [18], and a good introduction to quadratic forms may be found in O'Meara's book [17]. We will first reduce the questions at hand to questions about certain related (ternary) quadratic forms. Since $8T_x + 1$ is an odd square, multiplying by 8 and adding some positive integer will give a form $Q(x, y, z)$ which is a sum of squares with the restriction that some of x, y, and z must be odd. If we take $r_Q(n)$ to be the number of solutions to $Q(x, y, z) = n$ with the given restrictions on $x, y, z \in \mathbb{Z}$, then define

$$\theta_Q(\tau) := \sum_{n=0}^{\infty} r_Q(n) q^n,$$

where $q = e^{2\pi i\tau}$ with τ in the upper half plane. Since the number of solutions with z odd equals the number of solutions with z arbitrary minus the number of solutions with z even and since the form with z even gives another quadratic form, we get an inclusion/exclusion of theta series of quadratic forms. Let a ternary quadratic form $Q'(x, y, z)$ be given. Then it is well known that the theta series

$$\theta_{Q'}(\tau) := \sum_{n=0}^{\infty} r_{Q'}(n) q^n$$

is a modular form of weight 3/2, where $r_{Q'}(n)$ is the number of solutions to $Q'(x, y, z) = n$ with $x, y, z \in \mathbb{Z}$. The theta series splits naturally into the following three parts:

$$\theta_{Q'} = \theta_{gen(Q')} + \left(\theta_{spn(Q')} - \theta_{gen(Q')}\right) + \left(\theta_{Q'} - \theta_{spn(Q')}\right),$$

where the n-th coefficients of $\theta_{gen(Q')}$ and $\theta_{spn(Q')}$ are the weighted average of the number of representations of n by the genus and the spinor genus of Q', respectively. Furthermore, $\theta_{gen(Q')}$ is an Eisenstein series, $\theta_{spn(Q')} - \theta_{gen(Q')}$ is a cusp form in the space of 1-dimensional theta series, and $\theta_{Q'} - \theta_{spn(Q')}$ is a cusp form in the orthogonal complement of the space of 1-dimensional theta series. For a full description, see the survey paper of Schulze-Pillot [23]. We will then use the argument of Duke and Schulze-Pillot [6].

The coefficients of $\theta_{spn(Q')} - \theta_{gen(Q')}$ are supported at finitely many square classes. If $r_{Q',p^k}(n)$ is the number of solutions to $Q'(x, y, z) = n + p^k\mathbb{Z}$ with $x, y, z \in \mathbb{Z}/p^k\mathbb{Z}$, then the n-th coefficient of the Eisenstein series was shown by Siegel (cf. Jones [11]) to be

$$\prod_{p \text{ prime}} \lim_{k\to\infty} \frac{r_{Q',p^k}(n)}{p^{2k}}. \tag{2.1}$$

An anisotropic prime p is a prime for which the equation $Q'(x, y, z) = 0$ has no non-trivial solutions in $\mathbb{Z}_p$. Notice that for $q \neq p$, $r_{Q',q^k}(np^2) = r_{Q',q^k}(n)$, since p is invertible, and hence we have a bijection between solutions to $Q'(x, y, z) = np^2$ and $Q'(x', y', z') = n$ by taking $(x, y, z) \to p^{-1}(x, y, z)$. Because $Q'(x, y, z) = 0$, if n has sufficiently large divisibility by p (i.e., the order of n at p is large enough) then it is easy to check that $r_{Q',p^k}(np^2) = r_{Q',p^k}(n)$, and hence the np^{2k} coefficients of the Eisenstein series grow like a constant with respect to k.

When n has bounded divisibility at every anisotropic prime (i.e., the orders of n at anisotropic primes are bounded) equation (2.1), and hence the coefficients of the Eisenstein series, grow like a certain class number (cf. Jones [11, Theorem 86]), and hence are (ineffectively) $\gg n^{1/2-\epsilon}$ by the bound of Siegel [26]. The coefficients of the cusp forms in the orthogonal complement of 1-dimensional theta series (to which $\theta_{Q'} - \theta_{spn(Q')}$ belongs) were $\ll n^{1/2-1/28+\epsilon}$ as first obtained by Duke [5], extending

Iwaniec's result [10] (for coefficients of half integral weight $\geq 5/2$ modular forms) to the case of weight $3/2$ modular forms. Better bounds have been obtained by the amplification method on sums of special values of L-series as in Blomer, Harcos, and Michel [2], but the bound above is sufficient to guarantee that if n has bounded divisibility at the anisotropic primes and n is not in one of the finitely many square classes where the coefficients of $\theta_{spn(Q')} - \theta_{gen(Q')}$ are supported, the growth of the coefficients of the Eisenstein series $\theta_{gen(Q')}$ will overwhelm the growth of the coefficients of the cusp form $\theta_{Q'} - \theta_{gen(Q')}$, and hence the coefficients of $\theta_{Q'}$ will be positive for sufficiently large n, with bounded divisibility by the anisotropic primes, outside of these finitely many square classes. Moreover, if we take a weighted sum

$$\sum_{i=1}^{m} w_i \theta_{Q'_i}$$

(such as the inclusion/exclusion above) of finitely many such $\theta_{Q'_i}$ where $\theta_{gen(Q'_i)} = c_i \theta_{gen(Q')}$ and $\sum_{i=1}^{m} w_i c_i > 0$, then the resulting theta series will be

$$\left(\sum_{i=1}^{m} w_i c_i\right)\theta_{gen(Q')} + f_1 + f_2,$$

where $f_1 = \sum_{i=1}^{m} w_i(\theta_{spn(Q'_i)} - \theta_{gen(Q'_i)})$ and $f_2 = \sum_{i=1}^{m} w_i(\theta_{Q'_i} - \theta_{spn(Q'_i)})$. The bound of Duke [5] given above then shows that outside of the finitely many square classes where the coefficients of f_1 are supported, the n-th coefficient of this weighted average is positive for sufficiently large n with bounded divisibility at the anisotropic primes. The condition of the bounded divisibility at anisotropic primes will pose only a minor complication, and we will find in the end that for any asymptotically universal form the associated quadratic forms will never have an anisotropic prime $p \neq 2$, while conditions modulo 8 will guarantee that the coefficients which we are interested in automatically have bounded divisibility by 2.

We will thus be interested in determining which square classes of coefficients $t\mathbb{Z}^2$ are supported by $\theta_{spn(Q')} - \theta_{gen(Q')}$. Kneser [14] gave a necessary condition and later Schulze-Pillot [24] extended this to give necessary and sufficient conditions. For a quadratic form Q', there is an associated bilinear form $B(x,y) = (Q'(x+y) - Q'(x) - Q'(y))/2$. We will call V a (ternary) quadratic space over $\mathbb{Q}_2$ if it is a finite-dimensional vector space over $\mathbb{Q}_2$ with an associated bilinear form B. There is a quadratic form (over $\mathbb{Q}_2$) associated to V given by $Q'(x) = B(x,x)$ for every $x \in V$. Fix a $\mathbb{Z}_2$-lattice L. The quadratic form (over $\mathbb{Z}_2$) associated to L is $Q'(x) = B(x,x)$ with $x \in L$. In our case, the lattice will always have an orthogonal basis x_1, x_2, x_3 with $B(x_i, x_j) = 0$ when $i \neq j$. We will denote the $\mathbb{Z}_2$-lattice whose corresponding quadratic form is $ax^2 + by^2 + cz^2$ by $\langle a,b,c\rangle_2$.

We will denote isometries from V to V by $O(V)$. Let $O^+(V)$ be the subgroup of rotations consisting of isometries with determinant 1. We also use $O^+(L)$ to denote the rotations which fix L. Each rotation is the product of an even number of symmetries, where the symmetry τ_v with $v \in V$ is defined by

$$x \mapsto x - \frac{2B(x,v)}{Q'(v)}v.$$

The spinor norm mapping is the mapping $\theta(\sigma) = Q'(v_1)\cdots Q'(v_m)\mathbb{Q}_2^{\times 2}$ where $\sigma = \tau_{v_1}\cdots\tau_{v_m}$. The set $\theta(O^+(L))$ forms a subgroup of $\mathbb{Q}_2^{\times}/\mathbb{Q}_2^{\times 2}$. For the 2-adic lattice $L = L_2 = \langle a,b,c\rangle_2$, Earnest and Hsia determined this subgroup explicitly in [7].

Fix an imaginary quadratic field extension $K/\mathbb{Q}$ (in our cases, $K = \mathbb{Q}(i)$ or $K = \mathbb{Q}(\sqrt{-2})$) and a prime ideal β (of the ring O_K of algebraic integers in K) dividing 2. For convenience, we define

$$K_n := \mathbb{Q}(\sqrt{-n\mathcal{SF}(n)}).$$

We say that $\alpha \in \mathbb{Q}_2^\times/\mathbb{Q}_2^{\times 2}$ is a local norm at 2 (from the completion K_β to $\mathbb{Q}_2$) if $\alpha = x^2 + ny^2$ for some $x, y \in \mathbb{Q}_2$. We will denote the set of local norms at 2 by $N_2(K)$. Note that

$$N_2(\mathbb{Q}(i)) = \mathbb{Q}_2^{\times 2} \cup 5\mathbb{Q}_2^{\times 2} \cup 2\mathbb{Q}_2^{\times 2} \cup 10\mathbb{Q}_2^{\times 2}$$

and

$$N_2(\mathbb{Q}(\sqrt{-2})) = \mathbb{Q}_2^{\times 2} \cup 3\mathbb{Q}_2^{\times 2} \cup 2\mathbb{Q}_2^{\times 2} \cup 6\mathbb{Q}_2^{\times 2}.$$

Using explicit results of Earnest, Hsia, and Hung [8] based on Schulze-Pillot's classification of spinor exceptional square classes [24], we will reduce the question at hand to showing Kneser's necessary condition at the prime 2. The necessary condition of Kneser which we will need to show is that $\theta(O^+(L)) \subseteq N_2(K)$ (cf. [14]). We will use the explicit results of Earnest and Hsia [7] to determine when the necessary condition is satisfied.

For $a, b \in \mathbb{Q}_2^\times$, the Hilbert symbol $(a, b)_2 \in \{\pm 1\}$ takes the value 1 if and only if $ax^2 + by^2 = z^2$ for some $x, y, z \in \mathbb{Q}_2$ with x, y, z not all zero. We will need the following theorems.

Theorem 2.1 (Earnest and Hsia [7])**.** *Let U denote the group of units in $\mathbb{Z}_2$ and let $\alpha \in U$. Then*

$$\theta(O^+(\langle 1, 2^r\alpha\rangle_2)) = \begin{cases} \{\gamma \in \mathbb{Q}_2^\times : (\gamma, -2\alpha)_2 = 1\} & \text{if } r \in \{1, 3\}, \\ \{\gamma \in U\mathbb{Q}_2^{\times 2} : (\gamma, -\alpha)_2 = 1\} & \text{if } r = 2, \\ \mathbb{Q}_2^{\times 2} \cup \alpha\mathbb{Q}_2^{\times 2} \cup 5\mathbb{Q}_2^{\times 2} \cup 5\alpha\mathbb{Q}_2^{\times 2} & \text{if } r = 4, \\ \mathbb{Q}_2^{\times 2} \cup \alpha\mathbb{Q}_2^{\times 2} & \text{if } r \geq 5. \end{cases}$$

Furthermore, Earnest and Hsia [7, Theorem 2.2] showed that for the lattice $L_2 := \langle c', 2^r b', 2^s a'\rangle_2$, we have $\theta(O^+(L_2)) = \mathbb{Q}_2^\times$ if $\{r, s - r\} \cap \{1, 3\} \neq \emptyset$ and $\{r, s, s - r\} \cap \{2, 4\} \neq \emptyset$. If $0 < r < s$ and the conditions of Theorem 2.2 in [7] are not satisfied, they proved that $\theta(O^+(L_2))$ is equal to the union of the spinor norm restricted to 2-dimensional sublattices, allowing us to use the above theorem. Moreover, if $s \geq 5$ and $r \in \{0, s\}$, then their argument follows mutatis mutandis and will also allow us to reduce the problem to 2-dimensional sublattices.

Since our base field is $\mathbb{Q}_2$ and $K_\beta/\mathbb{Q}_2$ is ramified for $K = \mathbb{Q}(i)$ and $K = \mathbb{Q}(\sqrt{-2})$, we will only need the following restriction of the 2-adic conditions from Earnest, Hsia, and Hung's theorem.

Theorem 2.2 (Earnest, Hsia, and Hung [8])**.** *Let $a, b, c \in \mathbb{Z}^+$ and $K = \mathbb{Q}(\sqrt{-abc})$. Let $L_2 = \langle c', 2^r b', 2^s a'\rangle_2$ with $0 \leq r \leq s$, and let $t \in \mathbb{Z}^+$. Assume that $\theta(O^+(L_2)) \subseteq N_2(K)$, and define*

$$L' = \begin{cases} \langle 2^{r-2}c', 2^r b', 2^s a'\rangle_2 & \text{if } r + s \equiv v_2(t) \pmod 2, \\ \langle 2^{r-3}c', 2^r b', 2^s a'\rangle_2 & \text{otherwise.} \end{cases}$$

Consider the necessary and sufficient conditions given by Schulze-Pillot [24] for t to be a primitive spinor exception for the genus of the quadratic form $Q(x, y, z) = ax^2 + by^2 + cz^2$.

(1) *Set* $L'' := \langle 2^r c', 2^r b', 2^s a'\rangle_2$. *When* $r+s \equiv v_2(t) \pmod 2$, *the Schulze-Pillot conditions are not satisfied if and only if one of the following holds:*
   (a) r *is odd and* $v_2(t) \geq r-3$.
   (b) r *is even,* $\theta(O^+(L')) \not\subseteq N_2(K)$, *and*
$$(r \neq s \ \&\ v_2(t) \geq r-2) \quad or \quad (r = s \ \&\ v_2(t) \geq r).$$
   (c) r *is even,* $\theta(O^+(L')) \subseteq N_2(K)$, $\theta(O^+(L'')) \not\subseteq N_2(K)$ *and* $v_2(t) \geq r$.
   (d) r *is even,* $\theta(O^+(L')) \subseteq N_2(K)$, $\theta(O^+(L'')) \subseteq N_2(K)$ *and* $v_2(t) \geq s$.

(2) *When* $r+s \not\equiv v_2(t) \pmod 2$, *we have* $0 < r < s$, *and the Schulze-Pillot conditions are not satisfied if and only if one of the following holds:*
   (a) r *is even and* $v_2(t) \geq r-4$.
   (b) r *is odd,* $\theta(O^+(L')) \not\subseteq N_2(K)$ *and* $v_2(t) \geq r-3$.
   (c) r *is odd,* $\theta(O^+(L')) \subseteq N_2(K)$ *and* $v_2(t) \geq s-2$.

3. On asymptotically universal forms

In this section, we will show which forms are asymptotically universal, proving Theorems 1.1, 1.2, and 1.3. We will first need the following lemma.

Lemma 3.1. *Fix* $a, b, c \in \mathbb{Z}^+$ *with* $\gcd(a,b,c) = 1$. *Then*
$$Q(x,y,z) = ax^2 + by^2 + cz^2$$
represents every integer p*-adically for each odd prime* p *if and only if we have*
$$-ab \ R\ c', \ -ac \ R\ b', \ and \ -bc \ R\ a'. \tag{3.1}$$

Proof. It is well known that Q represents every integer p-adically for any odd prime p not dividing abc.

Let p be an odd prime divisor of abc. Without loss of generality, we assume that $p \mid c$. If $p \mid ab$, then Q clearly only represents all squares or all non-squares modulo p. Therefore, $p \nmid ab$.

Let a unit $u \in \mathbb{Z}_p$ be given. Suppose that there are $x, y \in \mathbb{Z}_p$ such that $ax^2+by^2 = pu$. If $x \in p\mathbb{Z}_p$, then we have $y \in p\mathbb{Z}_p$ and hence $u \in p\mathbb{Z}_p$, which contradicts the fact that u is a unit. Therefore both x and y must be units. Taking the Legendre symbol of both sides yields that
$$\left(\frac{b}{p}\right) = \left(\frac{by^2}{p}\right) = \left(\frac{pu-ax^2}{p}\right) = \left(\frac{-ax^2}{p}\right) = \left(\frac{-a}{p}\right).$$

Now assume that $(\frac{-ab}{p}) \neq 1$. Let a unit $u \in \mathbb{Z}_p$ be given. From the above, we know that $ax^2+by^2 = up$ does not have a solution. Suppose that there are $x, y, z \in \mathbb{Z}_p$ such that $ax^2+by^2+cz^2 = up$. Then
$$ax^2 + by^2 = \left(u - \frac{c}{p}z^2\right)p.$$
If $p^2 \mid c$, then $u - \frac{c}{p}z^2$ is also a unit, and it follows that up is not represented. In the case $p \| c$, without loss of generality we assume that the unit $\frac{c}{p}$ is a square. If u is not a square, then $u - \frac{c}{p}z^2$ must also be a unit, and it follows that up is not represented.

By the above, (3.1) is a necessary condition.

If $n \in \mathbb{Z}_p$ is represented, then so is np^2. Thus we only need to show that $(\frac{-ab}{p}) = 1$ implies that those $n \in \mathbb{Z}_p$ with $v_p(n) \in \{0,1\}$ are p-adically represented.

We have already shown that $(\frac{-ab}{p}) = 1$ if and only if those $n \in \mathbb{Z}_p$ with $v_p(n) = 1$ are represented, so we only need to prove that every unit is represented. Hence, it suffices to show that at least one square and one non-square are represented by Q. We will prove that $ax^2 + by^2$ represents every integer p-adically. If -1 is not a square, then

$$\left(\frac{b}{p}\right) = -\left(\frac{a}{p}\right),$$

and hence both squares and non-squares are represented. So we may assume that -1 is a square. For any unit $u \in \mathbb{Z}_p$, the form $ax^2 + by^2$ represents every integer p-adically if and only if $uax^2 + uby^2$ represents every integer p-adically. So, without loss of generality we may suppose that

$$\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right) = 1.$$

Since -1 is a square and we represent all squares by ax^2 (and also by^2), we must represent -1. We now argue inductively by noting that if $-m$ is a square, then $-m-1$ is represented by $ax^2 + by^2$ via taking $ax^2 = -m$ and $by^2 = -1$. If $-m-1$ is a non-square, then we are done; if $-m-1$ is a square, then we can continue the induction. Hence we must also represent a non-square, and the proof is concluded. □

We are now ready to prove Theorems 1.1, 1.2, and 1.3.

Proof of Theorem 1.1. Since $8T_x + 1 = (2x+1)^2$, the representation of n by $f(x, y, z)$
$= ax^2 + by^2 + cT_z$ is equivalent to the representation of $8n + c$ by

$$Q(x, y, z) = 8ax^2 + 8by^2 + cz^2$$

with z odd. The number of the latter representations equals the number of solutions with z arbitrary minus those with z even. As described in Section 2, every sufficiently large integer locally represented with bounded divisibility at the (finitely many) anisotropic primes of $Q' = Q$ or $Q'(x, y, 2z)$ are represented, outside of the finitely many spinor exceptional square classes for $Q'(x, y, z)$ or $Q'(x, y, 2z)$. Thus, if the local conditions are satisfied, then

$$\{8n + c : n \in E(f)\} \subseteq \left(\bigcup_{j=1}^{r} \bigcup_{p \text{ anisotropic}} \{n_j p^{2s} : s \in \mathbb{N}\}\right) \cup \left(\bigcup_{i=1}^{m} t_i \mathbb{Z}^2\right), \tag{3.2}$$

where p runs over the (finitely many) anisotropic primes, $n_1, \ldots, n_r$ are the finitely many "sporadic" natural numbers not represented by Q, and $t_1\mathbb{Z}^2, \ldots, t_m\mathbb{Z}^2$ are finitely many spinor exceptional square classes which may not be represented. Thus, $E(f)$ is a subset of a union of finitely many square classes, and hence its asymptotic density is zero.

We then see that the local conditions at any odd prime p are equivalent to those given in the theorem by Lemma 3.1 for Q. We will use the original form f to investigate the local condition at $p = 2$. A quick check shows that T_z represents every integer modulo 8, and Hensel's lemma then shows that T_z represents every integer 2-adically. Therefore, if c is odd, then cT_z represents every integer 2-adically. If $v_2(c) = 1$, then cT_z represents every even integer. Since $\gcd(a, b, c) = 1$, either a or b is odd, hence every integer is 2-adically represented. If $v_2(c) = 2$, then cT_z

represents every integer congruent to 4 mod 8. Hence, we must represent 1, 2, and 3 modulo 4 with $ax^2 + by^2$. Without loss of generality, we assume that b is odd. Then, b is congruent to 1 or 3 modulo 4, and so is by^2 whenever y is odd. If a is odd, then either $a \equiv b \pmod 4$, in which case $-b \pmod 4$ is not represented, or $a \equiv -b \pmod 4$, in which case $2 \pmod 4$ is not represented. Therefore, one sees $a \equiv 2 \pmod 4$, which is equivalent to $v_2(a) = 1$. Finally, if $v_2(c) \geq 3$, then $ax^2 + by^2$ cannot represent every integer modulo 8 because an odd square is always congruent to 1 mod 8, so the local conditions are not satisfied. □

Proof of Theorem 1.2. In this case the number of solutions to $n = f(x, y, z) = ax^2 + bT_y + cT_z$ equals the number of representations of $8n + b + c$ by

$$Q(x, y, z) = 8ax^2 + by^2 + cz^2$$

with y and z odd. Thus, we again only need to show that every integer is locally represented. The conditions given in the theorem for the odd primes are precisely those given by Lemma 3.1. For $p = 2$ we again use the fact that T_y and T_z represent every integer 2-adically. Note that if $v_2(b) \leq 1$ or $v_2(c) \leq 1$, then every 2-adic integer is represented because at least one of a, b, c must be odd. Also, if $2 \leq v_2(c) \leq v_2(b)$, then not every integer is represented modulo 4. □

Proof of Theorem 1.3. Clearly, $f(x, y, z) = aT_x + bT_y + cT_z$ represents the integer n if and only if

$$Q(x, y, z) = ax^2 + by^2 + cz^2$$

represents $8n + a + b + c$ with x, y, z all odd. Again the local conditions at the odd primes are given by Lemma 3.1. For the 2-adic conditions, we simply note that one of a, b, c is odd, so every 2-adic integer is represented. □

Proof of Corollary 1.4. We first note that if $b\ R\ a$, then $\left(\frac{b}{a}\right) = 1$. Thus, if the conditions given in Theorems 1.1, 1.2, or 1.3 hold, then (by the multiplicative property of Jacobi symbols) we have

$$1 = \left(\frac{-b'c'}{a'}\right)\left(\frac{-a'c'}{b'}\right)\left(\frac{-a'b'}{c'}\right)\left(\frac{2^r}{b'c'}\right)\left(\frac{2^s}{a'b'}\right)\left(\frac{2^t}{a'c'}\right), \tag{3.3}$$

where r, s, t are certain natural numbers.

By the law of quadratic reciprocity for Jacobi symbols,

$$\begin{aligned}&\left(\frac{-b'c'}{a'}\right)\left(\frac{-a'c'}{b'}\right)\left(\frac{-a'b'}{c'}\right)\\ =&\left(\frac{-1}{a'b'c'}\right)\cdot\left(\frac{b'}{a'}\right)\left(\frac{a'}{b'}\right)\cdot\left(\frac{c'}{a'}\right)\left(\frac{a'}{c'}\right)\cdot\left(\frac{c'}{b'}\right)\left(\frac{b'}{c'}\right)\\ =&(-1)^{\frac{a'-1}{2}+\frac{b'-1}{2}+\frac{c'-1}{2}}(-1)^{\frac{a'-1}{2}\cdot\frac{b'-1}{2}+\frac{a'-1}{2}\cdot\frac{c'-1}{2}+\frac{b'-1}{2}\cdot\frac{c'-1}{2}}\\ =&(-1)^{\frac{a'+1}{2}\cdot\frac{b'+1}{2}\cdot\frac{c'+1}{2}-\frac{a'-1}{2}\cdot\frac{b'-1}{2}\cdot\frac{c'-1}{2}-1}.\end{aligned}$$

Observe that $\frac{a'+1}{2}\cdot\frac{b'+1}{2}\cdot\frac{c'+1}{2}$ and $\frac{a'-1}{2}\cdot\frac{b'-1}{2}\cdot\frac{c'-1}{2}$ have opposite parity if and only if $a' \equiv b' \equiv c' \pmod 4$. So the product of three Jacobi symbols is 1 if and only if $a' \equiv b' \equiv c' \pmod 4$.

We finally deal with the 2-power part. If $a' \equiv b' \equiv -c' \pmod 8$, then

$$1 = \left(\frac{2}{a'b'}\right) = \left(\frac{2}{b'c'}\right) = \left(\frac{2}{a'c'}\right),$$

which concludes the first statement. We now note that if $\pm a' \equiv c' + 4 \pmod 8$, then $\left(\frac{2}{a'}\right) = -\left(\frac{2}{c'}\right)$. Therefore, in the cases $a' \equiv b' \equiv c' + 4 \pmod 8$ or $\pm a' \equiv -b' \equiv c' + 4 \pmod 8$ the Jacobi symbol from the 2-power part is $(-1)^{r+t}$, where r and t are as in equation (3.3). For (1), (4), and (5), we have $r = v_2(a) + 1$, while $r = v_2(a)$ in the cases (2) and (3). For (1), we have $t = v_2(b) + 1$, and otherwise $t = v_2(b)$. Thus for (1)-(3) we have $(-1)^{r+t} = (-1)^{v_2(a)+v_2(b)}$ and for (4)-(5) we have $(-1)^{r+t} = -(-1)^{v_2(a)+v_2(b)}$, from which we conclude the remaining two statements. □

Proof of Corollary 1.5. By Theorem 1.1, if $ax^2 + by^2 + 2cT_z$ or $ax^2 + cy^2 + 2bT_z$ is asymptotically universal, then we have (3.1). Similarly, if $ax^2 + 2cy^2 + bT_z$ or $ax^2 + 2by^2 + cT_z$ is asymptotically universal, then we have

$$-bc\ R\ a',\ -2ac\ R\ b',\ -2ab\ R\ c'. \tag{3.4}$$

Now assume that both (3.1) and (3.4) hold. We want to deduce a contradiction. (3.1) and (3.4) imply that $2\ R\ b'$ and $2\ R\ c'$. Recall that $v_2(b) \equiv v_2(c) \pmod 2$. So we have

$$-b'c'\ R\ a',\ -a'c'\ R\ b',\ -a'b'\ R\ c'. \tag{3.5}$$

It follows that

$$\left(\frac{-b'c'}{a'}\right) = \left(\frac{-a'c'}{b'}\right) = \left(\frac{-a'b'}{c'}\right) = 1.$$

Since $a' \equiv b' \equiv c' \pmod 4$ fails, as in the proof of Corollary 1.4 we have

$$\left(\frac{-b'c'}{a'}\right)\left(\frac{-a'c'}{b'}\right)\left(\frac{-a'b'}{c'}\right) = -1.$$

So a contradiction occurs. □

The following lemma gives a sufficient condition for a form not to be asymptotically universal. It will be helpful for our proofs in the next section.

Lemma 3.2. *Let $a, b, c \in \mathbb{Z}^+$. For*

$$f(x, y, z) = ax^2 + by^2 + cT_z, ax^2 + bT_y + cT_z, aT_x + bT_y + cT_z$$

we define $v_f = v_2(c), v_2(b + c), v_2(a + b + c)$, respectively. If $v_f \geq 3$, then f is not asymptotically universal.

Proof. Assume that $v_f \geq 3$ and f is asymptotically universal. We want to deduce a contradiction.

In the case $f = ax^2 + by^2 + cT_z$, by Theorem 1.1 we have $8 \nmid c$, which contradicts $v_f \geq 3$.

Now suppose that $f = ax^2 + bT_y + cT_z$. Since $4 \nmid b$ or $4 \nmid c$ by Theorem 1.2, (up to symmetry) the vector (b, c) modulo 8 is one of $(2, 6)$, $(5, 3)$, or $(1, 7)$. In the first case a must be odd, while in the remaining two cases we have $bc \equiv 7 \pmod 8$ and hence $\left(\frac{2}{bc}\right) = 1$. Therefore, Theorem 1.2 and equation (3.3) imply that $\left(\frac{-a'b'}{c'}\right)\left(\frac{-a'c'}{b'}\right)\left(\frac{-b'c'}{a'}\right) = 1$. However, the calculation from Corollary 1.4 shows that

$$\left(\frac{-a'b'}{c'}\right)\left(\frac{-a'c'}{b'}\right)\left(\frac{-b'c'}{a'}\right) = 1$$

if and only if $a' \equiv b' \equiv c' \pmod 4$. Since $b' \equiv 1 \pmod 4$ and $c' \equiv 3 \pmod 4$, we are led to a contradiction.

Finally we handle the case $f = aT_x + bT_y + cT_z$. By Theorem 1.3 and $v_f \geq 3$, the vector (a, b, c) modulo 8 is one of $(8, 1, 7)$, $(8, 3, 5)$, $(2, 5, 1)$, $(2, 3, 3)$, $(2, 7, 7)$, $(6, 1, 1)$, $(6, 5, 5)$, $(6, 3, 7)$, $(4, 1, 3)$, $(4, 5, 7)$. The cases $(8, 1, 7)$ and $(8, 3, 5)$ are covered above. For the cases $(2, 3, 3)$, $(2, 7, 7)$, $(4, 1, 3)$, $(4, 5, 7)$, $(6, 1, 1)$, $(6, 5, 5)$ we have

$$\left(\frac{-a'b'}{c'}\right)\left(\frac{-a'c'}{b'}\right)\left(\frac{-b'c'}{a'}\right) = -1 \quad \text{and} \quad \left(\frac{2^{v_2(a)}}{b'c'}\right) = 1,$$

while in the cases $(2, 5, 1)$ and $(6, 3, 7)$ we have

$$\left(\frac{-a'b'}{c'}\right)\left(\frac{-a'c'}{b'}\right)\left(\frac{-b'c'}{a'}\right) = 1 \quad \text{and} \quad \left(\frac{2}{b'c'}\right) = -1.$$

In view of (3.3), we get a contradiction. □

4. On almost universal forms

In this section we investigate almost universal forms. We will determine when asymptotically universal forms are not almost universal. We first consider sums with two squares.

Proof of Theorem 1.6. Assume the conditions of Theorem 1.1. Recall that n is represented by $f(x, y, z) = ax^2 + by^2 + cT_z$ if and only if $8n + c$ is represented by

$$Q(x, y, z) = 8ax^2 + 8by^2 + cz^2$$

with z odd. Since $v_2(c) \leq 2$ there are no representations of $8n + c$ by $Q(x, y, 2z)$ due to congruence conditions modulo 8; thus the odd condition can be removed. Therefore,

$$E(f) = \left\{\frac{n-c}{8} : \ n \equiv c \pmod{8}, \ Q(x, y, z) = n \text{ has no integral solution}\right\}. \tag{4.1}$$

Let $t\mathbb{Z}^2$ be a spinor exceptional square class for the genus of Q such that t is squarefree and $tx^2 \equiv c \pmod 8$ for some integer x. We will see below that $K = \mathbb{Q}(\sqrt{-tabc})$ will always be $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-2})$. Thus by the results of Earnest, Hsia, and Hung [8] t is a spinor exception for the genus because tx^2 satisfying the Schulze-Pillot conditions will imply that t satisfies the Schulze-Pillot conditions.

When t is not represented by the spinor genus of Q, Schulze-Pillot [24] showed that for every prime p splitting in K we have that tp^2 is not represented by the spinor genus of Q, and hence not by Q (see [25] for a full list of such properties). If t is represented by the spinor genus of Q, then for each prime p inert in K we have that tp^2 is not primitively represented by the spinor genus of Q [25]. Here a primitive representation means that $\gcd(x, y, z) = 1$. Thus, for any squarefree t represented by the spinor genus of Q but not represented by Q, tp^2 is not represented when p is inert in K, as the number of representations of tp^2 equals the number of (primitive) representations of t plus the number of primitive representations of tp^2, and both of these are zero. Hence, in either case we have seen that there are infinitely many integers in $t\mathbb{Z}^2$ not represented by Q so that $E(f)$ is infinite if such a t exists.

Next we show that if no such t exists, then $E(f)$ is finite. By equations (3.2) and (4.1), if there is no such t with $tx^2 \equiv c \pmod 8$ for some $x \in \mathbb{Z}$, then

$$\{8n + c : n \in E(f)\} \subseteq \left(\bigcup_{j=1}^{r} \bigcup_{p \text{ anisotropic}} \{n_j p^{2s} : s \in \mathbb{N}\} \right),$$

where $n_1, \ldots, n_r$ are "sporadic" exceptions. Note that if every integer is represented p-adically by the quadratic form $8ax^2+8by^2+cz^2$, then p is not anisotropic. Assume $p \mid c$ and fix an integer n. Clearly, any p-adic solution to $Q(x,y,z) = n$ gives a solution to $Q(px,py,pz) = np^2$. Since Q satisfies the condition of Lemma 3.1 and for any fixed $y \in \mathbb{Z}$ relatively prime to p the equation $ax^2 = np^2 - by^2$ has a solution with x relatively prime to p, there are more solutions to the equation $Q(x,y,z) = np^2$ than to the equation $Q(x,y,z) = n$; hence p is not anisotropic.

Thus, the only possible anisotropic prime is $p = 2$, and hence

$$\{8n + c : n \in E(f)\} \subseteq \bigcup_{j=1}^{r} \{n_j 2^{2s} : s \in \mathbb{N}\}.$$

As $v_2(c) \le 2$, we have

$$\{8n + c : n \in E(f)\} \subseteq \bigcup_{j=1}^{r} \{n_j 2^{2s} : s \in \{0,1,2\}\},$$

which shows that $E(f)$ is finite.

We now use Schulze-Pillot's classification [24] to determine the spinor exceptional square classes $t_i\mathbb{Z}^2$. Let a spinor exceptional square class $t\mathbb{Z}^2$ be given. Earnest, Hsia, and Hung showed that if an odd prime p is ramified in $K = \mathbb{Q}(\sqrt{-td})$, then $Q_p \cong \langle u_1, u_2p^r, u_3p^s\rangle$ with u_i units in $\mathbb{Z}_p$ and $0 < r < s$ (cf. [8, Theorem 1(b)]). However, since p divides at most one of a, b, c, this cannot occur. It follows that p is unramified in K, hence $K = K_{abc}$ or $K = K_{2abc}$.

Recall that $\mathcal{SF}(a'b'c')$ is the odd squarefree part of abc. Assume that a prime p dividing $\mathcal{SF}(a'b'c')$ is not split in K. Then, by Theorem 1(a) of Earnest, Hsia, and Hung [8], we have $Q_p \cong \langle u_1, u_2p^{2r}, u_3p^{2s}\rangle$ from the necessary condition given by Kneser [14]. But this would contradict the fact that $v_p(abc)$ is odd. Conversely, when p is split in K, Earnest, Hsia and Hung showed that the local conditions for t to be a spinor exception are satisfied (cf. [8]). If p is odd and $v_p(abc)$ is even, then [8, Theorem 1(a)] shows that $t \not\equiv 0 \pmod p$ satisfies the necessary and sufficient conditions. Thus, the only possible spinor exceptional square classes are given by $t = \mathcal{SF}(a'b'c')$ or $t = 2\mathcal{SF}(a'b'c')$. If $t \not\equiv 2^{-2s}c \pmod 8$ for some $s \in \mathbb{N}$ with $2s \le v_2(c)$, then this spinor exceptional square class will not occur in our consideration. Hence we conclude that $t = \mathcal{SF}(a'b'c)$. Since $tabc$ times a suitable square equals $aa'bb'c^2$, we have $K = K_{abc'}$.

From the above we see that every $p \mid \mathcal{SF}(a'b'c')$ must be split in K, which gives condition (2). If Q represents t, then Q also represents $t\mathbb{Z}^2$ (not necessarily primitively), and hence condition (3) is necessary.

We finally deal with the 2-adic conditions. Let β be a prime ideal of O_K dividing 2. Since $K = \mathbb{Q}(i)$ or $K = \mathbb{Q}(\sqrt{-2})$, the 2-adic completion $K_\beta/\mathbb{Q}_2$ is ramified. After division by common powers of 2, we get

$$Q_2 \cong \langle c', 2^r b', 2^s a'\rangle,$$

where $3 - v_2(c) + v_2(b) = r \le s = 3 - v_2(c) + v_2(a)$.

We now separate into cases depending on $v_2(c)$. First we consider the case where $v_2(c) = 2$. In this case, we divide the equation $Q(x,y,z) = 8n + c$ by 4 to find that representation of n by f is equivalent to representation of $2n+1$ by $Q'(x,y,z) = 2ax^2 + 2by^2 + c'z^2$. We recall by the conditions of Theorem 1.1 that $v_2(a) = 1$ and $v_2(b) = 0$. Thus, $L_2 \cong \langle c', 2b', 4a' \rangle$. Since $r \leq 3$ and $s \leq 2$, the conditions of Theorem 2.2 are always satisfied for $v_2(4t) = 2$. Therefore every sufficiently large integer is always represented in this case.

For the remaining case $\mathcal{SF}(a'b'c) = t \equiv c \pmod 8$, we conclude that $a'b' \equiv 1 \pmod{2^{3-v_2(c)}}$. Hence $a' \equiv b' \pmod{2^{3-v_2(c)}}$, which gives the first assertion of condition (1). Assume first that c is odd. When $r \geq 5$, Earnest and Hsia [7] showed that

$$\theta(O^+(\langle a,b,c\rangle_2)) = \theta(O^+(\langle ac, bc\rangle_2)).$$

Since $a' \equiv b' \pmod 8$ and scaling does not affect the spinor norm, the lattice on the right-hand side is equivalent to $\langle 1, 2^{s-r}\rangle_2$. If $s = r$, then this is precisely $N_2(\mathbb{Q}(i))$, as desired. Checking each case of Theorem 2.1 shows that $\theta(O^+(\langle 1, 2^{s-r}\rangle_2)) \subseteq N_2(K)$, since $K = \mathbb{Q}(i)$ if $s-r$ is even, and $K = \mathbb{Q}(\sqrt{-2})$ if $s-r$ is odd. Theorem 2.2 indicates that when $r \geq 5$, the sufficient conditions are also satisfied. Hence, if $4 \mid b$, then t is a spinor exception. For $2 \| b$ we have $5 \in \theta(O^+(\langle c, 2^r b'\rangle_2)) \notin N_2(\mathbb{Q}(\sqrt{-2}))$; it follows that $K = \mathbb{Q}(i)$ and hence s is even. But, when r and s have the same parity, none of the conditions of Theorem 2.2(1) is satisfied when $r \geq 4$; therefore t is a spinor exception. For $r = 3$, Theorem 2.1 implies that $K = \mathbb{Q}(\sqrt{-2})$ and $b \equiv c \pmod 8$, and hence s is even. If $s = 4$, then Theorem 2.2 of Earnest and Hsia [7] shows that $\theta(O^+(\langle a,b,c\rangle_2)) = \mathbb{Q}_2^\times$. Therefore, $v_2(a) \geq 3$ is odd and $8 \mid (b-c)$ in this case. But then Theorem 2.2(2)(c) is not satisfied since $s > 2$, so it follows that t is a spinor exception.

In the case $2 \mid c$, we have $2 \nmid b$. Thus we get $\langle c', 4b', 2^s a'\rangle_2$ after division by 2. In view of the sublattice $\langle c', 4b'\rangle_2$, we have $K = \mathbb{Q}(i)$, and hence $2 \mid s$. If $s = 2$, then taking the product of symmetries $\sigma = \tau_{2x_1+x_2+x_3}\tau_{x_1}$ gives $\theta(\sigma) = 4(c'+b'+a')c' \notin N_2(\mathbb{Q}(i))$ because each of a', b', c' must be congruent to 1 mod 4 by condition (2). Therefore $v_2(a) > 0$ is even so that $2 \mid a$ and $v_2(a) \equiv c \pmod 2$. In this case $L' = \langle c', 4b', 2^s a'\rangle_2$ where L' is as defined in Theorem 2.2(1), so $\theta(O^+(L') \subseteq N_2(K)$. None of the conditions in Theorem 2.2(1)(c)-(d) can be satisfied, so t is a spinor exception. □

Proof of Corollary 1.7. For any $n \in \mathbb{N}$ we have

$$\begin{aligned} & 2n+1 = 2ax^2 + 2by^2 + cz^2 \text{ for some } x,y,z \in \mathbb{Z} \\ \iff & 2n+1 = 2ax^2 + 2by^2 + c(2z+1)^2 \text{ for some } x,y,z \in \mathbb{Z} \\ \iff & n - \frac{c-1}{2} = ax^2 + by^2 + 4cT_z \text{ for some } x,y,z \in \mathbb{Z}. \end{aligned}$$

By Theorem 1.6 and Theorem 1.1,

$$\begin{aligned} & ax^2 + by^2 + 4cT_z \text{ is almost universal} \\ \iff & ax^2 + by^2 + 4cT_z \text{ is asymptotically universal} \\ \iff & 2\|ab,\ -8bc\ R\ a',\ -8ac\ R\ b', \text{ and } -ab\ R\ c. \end{aligned}$$

So the first part of Corollary 1.7 follows.

When $n \in \mathbb{N}$, clearly

$$\begin{aligned} &2n+1 = 2ax^2 + c(y^2+z^2) \text{ for some } x,y,z \in \mathbb{Z} \\ \iff &2n+1 = 2ax^2 + 4cy^2 + cz^2 \text{ for some } x,y,z \in \mathbb{Z}. \end{aligned}$$

In the case $b = 2c$,

$$\begin{aligned} &2\|ab,\ -2bc\ R\ a',\ -2ac\ R\ b',\ \text{and}\ -ab\ R\ c \\ \iff &c = 1,\ 2 \nmid a,\ \text{and}\ -1Ra'. \end{aligned}$$

So we also have the second part of Corollary 1.7. □

Proof of Corollary 1.8. In light of Theorem 1.1,

$$\begin{aligned} &ax^2 + by^2 + 2T_z \text{ is asymptotically universal} \\ \iff &-b\ R\ a' \text{ and } -a\ R\ b \\ \iff &ax^2 + y^2 + 2bT_z \text{ is asymptotically universal} \end{aligned}$$

and

$$\begin{aligned} &ax^2 + 2y^2 + bT_z \text{ is asymptotically universal} \\ \iff &-b\ R\ a' \text{ and } -2a\ R\ b \\ \iff &ax^2 + 2by^2 + T_z \text{ is asymptotically universal} \end{aligned}$$

Now assume that $-a\ R\ b$ and $-b\ R\ a'$. Then both $ax^2+by^2+2T_z$ and $ax^2+y^2+2bT_z$ are asymptotically universal. Recall that $\mathcal{SF}(a'b) = \mathcal{SF}(a')\mathcal{SF}(b)$ has a prime divisor $p \equiv 3 \pmod 4$. Whether $v_2(a)$ is even or odd, we cannot have both (1) and (2) of Theorem 1.6 for either of the two forms. It follows that $ax^2 + by^2 + 2T_z$ and $ax^2 + y^2 + 2bT_z$ are almost universal.

Suppose that $-2a\ R\ b$ and $-b\ R\ a'$. Then both $ax^2+2y^2+bT_z$ and $ax^2+2by^2+T_z$ are asymptotically universal. As $2b \equiv 2 \not\equiv 0 \pmod 4$ and not all prime divisors of $\mathcal{SF}(a'b)$ are congruent to 3 mod 4, we cannot have both (1) and (2) of Theorem 1.6 for either of the two forms. So $ax^2 + 2y^2 + bT_z$ and $ax^2 + 2by^2 + T_z$ must be almost universal. We are done. □

Proof of Corollary 1.9. (i) By Theorem 1.1,

$$\begin{aligned} &ax^2 + y^2 + T_z \text{ is asymptotically universal} \\ \iff &-2\ R\ a',\ \text{i.e.,}\ \left(\frac{-2}{p}\right) = 1 \text{ for each prime divisor } p \text{ of } a' \\ \iff &\text{all odd prime divisors of } a \text{ are congruent to 1 or 3 mod 8.} \end{aligned}$$

Similarly,

$$\begin{aligned} &ax^2 + 2y^2 + 2T_z \text{ is asymptotically universal} \\ \iff &\text{all prime divisors of } a \text{ are congruent to 1 or 3 mod 8.} \end{aligned}$$

Now suppose that $-2\ R\ a'$. As each prime $p \equiv 1, 3 \pmod 8$ can be written in the form $x^2 + 2y^2$ with $x, y \in \mathbb{Z}$, and

$$(x_1 + 2y_1^2)(x_2^2 + 2y_2^2) = (x_1x_2 - 2y_1y_2)^2 + 2(x_1y_2 + x_2y_1)^2,$$

we can write $\mathcal{SF}(a')$ in the form $x_0^2 + 2y_0^2$ with $x_0, y_0 \in \mathbb{Z}$ since all prime divisors of a' are congruent to 1 or 3 modulo 8. If $a' \equiv 1 \pmod 8$, then $\mathcal{SF}(a') \equiv 1 \pmod 8$ and hence y_0 must be even, so the equation $8(ax^2 + y^2) + z^2 = \mathcal{SF}(a')$ has a solution $(x,y,z)=(0,y_0/2,x_0)$, which violates (3) in Theorem 1.6 with $b=c=1$. If $a'\equiv 3 \pmod 8$, then we don't have (1) in Theorem 1.6 with $b=c=1$. Therefore, by Theorem 1.6, $ax^2+y^2+T_z$ must be almost universal. When a is odd, we have $4\nmid a$ and $v_2(2)\not\equiv 2 \pmod 2$; therefore $ax^2+2y^2+2T_z$ is almost universal by Theorem 1.6 with $c=2$.

(ii) By Theorem 1.1,

$$\begin{aligned} &ax^2+2y^2+T_z \text{ (or } ax^2+y^2+2T_z) \text{ is asymptotically universal}\\ \iff& -1\ R\ a',\ \text{i.e., } \left(\frac{-1}{p}\right)=1 \text{ for each prime divisor } p \text{ of } a'\\ \iff& \text{all odd prime divisors of } a \text{ are congruent to 1 mod 4.}\end{aligned}$$

Similarly,

$$\begin{aligned} &ax^2+4y^2+2T_z \text{ (or } ax^2+2y^2+4T_z) \text{ is asymptotically universal}\\ \iff& \text{all prime divisors of } a \text{ are congruent to 1 mod 4.}\end{aligned}$$

Below we assume that $-1\ R\ a'$. It is well known that each prime $p\equiv 1 \pmod 4$ is a sum of two squares (of integers) and

$$(x_1+y_1^2)(x_2^2+y_2^2)=(x_1x_2-y_1y_2)^2+(x_1y_2+x_2y_1)^2.$$

So we can write $\mathcal{SF}(a')$ in the form $x_0^2+y_0^2$ with x_0 odd and y_0 even (since all prime divisors of a' are congruent to 1 mod 4). Thus the equation $4(ax^2+y^2)+z^2=\mathcal{SF}(a')$ has a solution $(x,y,z)=(0,y_0/2,x_0)$, which violates (3) in Theorem 1.6 with $b=1$ and $c=2$. So $ax^2+y^2+2T_z$ is almost universal. If $a'\equiv 1 \pmod 8$, then $\mathcal{SF}(a')\equiv a'\not\equiv 5 \pmod 8$ and hence $4\mid y_0$, so the equation $8(ax^2+2y^2)+z^2=\mathcal{SF}(a')$ has an integral solution $(x,y,z)=(0,y_0/4,x_0)$, which violates (3) in Theorem 1.6 for the form $ax^2+2y^2+T_z$. If $a'\not\equiv 1 \pmod 8$, then we don't have (1) in Theorem 1.6 for the form $ax^2+2y^2+T_z$. Thus, in view of Theorem 1.6, $ax^2+2y^2+T_z$ is also almost universal.

Now we also assume that a is odd. Note that the equation $2(ax^2+2y^2)+z^2=\mathcal{SF}(a')$ has an integral solution $(x,y,z)=(0,y_0/2,x_0)$. Also, $v_2(4)\equiv v_2(a) \pmod 2$,

$$a\equiv \mathcal{SF}(a)\equiv 1 \pmod 8 \Longrightarrow 4(a0^2+4y^2)+z^2=\mathcal{SF}(a) \text{ for some } y,z\in\mathbb{Z},$$

and

$$a\equiv \mathcal{SF}(a)\equiv 5 \pmod 8 \Longrightarrow 4(ax^2+4y^2)+z^2=\mathcal{SF}(a) \text{ for no } x,y,z\in\mathbb{Z}.$$

Thus, by Theorem 1.6, the form $ax^2+2y^2+4T_z$ is almost universal, and $ax^2+4y^2+2T_z$ is almost universal if and only if $a\equiv 1 \pmod 8$.

The proof of is Corollary 1.9 is now complete. □

Proof of Corollary 1.10. By Theorem 1.1, $ax^2+3y^2+T_z$ (resp., $ax^2+y^2+3T_z$, $ax^2+2y^2+6T_z$, $ax^2+6y^2+2T_z$) is asymptotically universal if and only if both $-6\ R\ a'$ and $a\equiv 1 \pmod 3$ (resp., $a\equiv 2 \pmod 3$, $a\equiv 1 \pmod 6$, $a\equiv 5 \pmod 6$). Observe that $-6\ R\ a'$ if and only if for each odd prime divisor p of a we have

$$\left(\frac{2}{p}\right)=\left(\frac{-3}{p}\right)=\left(\frac{p}{3}\right),\ \text{i.e., } p\equiv 1,5,7,11 \pmod{24}.$$

For odd positive integers b and c not satisfying $a'\equiv b\equiv c \pmod 8$, by Theorem 1.6 the form $ax^2+bT_y+cT_z$ is almost universal if and only if it is asymptotically universal. Thus $ax^2+3y^2+T_z$ (or $ax^2+y^2+3T_z$) is almost universal if and only if it is asymptotically universal. If a is odd, then $4 \nmid a$ and $v_2(2) = v_2(6) = 1 \not\equiv 6 \equiv 2 \pmod 2$; thus by Theorem 1.6 the form $ax^2 + 2T_y + 6T_z$ (or $ax^2 + 6y^2 + 2T_z$) is almost universal if and only if it is asymptotically universal.

Combining the above, we have completed the proof of Corollary 1.10. □

Proof of Corollary 1.11. Let $k, l \in \mathbb{N}$ with $k \geq l$. By Theorem 1.1, the form $2^k x^2 + 2^l y^2 + mT_z$ is asymptotically universal if and only if $-2^{k+l}\ R\ m'$ and

$$4 \nmid m \quad \text{or} \quad (4 \| m \ \&\ k = 1 \ \&\ l = 0).$$

Assume that $2^k x^2 + 2^l y^2 + mT_z$ is asymptotically universal. As $v_2(m) < 3$ and $2 \nmid \mathcal{SF}(m')$, the equation

$$2^{3-v_2(m)}(2^k x^2 + 2^l y^2) + m'z^2 = \mathcal{SF}(m')$$

has no integral solution if and only if m' is not squarefree (i.e., $\mathcal{SF}(m') < m'$). Thus, by Theorem 1.6, the form $2^k x^2 + 2^l y^2 + mT_z$ is not almost universal if and only if $k > 0$ and $4 \nmid m$ and

$$\begin{cases} l \leq 1 \implies k \equiv m \pmod 2, \\ l = 0 \ \&\ 2 \nmid m \implies k \geq 3 \ \&\ m \equiv 1 \pmod 8). \end{cases}$$

In view of the above, we have the desired results in Corollary 1.11. □

Proof of Corollary 1.12. By Theorem 1.1,

$$\begin{aligned} & ax^2 + 6^3 y^2 + T_z \text{ is asymptotically universal} \\ \iff & -2^4 3^3\ R\ a' \text{ and } -2a\ R\ 3^3 \\ \iff & -3\ R\ a' \text{ and } -2a\ R\ 3 \\ \iff & a \equiv 1 \pmod 3 \text{ and } \left(\frac{p}{3}\right) = \left(\frac{-3}{p}\right) = 1 \text{ for each prime divisor } p \text{ of } a' \\ \iff & \text{all prime divisors of } a' \text{ are congruent to 1 mod 3, and } 2 \mid v_2(a). \end{aligned}$$

Also,

$$\begin{aligned} & ax^2 + 2 \cdot 5^3 y^2 + T_z \text{ is asymptotically universal} \\ \iff & -2^2 5^3\ R\ a' \text{ and } -2a\ R\ 5^3 \\ \iff & -5\ R\ a' \text{ and } -2a\ R\ 5 \\ \iff & a \equiv \pm 2 \pmod 5 \text{ and } \left(\frac{-5}{p}\right) = 1 \text{ for each prime divisor } p \text{ of } a'. \end{aligned}$$

For an odd prime p, clearly

$$\left(\frac{-5}{p}\right) = 1 \iff \left(\frac{-1}{p}\right) = \left(\frac{p}{5}\right) \iff p \equiv 1, 3, 7, 9 \pmod{20} \iff 2 \,\Big|\, \left\lfloor \frac{p}{10} \right\rfloor.$$

(i) Under the supposition, $ax^2 + 216y^2 + T_z$ is asymptotically universal by the above. If $8(ax^2 + 216y^2) + z^2 = \mathcal{SF}(3^3 a') = 3\mathcal{SF}(a')$ for some $x, y, z \in \mathbb{Z}$, then we must have $x = 0$ (since $8a > 3a$) and $3 \mid z$, which contradicts the fact that $3 \nmid \mathcal{SF}(a')$. So the equation $8(ax^2+216y^2)+z^2 = \mathcal{SF}(3^3 a')$ has no integral solutions. Applying Theorem 1.6 we find that $ax^2 + 216y^2 + T_z$ is not almost universal if and only if $a' \equiv 3^3 \pmod 8$ and all prime divisors of $\mathcal{SF}(3^3 a') = 3\mathcal{SF}(a')$ are congruent to 1 or 3 mod 8. Since $\mathcal{SF}(a') \equiv a' \pmod 8$ and each prime divisor of a' is congruent to 1 mod 3, the desired result follows.

(ii) Under the assumption, $a = 2^{v_2(a)}a' \equiv \pm 2 \pmod 5$, and hence $ax^2 + 250y^2 + T_z$ is asymptotically universal. Note that the equation $8(ax^2 + 250y^2) + z^2 = \mathcal{SF}(5^3a') = 5\mathcal{SF}(a')$ has no integral solutions. In view of Theorem 1.6, $ax^2 + 250y^2 + T_z$ is not almost universal if and only if $a' \equiv 5^3 \equiv 5 \pmod 8$ and all prime divisors of $\mathcal{SF}(5^3a') = 5\mathcal{SF}(a')$ are congruent to 1 mod 4. Since $a' \equiv \pm 1 \pmod{10}$ and each prime divisor of a' is congruent to one of 1, 3, 7, 9 modulo 20, we finally obtain the desired result. □

Proof of Theorem 1.13. As in Theorem 1.6, f will not be almost universal only if there is a relevant anisotropic prime or a spinor exceptional square class with the correct congruence conditions modulo 8 for one of the quadratic forms occuring in the inclusion/exclusion of theta series

$$\theta_{Q(x,y,z)} := \theta_{Q'(x,y,z)} - \theta_{Q'(x,2y,z)} - \theta_{Q'(x,y,2z)} + \theta_{Q'(x,2y,2z)},$$

where $Q'(x,y,z) = 8ax^2 + by^2 + cz^2$. We will first show that there are no relevant anisotropic primes. The conditions given by Theorem 1.2 imply that every odd prime p is not anisotropic. By Lemma 3.2, the prime 2 is never relevant because the congruence condition implies that the 2-adic order is at most two.

Also as in Theorem 1.6, the local conditions at each odd prime imply that the only possible spinor exceptional square classes are $t\mathbb{Z}^2$ with $t = \mathcal{SF}(a'b'c')$ or $t = 2\mathcal{SF}(a'b'c')$. Moreover, the sufficient local conditions for the odd primes are satisfied if and only if every prime divisor of $\mathcal{SF}(a'b'c')$ is split in $K = \mathbb{Q}(\sqrt{-2abct})$.

If t is a spinor exception for the genus of $Q'(x,y,z)$, then t is a spinor exception for the genus of $Q'(x,2y,z)$ and condition (3) implies that t is represented the same number of times by each quadratic form. If t is not represented by the spinor genus of Q', then tp^2 is also not (primitively) represented, where p is an odd prime split in K. If t is represented by the spinor genus of Q', then tp^2 is not primitively represented, where p is an inert prime. In either case tp^2 will also clearly not be primitively represented by $Q'(x,2y,z)$, so tp^2 is not represented by Q. Also, if t is not a spinor exception for $Q'(x,y,z)$ or $Q'(x,2y,z)$, then $E(f)$ is finite. Therefore, for $E(f)$ to be infinite, it is sufficient that t is a spinor exception for the genus of Q', while it is necessary that t is a spinor exception for the genus of $Q'(x,2y,z)$.

We now break into cases depending on $v_2(b+c)$. Since $v_2(b+c) < 3$ by Lemma 3.2, we begin with the case $4 \| b+c$. Without loss of generality we assume $v_2(c) \le v_2(b)$. Since $v_2(c) < 2$ by local conditions, congruence conditions imply in this case that $v_2(b) = v_2(c) \le 1$. If $v_2(c) = v_2(b) = 1$, then a is odd, and after division by common 2-powers we get $L = \langle c', b', 4a'\rangle_2$, so Theorem 2.2(1) is not satisfied because $v_2(4t) = 2 \ge s$. If $v_2(b) = v_2(c) = 0$, then congruence conditions, without loss of generality, give ($b \equiv 1 \pmod 8$ & $c \equiv 3 \pmod 8$) or ($b \equiv 5 \pmod 8$ & $c \equiv 7 \pmod 8$). In the second case not all prime divisors of $\mathcal{SF}(a'b'c')$ split in K. In the first case we must have $K = \mathbb{Q}(\sqrt{-2})$, and hence s is odd. But then $v_2(2^st)$ is odd and Theorem 2.2(2) implies that $0 = v_2(b) > 0$, thus t is not a spinor exception.

For the remaining cases we note that $t \equiv b + c \pmod 8$, so we must have $v_2(t) = v_2(b+c)$. Conditions (1), (2), and (3) now follow immediately. First consider $v_2(b) \ge 5$ odd. Then $\theta_Q = \theta_{Q'(x,y,z)} - \theta_{Q'(x,2y,z)}$, where $Q'(x,y,z) = 8ax^2 + by^2 + cz^2$. If c is even, then we have $L_2 = \langle c', 4a', 2^{r-1}b'\rangle_2$ and $K = \mathbb{Q}(i)$. In this case, Earnest and Hsia proved that the spinor norm can be considered only on 2×2 sublattices. Since $r - 1$ is even, Theorem 2.1 shows that $\theta(O^+(L_2)) \subseteq N_2(K)$. Moreover, Theorem 2.2(1)(c)-(d) cannot be satisfied, so t is a spinor exception. If c is odd and

$4 \mid a$, then $r \geq 5$, so the spinor norm again equals the spinor norm on 2-dimensional sublattices. Since $\mathcal{SF}(a'b'c') \equiv (b+c)' \equiv c' \pmod 8$, we have $a' \equiv b' \pmod 8$. Therefore, Theorem 2.1 shows that the spinor norm on each sublattice gives a subset of $N_2(K)$. Since $r \geq 5$ none of the conditions of Theorem 2.2 is satisfied, and hence t is a spinor exception. For $r = v_2(a) + 3 < 5$, we have $\langle c', 2^r a', 2^s b'\rangle_2$. Applying Theorem 2.1 to the the sublattice $\langle c', 2^r a'\rangle_2$, we get $K = \mathbb{Q}(\sqrt{-2^r})$, and it follows that s must be even. Therefore t is not a spinor exception for $Q'(x, y, z)$ or $Q'(x, 2y, z)$.

For $v_2(b) \geq 6$ even and c even, in view of the sublattice $\langle 2c', 8a\rangle_2$ and Theorem 2.1, we have $K = \mathbb{Q}(i)$, which implies that $v_2(b)$ must be odd. For $v_2(b) \geq 6$ even and c odd, Earnest and Hsia showed that we may reduce the problem to 2-dimensional sublattices. If a is odd, then the sublattice $\langle c, 8a\rangle_2$ gives the set $\{\gamma : (\gamma, -2ac)_2 = 1\}$, which is a subgroup of $N_2(\mathbb{Q}(\sqrt{-2}))$ if and only if $a \equiv c \pmod 8$. Theorem 2.2(2)(c) shows that t is a spinor exception in this case, as $s > 2$. When a is even, we again note that $a' \equiv b' \pmod 8$ by condition (1) and that Theorem 2.1 implies that $\theta(O^+(\langle 8a, b, c\rangle_2)) \subseteq N_2(K)$.

For $v_2(b) < 3$, inclusion/exclusion gives $\theta_Q = \theta_{Q'}$. For b odd, the sublattice $\langle 1, bc\rangle_2$ gives the spinor norm $Q(x_1 + 2x_2)Q(x_1)\mathbb{Q}_2^{\times 2} = 5\mathbb{Q}^{\times 2} \notin N_2(\mathbb{Q}(\sqrt{-2}))$, so $K = \mathbb{Q}(i)$ and $v_2(a)$ is even. We then note that condition (1) gives $1 = \mathcal{SF}(a'b'c') \equiv (b+c)' \pmod 4$, so that $b \equiv c \pmod 8$. For $s = 3 + v_2(a) \geq 5$, the problem is now reduced to considering 2-dimensional sublattices, and we are done since $\theta(O^+(\langle 1, 1\rangle_2)) = N_2(\mathbb{Q}(i))$ and $a' \equiv 1 \pmod 4$ by condition (2). In this case $L'' = L_2$ and $\theta(O^+(L')) \subseteq N_2(K)$, where L' and L'' are as in Theorem 2.2(1), so that condition (1)(d) is not satisfied and t is a spinor exception. When $s = 3$, Theorem 2.1 shows that $\theta(O^+(\langle 8a, b, c\rangle_2)) \not\subseteq N_2(\mathbb{Q}(i))$. For $v_2(b) = 1$ Theorem 2.1 implies that $\theta(O^+(\langle 2b', c'\rangle_2)) \subseteq K$ if and only if $K = \mathbb{Q}(\sqrt{-2})$ and $b' \equiv c' \pmod 8$. $K = \mathbb{Q}(\sqrt{-2})$ is equivalent to $2 \nmid v_2(a)$ (which implies that $s \in \{4, 6, \ldots\}$), while $b' \equiv c' \pmod 8$ follows from $2b' + c \equiv \mathcal{SF}(a'b'c') \pmod 8$, as each of a', b', c' is congruent to 1 or 3 mod 8. For $s > 4$ we are led to 2-dimensional lattices and Theorem 2.1 implies that $\theta(O^+(\langle 8a, b, c\rangle_2)) \subseteq N_2(\mathbb{Q}(\sqrt{-2}))$ while one sees that none of the conditions of Theorem 2.2(2) can be satisfied, so t is a spinor exception. For $s = 4$, Theorem 2.2(2)(a) is satisfied, so t cannot be a spinor exception.

When $v_2(b) = 2$, Theorem 2.1 implies that $K = \mathbb{Q}(i)$ and hence $v_2(a)$ is odd, and Earnest and Hsia [7] showed that we may again consider the spinor norm on 2-dimensional sublattices to get $\theta(O^+(\langle 8a, b, c\rangle_2)) \subseteq N_2(K)$. The conditions in Theorem 2.2(1)(c)-(d) are not satisfied and $L' = L_2$, so t is a spinor exception.

For $v_2(b) = 3$, if t is a spinor exception for $Q'(x, y, z)$, then Theorem 2.1 for the sublattice $\langle 1, 8b'c'\rangle_2$ implies that $K = \mathbb{Q}(\sqrt{-2})$ and $b' \equiv c' \pmod 8$. Hence $v_2(a)$ must be odd. If $v_2(a) = 1$, then Theorem 2.2 of Earnest and Hsia [7] implies that $\theta(O^+(\langle 8a, b, c\rangle_2)) = \mathbb{Q}_2^{\times}$. For $v_2(a) > 1$ odd we may again consider only 2-dimensional sublattices and Theorem 2.2(2)(c) is satisfied, so t is a spinor exception for $Q'(x, y, z)$. Finally, the property that t is a spinor exception for $Q'(x, 2y, z)$ is equivalent to the case where $r = 5$, which was covered above.

For $v_2(b) = 4$, if t is a spinor exception for $Q'(x, y, z)$, then Theorem 2.1 implies that $K = \mathbb{Q}(i)$, and hence $v_2(a)$ is odd. Since $\mathcal{SF}(a'b'c') \equiv (b+c)' \equiv c' \pmod 8$, we have $a' \equiv b' \pmod 8$ by condition (1), and thus $\theta(O^+(\langle 8a, b, c\rangle_2)) \subseteq N_2(K)$. Moreover, none of the conditions of Theorem 2.2(1) is satisfied, so t is a spinor exception. Finally, the property that t is a spinor exception for $Q'(x, 2y, z)$ is equivalent to the case where $r = 6$, which was covered above. □

Proof of Corollary 1.14. (i) By Theorem 1.2, the form $ax^2 + 2T_y + T_z$ is asymptotically universal if and only if $-2\ R\ a'$, i.e., each prime divisor of a' is congruent to 1 or 3 mod 8.

Now assume that $-2\ R\ a'$. As we mentioned before, $\mathcal{SF}(a') = 2y^2 + z^2$ for some $y, z \in \mathbb{Z}$. Clearly z is odd. If $\mathcal{SF}(a') \equiv 2 + 1 \pmod{2^3}$, then y must be odd. Thus we cannot have both (1) and (3) in Theorem 1.13 with $b = 2$ and $c = 1$. Therefore $ax^2 + 2T_y + T_z$ is almost universal.

(ii) By Theorem 1.2, the form $ax^2 + 4T_y + T_z$ is asymptotically universal if and only if $-1\ R\ a'$, i.e., each prime divisor of a' is congruent to 1 mod 4.

Now assume that $-1\ R\ a'$. Then $\mathcal{SF}(a') = 4y^2 + z^2$ for some $y, z \in \mathbb{Z}$. If $\mathcal{SF}(a') \equiv 4 + 1 \pmod{2^3}$, then y must be odd. So we cannot have both (1) and (3) in Theorem 1.13 with $b = 4$ and $c = 1$. It follows that $ax^2 + 4T_y + T_z$ is almost universal. We are done. □

Proof of Corollary 1.15. Set $f_k(x, y, z) = 2^k(x^2 + T_y) + mT_z$ for $k = 0, 1, 2, \ldots$. By Theorem 1.2, the form f_k is asymptotically universal if and only if $-2\ R\ m'$, and $2 \nmid m$ when $k > 0$.

Assume that f_k is asymptotically universal. Then all prime divisors of m' are congruent to 1 or 3 mod 8; thus $m' \equiv 1, 3 \pmod 8$. Note that the equation

$$8 \times 2^k x^2 + 2^k y^2 + mz^2 = 2^{v_2(2^k+m)}\mathcal{SF}(m')$$

has no integral solutions with yz odd, since the right-hand side of the equation is smaller than $2^k + m$.

Case 1. $k = 0$. When $a = c = 1$ and $b = m$, we obviously have $v_2(a) = 0$, $2 \nmid ac$ and $a \equiv c \pmod 8$. Thus, if $v_2(m) \neq 4, 6, \ldots$, then f_0 is almost universal by Theorem 1.13 for the form $x^2 + mT_y + T_z$.

Now suppose $v_2(m) \in \{4, 6, \ldots\}$. Then $\mathcal{SF}(m) \equiv m+1 \pmod 2$. Also, $4 \mid m+1$, $v := v_2(m+1) = 0$, $m' \equiv \mathcal{SF}(m') \pmod 8$ and $(m+1)' = m+1 \equiv 1 \pmod 8$. By Theorem 1.13 for the form $x^2 + mT_y + T_z$, f_0 is almost universal if $m' \equiv 3 \pmod 8$, and f_0 is not almost universal if $m' \equiv 1 \pmod 8$ and $v_2(m) \neq 4$.

Case 2. $k > 0$. In this case, m is odd. If $k \in \{1, 2\}$, then f_k is almost universal since (4) in Theorem 1.13 does not hold for $a = b = 2^k$ and $c = m$. For $k \geq 3$, clearly $4 \nmid 2^k + 1$ and $\mathcal{SF}(m') = \mathcal{SF}(m) \equiv m \equiv 2^k + m = (2^k + m)' \pmod 8$. Applying Theorem 1.13, we find that f_k is not almost universal if $k > 5$, or $k = 3$ and $m \equiv 1 \pmod 8$.

Combining the above, we have completed the proof. □

Proof of Corollary 1.16. Define $g_k(x, y, z) = 2^k(x^2 + 2T_y) + mT_z$ for $k \in \mathbb{N}$. By Theorem 1.2, the form g_k is asymptotically universal if and only if $-1\ R\ m'$ (i.e., all prime divisors of m' are congruent to 1 mod 4), and $2 \nmid m$ when $k > 0$.

Suppose that $-1\ R\ m'$. Clearly the equation

$$8 \times 2^k x^2 + 2^{k+1} y^2 + mz^2 = 2^{v_2(2^{k+1}+m)}\mathcal{SF}(m')$$

has no integral solutions with y and z odd (since the right-hand side of the equation is smaller than $2^{k+1}+m$). Note also that if $2 \nmid m$ or $2 \nmid v_2(m)$, then $\mathcal{SF}(2^k 2^{k+1} m) =$

$\mathcal{SF}(2m) \not\equiv 2^{k+1} + m \pmod 2$, and hence, (2) in Theorem 1.13 holds for the form g_k.

Case 1. $k = 0$. Since $v_2(1) - v_2(2) \neq 2, 4, 6, \ldots$, if $v_2(m) \le v_2(2) = 1$ (i.e., $4 \nmid m$), then $x^2 + 2T_y + mT_z$ is almost universal by Theorem 1.13. For $a = 1$, $b = m$ and $c = 2$, clearly $v_2(a) \neq 1, 3, 5, \ldots$, and $2 \nmid a$ and $a \not\equiv c \pmod 8$. So, by Theorem 1.13 for the form $x^2+mT_y+2T_z$, f_0 is also almost universal when $v_2(m) \in \{2, 4, 6, \ldots\}$. In the case $v_2(m) \in \{3, 5, \ldots\}$, clearly $\mathcal{SF}(m') \equiv m' \equiv 1 \equiv m/2+1 = (m+2)' \pmod 4$, and hence we have (1)-(4) in Theorem 1.13 for the form $x^2+mT_y+2T_z$. So Theorem 1.13 implies that g_0 is not almost universal if $v_2(m) \in \{5, 7, \ldots\}$.

Case 2. $k = 1$. As $v_2(m + 4) = 0$ and $\mathcal{SF}(2'4'm') = \mathcal{SF}(m) \equiv m \not\equiv (m + 4)' \pmod{2^3}$, $g_1(x, y, z) = 2x^2 + 4T_y + mT_z$ is almost universal by Theorem 1.13.

Case 3. $k \ge 2$. In this case, we have (1)-(3) in Theorem 1.13 with $a = 2^k$, $b = 2^{k+1}$ and $c = m$. Note also that $4 \mid 2^k$. So, by Theorem 1.13, g_k is not almost universal if $k > 2$.

In view of the above, we have proved both (i) and (ii) in Corollary 1.16. □

Now we turn to the proof of Theorem 1.17.

Proof of Theorem 1.17. We again start by considering anisotropic primes, again arriving at the fact that only $p = 2$ is possible. However, Lemma 3.2 implies bounded divisibility at $p = 2$ by the congruence conditions, so there are no relevant anisotropic primes.

We now determine when $t = \mathcal{SF}(a'b'c')$ or $t = 2\mathcal{SF}(a'b'c')$ is a spinor exception. For $E(f)$ to be infinite, it is sufficient that t is a spinor exception for $Q'(x, y, z)$, while it is necessary that t is a spinor exception for one of the quadratic forms in the inclusion/exclusion.

We will break into cases depending on $v := v_2(a+b+c)$. For $v = 2$ and $v_2(a) < 3$ we have $\theta_Q = \theta_{Q'}$ and $v_2(4t) = 2 \ge s = v_2(a)$, so none of the conditions of Theorem 2.2 is satisfied and t is not a spinor exception. When $v = 2$ and $v_2(a) \ge 3$, we have $4 \mid b+c$. But then, we may assume that $b \equiv 3 \pmod 4$ without loss of generality, as b and c are both odd. Thus $K = \mathbb{Q}(\sqrt{-2})$ since every prime divisor of b must split in K, and hence $v_2(tabc)$ is odd. However, Theorem 2.2(2) implies that $v_2(tabc)$ must be even because $r = v_2(b) = 0$.

We now must have $v \le 1$, $t = 2^v\mathcal{SF}(a'b'c') \equiv (a + b + c) \pmod 8$, and $K = K_{2^v abc}$. This gives conditions (1), (2), and (3).

For $v_2(b) \ge 5$ we have $\mathcal{SF}(a'b'c') \equiv (a+b+c)' \equiv c' \pmod 8$, so $a' \equiv b' \pmod 8$. Again we are led to 2-dimensional sublattices, and it follows that $O^+(\langle a, b, c\rangle_2) \subseteq N_2(K)$, while none of the conditions of Theorem 2.2 is satisfied, so t is a spinor exception.

For $v_2(b) < 3$ we have $\theta_Q = \theta_{Q'(x,y,z)} - \theta_{Q'(2x,y,z)}$. For b odd the sublattice $\langle b, c\rangle_2$ gives $b \equiv c \pmod 8$ and $K = \mathbb{Q}(i)$, so that $v_2(a)$ is odd. But $b \equiv c \pmod 8$ automatically by condition (1). If $s = v_2(a) \le 3$, then Theorem 2.2 of Earnest and Hsia [7] implies that $\theta(O^+(\langle a, b, c\rangle_2)) = \mathbb{Q}_2^\times$ so that t is not a spinor exception for Q'. For $s \ge 5$ Earnest and Hsia showed that we may reduce to 2-dimensional sublattices, so that $\theta(O^+(\langle a, b, c\rangle_2)) \subseteq N_2(K)$. We then verify with Theorem 2.1 that the Kneser condition is satisfied for L' and L'' as defined in Theorem 2.2(1). In this case condition (1)(d) of Theorem 2.2 is not satisfied, so t is a spinor exception for Q'. For $s = 3$ and x even, the situation is similar to the case $s = 5$, thus the above argument shows that t is a spinor exception for $Q'(2x, y, z)$.

When $v_2(b) = 1$, Theorem 2.1 for the sublattice $\langle c, 2b'\rangle_2$ implies that $K = \mathbb{Q}(\sqrt{-2})$ and $b' \equiv c' \pmod 8$ (which is already satisfied by (1)). Thus, s is even. If $s \leq 4$, then Theorem 2.1 for the sublattice $\langle c, 2^s a'\rangle_2$ gives $5 \in \theta(O^+(\langle a, b, c\rangle_2))$, so t is not a spinor exception. For $s > 4$ we again split into 2-dimensional sublattices, and the Kneser condition is satisfied by Theorem 2.1. The Kneser condition for L' as defined in Theorem 2.2(2) is satisfied and condition 2(c) is not satisfied, so t is a spinor exception. Again when $s = 4$ we have $s' = 6$ for $Q'(2x, y, z)$.

When $v_2(b) = 2$, Theorem 2.1 for the sublattice $\langle c, 4b'\rangle_2$ implies that $K = \mathbb{Q}(i)$, and hence $s \geq 2$ is even. For $s \geq 4$ we may reduce to 2-dimensional sublattices, and the Kneser condition is satisfied by Theorem 2.1. Theorem 2.2(1)(c)-(d) are not satisfied, so t is a spinor exception. For $s = 2$ we have $Q(2x_1+x_2+x_3)Q(x_1)/4 \equiv 3 \pmod 4$, so that we don't have a spinor exception for Q' in this case. However, in the case $s = 2$, for $Q'(2x, y, z)$, it follows that y is even by congruence considerations and $r' = s' = 4$. As we will show later, $Q'(2x, y, z)$ has the spinor exception t in this case.

When $v_2(b) = 3$, by Theorem 2.1 for the sublattice $\langle c, 8b'\rangle_2$, we have $K = \mathbb{Q}(\sqrt{-2})$ and hence $2 \mid s$, as well as $b' \equiv c' \pmod 8$. For $s = 4$, Theorem 2.2 of Earnest and Hsia [7] implies that $\theta(O^+(L_2)) = \mathbb{Q}_2^\times$, so t is not a spinor exception. When $s \geq 6$ we may again consider only 2-dimensional sublattices, and the Kneser condition is satisfied by Theorem 2.1. Theorem 2.2(2)(c) is not satisfied, so t is a spinor exception. For $s = 4$ and x even we get a form with $s' = 6$ and argue as above. For x, y even we get $r' = v_2(b) + 2 \geq 5$, so that t is a spinor exception.

Finally we deal with the case $v_2(b) = 4$. For x, y even, we have $r' \geq 5$, so t is a spinor exception for $Q'(2x, 2y, z)$. Theorem 2.1 for the sublattice $\langle c, 16b'\rangle_2$ implies that $K = \mathbb{Q}(i)$, and hence $2 \mid s$. But condition (1) implies that $a' \equiv b' \pmod 8$, so we find that for s even, $\theta(O^+(\langle a, b, c\rangle_2)) \subseteq N_2(K)$. Since $r = 4$ and $s \geq 4$ is even, none of the conditions of Theorem 2.2(1) is satisfied, and t is a spinor exception for $Q'(x, y, z)$. □

Proof of Corollary 1.18. By Theorem 1.3, the form $ax^2+2T_y+T_z$ is asymptotically universal if and only if $-2\ R\ a'$, i.e., each odd prime divisor of a is congruent to 1 or 3 modulo 8.

Below we assume that $-2\ R\ a'$. As $\min\{v_2(a), v_2(2)\} \leq 1$, (4) in Theorem 1.17 for the form $ax^2 + 2y^2 + T_z$ just says that $v_2(a) - v_2(2) \in \{3, 5, \ldots\}$.

Now suppose that $v_2(a) \in \{4, 6, \ldots\}$. Then $v := v_2(a + 2 + 1) = 0$ and

$$\mathcal{SF}(a'2'1') \equiv (a+3)' \pmod 8 \iff a' \equiv a + 3 \equiv 3 \pmod 8.$$

Also, $\mathcal{SF}(2a) = 2\mathcal{SF}(a') \not\equiv a + 2 + 1 \pmod 2$, and for any odd integers x, y, z we have $ax^2 + 2y^2 + z^2 > a \geq \mathcal{SF}(a'2'1')$.

In view of the above, (1)-(4) in Theorem 1.17 for the form $ax^2 + 2y^2 + T_z$ are all valid if and only if $v_2(a) \in \{4, 6, \ldots\}$ and $a' \equiv 3 \pmod 8$. Thus, by Theorem 1.17, if $ax^2 + 2y^2 + T_z$ is not almost universal, then we must have $a' \equiv 3 \pmod 8$ and $v_2(a) \in \{4, 6, \ldots\}$; when $v_2(a) \neq 4$ (i.e., $v_2(a) - v_2(2) \neq 3$) the converse also holds.

Combining the above we finally obtain the desired result. □

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Equidistribution of Heegner points and ternary quadratic forms

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Abstract We prove new equidistribution results for Galois orbits of Heegner points with respect to single reduction maps at inert primes. The arguments are based on two different techniques: primitive representations of integers by quadratic forms and distribution relations for Heegner points. Our results generalize an equidistribution result with respect to a single reduction map established by Cornut and Vatsal in the sense that we allow both the fundamental discriminant and the conductor to grow. Moreover, for fixed fundamental discriminant and variable conductor, we deduce an effective surjectivity theorem for the reduction map from Heegner points to supersingular points at a fixed inert prime. Our results are applicable to the setting considered by Kolyvagin in the construction of the Heegner points Euler system.

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1 Introduction

Uniform distribution of Galois orbits of Heegner points with respect to reduction maps was the key step in the argument of Cornut and Vatsal for the proof of Mazur's conjecture on the non-triviality of Heegner points over the p-adic anticyclotomic tower (see [24] for the statement; [3,36,37] and [5] for the proofs). Both Cornut and Vatsal used ergodic theory techniques based on Ratner's theorem for unipotent flows

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on p-adic Lie groups (see [31]) in order to prove the results for simultaneous reduction maps (i.e., maps that reduce simultaneously n-tuples of Galois conjugates of Heegner points modulo a fixed inert prime ℓ). Due to the p-adic nature of the ergodic techniques, one needs to fix the fundamental discriminant and vary the conductor p-adically.

This paper proves a more general equidistribution result in the case of a single reduction map, in the sense that both the fundamental discriminant and the conductor are allowed to vary and the only assumption on the conductor is that it is prime to the level of the modular curve. Our arguments are based on equidistribution of primitive representations of integers by quadratic forms in genera, as well as distribution relations of Heegner points and Hecke eigenvalue bounds.

Similar results have been established by Michel [25], Harcos–Michel [19] (see also [11, Thm. 4.6]) using subconvexity results for Rankin–Selberg L-functions of automorphic forms twisted by ring class characters.

Our method naturally leads to effective surjectivity results for sufficiently large Galois orbits with respect to reduction maps in the case when the fundamental discriminant is fixed and the conductor varies.

1.1 Motivation

Let $N \geq 1$ be an integer and let $X_0(N)_{/\mathbb{Q}}$ be the modular curve associated to the congruence subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$. Let ℓ be a prime such that $(\ell, N) = 1$ and let $\mathcal{D}_N$ be the set of all fundamental discriminants $D < 0$ such that every prime factor of N is split in $K_D := \mathbb{Q}(\sqrt{D})$ and such that ℓ is inert in K_D. Let Ω_N be the set of all pairs (D, c), where $D \in \mathcal{D}_N$ and $(c, N) = 1$.

Fix an embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$ (i.e., a prime in $\overline{\mathbb{Q}}$ lying above ℓ). Let $(D, c) \in \Omega_N$ and let $\mathcal{O}_{D,c}$ be the order of conductor c in the quadratic imaginary field $K_D = \mathbb{Q}(\sqrt{D})$. Fix an ideal $\mathfrak{n}_D \subset \mathcal{O}_{D,1}$ for which $\mathcal{O}_{D,1}/\mathfrak{n}_D \cong \mathbb{Z}/N\mathbb{Z}$. For $(c, N) = 1$, $\mathfrak{n}_{D,c} := \mathfrak{n}_D \cap \mathcal{O}_{D,c}$ is an invertible ideal of $\mathcal{O}_{D,c}$. Consider the point $x_c = [\mathbb{C}/\mathcal{O}_{D,c} \to \mathbb{C}/\mathfrak{n}_{D,c}^{-1}] \in X_0(N)(\overline{\mathbb{Q}})$. By the theory of complex multiplication, it is defined over the ring class field $K_D[c]$ of conductor c for K_D. We refer to that point as the higher Heegner point of conductor c. Let $\Gamma_{D,c} := \{\sigma x_c : \sigma \in \mathrm{Gal}(K_D[c]/K_D)\}$ be the corresponding Galois orbit. The fixed embedding ι gives us a prime in $K_D[c]$ above ℓ. The choice of the embedding defines a reduction map

$$\mathrm{red}_\ell : X_0(N)(K_D[c]) \hookrightarrow X_0(N)(K_D[c]_\ell) = X_0(N)(\mathcal{O}_{K_D[c]_\ell}) \xrightarrow{\text{mod } \ell} X_0(N)(\overline{\mathbb{F}}_\ell)$$

where $\mathcal{O}_{K_D[c]_\ell}$ is the ring of integers of the completion $K_D[c]_\ell$ (the equality in the middle follows from the valuative criterion of properness). Moreover, since ℓ is inert in K_D, then CM points for K_D reduce to supersingular points modulo ℓ (see [7]). Let $X_0(N)^{\mathrm{SS}}_{/\overline{\mathbb{F}}_\ell}$ be the set of supersingular points on $X_0(N)$ modulo ℓ. It is well-known that these points are defined over $\mathbb{F}_{\ell^2}$. We will prove an equidistribution theorem according to which as $d_c := -Dc^2 \to \infty$, every $s \in X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}$ will have the same number of preimages in $\Gamma_{D,c}$ under red_ℓ.

1.2 Outline of the proof techniques

Before we state our result rigorously in terms of probability measures on the finite set $X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}$, we will outline the three main steps of the proof. The first one is a theorem of Cornut and the first author establishing a natural correspondence between the set of Heegner points of conductor c on $X_0(N)$ reducing to a given supersingular point $s \in X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}$ and the set of conjugacy classes of optimal embeddings of the quadratic order that is isomorphic to the endomorphism ring of these Heegner points into the Eichler order that is the endomorphism ring of the supersingular point s. We state the precise correspondence in Sect. 2 and interpret the adelic result in our context.

The second step consists of establishing a precise correspondence between such optimal embeddings and primitive representations of the non-fundamental discriminant $d_c = -Dc^2$ by a certain ternary quadratic form Q_s associated to the point $s \in X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}$. More precisely, Q_s is the form determined by the reduced norm on a free $\mathbb{Z}$-module of rank 3 that is a submodule of the trace 0 submodule of the endomorphism ring $\mathrm{End}(s)$ (see Sect. 4.1 for details).

The third and the most technical step is related to equidistribution of primitive representations within the genus of the quadratic form Q_s. The genus subdivides further into classes (the spinor genera) for which such an equidistribution is known due to the previous work of Schulze-Pillot and Duke–Schulze-Pillot. The difficulty of our argument lies in deducing equidistribution within the genus from the equidistribution within the spinor genera. The key idea is to analyze the Fourier coefficients of two theta series—one that is associated to the genus of Q_s and another one, associated to the spinor genus of Q_s. Our key Proposition 4.7 establishes the equality of the genus and the spinor genus masses away from primes dividing ℓ and the level N. The proof is based on a combination of arguments using known results about these theta series, Vatsal's equidistribution result [36, Thm. 1.5] (for single reduction maps and p-adically growing conductor that is established using the theory of random walks and that uses no ergodic theory methods) and the analytic class number formula. Finally, we deduce our desired equidistribution from known bounds on Fourier coefficients of cuspidal forms of half-integral weight originally proved by Iwaniec and extended by Duke to the case of weight $3/2$. This then establishes equidistribution of optimal embeddings (Theorem 1.2) which via the correspondence from the first step translates into an equidistribution of Heegner points (Theorem 1.1). In order to state these theorems formally, we will now define the necessary probability measure on $X_0(N)^{\mathrm{SS}}_{/\overline{\mathbb{F}}_\ell}$.

1.3 A canonical measure on $X_0(N)^{\mathrm{SS}}_{/\overline{\mathbb{F}}_\ell}$

Let $s \in X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}$ be a supersingular point. Then s is represented by a pair $(\tilde{E}, \tilde{C})$ of a supersingular elliptic curve $\tilde{E}_{/\overline{\mathbb{F}}_\ell}$ and a cyclic subgroup $\tilde{C}$ of $\tilde{E}$ of order N. Following [32, Sect. 3], we refer to the pair $\mathbb{E} = (\tilde{E}, \tilde{C})$ as an *enhanced elliptic curve* over $\mathbb{F}_{\ell^2}$. Homomorphisms of enhanced elliptic curves are defined in the obvious way.

In particular, one could talk about endomorphisms and automorphisms of enhanced elliptic curves.

Let $\mathbb{E} = (\tilde{E}, \tilde{C})$ be an enhanced elliptic curve representing the point s. The endomorphism algebra $\mathrm{End}(\tilde{E}) \otimes \mathbb{Q}$ is isomorphic to the unique quaternion algebra $B_{\ell,\infty}$ ramified precisely at ℓ and ∞. The endomorphism ring $\mathrm{End}(\tilde{E})$ is a maximal order in $B_{\ell,\infty}$ and the ring $\mathrm{End}(\mathbb{E})$ is an Eichler order of level N. Indeed, if $\lambda : \tilde{E} \to \tilde{E}/\tilde{C}$ is the quotient map, then $\mathrm{End}(\tilde{E}/\tilde{C})$ can be viewed as a subring of $B_{\ell,\infty}$ via the map $\sigma \in \mathrm{End}(\tilde{E}/\tilde{C}) \mapsto \lambda^{-1}\sigma\lambda$. Then $\mathrm{End}(\mathbb{E})$ is the intersection of the two maximal orders $\mathrm{End}(\tilde{E})$ and $\mathrm{End}(\tilde{E}/\tilde{C})$. Let R_s denote this Eichler order and let $w_s := \# R_s^\times$. We can use w_s to define a canonical measure μ_{can} on $X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}$ by

$$\mu_{\mathrm{can}}(s) := \frac{1/w_s}{\sum_{s' \in X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}} 1/w_{s'}}.$$

1.4 Main results

1.4.1 Equidistribution of Heegner points

We can now state the main result of the paper. For $(D, c) \in \Omega_N$, define a measure $\mu_{D,c}$ on the finite set $X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}$ by

$$\mu_{D,c}(s) := \frac{\#\{x \in \Gamma_{D,c} : \mathrm{red}_\ell(x) = s\}}{\#\Gamma_{D,c}}, \quad s \in X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}.$$

Theorem 1.1 *The weak-* limit* $\lim_{\substack{-Dc^2 \to \infty, \\ (D,c) \in \Omega_N}} \mu_{D,c}$ *exists and equals* μ_{can}.

Remark 1 To say that the weak-* limit of a sequence of measures $\{\mu_n\}$ on a finite set X exists and converges to a measure μ on X means that for each function $f : X \to \mathbb{R}$, the limit $\lim_{n\to\infty} \int_X f d\mu_n$ exists and equals $\int_X f d\mu$.

1.4.2 Equidistribution of Gross points on the definite quaternion algebra $B_{\ell,\infty}$

The curve $X_0(N)_{/\mathbb{Q}}$ can be viewed as a Shimura curve for the quaternion algebra $M_2(\mathbb{Q})$, and thus, Heegner points can be regarded as CM points on the indefinite quaternion algebra $M_2(\mathbb{Q})$. In the case of a totally definite quaternion algebra (e.g., $B_{\ell,\infty}$), the analogues of Heegner points (also known as Gross points) were studied in detail by Gross [15].

Let G' be the algebraic group associated to $B_{\ell,\infty}^\times$, let R be an Eichler order of $B_{\ell,\infty}$ of level N and let $I_1, \ldots, I_h$ be left ideals representing the left ideal classes (corresponding to the double quotient $G'(\mathbb{Q})\backslash G'(\mathbb{A}_f)/\widehat{R}^\times$). Let $R_1, \ldots, R_h$ be the associated Eichler orders. Given a conductor c, the points of conductor c are simply pairs $(f : \mathcal{O}_c \hookrightarrow R_{i/R_i^\times}, R_i)$ of one of these orders R_i and an $R_i^\times$-conjugacy classes of optimal embeddings $f : \mathcal{O}_c \to R_i$. Recall that $f : \mathcal{O}_c \to R$ is optimal if $f(K) \cap R = \mathcal{O}_c$ (we have extended f to an embedding $f : K \to B_{\ell,\infty}$).

Let $\tilde{\mu}_{D,c}([I_i])$ be the number of Gross points (f, R_i) of conductor c divided by the total number of Gross points of conductor c. Then $\tilde{\mu}_{D,c}$ is a probability measure on $G'(\mathbb{Q})\backslash G'(\mathbb{A}_f)/\widehat{R}^\times$. There is a canonical measure on $G'(\mathbb{Q})\backslash G'(\mathbb{A}_f)/\widehat{R}^\times$ defined as

$$\tilde{\mu}_{\text{can}}([I_k]) := \frac{1/w_k}{\sum_{i=1}^h 1/w_i}.$$

Theorem 1.2 *The weak-* limit* $\lim_{\substack{-Dc^2\to\infty,\\ (D,c)\in\Omega_N}} \tilde{\mu}_{D,c}$ *exists and equals* $\tilde{\mu}_{\text{can}}$.

Remark 2 A similar statement (for trivial conductor $c = 1$) has already been established by Michel [25, Thm.3] using subconvexity bounds for L-functions and independently by Elkies, Ono, and Yang [12, Theorem 1.2].

Remark 3 Both Theorems 1.1 and 1.2 hold in greater generality for CM points on indefinite and totally definite quaternion algebras, respectively, with respect to more general reduction maps at several primes. The more general statements will be the subject of a forthcoming paper.

Remark 4 We will see in Sect. 2.5 that the canonical measures μ_{can} and $\tilde{\mu}_{\text{can}}$ indeed coincide.

1.4.3 Congruences for Hilbert class polynomials under the U-operator

Recall that for a function with Fourier expansion $f(z) = \sum_{n\geq 0} a(n)q^n$ the operator $U(\ell)$ is defined by $f(z)|U(\ell) := \sum_{n\geq 0} a(\ell n)q^n$. Elkies, Ono and Yang were interested in the equidistribution of Heegner points with respect to reduction maps which they used to study a certain congruence for the Hilbert class polynomial under the U-operator. In particular, combining the case $N = 1$ of Theorem 1.1 with [12, Thm. 2.3 (1)] gives the following immediate corollary (the case $c = 1$ is [12, Thm. 1.1]):

Corollary 1.3 *Let $H_{D,c} \in \mathbb{Z}[x]$ be the polynomial whose roots are precisely the j-invariants of those elliptic curves with CM by $\mathcal{O}_{D,c}$. Let ℓ be a prime which is non-split in $\mathcal{O}_{D,c}$. Then for $d_c = -Dc^2$ sufficiently large (depending on ℓ) there exists a polynomial $P_{D,c,\ell} \in \mathbb{Z}[x]$ such that*

$$H_{D,c}(j(z))|U(\ell) \equiv P_{D,c,\ell}(j(z)) \pmod{\ell}.$$

1.4.4 Effective surjectivity of red_ℓ

One consequence of both Theorems 1.1 and 1.2 is the fact that for sufficiently large discriminant $d_c = -Dc^2$, the reduction map from CM points of conductor c to supersingular points is surjective. It is natural to ask whether this theorem can be made effective. The ineffectiveness of one of the ingredients used in our argument, Siegel's lower bound on the class number, prevents us from establishing an effective result when both D and c vary. Yet, fixing the fundamental discriminant D and varying the conductor c, one can establish effective surjectivity theorems (see Theorem 6.1 and Lemma 6.2).

2 Heegner points and optimal embeddings

Let $s \in X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}$ be a supersingular point modulo ℓ. In this section we will establish a one-to-one correspondence between

$$\left\{ \begin{array}{c} \text{Heegner points } x \text{ on } X_0(N) \text{ of conductor } c \\ \text{reducing to } s \in X_0(N)^{\mathrm{SS}}_{/\overline{\mathbb{F}}_\ell} \end{array} \right\}$$
$$\Longleftrightarrow \left\{ \begin{array}{c} R_s^\times - \text{conjugacy classes of conjugate pairs of} \\ \text{optimal embeddings } \mathcal{O}_{D,c} \hookrightarrow R_s \end{array} \right\} \tag{2.1}$$

For $c = 1$, the above correspondence is known as Deuring lifting theorem (see [7]) and has been subsequently refined (as a correspondence) by Gross and Zagier [16, Prop. 2.7]. We will deduce the correspondence from a recent result of the first author and Cornut [2].

2.1 Galois orbits of Heegner points

We start by proving that there are exactly $2^{\nu(N)}$ Galois orbits of Heegner points of conductor c, where $\nu(N)$ is the number of distinct prime divisors of N.

Lemma 2.1 *Suppose that* $(c, N) = 1$. *Then there are exactly* $2^{\nu(N)}$ *Galois orbits of Heegner points of conductor c on* $X_0(N)$ *and each of these orbits has size* $\#\,\mathrm{Pic}(\mathcal{O}_{D,c})$.

Proof Consider the set of all Heegner points of conductor c on $X_0(N)$. They could be described as pairs $([\mathfrak{a}], \mathfrak{n})$ of an ideal class $[\mathfrak{a}]$ and an ideal $\mathfrak{n} \subset \mathcal{O}_K$ with the property that $\mathcal{O}_K/\mathfrak{n} \cong \mathbb{Z}/N\mathbb{Z}$. The last property is equivalent to the fact that $\mathfrak{n}$ is primitive of norm N ($\mathfrak{n}$ being primitive means that there is no rational prime number dividing $\mathfrak{n}$). Equivalently, if $N = p_1^{e_1} \dots p_t^{e_t}$ are the distinct prime divisors of N, we want $\mathfrak{n} = \mathfrak{p}_1^{e_1} \dots \mathfrak{p}_t^{e_t}$, where $\mathfrak{p}_i$ is one of the primes of $\mathcal{O}_K$ above p_i (indeed, if both $\mathfrak{p}_i$ and $\overline{\mathfrak{p}}_i$ occur then $\mathfrak{n}$ would be divisible by p_i and hence, would not be primitive). □

2.2 Modular curves and Shimura curves

We recall some basic facts that link modular and Shimura curves:

2.2.1 Modular curves as adelic quotients of $\mathrm{SL}_2(\mathbb{A}_f)$

Let U be a compact open subgroup of $\mathrm{SL}_2(\mathbb{A}_f)$ and let $\Gamma = \mathrm{SL}_2(\mathbb{Q}) \cap U$. It is known that Γ is a general congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Conversely, if Γ is a general congruence subgroup, let $U = U(\Gamma)$ be the closure of Γ in $\mathrm{SL}_2(\mathbb{A}_f)$. The group $\mathrm{SL}_2(\mathbb{Q})$ admits a left action on $\mathfrak{h}$ by linear fractional transformations and a left action on $\mathrm{SL}_2(\mathbb{A}_f)$ by left multiplication. Thus, $\mathrm{SL}_2(\mathbb{Q})$ acts on the left on $\mathfrak{h} \times \mathrm{SL}_2(\mathbb{A}_f)$. Moreover, U has a right action on $\mathfrak{h} \times \mathrm{SL}_2(\mathbb{A}_f)$ by acting trivially on $\mathfrak{h}$ and by right multiplication on $\mathrm{SL}_2(\mathbb{A}_f)$. Strong approximation (see [38, p.81]) gives a homeomorphism

$$Y(\Gamma) := \Gamma\backslash\mathfrak{h} \to \mathrm{SL}_2(\mathbb{Q})\backslash\mathfrak{h} \times \mathrm{SL}_2(\mathbb{A}_f)/U, \quad z \mapsto [z, 1].$$

This allows us to identify $Y(\Gamma)$ with a double adelic quotient of $\mathrm{SL}_2(\mathbb{A}_f)$.

2.2.2 *Transition from* $\mathrm{SL}_2(\mathbb{A}_f)$ *to* $\mathrm{GL}_2(\mathbb{A}_f)$

Let H be a compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$. We start from the observation that the double quotient

$$\mathbb{Q}^\times\backslash\{\pm 1\} \times \mathbb{A}_f^\times / \det(H)$$

is compact and discrete, and hence, finite. Here, $\mathbb{Q}^\times$ acts on $\{\pm 1\}$ and $\mathbb{A}_f$ on the left and $\det(H)$ acts on $\mathbb{A}_f$ on the right. Next, $\mathrm{GL}_2(\mathbb{R})$ acts on $\mathbb{C} - \mathbb{R} = \mathfrak{h}^+ \cup \mathfrak{h}^-$ in an analogous way to the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathfrak{h}$. Thus, we can consider the adelic quotient

$$\mathrm{Sh}_H = \mathrm{GL}_2(\mathbb{Q})\backslash(\mathbb{C} - \mathbb{R}) \times \mathrm{GL}_2(\mathbb{A}_f)/H$$

and the map

$$\mathrm{GL}_2(\mathbb{Q})\backslash(\mathbb{C} - \mathbb{R}) \times \mathrm{GL}_2(\mathbb{A}_f)/H \to \mathbb{Q}^\times\backslash\{\pm 1\} \times \mathbb{A}_f^\times / \det(H)$$

that sends $\phi : [z, g] \mapsto [\mathrm{sgn}(\mathrm{Im}(z)), \det(g)]$. The fiber of that map over $[+1, 1]$ is isomorphic to

$$\mathrm{SL}_2(\mathbb{Q})\backslash\mathfrak{h} \times \mathrm{SL}_2(\mathbb{A}_f)/(H \cap \mathrm{SL}_2(\mathbb{A}_f)),$$

which is precisely the modular curve $Y(\Gamma)$, where $\Gamma = H \cap \mathrm{SL}_2(\mathbb{Q})$. Moreover, we can also describe the entire curve Sh_H in terms of modular curves. Let $b_1, \ldots, b_n$ be representatives of $\mathbb{Q}^{>0}\backslash\mathbb{A}_f^\times$ in $\mathbb{A}_f^\times$. Let $a_1, \ldots, a_n$ be elements of $\mathrm{GL}_2(\mathbb{A}_f)$ such that $\det(a_i) = b_i$. Then $\Gamma_i = \mathrm{SL}_2(\mathbb{Q}) \cap a_i K a_i^{-1}$ is a congruence subgroup. Moreover, the maps

$$\Gamma_i\backslash\mathfrak{h} \to \mathrm{GL}_2(\mathbb{Q})\backslash(\mathbb{C} - \mathbb{R}) \times \mathrm{GL}_2(\mathbb{A}_f)/H$$

given by $[h_i] \mapsto [h_i, a_i]$ define a homeomorphism

$$\coprod_{i=1}^{n} \Gamma_i\backslash\mathfrak{h} \cong \mathrm{GL}_2(\mathbb{Q})\backslash(\mathbb{C} - \mathbb{R}) \times \mathrm{GL}_2(\mathbb{A}_f)/H =: \mathrm{Sh}_H.$$

2.3 Adelic description of CM points

2.3.1 *CM points on the Shimura curve* Sh_H

Fix an embedding $K \hookrightarrow M_2(\mathbb{Q})$. This gives us an embedding $T \hookrightarrow \mathrm{GL}_2$, where $T := \mathrm{Res}_{K/\mathbb{Q}} K^\times$. Consider the set $\mathrm{CM}(\mathrm{GL}_2, H)$ of all points of the form $[g, h] \in \mathrm{Sh}_H$ whose stabilizer is a torus isomorphic to $K^\times$. It is easy to verify that an element $z \in \mathbb{C}\backslash\mathbb{R}$ is in K if and only if $\mathrm{Stab}_{\mathrm{GL}_2(\mathbb{Q})}(z)$ is isomorphic to $\mathrm{Res}_{K/\mathbb{Q}} K^\times = T$. This allows us to conclude that $\mathrm{CM}(\mathrm{GL}_2, H)$ admits an adelic description as the double quotient $T(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_f)/H$. Indeed, a point in $\mathrm{CM}(\mathrm{GL}_2, H)$ is represented by a pair $[z, g]$, where $z \in \mathbb{C}\backslash\mathbb{R}$ is in K and $g \in \mathrm{GL}_2(\mathbb{A}_f)$. Since all $z \in K$ are $\mathrm{GL}_2(\mathbb{Q})$-conjugates and since the stabilizer of each z in $\mathrm{GL}_2(\mathbb{Q})$ is isomorphic to $T(\mathbb{Q})$, we obtain

$$\mathrm{CM}(\mathrm{GL}_2, H) \cong T(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_f)/H.$$

2.3.2 *Conductors of CM points*

Here, we assume that $R = (R', R'')$ is an oriented Eichler order of $M_2(\mathbb{Q})$ of level N (i.e., R' and R'' are maximal orders and $R = R' \cap R''$) and consider the Shimura curve Sh_H, where $H = \widehat{R}^\times$. Consider the two degeneracy maps

$$\delta' : \mathrm{CM}(\mathrm{GL}_2, H) \to T(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_f)/\widehat{R'}^\times$$

and

$$\delta'' : \mathrm{CM}(\mathrm{GL}_2, H) \to T(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_f)/\widehat{R''}^\times.$$

Given a CM point $x \in T(\mathbb{Q})\backslash G(\mathbb{A}_f)/\widehat{R}^\times$ such that $x = [g]$, let x' and x'' be the images of x in $T(\mathbb{Q})\backslash G(\mathbb{A}_f)/\widehat{R'}^\times$ and $T(\mathbb{Q})\backslash G(\mathbb{A}_f)/\widehat{R''}^\times$, respectively. The stabilizer

$$\mathrm{Stab}_{\widehat{K}^\times}(x') = \widehat{K}^\times \cap g\widehat{R'}^\times g^{-1} = \widehat{\mathcal{O}(x')}^\times$$

for some order $\mathcal{O}(x') \subseteq \mathcal{O}_K$. Let $c(x')$ be the conductor of that order. Similarly, we obtain an integer $c(x'')$ for R''. The conductor $\mathbf{c}(x)$ is then defined as

$$\mathbf{c}(x) := \mathrm{lcm}(c(x'), c(x'')).$$

Remark 5 Note that if q is a prime that divides one of $c(x')$ and $c(x'')$, but not the other one, then q necessarily divides N. This shows that if $(c, N) = 1$, all CM points of conductor c will be in fact Heegner points (i.e., $c(x') = c(x'')$).

Remark 6 For $\Gamma = \Gamma_0(N)$, i.e., for the modular curve $X_0(N)$, these degeneracy maps correspond precisely to the two degeneracy maps $\delta_1, \delta_N : X_0(N) \to X(1)$ that map $[E, C]$ to $[E]$ and $[E/C]$, respectively.

Remark 7 If $X = X_0(N)$, a CM point $[\tau] \in \Gamma_0(N)\backslash\mathfrak{h}$ would correspond to the pair of N-isogenous CM elliptic curves $E' = \mathbb{C}/\langle 1, \tau\rangle$ and $E'' = \mathbb{C}/\langle 1, N\tau\rangle$. Then $\mathcal{O}' = \mathrm{End}(E')$ and $\mathcal{O}'' = \mathrm{End}(E'')$ are both orders in $K = \mathbb{Q}(\sqrt{-D})$. Let c' and c'' be their conductors, respectively. The conductor of the point $[E', E'']$ is then $\mathbf{c}([E', E'']) = \mathrm{lcm}(c', c'')$.

2.4 Optimal embeddings and Gross points

Let $B_{\ell,\infty}$ be the unique quaternion algebra ramified precisely at ℓ and ∞ and let $G' := B_{\ell,\infty}^{\times}$ be the corresponding algebraic group. Let $R_1, \ldots, R_h$ be the Eichler orders of level N defined in Sect. 1.4.

Lemma 2.2 *The set of pairs* $(f : \mathcal{O} \hookrightarrow R_{i/R_i^{\times}}, [R_i])$ *of an ideal class* $[R_i]$ *of* $G'(\mathbb{Q})\backslash G'(\mathbb{A}_f)/\widehat{R}^{\times}$ *and a* $R_i^{\times}$*-conjugacy class of optimal embeddings* $f : \mathcal{O} \hookrightarrow R_{i/R_i^{\times}}$ *for some quadratic order* $\mathcal{O}$ *in* K *is in one-to-one correspondence with the double adelic quotient*

$$T(\mathbb{Q})\backslash G'(\mathbb{A}_f)/\widehat{R}^{\times}.$$

Proof Given an order R_i representing an ideal class $[R_i]$, the set of $R_i^{\times}$-conjugacy classes of optimal embeddings $f : \mathcal{O} \hookrightarrow R_i$ is in bijection with $T(\mathbb{Q})\backslash G'(\mathbb{Q})$ (since all the embeddings of K into $B_{\ell,\infty}$ are conjugate). Therefore, the set of the desired pairs is in bijection with

$$T(\mathbb{Q})\backslash G'(\mathbb{Q}) \times G'(\mathbb{Q})\backslash G'(\mathbb{A}_f)/\widehat{R}^{\times} \cong T(\mathbb{Q})\backslash G'(\mathbb{A}_f)/\widehat{R}^{\times}.$$

Remark 8 If $H' = \widehat{R}^{\times}$ then we can view the set

$$\mathrm{CM}(G', H') := T(\mathbb{Q})\backslash G'(\mathbb{A}_f)/H'$$

as the analogue of set of CM points for the pair (G', H').

2.5 Adelic description of supersingular points

The set $X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}$ is in bijection with the double quotient $\mathcal{X}(G', H') := G'(\mathbb{Q})\backslash G'(\mathbb{A}_f)/\widehat{R'}^{\times}$ where $H' = \widehat{R'}^{\times}$ for R' being an Eichler order of level N for $B_{\ell,\infty}$ that is the ring of endomorphisms of a fixed enhanced supersingular elliptic curve $\mathbb{E}_0 = (\tilde{E}_0, \tilde{C}_0)$. We briefly summarize the bijection and refer the reader to [32, Prop. 3.3] for the details.

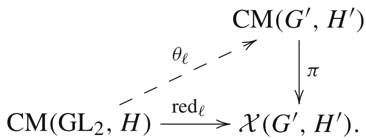
Let $\mathbb{E}$ be any enhanced elliptic curve and take an endomorphism $\lambda \in \mathrm{Hom}(\mathbb{E}, \mathbb{E}_0) \otimes \mathbb{Q}$ (here, we use the fact that there is a single isogeny class of supersingular elliptic curves). One could use λ to identify the adelic Tate module $\widehat{T}(\mathbb{E})$ with a sublattice of $\widehat{V}(\mathbb{E}_0)$. This means that there is a unique element $g \in G'(\mathbb{A}_f)/\widehat{R'}^{\times}$ that sends this sublattice to $\widehat{T}(\mathbb{E}_0)$. Since g is dependent on the choice of λ, it makes sense only in $G'(\mathbb{Q})\backslash G'(\mathbb{A}_f)/\widehat{R'}^{\times}$. This gives us a bijection

$$\varphi : X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}} \to \mathcal{X}(G', H') := G'(\mathbb{Q})\backslash G'(\mathbb{A}_f)/\widehat{R'}^{\times}.$$

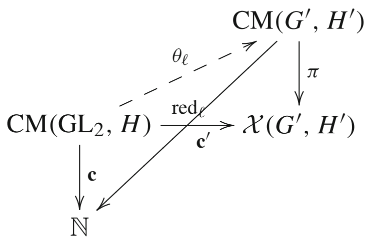
2.6 Heegner points on definite and indefinite quaternion algebras

The probability measures μ_c are defined in terms of the cardinalities $|\operatorname{red}_\ell^{-1}(s)\cap\Gamma_{D,c}|$. Let $s \in X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}$ be a supersingular point and let $h(\mathcal{O}_{D,c}, R_s)$ be the number of $R_s^{\times}$-conjugacy classes of optimal embeddings $\mathcal{O}_{D,c} \hookrightarrow R_s$.

Let R be an Eichler order of B of level N. As explained in [2], there is an Eichler order R' of $B_{\ell,\infty}$ associated to R. Let $H = \widehat{R}^{\times} \subset \mathrm{GL}_2(\mathbb{A}_f)$ and let $H' = \widehat{R'}^{\times} \subset G'(\mathbb{A}_f)$. For the reduction map $\operatorname{red}_\ell\colon\ \mathrm{CM}(\mathrm{GL}_2, H) \to \mathcal{X}(G', H')$, Cornut and the first author [2, Sect. 2] construct a Galois and Hecke-equivariant lifting θ_ℓ of the reduction map red_ℓ making the following diagram commutative:



One can then look at the following diagram:



The correspondence between CM points on the definite and the indefinite algebras is the following theorem that is an easy consequence of [2, Thm. 3.1]:

Theorem 2.3 (Cornut–Jetchev) *Let c be integer satisfying $(c, N\ell) = 1$ and let $s \in \mathcal{X}(G', H')$ be a supersingular point. Then θ_ℓ induces a bijection on the set $\mathbf{c}^{-1}(c) \cap \operatorname{red}_\ell^{-1}(s)$ onto the set of conjugacy pairs of optimal embeddings inside $\mathbf{c}'^{-1}(c) \cap \pi^{-1}(s)$. In particular,*

$$2\left|\mathbf{c}^{-1}(c) \cap \operatorname{red}_\ell^{-1}(s)\right| = \left|\mathbf{c}'^{-1}(c) \cap \pi^{-1}(s)\right|.$$

We will apply the above theorem together with the above adelic interpretations of CM points, optimal embeddings and supersingular points to deduce the following corollary:

Corollary 2.4 *We have*

$$h(\mathcal{O}_{D,c}, R_s) = 2^{\nu(N)+1} \left|\{x \in \Gamma_{D,c} \colon \mathrm{red}_\ell(x) = s\}\right|.$$

Proof By Theorem 2.3 and the adelic interpretation of CM points on the definite and the indefinite algebras as well as the adelic desciption of the supersingular points, the subset of CM points on $X_0(N)$ of conductor c reducing to a fixed supersingular point s is in bijection with the $R_s^\times$-conjugacy classes of conjugacy pairs of optimal embeddings $f : \mathcal{O}_c \hookrightarrow R_s$. Since $(c, N) = 1$, all CM points on $X_0(N)$ are Heegner points and by Lemma 2.1 there are exactly $2^{\nu(N)}$ such orbits.

The corollary shows that

$$\mu_c(s) = \frac{\left|\{x \in \Gamma_{D,c} \colon \mathrm{red}_\ell(x) = s\}\right|}{\left|\Gamma_{D,c}\right|} = \frac{h(\mathcal{O}_{D,c}, R_s)}{2^{\nu(N)+1}\left|\mathrm{Pic}(\mathcal{O}_{D,c})\right|}.$$

In Sect. 4, the number $h(\mathcal{O}_{D,c}, R_s)$ will be related to primitive representations of $d_c = -Dc^2$ by a certain quadratic form associated to R_s.

3 Modular forms of half-integral weight and Shimura correspondence

Let λ be a non-negative integer and consider the space $M_{\lambda+\frac{1}{2}}(\Gamma_0(4M), \chi)$ of modular forms of weight $\lambda + \frac{1}{2}$. Let $S_{\lambda+\frac{1}{2}}(\Gamma_0(4M), \chi)$ be the space of cusp forms. Let $q := e^{2\pi i z}$ and ψ be an odd Dirichlet character of conductor $r(\psi)$. We will refer to the form

$$h_{\psi,t}(z) := \sum_{m\geq 1} \psi(m) m e^{2\pi i t m^2 z} = \sum_{m\geq 1} \psi(m) m q^{tm^2} \in S_{3/2}(4r(\psi)^2, \psi \cdot \chi_{-4}) \tag{3.1}$$

as a *one-dimensional theta series*. Due to the exceptional behaviour of these forms, we will often decompose $S_{3/2}(4M)$ into the subspace spanned by one-dimensional theta series and the orthogonal complement of this space under the Petersson inner product, and then investigate each separately.

Suppose that $g(z) \in S_{\lambda+\frac{1}{2}}(\Gamma_0(4M), \chi)$. Let t be a positive square-free integer and let

$$\psi_t(n) := \chi(n) \left(\frac{-1}{n}\right)^\lambda \left(\frac{t}{n}\right).$$

Suppose that the complex numbers $A_t(n)$ are defined by

$$\sum_{n=1}^\infty \frac{A_t(n)}{n^s} := L(s - \lambda + 1, \psi_t) \cdot \sum_{n=1}^\infty \frac{b(tn^2)}{n^s}.$$

Shimura then proved that the tth *Shimura correspondence* $S_{t,\lambda}(g(z)) := \sum_{n=1}^{\infty} A_t(n)q^n$ is a modular form in $M_{2\lambda}(\Gamma_0(2N), \chi^2)$ of weight 2λ.

Kohnen then defined a subspace $S^+_{\lambda+\frac{1}{2}}(\Gamma_0(4M))$, referred to as Kohnen's plus space, consisting of forms $g(z)$ of weight $\lambda + \frac{1}{2}$ on $\Gamma_0(4M)$ with Fourier coefficients of the form

$$g(z) = \sum_{(-1)^\lambda n \equiv 0,1 \mod 4} b(n)q^n.$$

In this space Kohnen extended the definition of the Shimura correspondence $S_{t,\lambda}$ to $t' := (-1)^\lambda D$ where D is a fundamental discriminant. For $D \equiv 1 \pmod 4$ we take $S_{t',\lambda} := S_{t,\lambda}$ as previously defined and for $D \equiv 0 \pmod 4$ we take $S_{t',\lambda} := S_{t,\lambda}|U(4)$. Kohnen's plus space decomposes into new and old subspaces as follows:

$$S^+_{\lambda+\frac{1}{2}}(\Gamma_0(4M)) = S^{\text{new}}_{\lambda+\frac{1}{2}}(\Gamma_0(4M)) \oplus S^{\text{old}}_{\lambda+\frac{1}{2}}(\Gamma_0(4M)).$$

Kohnen used this decomposition and the Shimura correspondences

$$S_{t',\lambda} : S^{\text{new}}_{\lambda+\frac{1}{2}}(\Gamma_0(4M)) \to S_{2\lambda}(\Gamma_0(N))$$

to prove that there exists a finite linear combination of $S_{t',\lambda}$'s which provides an isomorphism

$$S : S^{\text{new}}_{\lambda+\frac{1}{2}}(\Gamma_0(4M)) \to S_{2\lambda}(\Gamma_0(N)) \tag{3.2}$$

that is Hecke equivariant. The image of a half-integral weight Kohnen newform in $S^{\text{new}}_{\lambda+\frac{1}{2}}(\Gamma_0(4M))$ is a newform in $S^{\text{new}}_{2\lambda}(\Gamma_0(M))$ whose Hecke eigenvalues are the same.

4 Equidistribution and ternary quadratic forms

In order to prove the main theorem, we establish the correspondence between optimal embeddings and primitive representations in Sect. 4.1 by associating a quadratic form Q_s to the Eichler order R_s. We then compute the discriminant of that quadratic form. We introduce the theta series θ_{Q_s} associated to Q_s, as well as the series $\theta_{\text{gen}(Q_s)}$ and $\theta_{\text{spn}(Q_s)}$ associated to the genus $\text{gen}(Q_s)$ and the spinor genus $\text{spn}(Q_s)$ of Q_s, respectively. Finally, using that the form $\theta_{\text{gen}(Q_s)} - \theta_{\text{spn}(Q_s)}$ is in the space spanned by one-dimensional theta series, we are able to prove that the coefficients of $\text{gen}(Q_s)$ and $\text{spn}(Q_s)$ coincide away from the primes dividing $N\ell$. We use bounds on Fourier coefficients of modular forms of half-integral weight that lie in the the orthogonal complement (under the Petersson inner product) of the space spanned by one-dimensional theta series (due to Iwaniec and Duke) to conclude the proof of Theorems 1.1 and 1.2.

4.1 Optimal embeddings and primitive representations by ternary quadratic forms

Let $s \in X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}$ and let $R_s := \mathrm{End}(s)$ be the ring of endomorphisms of s. Recall the notation $w_s := \#R_s^\times$ and $u_{D,c} := \#\mathcal{O}_{D,c}$.

4.1.1 A ternary quadratic form associated to an Eichler order

Let $V \subset R_s$ be the set of elements of trace zero. Following [15, pp. 171–172] define

$$G_s := (2R_s + \mathbb{Z}) \cap V.$$

The $\mathbb{Z}$-module G_s is free of rank 3. Define a quadratic form $Q_s : G_s \to \mathbb{Q}$ by

$$Q_s(b) := \mathrm{nr}(b).$$

4.1.2 Correspondence between optimal embeddings and primitive representations

Let $f : \mathcal{O}_{D,c} \hookrightarrow R_s$ be an embedding (not necessarily optimal) and let $\beta := f(\sqrt{-d_c})$. Notice that $\mathrm{Tr}(\beta) = 0$ and $\mathrm{nr}(\beta) = d_c$. We claim that $\beta \in G_s$. Indeed, since $\mathcal{O}_{D,c} = \mathbb{Z} + \frac{d_c+\sqrt{-d_c}}{2}\mathbb{Z}$, it follows that $2f\left(\frac{d_c+\sqrt{-d_c}}{2}\right) = d_c + \beta$, i.e.,

$$\beta \equiv -d_c \mod 2R_s.$$

Therefore, $\beta \in (\mathbb{Z} + 2R_s) \cap V = G_s$, i.e., $Q_s(\beta) = d_c$ is a representation.

Conversely, suppose that $\beta \in G_s$ and $Q_s(\beta) = d_c$. We claim that $\beta \equiv -d_c \mod 2R_s$. Indeed, let $\beta = \gamma + 2r$ for some $\gamma \in \mathbb{Z}$ and $r \in R_s$. Then

$$d_c = Q_s(\beta) = \mathrm{nr}(\beta) = \beta\overline{\beta} = -\beta^2 = -(\gamma + 2r)^2 \equiv -\gamma^2 \mod 4R_s. \quad (4.1)$$

Thus,

$$\beta = \gamma + 2r \equiv (\gamma + \gamma^2) - \gamma^2 \equiv d_c \equiv -d_c \mod 2R_s.$$

Now, we can define an embedding $f : \mathcal{O}_{D,c} \hookrightarrow R_s$ by

$$f\left(\frac{d_c + \sqrt{-d_c}}{2}\right) := \frac{d_c + \beta}{2} \in R_s.$$

We next show under the established correspondence that optimal embeddings correspond to primitive representations.

Lemma 4.1 *The embedding f is optimal if and only if the representation $Q_s(\beta) = d_c$ is primitive.*

Proof Suppose that the representation $Q_s(\beta) = d_c$ is non-primitive. We will show that f is not an optimal embedding. Indeed, let $\beta = k\alpha$ for some $k \in \mathbb{Z}$ and $\alpha \in G_s$. Then $\mathrm{nr}(\alpha) = \frac{d_c}{k^2}$. Let $d = \frac{d_c}{k^2}$. Consider the element $\gamma = \frac{d+\alpha}{2}$. We claim that $\gamma \in f(K_D) \cap R_s$, but $\gamma \notin f(\mathcal{O}_{D,c})$ which would imply that f is a non-optimal embedding. First, $\gamma = \frac{1}{k^2} f\left(\frac{d_c+\sqrt{d_c}}{2}\right) \in f(\mathcal{O}_{D,c}) \otimes \mathbb{Q}$. Let $\alpha = a + 2r$ for $a \in \mathbb{Z}$ and $r \in R_s$. Then $d = \mathrm{nr}(\alpha) = -\alpha^2 \equiv -a^2 \mod 2R_s$. Thus, $\alpha = a + 2r \equiv -a^2 \equiv d \equiv -d \mod 2R_s$, i.e., $\gamma \in R_s$. Next, we show that $\gamma \notin f(\mathcal{O}_{D,c})$. If $k \neq 2$ then $\gamma = \frac{d+\alpha}{2} = \frac{d_c+\beta+dk-dk^2}{2k}$. Since $d_c + \beta = 2f(w) \notin kf(\mathcal{O}_{D,c})$ then $\gamma \notin f(\mathcal{O}_{D,c})$. If $k = 2$ then $\gamma = \frac{d_c+\beta-2d}{4}$. Since $d_c + \beta - 2d = f(2w - 2d) \notin 4f(\mathcal{O}_{D,c})$ we obtain the same statement. Thus, $\gamma \in f(K_D) \cap R_s$, but $\gamma \notin f(\mathcal{O}_{D,c})$, i.e., the embedding is not optimal.

Conversely, suppose that $f : \mathcal{O}_{D,c} \hookrightarrow R_s$ is a non-optimal embedding. Let $\mathcal{O} = (f(\mathcal{O}_{D,c}) \otimes \mathbb{Q}) \cap R_s$. It follows that $\mathcal{O} \cong \mathcal{O}_{D,c'}$, where $c = kc'$ for some $k > 1$. Now, we can choose $\alpha \in (2\mathcal{O} + \mathbb{Z}) \cap V$, such that $Q_s(\alpha) = -Dc'^2$. Since $(\mathbb{Z} + 2\mathcal{O}) \cap V$ is a free $\mathbb{Z}$-module of rank 1, we obtain $\beta = k\alpha$, i.e., the representation $Q_s(\beta) = -Dc^2$ is not primitive. This proves the lemma.

Thus, we have proved the following:

Proposition 4.2 *There is a $\frac{w_s}{u_{D,c}}$-to-one correspondence between primitive representations of the integer $d_c = -Dc^2$ by Q_s and optimal embeddings $f : \mathcal{O}_{D,c} \hookrightarrow R_s$.*

4.2 The discriminant of Q_s

For what follows, we will need the discriminant of the quadratic form Q_s.

Lemma 4.3 *The discriminant D_{Q_s} of the quadratic form Q_s is equal to $4N^2\ell^2$.*

Proof Let $p \neq \ell$ be a prime and let $v_p(N) =: n$. Since R_s is an Eichler order of level N, we know that $R_s \otimes \mathbb{Z}_p$ is an Eichler order of level p^n of two-by-two matrices over $\mathbb{Z}_p$. In particular (up to conjugation) we have

$$R_s \otimes \mathbb{Z}_p = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}.$$

Therefore,

$$G_s \otimes \mathbb{Z}_p = \left[2\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} + \mathbb{Z}_p\right] \cap V = \left\{\begin{pmatrix} a & 2b \\ 2p^n c & -a \end{pmatrix} : a, b, c \in \mathbb{Z}_p\right\}.$$

But then the local quadratic form $Q_{s,p} := Q_s \otimes \mathbb{Z}_p$ is given by

$$Q_{s,p}(a,b,c) = \begin{vmatrix} a & 2b \\ 2p^n c & -a \end{vmatrix} = -a^2 - 4p^n bc.$$

The corresponding matrix for the quadratic form $Q_{s,p}$ is then

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -2p^n \\ 0 & -2p^n & 0 \end{pmatrix}. \tag{4.2}$$

The determinant of this matrix is $-4p^{2n}$. Therefore, if $p \neq 2$ then 4 is a unit and the contribution to the determinant of Q_s is p^{2n}, while if $p = 2$ the contribution is $4p^{2n}$.

Now consider the case $p = \ell \neq 2$. In this case, we have $R_s \otimes \mathbb{Z}_p$ is the unique maximal order of the unique division algebra, with $\mathbb{Z}_p$-basis $(1, \alpha, \beta, \gamma)$ satisfying $\alpha^2 = -p$, $\beta^2 = -1$ and $\gamma = \alpha\beta = -\beta\alpha$. But then $G_s \otimes \mathbb{Z}_p$ has basis $(2\alpha, 2\beta, 2\gamma)$. We obtain the quadratic form

$$Q_{s,p}(2a\alpha + 2b\beta + 2c\gamma) = 4pa^2 + 4b^2 + 4pc^2, \tag{4.3}$$

which is diagonal with discriminant $64p^2$, contributing p^2 to the discriminant.

For $p = \ell = 2$ we note that since the Eichler order is locally isomorphic to the (unique) maximal order, Gross [15, p. 177] has shown that for $\alpha^2 = \beta^2 = \gamma^2 = -1$ with $\gamma = \alpha\beta = -\beta\alpha$,

$$G_s \otimes Z_p = \{a\alpha + (a+2b)\beta + (a+2c)\gamma : a, b, c \in \mathbb{Z}_p\}.$$

Thus the p-adic quadratic form is given by

$$Q_{s,p}(a,b,c) = -(3a^2 + 4ab + 4ac + 4b^2 + 4c^2),$$

with corresponding matrix

$$\begin{pmatrix} 3 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 4 \end{pmatrix}. \tag{4.4}$$

The determinant of this matrix is 16, and hence contributes $16 = 4\ell^2$ to the discriminant.

4.3 The theta series associated to Q_s

Consider the theta series

$$\theta_{Q_s} := \sum_{\beta \in G_s} q^{Q_s(\beta)} = \sum_{d \geq 1} a_s(d) q^d.$$

Since $-Q_s(\beta) \equiv 0, 1 \mod 4$, we obtain that $a_s(d) \neq 0$ only if $-d$ is a discriminant, i.e., $-d \equiv 0, 1 \mod 4$. Thus,

$$\theta_{Q_s} = \sum_{\beta \in G_s} q^{Q_s(\beta)} = \sum_{-d \equiv 0,1 \mod 4} a_s(d) q^d.$$

Recall the definition of Kohnen's plus space $M^+_{3/2}(\Gamma_0(4M))$ from Sect. 3.

Lemma 4.4 *We have* $\theta_{Q_s} \in M^+_{3/2}(\Gamma_0(4N\ell))$.

Proof Let A be the matrix corresponding to Q_s. It is well known that $\theta_{Q_s} \in M^+_{3/2}(\Gamma_0(4M))$, where M is the minimal positive integer, such that $4MA^{-1}$ has coefficients that are even integers (see [10, p. 39]). Since A^{-1} has rational coefficients, it suffices to check that each coefficient of $4MA^{-1}$ has non-negative p-adic valuation for each p. We then explicitly compute the inverse of Eqs. (4.2–4.4) to check that it has even integral coefficients when we multiply by $4p^{v_p(N\ell)}$. □

4.4 The theta series associated to the genus and the spinor genus of Q_s

Let Q be a ternary quadratic form. Let $\mathrm{gen}(Q)$ be the genus of Q and let $\mathrm{spn}(Q)$ be the spinor genus of Q (see [28, Ch. X] for the definitions). Let $\mathrm{loc}(Q)$ be the set of all integers n that are everywhere locally represented by Q. Let $r(Q, n)$ (resp. $r^*(Q, n)$) be the number of representations (resp. primitive representations) of n by Q. Let w_Q be the number of automorphs of Q (see [22] for the definition).

4.4.1 Theta series associated to gen(Q)

Let

$$r(\mathrm{gen}(Q), n) := \frac{\sum_{Q' \in \mathrm{gen}(Q)} r(Q', n)/w_{Q'}}{\sum_{Q' \in \mathrm{gen}(Q)} 1/w_{Q'}}. \quad (4.5)$$

Similarly, define

$$r^*(\mathrm{gen}(Q), n) := \frac{\sum_{Q' \in \mathrm{gen}(Q)} r^*(Q', n)/w_{Q'}}{\sum_{Q' \in \mathrm{gen}(Q)} 1/w_{Q'}}.$$

We define the theta series associated to $\mathrm{gen}(Q)$ as

$$\theta_{\mathrm{gen}(Q)} := \sum_{n \geq 1} r(\mathrm{gen}(Q), n) q^n.$$

By calculating local densities, Jones [22, Thm. 86] has shown that for $d_c = -Dc^2$

$$r^*(\mathrm{gen}(Q_s), d_c) = C \frac{h(-\Delta d_c)}{u_{\Delta D, c}}. \quad (4.6)$$

Here, Δ denotes the discriminant D_{Q_s} of Q_s divided by the square of the greatest common divisor of the determinants of all two-by-two minors of the matrix corresponding to Q_s, and C only depends on the Legendre symbol $(\frac{d_c}{D_{Q_s}})$. One can calculate Δ p-adically using Eqs. (4.2–4.4) to show that the greatest common divisor of the determinants of all two-by-two minors is precisely $\sqrt{D_{Q_s}}$. Thus, $\Delta = 1$.

4.4.2 Theta series associated to spn(Q)

We define the theta series associated to the spinor genus in a similar way. First, let

$$r(\mathrm{spn}(Q), n) := \frac{\sum_{Q' \in \mathrm{spn}(Q)} r(Q', n)/w_{Q'}}{\sum_{Q' \in \mathrm{spn}(Q)} 1/w_{Q'}}. \tag{4.7}$$

Similarly, let

$$r^*(\mathrm{spn}(Q), n) := \frac{\sum_{Q' \in \mathrm{spn}(Q)} r^*(Q', n)/w_{Q'}}{\sum_{Q' \in \mathrm{spn}(Q)} 1/w_{Q'}}.$$

We also define

$$\theta_{\mathrm{spn}(Q)} := \sum_{n \geq 1} r(\mathrm{spn}(Q), n) q^n.$$

The theta series $\theta_{\mathrm{gen}(Q)}$ and $\theta_{\mathrm{spn}(Q)}$ are in the same space as θ_Q (by (4.5) and (4.7) and the fact that $\theta_{Q'}$ are in the same space as Q for all $Q' \in \mathrm{gen}(Q)$; see also [17, p. 366]).

4.5 Equidistribution in terms of quadratic forms

In light of the correspondence obtained in Proposition 4.2, the required equidistribution results (Theorems 1.1 and 1.2) are equivalent to showing that

$$\lim_{\substack{(D,c) \in \Omega_N \\ d_c \to \infty}} \frac{r^*(Q_s, d_c) u_{D,c}}{2^{\nu(N)+1} \# \Gamma_{D,c}} = w_s \mu_{\mathrm{can}}(s). \tag{4.8}$$

This result will be equivalent to showing that the limit

$$f(s) := \lim_{\substack{(D,c) \in \Omega_N \\ d_c \to \infty}} \frac{r^*(Q_s, d_c) u_{D,c}}{\# \Gamma_{D,c}}. \tag{4.9}$$

exists and is independent of the supersingular point s. Here, recall that $d_c := -Dc^2$.

First, note that $\theta_{Q_s} - \theta_{\mathrm{spn}(Q_s)}$ is a modular form of weight $3/2$ that lies in the orthogonal complement of the space of one-dimensional theta series under the Petersson inner product [35]. Duke's bound for the Fourier coefficients of such forms [9], extending the work of Iwaniec [20] to forms of weight 3/2, combined with Möbius inversion, implies that

$$r^*(\mathrm{spn}(Q_s), d_c) - r^*(Q_s, d_c) = O(d_c^{\frac{13}{28}+\epsilon}).$$

Siegel's lower bound for the class number [34] (see also [4, p. 149]) implies that $\#\Gamma_{D,c} \gg d_c^{\frac{1}{2}-\epsilon}$, so

$$\frac{r^*(Q_s, d_c)u_{D,c}}{\#\Gamma_{D,c}} = \frac{r^*(\mathrm{spn}(Q_s), d_c)u_{D,c}}{\#\Gamma_{D,c}} + O(d_c^{-\frac{1}{28}+\epsilon}). \tag{4.10}$$

Thus, we only need to show independence and convergence of the limit for each spinor genus. Since $r^*(\mathrm{gen}(Q_s), n)$ is independent of s by definition, it will be natural to compare $r^*(\mathrm{gen}(Q_s), n)$ with $r^*(\mathrm{spn}(Q_s), n)$ in order to determine the desired independence.

In particular, we have the following.

Lemma 4.5 *The limit*

$$\lim_{k\to\infty} \frac{r^*(\mathrm{gen}(Q_s), -Dp^{2k})u_{D,p^k}}{\#\Gamma_{D,p^k}}$$

exists and is independent of p and s.

Proof The independence on s is clear from the definition of $\mathrm{gen}(Q_s)$. We will apply (4.6) to $n = -Dp^{2k}$. First, recall (see [4, Cor. 7.28, p. 148]) that for any discriminant $D_0 < 0$ and any prime p we have

$$h(-D_0p^{2k}) = Cp^k\left(1 - \frac{1}{p}\left(\frac{-D_0}{p}\right)\right) \cdot \frac{u_{D_0,p^k}}{u_{D_0,1}} h(-D_0). \tag{4.11}$$

Equation (4.11) with $D_0 = D$ allows us to express $r^*(\mathrm{gen}(Q_s), -Dp^{2k})$ and $\#\Gamma_{D,p^k}$ as

$$\begin{aligned} r^*(\mathrm{gen}(Q_s), -Dp^{2k}) &= \frac{c_D h(-Dp^{2k})}{u_{D,p^k}} \\ &= c_d C p^k\left(1 - \frac{1}{p}\left(\frac{-D}{p}\right)\right)\frac{h(-D)}{u_{D,1}}, \end{aligned} \tag{4.12}$$

$$\frac{\#\Gamma_{D,p^k}}{u_{D,p^k}} = p^k\left(1 - \frac{1}{p}\left(\frac{-D}{p}\right)\right)\frac{\#\Gamma_{D,1}}{u_{D,1}}. \tag{4.13}$$

Hence, for $k \geq 1$ we obtain

$$\frac{r^*(\mathrm{gen}(Q_s), -Dp^{2k})u_{D,p^k}}{\#\Gamma_{D,p^k}} = c_D C \frac{h(-D)}{\#\Gamma_{D,1}}. \tag{4.14}$$

The result follows since the right-hand side of (4.14) is independent of k and p.

We now define the restricted limit

$$f_{D,p}(s) := \lim_{k\to\infty} \frac{r^*(Q_s, -Dp^{2k})u_{D,p^k}}{\#\Gamma_{D,p^k}} = \lim_{k\to\infty} \frac{r^*(\mathrm{spn}(Q_s), -Dp^{2k})u_{D,p^k}}{\#\Gamma_{D,p^k}}. \tag{4.15}$$

The equidistribution result of Vatsal [36, Thm. 1.5] combined with Proposition 4.2 states the following:

Lemma 4.6 (Vatsal) *For every $s \in X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}$, fundamental discriminant $D < 0$ and $p \nmid N\ell$ we have*

$$f_{D,p}(s) = w_s \mu_{\mathrm{can}}(s). \tag{4.16}$$

We will now use Eq. (4.16) to rewrite $r^*(\mathrm{spn}(Q_s), n)$ in terms of $r^*(\mathrm{gen}(Q_s), n)$ and then use Eq. (4.10) to show that $f(s)$ exists and is independent of s.

Proposition 4.7 *Let $s \in X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}$ and $n = -Dc^2$, where $D < 0$ is a fundamental discriminant. Assume that $(c, N\ell) = 1$. Then*

$$r^*(\mathrm{gen}(Q_s), n) = r^*(\mathrm{spn}(Q_s), n).$$

Let $a_m := r(\mathrm{gen}(Q_s), m) - r(\mathrm{spn}(Q_s), m)$. According to a result of Schulze-Pillot [35] as well as Flicker [13], Niwa [26], Cipra [1] and others, $\theta_{\mathrm{gen}(Q_s)} - \theta_{\mathrm{spn}(Q_s)}$ belongs to the subspace of cuspidal forms of weight 3/2 spanned by the one-dimensional theta series (see also [18]). Note that the Fourier coefficients of the one-dimensional theta series $h_{\psi,t}(z)$ defined in Eq. (3.1) vanish outside the square class $t\mathbb{Z}^2$. Let

$$\theta_{\mathrm{gen}(Q_s)} - \theta_{\mathrm{spn}(Q_s)} = \sum_{\psi,t} c_{\psi,t} h_{\psi,t}. \tag{4.17}$$

Let $h_{\psi,t}$ be one of the one-dimensional theta series in (4.17) and let $4M = 4N\ell$ be the level of $\theta_{\mathrm{gen}(Q_s)} - \theta_{\mathrm{spn}(Q_s)}$. The transformation law for modular forms of level $4M$ with Nebentypus χ implies that $\psi(mn) = \psi(n)\chi_t(m)$ for every $(m, M) = 1$, where $\chi_t(m) = \chi(m)\left(\frac{-t}{m}\right)$ (see [35, p. 285]). In addition, if $h_{\psi,t} \neq 0$ then $4t \mid M$ (see, e.g., [35, Kor. 2]).

Lemma 4.8 *Let $-d > 0$ be the smallest positive integer satisfying the following two conditions*:

1. *If* $d = Dc^2$*, where* $D < 0$ *is a fundamental discriminant then* c *is prime to* $N\ell$*;*
2. $a_{-d} \neq 0$.

Then $c = 1$ *and* $d = D$ *is a fundamental discriminant.*

Proof It follows from (4.17) and Lemma 4.4 [since $(m, M) = 1$] that $\psi(m) = \chi_t(m)$. Hence,

$$a_{-d} = \sum_{-d=tm^2,\psi} c_{\psi,t}\psi(m)m = \sum_{-d=tm^2} \chi_t(m)m \sum_{\psi} c_{\psi,t} \neq 0.$$

Hence there exists t satisfying $\sum_{\psi} c_{\psi,t} \neq 0$. Choose the minimal t with this property and observe that $a_t = \sum_{t',t=t'm^2} \sum_{\psi} c_{\psi,t'}\psi(m)m = \sum_{\psi} c_{\psi,t} \neq 0$. Hence, $t = -d$ and

$$a_{-d} = \sum_{-d=tm^2,\psi} c_{\psi,t}h_{\psi,t} = \sum_{\psi} c_{\psi,-d}h_{\psi,-d} \neq 0.$$

Now, if the conductor c of d were not equal to 1, it would have divided M and hence would not have been prime to $N\ell$. Thus, the only possibility is that $c = 1$ and $d = D$ is a fundamental discriminant.

Lemma 4.9 *Let* $D < 0$ *be the fundamental discriminant from Lemma 4.8 and* $(c, N\ell) = 1$*. If* D_{Q_s} *is the discriminant of the quadratic form* Q_s *then*

$$a_{-Dc^2} = c\left(\frac{DD_{Q_s}}{c}\right)a_{-D}.$$

In particular, for $c = p^k$ *we have*

$$a_{-Dp^{2k}} = p^k\left(\frac{DD_{Q_s}}{p}\right)^k a_{-D}.$$

Proof Note that $a_{-Dc^2} = \sum_{-Dc^2=tm^2,\psi} c_{\psi,t}\psi(m)m$. We know that if $t = -D(c')^2$ for some $c' > 1$ then $h_{t,\psi} = 0$ (since $(c', M) = 1$). Hence,

$$a_{-Dc^2} = \sum_{\psi} c_{\psi,D}\psi(c)c = c\chi_D(c)\sum_{\psi} c_{\psi,D} = c\left(\frac{DD_{Q_s}}{c}\right)a_{-D}.$$

Proof of Proposition 4.7 We will prove the statement by contradiction. Assume the contrary and let n be the smallest integer whose square part is prime to $N\ell$ and such that $r(\mathrm{gen}(Q_s), n) \neq r(\mathrm{spn}(Q_s), n)$. Lemma 4.8 implies that if $-n = Dc^2$ for a fundamental discriminant D and a conductor c then $c = 1$ and $n = -D$. Let $p \nmid N\ell$ be a prime for which $\left(\frac{DD_{Q_s}}{p}\right) = -1$. We will show that under these assumptions the limit

$f_{-D,p}(s)$ does not exist, contradicting Lemma 4.6. Using Lemma 4.9 and equation (4.13), we have

$$f_{-D,p}(s) = \lim_{k\to\infty}\left(\frac{\left(r^*(\mathrm{spn}(Q_s), -Dp^{2k}) - r^*(\mathrm{gen}(Q_s), -Dp^{2k})\right) u_{D,p^k}}{\#\Gamma_{D,p^k}} + \frac{r^*(\mathrm{gen}(Q_s), -Dp^{2k}) u_{D,p^k}}{\#\Gamma_{D,p^k}}\right)$$

$$= \lim_{k\to\infty}\left(-\frac{a_{-Dp^{2k}}}{\#\Gamma_{D,p^k}/u_{D,p^k}} + \frac{r^*(\mathrm{gen}(Q_s), -Dp^{2k}) u_{D,p^k}}{\#\Gamma_{D,p^k}}\right)$$

$$= \lim_{k\to\infty}\left(-\frac{a_{-D}p^k\left(\frac{DD_{Q_s}}{p}\right)^k}{p^k\left(1-\frac{1}{p}\left(\frac{D}{p}\right)\right)\#\Gamma_{D,1}/u_{D,1}} + \frac{r^*(\mathrm{gen}(Q_s), -Dp^{2k}) u_{D,p^k}}{\#\Gamma_{D,p^k}}\right)$$

$$= \lim_{k\to\infty}\left(-\frac{a_{-D}(-1)^k}{\left(1-\frac{1}{p}\left(\frac{D}{p}\right)\right)\#\Gamma_{D,1}/u_{D,1}} + \frac{r^*(\mathrm{gen}(Q_s), -Dp^{2k}) u_{D,p^k}}{\#\Gamma_{D,p^k}}\right).$$

However, the limit

$$\lim_{k\to\infty}\frac{r^*(\mathrm{gen}(Q_s), -Dp^{2k}) u_{D,p^k}}{\#\Gamma_{D,p^k}}$$

exists by Lemma 4.5. Therefore, if the limit $f_{-D,p}(s)$ exists, then the limit

$$\lim_{k\to\infty} -\frac{a_{-D}(-1)^k}{\left(1-\frac{1}{p}\left(\frac{D}{p}\right)\right)\#\Gamma_{D,1}/u_{D,1}}$$

must also exist. But $a_{-D} \neq 0$ and the only dependence on k is the term $(-1)^k$, leading to a contradiction.

Remark 9 After submission, Schulze-Pillot has noticed that one may use the explicit calculations in Sect. 4 to establish that the spinor norm is trivial away from the prime ℓ. A further analysis leads to the conclusion that the spinor norm is trivial globally, and hence there is only one spinor genera, explaining the equality established in Proposition 4.7 for the genera and the spinor genera of this wide class of ternary quadratic forms.

4.6 Proof of the main theorem

We are now ready to prove Theorems 1.1 and 1.2. Let $D < 0$ be a fundamental discriminant and let c be an integer with $(c, N\ell) = 1$. Define

$$g_{D,c}(s) := \frac{r^*(Q_s, -Dc^2)u_{D,c}}{\#\Gamma_{D,c}}$$

and

$$h_{D,c} := \frac{r^*(\mathrm{gen}(Q_s), -Dc^2)u_{D,c}}{\#\Gamma_{D,c}}.$$

Note that $h_{D,c}$ is independent of s. Proposition 4.7 combined with Eq. (4.10) gives

$$g_{D,c}(s) = h_{D,c} + O_s\left((-Dc^2)^{-\frac{1}{28}+\epsilon}\right).$$

We now divide by w_s and sum over all $s' \in X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}$. Recall from Proposition 4.2 that

$$\frac{g_{D,c}(s')\#\Gamma_{D,c}}{w_{s'}} = r^*(Q_{s'}, -Dc^2)\frac{u_{D,c}}{w_{s'}}$$

is the number of optimal embeddings of $\mathcal{O}_{D,c}$ into $R_{s'}$. Summing over all $s' \in X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}$ thus gives $\#\Gamma_{D,c}$. Hence, we have

$$1 = \sum_{s'\in X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}} \frac{g_{D,c}(s')}{w_{s'}} = h_{D,c} \sum_{s'\in X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}} \frac{1}{w_{s'}} + O\left((-Dc^2)^{-\frac{1}{28}+\epsilon}\right). \tag{4.18}$$

Therefore

$$h_{D,c} = \frac{1}{\sum_{s'\in X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}} 1/w_{s'}} + O\left((-Dc^2)^{-\frac{1}{28}+\epsilon}\right).$$

Thus the limit $\lim_{-Dc^2\to\infty} h_{D,c}$ exists, and we obtain

$$\begin{aligned} f(s) &= \lim_{-Dc^2\to\infty} g_{D,c}(s) = \lim_{-Dc^2\to\infty}\left[h_{D,c}(s) + O\left((Dc^2)^{-\frac{1}{28}+\epsilon}\right)\right] \\ &= \frac{1}{\sum_{s'\in X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}} 1/w_{s'}} = w_s\mu_{\mathrm{can}}(s). \end{aligned}$$

But this is precisely Eq. (4.8), and hence we obtain Theorem 1.1.

Remark 10 Due to dependence on Siegel's lower bound for the class number, Theorem 1.1 is ineffective. However, if we fix a fundamental discriminant $D < 0$ and only vary the conductor c, then this result becomes effective due to known growth of the class number in a fixed square class. Moreover, in a fixed square class the $-D$th Shimura correspondence implies that the difference

$$a_c(s) := g_{D,c}(s) - h_{D,c}$$

are coefficients of a weight 2 cusp form. Using Deligne's optimal bound, the error term can be improved to $O(c^{-1/2+\epsilon})$. Therefore, the error can be written as

$$O((-D)^{-\frac{1}{28}+\epsilon}c^{-\frac{1}{2}+\epsilon}).$$

5 Distribution relations method

In this section, we establish equidistribution when the fundamental discriminant $D < 0$ is fixed and the conductor varies using an alternative argument based on the distribution relations for Heegner points and Hecke eigenvalue bounds.

5.1 An easier equidistribution theorem

Here, we only consider a special infinite set of conductors c and a fixed fundamental discriminant $D < 0$. Let $\mathcal{P}$ be the set of all primes $r \nmid N$, such that r is inert in K. Let $\mathcal{I}$ be the set of all integers that are square-free products of primes in $\mathcal{P}$. Note that $\Lambda \subset \mathcal{I}$. Under the same hypothesis as before, we will prove the following statement:

Theorem 5.1 *Given a Galois orbit* $\Gamma_{D,c}$ *let* $\mu_{D,c}$ *be the measure on* $X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}$ *defined as in Theorem 1.1. Then* $\lim_{\substack{c\to\infty,\\ c\in\mathcal{I}}} \mu_{D,c} = \mu_{\mathrm{can}}$.

Remark 11 The assumption that $c \in \mathcal{I}$ is not necessary and the argument in the more general case is exactly the same, except for the more technical form of the distribution relations. Here, we prove only the less technical statement where the distribution relations are easier to work with (see Sect. 5.2).

5.2 Distribution relations

Let $X_c \in \mathrm{Div}(X_0(N))$ be defined as $X_c := \sum_{\sigma\in\mathrm{Gal}(K[c]/K)}(x_c^\sigma)$. We will prove the following distribution relation:

Lemma 5.2 *For any prime number* ℓ *which is inert in* K *and any positive integer* c *coprime to* ℓ, *the following distribution relation holds*:

$$X_{c\ell} = T_\ell X_c.$$

Proof Let S be a set of coset representatives for $\mathrm{Gal}(K[c\ell]/K[c])/\mathrm{Gal}(K[c]/K)$. The distribution relation for Heegner points [14, §6] is the following equality of divisors of degree $\ell+1$ on $X_0(N)$:

$$\mathrm{Tr}_{K[\ell c]/K[c]}(x_{c\ell}^{\sigma}) = T_\ell(x_c^{\sigma}), \quad \sigma \in \mathrm{Gal}(K[c\ell]/K),$$

i.e.,

$$\sum_{\tau \in \mathrm{Gal}(K[c\ell]/K[c])} (x_{c\ell}^{\sigma\tau}) = T_\ell(x_c^{\sigma}), \quad \sigma \in \mathrm{Gal}(K[c\ell]/K).$$

Hence,

$$\sum_{\sigma \in S} \sum_{\tau \in \mathrm{Gal}(K[c\ell]/K[c])} (x_{c\ell}^{\sigma\tau}) = \sum_{\sigma \in S} T_\ell(x_c^{\sigma}), \quad \sigma \in \mathrm{Gal}(K[c\ell]/K),$$

which implies

$$X_{c\ell} = T_\ell X_c.$$

5.3 Proof of the main theorem

Proof of Theorem 5.1 First, we note that the reduction map $\mathrm{red}_\ell : X_0(N)_{/\mathbb{Q}} \to X_0(N)_{/\mathbb{F}_\ell}$ defined in Sect. 1 is Hecke equivariant. Thus,

$$\mathrm{red}_\ell(X_{cr}) = T_r\, \mathrm{red}_\ell(X_c).$$

Next, $\mathrm{red}_\ell(X_{cr})$ and $\mathrm{red}_\ell(X_c)$ belong to the subgroup $\mathrm{Div}^{\mathrm{SS}}(X_0(N)_{/\overline{\mathbb{F}}_\ell})$ of divisors supported on the supersingular points of $X_0(N)_{/\overline{\mathbb{F}}_\ell}$. The Hecke algebra $\mathbb{T}_{N\ell}$ acts on the vector space $V_{\mathrm{SS}} = \mathrm{Div}^{\mathrm{SS}}(X_0(N)_{/\overline{\mathbb{F}}_\ell}) \otimes \overline{\mathbb{Q}}$ via its ℓ-new quotient $\mathbb{T}_{N\ell}^{\ell-\mathrm{new}}$ (see [30] or [33]). Let

$$V_{\mathrm{SS}} = V_{\mathrm{Eis}} \oplus \left(\bigoplus_f V_f \right)$$

be the eigenspace decomposition of V, where f ranges over all normalized eigenforms $f \in S_2^{\ell-\mathrm{new}}(\Gamma_0(N\ell))$,

$$V_f = \{v \in V_{\mathrm{SS}} \ : \ T_r v = a_r(f) v \text{ for all primes } r\},$$

and

$$V_{\mathrm{Eis}} = \{v \in V_{\mathrm{SS}} \ : \ T_r v = (r+1) v \text{ for all primes } r\}.$$

Here, $a_r(f)$ denotes the rth Fourier coefficient of the eigenform f.

Let $Y_c = \frac{1}{\#\operatorname{Pic}(\mathcal{O}_c)} \operatorname{red}_\ell(X_c) \in V_{\mathrm{SS}}$. It is easy to see that $Y_c = \sum_{s \in X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}} \mu_c(s) \cdot (s)$. We can write the decomposition of Y_c as

$$Y_c = Y_{c,\mathrm{Eis}} + \sum_f Y_{c,f}, \;\; Y_{c,f} \in V_f, \;\; Y_{c,\mathrm{Eis}} \in V_{\mathrm{Eis}}.$$

The distribution relation from Lemma 5.2 implies that $\#\operatorname{Pic}(\mathcal{O}_{cr}) Y_{cr} = \#\operatorname{Pic}(\mathcal{O}_c) T_r Y_c$. Since $\#\operatorname{Pic}(\mathcal{O}_{cr}) = (r+1)\#\operatorname{Pic}(\mathcal{O}_c)$ then

$$Y_{cr} = \frac{1}{r+1} T_r Y_c.$$

We use this equality to obtain $Y_{cr,\mathrm{Eis}} = Y_{c,\mathrm{Eis}}$ and $Y_{cr,f} = \frac{a_r(f)}{r+1} Y_{c,f}$ for any normalized eigenform $f \in S_2^{\ell-\mathrm{new}}(\Gamma_0(N\ell))$.

The Ramanujan–Petersson conjecture then implies that

$$\frac{a_r(f)}{r+1} \leq \frac{2r^{1/2}}{r+1} \leq \frac{2}{r^{1/2}}.$$

Thus, we obtain by induction on the number of prime divisors of c that

$$Y_c = Y_{1,\mathrm{Eis}} + O(c^{-1/2}).$$

This means that $\lim_{\substack{c\to\infty,\\ c\in\mathcal{I}}} Y_c = Y_{1,\mathrm{Eis}}$.

Finally, one uses the result from Sect. 5.4 to conclude that $Y_{1,\mathrm{Eis}}$ is equal to the divisor associated to the canonical measure μ_{can}.

5.4 The divisor $Y_{1,\mathrm{Eis}}$

Let

$$D_{\mu_{\mathrm{can}}} := \sum_{s \in X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}} \mu_{\mathrm{can}}(s) \cdot (s) \in V_{\mathrm{SS}}.$$

It is well-known (see, e.g., [36, Lem. 2.5]) that the divisor $D_{\mu_{\mathrm{can}}}$ is Eisenstein. In other words,

$$T_r D_{\mu_{\mathrm{can}}} = (r+1) D_{\mu_{\mathrm{can}}}$$

for every prime $(r, N) = 1$. Next, we verify that $D_{\mu_{\mathrm{can}}}$ is the same as the Eisenstein part $Y_{1,\mathrm{Eis}}$ of Y_1:

Lemma 5.3 *We have*

$$D_{\mu_{\mathrm{can}}} = Y_{1,\mathrm{Eis}}.$$

Proof First, note that $\deg(D_{\mu_{\text{can}}}) = 1 = \deg(Y_1)$. Furthermore, a divisor $D \in \text{Div}(X_0(N)^{\text{SS}}_{/\mathbb{F}_{\ell^2}}) \otimes \overline{\mathbb{Q}}$ is cuspidal if and only if it has degree zero (see e.g., [33]). Thus, $\deg(Y_{1,\text{cusp}}) = 0$ and hence, $\deg(Y_{1,\text{Eis}}) = 1$.

Next, consider the exact sequence

$$0 \to \text{Div}^0(X_0(N)^{\text{SS}}) \to \text{Div}\left(X_0(N)^{\text{SS}}_{/\mathbb{F}_{\ell^2}}\right) \xrightarrow{\deg} \mathbb{Z} \to 0,$$

and look at the divisor $D = D_{\mu_{\text{can}}} - Y_{1,\text{Eis}}$. We know that $\deg(D) = 0$ and hence, D is cuspidal. At the same time, D is Eisenstein. If $D \neq 0$ then one would obtain a contradiction using the Hecke eigenvalue bounds for cusp forms. Thus, $D = 0$ and hence, $Y_{1,\text{Eis}} = D_{\mu_{\text{can}}}$.

6 Effective surjectivity results

We have seen in Theorem 1.1 that $\mu_{D,c} \to \mu_{\text{can}}$ as $d_c := -Dc^2 \to \infty$. In particular, for sufficiently large d_c we have $\mu_{D,c}(s) > 0$ for every $s \in X_0(N)^{\text{SS}}_{/\mathbb{F}_{\ell^2}}$, giving surjectivity of the reduction red_ℓ from $\Gamma_{D,c}$ to $X_0(N)^{\text{SS}}_{/\mathbb{F}_{\ell^2}}$. Here, we discuss effective versions of this surjectivity result.

Recall that the proof of Theorem 1.1 uses Siegel's lower bound on the class number [see (4.10)]. Siegel's bound $\#\Gamma_{D,c} \gg_{c,\varepsilon} D^{\frac{1}{2}-\varepsilon}$ is ineffective due to the fact that Siegel proved this result by first assuming the truth of GRH for Dirichlet L-functions and then proved the bound again with a different implied constant depending on the location of a possible Siegel zero [34]. The best known effective results are due to Oesterlé [27], but the growth obtained is only logarithmic in D. Hence, the surjectivity will be ineffective whenever we allow the fundamental discriminant to vary.

Thus, we fix a fundamental discriminant $D < 0$. Given a supersingular point $s \in X_0(N)^{\text{SS}}_{/\mathbb{F}_{\ell^2}}$, decompose θ_{Q_s} as

$$\theta_{Q_s} - \theta_{\text{spn}(Q_s)} = \sum_{i=1}^{r} b_i g_i, \tag{6.1}$$

where $b_i \in \mathbb{C}$ and $\{g_1, \dots, g_r\}$ is a fixed set of cuspidal Hecke eigenforms in the orthogonal complement (under the Petersson inner product) of the space spanned by one-dimensional theta series of weight $3/2$. We will denote the dth coefficient of g_i by $a_{g_i}(d)$ and the $-D$th Shimura correspondence (recall the extended definition in Sect. 3 given by Kohnen for fundamental discriminants) by $G_i := S_{-D,1}(g_i)$. Denote the number of distinct prime divisors of c by $\upsilon(c)$.

The following theorem establishes an effective bound for c [depending on the decomposition (6.1) and the fundamental discriminant $-D$] beyond which the preimage $\text{red}_\ell^{-1}(s)$ is non-empty. Taking the maximum occurring bound over all $s' \in X_0(N)^{\text{SS}}_{/\mathbb{F}_{\ell^2}}$ gives a bound depending only on N, ℓ and D beyond which surjectivity must hold.

Theorem 6.1 *Let $c > 2$ be an integer prime to $N\ell$ that satisfies the following inequality*

$$\frac{c^{1/2}}{2^{2v(c)+1}\sigma_0(c)\log c} > \frac{1}{\log 2}\frac{u_{D,1}}{\#\Gamma_{D,1}}\left(\sum_{i=1}^{r}\left|b_i a_{g_i}(-D)\right|\right)\left(\sum_{s'\in X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}} 1/w_{s'}\right).$$

Then the reduction map $\mathrm{red}_\ell : \Gamma_{D,c} \to X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}$ *satisfies* $\mathrm{red}_\ell^{-1}(s) \neq \varnothing$.

Proof For the θ-series θ_{Q_s} we have the decomposition

$$\theta_{Q_s}(z) = E(z) + H(z) + f(z),$$

where $E(z)$ is an Eisenstein series, $H(z)$ is in the space spanned by one-dimensional theta series of weight 3/2, and $f(z)$ is a cusp form in the orthogonal complement of the space spanned by one-dimensional theta series (see [18, p. 156]). Moreover, from the work of Schulze-Pillot [35], we know that $E(z) = \theta_{\mathrm{gen}(Q_s)}(z)$ and $H(z) = \theta_{\mathrm{spn}(Q_s)}(z) - \theta_{\mathrm{gen}(Q_s)}(z)$.

Let $a(n) := r(Q_s, n) - r(\mathrm{spn}(Q_s), n)$ be the nth Fourier coefficient of the form $f(z)$ and let $a^*(n) := r^*(Q_s, n) - r^*(\mathrm{spn}(Q_s), n)$. Let $n > 0$ be an integer satisfying $(n, N\ell) = 1$. We know by Proposition 4.7 that $r(\mathrm{gen}(Q_s), n) = r(\mathrm{spn}(Q_s), n)$. Therefore,

$$r(Q_s, n) = r(\mathrm{gen}(Q_s), n) + (r(Q_s, n) - r(\mathrm{gen}(Q_s), n)) = r(\mathrm{gen}(Q_s), n) + a(n).$$

Next, if $n = tc^2$ where t is square-free, Möbius inversion gives us

$$r^*(Q_s, n) = \sum_{c'|c}\mu(c')r(Q_s, n/c'^2) = r^*(\mathrm{gen}(Q_s), n) + \sum_{c'|c}\mu(c')a(n/c'^2). \tag{6.2}$$

By Proposition 4.2, we know that $s \in X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}$ is in the image of $\mathrm{red}_\ell : \Gamma_{D,c} \to X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}$ if and only if Q_s primitively represents $d_c = -Dc^2$. Thus, s is not in the image of the reduction map if and only if $r^*(Q_s, d_c) = 0$, i.e., if and only if

$$r^*(\mathrm{gen}(Q_s), d_c) = -\sum_{c'|c}\mu(c')a(d_c/c'^2). \tag{6.3}$$

The left-hand side can be computed using Jones' formula and [4, Cor. 7.28, p. 148] as it was applied previously for (4.12). We obtain

$$r^*(\mathrm{gen}(Q_s), d_c)\sum_{s'\in X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}} 1/w_{s'} = \frac{\#\Gamma_{D,c}}{u_{D,c}} = \left(\sum_{c'|c}\mu(c')\left(\frac{D}{c'}\right)\frac{c}{c'}\right)\frac{\#\Gamma_{D,1}}{u_{D,1}}. \tag{6.4}$$

For the right-hand side of (6.3), we would like to express the Fourier coefficient $a(d_c)$ in terms of $a(-D)$. This cannot be done directly for an arbitrary cusp form f in the orthogonal complement of the space of one-dimensional theta series, but could be achieved if f were an eigenform (due to the recurrence relations of the Hecke operators). In order to get such a relation, we write

$$f(z) = \sum_{i=1}^{r} b_i g_i(z),$$

where g_i's are Hecke eigenforms of weight $3/2$ whose images G_i under the $-D$th Shimura correspondence $S_{-D,1}$ are normalized Hecke eigenforms.

Decomposing $\theta_{Q_s} - \theta_{\mathrm{gen}(Q_s)}$ gives

$$a^*(d_c) = \sum_{i=1}^{r} b_i \sum_{c'|c} \mu(c') a_{g_i}\left(d_{c/c'}\right). \tag{6.5}$$

If $g := g_i$ is a Hecke eigenform, the $-D$th Shimura correspondence $G := S_{-D,1}(g) \in S_2(\Gamma_0(N\ell))$ is also a Hecke eigenform. Assume further that G is normalized so that $a_G(1) = 1$. By the multiplicity one theorem for forms of weight 2, there exists a newform $\widetilde{G} \in S_2(\Gamma_0(M))$ for some $M \mid N\ell$ such that $G = \sum_{d|\frac{N\ell}{M}} C_d \widetilde{G}|V(d)$ for some constants C_d (with $C_1 = 1$). Here, the operator $V(d)$ corresponds to one of the degeneracy maps (see e.g., [29, p. 28] for the definition). Notice that for $(c, N\ell) = 1$, the cth coefficient of G corresponds to the cth coefficient of the newform $\widetilde{G}$. Since c is relatively prime to the level, the cth coefficient of $\widetilde{G}$ is determined by the eigenvalues under the Hecke operators.

Using this connection and the definition of the $-D$th Shimura correspondence to evaluate the coefficients of $\widetilde{G}$ (using the fact that $\widetilde{G}$ is normalized), the second author [23, equation (4.2)] has shown for $c = p^m$ relatively prime to $FN\ell$,

$$\begin{aligned} a_g(d_{cF}) &= a_g(d_F)\left(a_G(p^m) - \left(\frac{-D}{p}\right) a_G(p^{m-1})\right) \\ &= a_g(d_F) \sum_{c'|c} \mu(c') \left(\frac{-D}{c'}\right) a_G\left(\frac{c}{c'}\right). \end{aligned}$$

Here we have rewritten the right hand side so that extending by multiplicativity, it follows that

$$a_g(d_c) = a_g(d_1) \sum_{c'|c} \mu(c') \left(\frac{D}{c'}\right) a_G\left(\frac{c}{c'}\right).$$

Substituting this in equation (6.5) gives the identity

$$a^*(d_c) = \sum_{i=1}^{r} b_i a_{g_i}(d_1) \sum_{c'|c} \sum_{c''|\frac{c}{c'}} \mu(c')\mu(c'') \left(\frac{D}{c''}\right) a_{G_i}\left(\frac{c}{c'c''}\right). \tag{6.6}$$

Thus we have established that the supersingular point $s \in X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}$ is not in the image of red_ℓ from $\Gamma_{D,c}$ if and only if

$$\frac{1}{\sum_{s' \in X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}}} 1/w_{s'}} \left(\sum_{c' \mid c} \mu(c') \left(\frac{D}{c'}\right) \frac{c}{c'} \right) \frac{\#\Gamma_{D,1}}{u_{D,1}}$$
$$= -\sum_{i=1}^{r} b_i a_{g_i}(d_1) \sum_{c' \mid c} \sum_{c'' \mid \frac{c}{c'}} \mu(c')\mu(c'') \left(\frac{D}{c''}\right) a_{G_i}\left(\frac{c}{c'c''}\right). \tag{6.7}$$

Now consider the Euler φ-function $\varphi(c) := \#\{m < c : (m,c) = 1\}$. Then

$$\sum_{c' \mid c} \mu(c') \left(\frac{D}{c'}\right) \frac{c}{c'} \geq \varphi(c),$$

since the inequality holds for c being a prime power and both functions are multiplicative. We can then use the explicit elementary bound $\varphi(c) \geq \frac{\log 2}{2} \frac{c}{\log c}$ for $c > 2$ (cf. [21, p. 9]).

We next pull the absolute value inside the sum on the right hand side of (6.7) and use Deligne's optimal bound [6] for integer weight cusp forms from the proof of the Weil conjectures, namely $|a_{G_i}(n)| \leq \sigma_0(n) n^{\frac{1}{2}}$. Since $\#\{c' \mid c : \mu(c') \neq 0\} = 2^{v(c)}$ and $\sigma_0(c') \leq \sigma_0(c)$ for $c' \mid c$, we have

$$|a^*(d_c)| \leq 2^{2v(c)} \sigma_0(c) c^{\frac{1}{2}} \sum_{i=1}^{r} |b_i a_{g_i}(D)|, \tag{6.8}$$

giving the assertion.

In the case when $\#X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}} = 2$ we obtain an explicit bound independent of D beyond which surjectivity holds. Let $m_s = \max\left(1, \frac{w_{s'}}{w_s}\right)$.

Lemma 6.2 *If* $\#X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}} = 2$ *then the inequality*

$$\varphi(c) > m_s 2^{2v(c)} \sigma_0(c) c^{\frac{1}{2}} \tag{6.9}$$

implies that the reduction red_ℓ *on* $\Gamma_{D,c}$ *is surjective for any fundamental discriminant* $D < 0$.

Proof Let $X_0(N)^{\mathrm{SS}}_{/\mathbb{F}_{\ell^2}} = \{s, s'\}$. Recall that

$$\theta_{\mathrm{gen}(Q_s)} = \frac{\frac{1}{w_s}\theta_{Q_s} + \frac{1}{w_{s'}}\theta_{Q_{s'}}}{1/w_s + 1/w_{s'}}.$$

By Siegel's theorem (see [8, Thm. 2(ii)]) there is a Hecke eigenform g such that $\theta_{Q_s} = \theta_{\mathrm{gen}(Q_s)} + g$ and $\theta_{Q_{s'}} = \theta_{\mathrm{gen}(Q_s)} - \frac{w_{s'}}{w_s} g$. Since $r(Q_s, |D|) \geq 0$ and $r(Q_{s'}, |D|) \geq 0$, we have $|a_g(|D|)| \leq \max(1, \frac{w_{s'}}{w_s}) r(\mathrm{gen}(Q_s), |D|)$. The lemma then follows immediately by combining Eqs. (6.4) and (6.8) with $b_1 g_1 = g$ after canceling $r(\mathrm{gen}(Q_s), |D|)$ on both sides.

Let $G = S_{-D,1}(g)$ be the $-D$th Shimura correspondence of g as defined in Sect. 3. Define

$$r_c := \frac{\sum_{c'|c} \mu(c') \left(\frac{-D}{c'}\right) \frac{c}{c'}}{\left|\sum_{c'|c} \mu(c') \sum_{c''|\frac{c}{c'}} \mu(c'') \left(\frac{-D}{c''}\right) a_G\left(\frac{c}{c'c''}\right)\right|},$$

where we take $r_c = \infty$ by convention if the denominator is zero, and

$$\widetilde{r}_c := \frac{\varphi(c)}{2^{2v(c)}\sigma_0(c)c^{\frac{1}{2}}}.$$

By Eqs. (6.7) and (6.8) if $r_c > m_s$ or $\widetilde{r}_c > m_s$ then s is in the image of red_ℓ. Note that both r_c and $\widetilde{r}_c$ are multiplicative and $r_c \geq \widetilde{r}_c$. For $c = p^m$ we have

$$\widetilde{r}_c = \frac{p^{\frac{m}{2}-1}(p-1)}{4(m+1)}.$$

For $p \geq 5$, $\widetilde{r}_c$ is increasing as a function of m, whereas for $p < 5$ it is increasing for $m > 2$. For a constant a and $m = 1$ the inequality $\widetilde{r}_c > a$ is satisfied for

$$p > P_a := \left(\frac{4a + \sqrt{16a^2 + 4}}{2}\right)^2.$$

For $p \leq P_a$ we use the fact that $\widetilde{r}_c$ is increasing exponentially as a function of m to obtain a bound $M_{p,a}$ such that $m > M_{p,a}$ implies that $\widetilde{r}_c > a$. Therefore, there are only finitely many choices for the pair (p, m) with $m \geq 1$ for which $r_{p^m} \leq a$. Let

$$C_a = \{(p, m) : r_{p^m} \leq a\}.$$

Computing r_{p^m} explicitly for $m \leq M_{p,a}$ allows us to explicitly calculate C_a.

We first follow the above argument with $a = 1$ to show that

$$r_{\min} := \prod_p \min_{m \geq 0} r_{p^m}$$

is well defined and satisfies $r_c \geq r_{\min}$ for every c. We will now use the above bounds with $a := \frac{m_s}{r_{\min}}$. Let c be an arbitrary integer such that $r_c \leq m_s$. Write $c = p^m c'$ with $(p, c') = 1$. By multiplicativity we have

$$m_s \geq r_c = r_{c'} r_{p^m} \geq r_{\min} r_{p^m}.$$

Therefore, $r_{p^m} \leq a$, so $(p, m) \in C_a$, and it follows that

$$c \mid \prod_{(p,m)\in C_a} p^m.$$

We can refine this argument by recursively computing

$$S_v := \{c : v(c) = v, r_c \leq m_s\}.$$

For $c' \in S_v$, consider

$$a' := a \frac{\prod_{(p,c')=1} \min_{m\geq 0} r_{p^m}}{r_{c'}}.$$

Then for $c = p^m c'$ with $(p, c) = 1$, $r_c \in S_{v+1}$ if and only if $(p, m) \in C_{a'}$. Constructing the resulting tree in this manner allows us to terminate the depth-first search when $C_{a'}$ is empty.

Proceeding in this manner, we obtain for $\ell = 11$ and $N = 1$ exactly 116 possible values of c in the union of all S_v, the largest of which is 5,124. For $\ell = 17$ and $N = 1$ there are 93 possible values of c, the largest of which is 3,990, and for $\ell = 19$ and $N = 1$ there are 165 possible values of c, the largest of which is 8,502.

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LOCALLY HARMONIC MAASS FORMS AND THE KERNEL OF THE SHINTANI LIFT

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In memory of Marvin Knopp

Abstract. In this paper we define a new type of modular object and construct explicit examples of such functions. Our functions are closely related to cusp forms constructed by Zagier [29] which played an important role in the construction by Kohnen and Zagier [22] of a kernel function for the Shimura and Shintani lifts between half-integral and integral weight cusp forms. Although our functions share many properties in common with harmonic weak Maass forms, they also have some properties which strikingly contrast those exhibited by harmonic weak Maass forms. As a first application of the new theory developed in this paper, one obtains a new proof of the fact that the even periods of Zagier's cusp forms are rational as an easy corollary.

1. Introduction and statement of results

For an integer $k > 1$ and a discriminant $D > 0$, define

$$f_{k,D}(\tau) := \frac{D^{k-\frac{1}{2}}}{\binom{2k-2}{k-1}\pi} \sum_{\substack{a,b,c\in\mathbb{Z}\\ b^2-4ac=D}} \left(a\tau^2+b\tau+c\right)^{-k}, \tag{1.1}$$

where $\tau \in \mathbb{H}$. This function was introduced by Zagier [29] in connection with the Doi-Naganuma lift (between modular forms and Hilbert modular forms) and lies in the space S_{2k} of (classical, holomorphic) cusp forms of weight $2k$ for $\Gamma_1 := \mathrm{SL}_2(\mathbb{Z})$. One may also realize $f_{k,D}$ as a certain linear combination of hyperbolic Poincaré series whose construction is due to Petersson [25].

The functions $f_{k,D}$ (and certain variations of them) play an important role in the theory of modular forms of half-integral weight. Indeed, as shown in [22] and later in [21], they are the Fourier coefficients of holomorphic kernel functions for the Shimura [27] (resp. Shintani [28]) lifts between half-integral and integral weight cusp forms. More precisely, for $\tau, z \in \mathbb{H}$, define

$$\Omega(\tau,z) := \sum_{0<D\equiv 0,1 \pmod 4} f_{k,D}(\tau)\, e^{2\pi i D z}. \tag{1.2}$$

Then Ω is a modular form of weight $2k$ in the variable τ and weight $k+\frac{1}{2}$ in the variable z. Furthermore, integrating Ω against a cusp form f of weight $2k$ (resp. $k+\frac{1}{2}$) with respect to the first (resp. second) variable is the Shintani (resp. Shimura) lift of f.

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In a different way, they also give important examples of modular forms with rational periods, as studied in [23]. In this paper, we construct a new type of modular object which both closely resembles and is connected to $f_{k,D}$ through differential operators which naturally occur in the theory of harmonic weak Maass forms (see Theorem 1.2). The resulting functions also give a new explanation and a new proof of the rationality of the even periods of $f_{k,D}$ for k even (see Theorem 1.4). We expect that these new objects will have further important applications to the theory of modular forms.

Before introducing these new modular objects, we first recall that a weight $2-2k$ *harmonic weak Maass form* is a real analytic function $\mathcal{F}$ which satisfies weight $2-2k$ modularity, is annihilated by the weight $2-2k$ *hyperbolic Laplacian*

$$\Delta_{2-2k} := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+i\left(2-2k\right)y\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$$

and has at most exponential growth at $i\infty$. Here and throughout $\tau\in\mathbb{H}$ is written as $\tau=x+iy$, $x,y\in\mathbb{R}$ with $y>0$. The theory of harmonic weak Maass forms has proven useful in many areas including combinatorics, number theory, physics, Lie theory, probability theory, and knot theory. To name a few examples, harmonic weak Maass forms have played a role in understanding Ramanujan's mock theta functions [32], in proving asymptotics and congruences in partition theory [6, 8], in relating character formulas of Kac and Wakimoto [17] to automorphic forms [7], in the study of metastability thresholds for bootstrap percolation models [2, 4], in the quantum theory of black holes [12], in studying the elliptic genera of $K3$ surfaces [16], and in the study of central values of L-series and their derivatives [10].

Bruinier and Funke [9] have shown that for every $f\in S_{2k}$, there exists a weight $2-2k$ harmonic weak Maass form $\mathcal{F}$ which is related to f through the anti-holomorphic operator $\xi_{2-2k}:=2iy^{2-2k}\overline{\frac{d}{d\overline{\tau}}}$ by $\xi_{2-2k}\left(\mathcal{F}\right)=f$. Such an $\mathcal{F}$ may be constructed via parabolic Poincaré series [5]. In particular, although we therefore know that such a lift of $f_{k,D}$ exists, it would be desirable to constuct a particular lift which resembles the shape (1.1) and is also related to hyperbolic Poincaré series. The construction of such a function analogous to (1.1) leads to a new class of automorphic objects which are the topic of this paper. To describe the resulting object, we first require some notation. Let

$$\psi\left(v\right):=\frac{1}{2}\beta\left(v;k-\frac{1}{2},\frac{1}{2}\right) \tag{1.3}$$

be a special value of the incomplete β-function, which is defined for $s,w\in\mathbb{C}$ satisfying $\mathrm{Re}\left(s\right)$, $\mathrm{Re}\left(w\right)>0$ by $\beta\left(v;s,w\right):=\int_0^v u^{s-1}\left(1-u\right)^{w-1}du$ (for some properties, see p. 263 and p. 944 of [1]). The function ψ may be written in a variety of forms, but we choose this representation because it generalizes to other weights (see (3.8) for another useful representation). Denote the set of integral binary quadratic forms $[a,b,c](X,Y):=aX^2+bXY+cY^2$ of discriminant D by $\mathcal{Q}_D:=\left\{[a,b,c]:b^2-4ac=D,\ a,b,c\in\mathbb{Z}\right\}$. For some technical reasons, we will restrict in the following to the case where D is a non-square discriminant. For $\tau\in\mathbb{H}$ we set

$$\mathcal{F}_{1-k,D}\left(\tau\right):=\frac{D^{\frac{1}{2}-k}}{\binom{2k-2}{k-1}\pi}\sum_{Q=[a,b,c]\in\mathcal{Q}_D}\operatorname{sgn}\left(a\left|\tau\right|^2+bx+c\right)Q\left(\tau,1\right)^{k-1}\psi\left(\frac{Dy^2}{\left|Q\left(\tau,1\right)\right|^2}\right). \tag{1.4}$$

Remark. After presenting the results of this paper, Zagier has informed us that he has independently investigated (in unpublished work) examples similar to (1.4) for some small k (in cases where there are no cusp forms in S_{2k}). In these cases, as we will see in Theorem 1.3, the function (1.4) is locally equal to a polynomial. Zagier's investigation of these functions was initiated by a question posed by physicists. It would be interesting to investigate what our new theory yields in physics. After viewing a preliminary version of this paper, Bruinier pointed out to the authors that his Ph.D. student Martin Hövel is also studying a related function in his upcoming thesis. Hövel's construction appears to have connections to the case when $k = 1$ (i.e., weight 0) which is excluded in our study, while his ongoing work does not include the case $k > 1$ which is investigated in this paper.

Before relating $\mathcal{F}_{1-k,D}$ and $f_{k,D}$, we investigate the functions $\mathcal{F}_{1-k,D}$ themselves a bit closer. We put

$$E_D := \left\{ \tau = x + iy \in \mathbb{H} : \exists a, b, c \in \mathbb{Z},\ b^2 - 4ac = D,\ a\left|\tau\right|^2 + bx + c = 0 \right\}. \tag{1.5}$$

The group Γ_1 acts on this set, and E_D is a union of closed geodesics (Heegner cycles) projecting down to finitely many on the compact modular curve. The set E_D naturally partitions $\mathbb{H}$ into (open) connected components (see Lemma 5.1). Owing to the sign in the definition of $\mathcal{F}_{1-k,D}$, the functions $\mathcal{F}_{1-k,D}$ exhibit discontinuities when crossing from one connected component to another, with the value of the limits from either side differing by a polynomial. The functions $\mathcal{F}_{1-k,D}$ hence exhibit what is known as wall crossing behavior. Wall crossing behavior has recently been extensively studied due to its appearance in the quantum theory of black holes in physics (see e.g. [12]). Although $\mathcal{F}_{1-k,D}$ is not a harmonic weak Maass form, it exhibits many similar properties. Outside of the exceptional set E_D, the functions $\mathcal{F}_{1-k,D}$ are locally annihilated by Δ_{2-2k} and satisfy weight $2-2k$ modularity. We hence call them *locally harmonic Maass forms* with exceptional set E_D (see Section 2 for a full definition).

Theorem 1.1. *For $k > 1$ and $D > 0$ a non-square discriminant, the function $\mathcal{F}_{1-k,D}$ is a weight $2-2k$ locally harmonic Maass form with exceptional set E_D.*

Although $\mathcal{F}_{1-k,D}$ exhibits some behavior which is similar to harmonic weak Maass forms, it also has some other surprising properties. The differential operator $\mathcal{D}^{2k-1}$ (where $\mathcal{D} := \frac{1}{2\pi i}\frac{d}{d\tau}$) also plays a central role in the theory of harmonic weak Maass forms (see e.g., [11]). However, a harmonic weak Maass form cannot map to a cusp form under both ξ_{2-2k} and $\mathcal{D}^{2k-1}$, as is well known. Due to discontinuities along the exceptional set E_D, our function $\mathcal{F}_{1-k,D}$ is actually allowed to (locally) map to a constant multiple of $f_{k,D}$ under both operators.

Theorem 1.2. *Suppose that $k > 1$ and $D > 0$ is a non-square discriminant. Then for every $\tau \in \mathbb{H} \setminus E_D$, the function $\mathcal{F}_{1-k,D}$ satisfies*

$$\begin{aligned} \xi_{2-2k}\left(\mathcal{F}_{1-k,D}\right)(\tau) &= D^{\frac{1}{2}-k} f_{k,D}(\tau), \\ \mathcal{D}^{2k-1}\left(\mathcal{F}_{1-k,D}\right)(\tau) &= -\frac{(2k-2)!}{(4\pi)^{2k-1}} D^{\frac{1}{2}-k} f_{k,D}(\tau). \end{aligned}$$

The aforementioned discontinuities of $\mathcal{F}_{1-k,D}$ along E_D are captured by very simple functions, which are given piecewise as polynomials. The functions $\mathcal{F}_{1-k,D}$ are formed by adding these (piecewise) polynomials to real analytic functions which induce the image of $\mathcal{F}_{1-k,D}$ under the operators ξ_{2-2k} and $\mathcal{D}^{2k-1}$ given in Theorem 1.2. Indeed, in the theory of harmonic weak Maass forms, the function $f_{k,D}$ has a natural (real analytic) preimage under ξ_{2-2k} (resp. $\mathcal{D}^{2k-1}$) called the non-holomorphic (resp. holomorphic) Eichler integral. To be more precise, as in [31], for $f(\tau) = \sum_{n=1}^{\infty} a_n q^n \in S_{2k}$ ($q = e^{2\pi i \tau}$) we define the *non-holomorphic Eichler integral of f* by

$$f^*(\tau) := (2i)^{1-2k} \int_{-\overline{\tau}}^{i\infty} f^c(z)\,(z+\tau)^{2k-2}\,dz, \tag{1.6}$$

where $f^c(\tau) := \overline{f(-\overline{\tau})}$ is the cusp form whose Fourier coefficients are the conjugates of the coefficients of f. We likewise define the *(holomorphic) Eichler integral of f* by

$$\mathcal{E}_f(\tau) := \sum_{n=1}^{\infty} \frac{a_n}{n^{2k-1}} q^n. \tag{1.7}$$

Hence, combining Theorem 1.2 with the wall crossing behavior mentioned earlier in the introduction, we are able to obtain a certain type of expansion for $\mathcal{F}_{1-k,D}$.

Theorem 1.3. *Suppose that $k > 1$, $D > 0$ is a non-square discriminant, and $\mathcal{C}$ is one of the connected components partitioned by E_D. Then there exists a polynomial $P_{\mathcal{C}}$ of degree at most $2k-2$ such that for all $\tau \in \mathcal{C}$,*

$$\mathcal{F}_{1-k,D}(\tau) = P_{\mathcal{C}}(\tau) + D^{\frac{1}{2}-k} f^*_{k,D}(\tau) - D^{\frac{1}{2}-k} \frac{(2k-2)!}{(4\pi)^{2k-1}} \mathcal{E}_{f_{k,D}}(\tau).$$

Remark. According to [20], one can obtain an exact formula for the coefficients of $f_{k,D}$ in terms of infinite sums involving Salié sums and J-Bessel functions. For more details of the proof, see Theorem 3.1 of [24].

The polynomials $P_{\mathcal{C}}$ occurring in Theorem 1.3 lead to a new proof of the rationality of the even periods of $f_{k,D}$. Denoting the even part of the period polynomial of $f \in S_{2k}$ by $r^+(f; X)$ (see Section 8 for a full definition), we provide a new proof of the following result of the third author and Zagier [23].

Theorem 1.4. *Suppose that $D > 0$ is a non-square discriminant and $k > 1$ is even. Then the even part of the period polynomial of $f_{k,D}$ satisfies*

$$r^+(f_{k,D}; X) \equiv 2 \sum_{\substack{[a,b,c] \in \mathcal{Q}_D \\ a<0<c}} \left(aX^2 + bX + c\right)^{k-1} \left(\mathrm{mod}\ \left(X^{2k-2} - 1\right)\right). \tag{1.8}$$

Remarks.

(1) By the congruence we mean that the left and right hand sides differ by a constant multiple of $X^{2k-2} - 1$. The theorem of the third author and Zagier explicitly supplies the implied constant, which is a ratio of Bernoulli numbers times a certain class number. We also note that the sum in (1.8) is finite, which follows from reduction theory.

(2) The period polynomials in (1.8) also appear in Theorem 3 of [14] as the error to modularity of certain holomorphic functions. Instead of being defined in terms of hyperbolic Poincaré series, these functions are defined coefficient-wise by cycle integrals. It would be interesting to further investigate this relation.

The Hecke algebra naturally decomposes S_{2k} into one dimensional simultaneous eigenspaces for all Hecke operators. The action of the Hecke operators on $f_{k,D}$ is easily computed and particularly simple, namely, for a prime p

$$f_{k,D}\Big|_{2k} T_p = f_{k,Dp^2} + p^{k-1}\left(\frac{D}{p}\right) f_{k,D} + p^{2k-1} f_{k,\frac{D}{p^2}},$$

where T_p is the p-th Hecke operator acting on translation invariant functions (see (9.1) for a definition). Note that the right hand side of the above formula reflects the action of the half-integral weight Hecke operator T_{p^2} (when the subscript D is taken to denote the D-th coefficient). This is no accident, owing to the fact that $f_{k,D}$ is the D-th Fourier coefficient of the kernel function Ω (defined in (1.2)) in the z variable and the Hecke operators commute with the Shimura and Shintani lifts. This connection between the integral and half-integral weight Hecke operators on the functions $f_{k,D}$ extends to the functions $\mathcal{F}_{1-k,D}$.

Theorem 1.5. *Suppose that $k > 1$, $D > 0$ is a non-square discriminant, and p is a prime. Then*

$$\mathcal{F}_{1-k,D}\Big|_{2-2k} T_p = \mathcal{F}_{1-k,Dp^2} + p^{-k}\left(\frac{D}{p}\right)\mathcal{F}_{1-k,D} + p^{1-2k}\mathcal{F}_{1-k,\frac{D}{p^2}}, \tag{1.9}$$

where $\mathcal{F}_{1-k,\frac{D}{p^2}} = 0$ if $p^2 \nmid D$.

Remark. The fact that the right hand side of (1.9) looks like the formula for the half-integral weight $\frac{3}{2} - k$ Hecke operator hints towards a connection between integral weight $2 - 2k$ and half-integral weight $\frac{3}{2} - k$ objects, mirroring the behavior for weight $2k$ and $k + \frac{1}{2}$ cusp forms coming from the Shintani and Shimura lifts. In light of this, there could be some relation with the results in [13] in the case $k = 1$, which is not considered in this paper.

The paper is organized as follows. In Section 2 we give some background and a formal definition of locally harmonic Maass forms. In Section 3 we explain the interpretation of $\mathcal{F}_{1-k,D}$ as a (linear combination of) hyperbolic Poincaré series. We next show compact convergence in Section 4. Section 5 is devoted to a discussion about the exceptional set E_D. Section 6 is devoted to proving Theorem 1.2. The expansion given in Theorem 1.3 is proven in Section 7. Combining this with the results of the previous sections, we conclude Theorem 1.1. In Section 8 we connect the polynomials $P_{\mathcal{C}}$ from Theorem 1.3 to the period polynomial of $f_{k,D}$ in order to prove Theorem 1.4. We conclude the paper with the proof of Theorem 1.5 in Section 9.

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2. Harmonic weak Maass forms and locally harmonic Maass forms

In this section, we recall the definition of harmonic weak Maass forms and introduce a formal definition of locally harmonic Maass forms. A good background reference for harmonic weak Maass forms is [9]. As usual, we let $|_{2k}$ denote the *weight $2k \in 2\mathbb{Z}$ slash-operator*, defined for $f : \mathbb{H} \to \mathbb{C}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1$ by

$$f\Big|_{2k}\gamma(\tau) := (c\tau + d)^{-2k} f(\gamma\tau),$$

where $\gamma\tau := \frac{a\tau+b}{c\tau+d}$ is the action by fractional linear transformations.

For $k \in \mathbb{N}$, a *harmonic weak Maass form* $\mathcal{F} : \mathbb{H} \to \mathbb{C}$ of weight $2 - 2k$ for Γ_1 is a real analytic function satisfying:

(1) $\mathcal{F}|_{2-2k}\gamma(\tau) = \mathcal{F}(\tau)$ for every $\gamma \in \Gamma_1$,
(2) $\Delta_{2-2k}(\mathcal{F}) = 0$,
(3) $\mathcal{F}$ has at most linear exponential growth at $i\infty$.

As noted in the introduction, the differential operators ξ_{2-2k} and $\mathcal{D}^{2k-1}$ naturally occur in the theory of harmonic weak Maass forms. More precisely, for a harmonic weak Maass form $\mathcal{F}$, one has $\xi_{2-2k}(\mathcal{F}), \mathcal{D}^{2k-1}(\mathcal{F}) \in M_{2k}^!$, the space of weight $2k$ weakly holomorphic modular forms (i.e., those meromorphic modular forms whose poles occur only at the cusps). It is well known that the operator ξ_{2-2k} commutes with the group action of $\mathrm{SL}_2(\mathbb{R})$. Moreover, by Bol's identity

([26], see also [15] or [11], for a more modern usage), the operator $\mathcal{D}^{2k-1}$ also commutes with the group action of $\mathrm{SL}_2(\mathbb{R})$. Furthermore, a direct calculation shows that

$$\Delta_{2-2k} = -\xi_{2k}\xi_{2-2k}. \tag{2.1}$$

Each harmonic weak Maass form $\mathcal{F}$ naturally splits into a holomorphic part and a non-holomorphic part. Indeed, in the special case that $\xi_{2-2k}(\mathcal{F}) = f \in S_{2k}$ (which is the only case relevant to this paper), one can show that $\mathcal{F} - f^*$ is holomorphic on $\mathbb{H}$, where f^* was defined in (1.6). We hence call f^* the *non-holomorphic part* of $\mathcal{F}$ and $\mathcal{F} - f^*$ the *holomorphic part.* While the holomorphic part is obviously annihilated by ξ_{2-2k}, an easy calculation shows that the non-holomorphic part is annihilated by $\mathcal{D}^{2k-1}$. From this one also immediately sees that $\mathcal{D}^{2k-1}(\mathcal{F}) = \mathcal{D}^{2k-1}(\mathcal{F} - f^*)$ is holomorphic.

We next define the new automorphic objects which we investigate in this paper. A weight $2-2k$ *locally harmonic Maass form* for Γ_1 with *exceptional set* E_D (defined in (1.5)) is a function $\mathcal{F} : \mathbb{H} \to \mathbb{C}$ satisfying:

(1) For every $\gamma \in \Gamma_1$, $\mathcal{F}\big|_{2-2k}\gamma = \mathcal{F}$.
(2) For every $\tau \in \mathbb{H} \setminus E_D$, there is a neighborhood N of τ in which $\mathcal{F}$ is real analytic and $\Delta_{2-2k}(\mathcal{F}) = 0$.
(3) For $\tau \in E_D$ one has
$$\mathcal{F}(\tau) = \frac{1}{2}\lim_{w\to 0^+}\left(\mathcal{F}(\tau + iw) + \mathcal{F}(\tau - iw)\right) \qquad (w \in \mathbb{R}).$$
(4) For $\tau \to i\infty$, $\mathcal{F}(\tau)$ is bounded.

Since the theory of harmonic weak Maass forms has proven so fruitful, it might be interesting to further investigate the properties of functions in the space of locally harmonic Maass forms.

3. Locally harmonic Maass forms and hyperbolic Poincaré series

In this section, we define Petersson's more general hyperbolic Poincaré series [25], which span the space S_{2k}, and describe their connection to (1.1). In addition, we define a weight $2-2k$ locally harmonic hyperbolic Poincaré series which basically maps to Petersson's hyperbolic Poincaré series under both ξ_{2-2k} and $\mathcal{D}^{2k-1}$ (see Proposition 6.1).

Suppose that $D > 0$ is a non-square discriminant and $\mathcal{A} \subseteq \mathcal{Q}_D$ is a *narrow equivalence class* of integral binary quadratic forms (that is, there exists $Q_0 \in \mathcal{Q}_D$ such that $\mathcal{A} =: [Q_0]$ consists of precisely those $Q \in \mathcal{Q}_D$ which are Γ_1-equivalent to Q_0). One defines

$$f_{k,D,\mathcal{A}}(\tau) := \frac{(-1)^k D^{k-\frac{1}{2}}}{\binom{2k-2}{k-1}\pi} \sum_{[a,b,c]\in\mathcal{A}} \left(a\tau^2 + b\tau + c\right)^{-k} \in S_{2k}. \tag{3.1}$$

In the spirit of (1.4), we define

$$\mathcal{F}_{1-k,D,\mathcal{A}}(\tau) := \frac{(-1)^k D^{\frac{1}{2}-k}}{\binom{2k-2}{k-1}\pi} \sum_{Q=[a,b,c]\in\mathcal{A}} \operatorname{sgn}\left(a|\tau|^2 + bx + c\right) Q(\tau,1)^{k-1}\psi\left(\frac{Dy^2}{|Q(\tau,1)|^2}\right), \tag{3.2}$$

where ψ was given in (1.3). We shall see in Theorem 7.4 that $\mathcal{F}_{1-k,D,\mathcal{A}}$ is a locally harmonic Maass form with exceptional set E_D.

As alluded to in the introduction, (3.1) is not the definition given by Petersson (in fact, the definition (3.1) was given in [21, 22]). Since we make use of Petersson's definition repeatedly throughout the paper, we now describe Petersson's construction and give the link between the two definitions. Let η, η' be real conjugate *hyperbolic fixed points* of $\mathrm{SL}_2(\mathbb{R})$ (that is, there exists a matrix $\gamma \in \mathrm{SL}_2(\mathbb{R})$ fixing η and η'). We call such a pair of points a *hyperbolic pair.* Denote the group of matrices in Γ_1 fixing η and η' by Γ_η. The group $\Gamma_\eta/\{\pm I\}$ is an infinite cyclic subgroup of $\Gamma_1/\{\pm I\}$ and is generated by

$$g_\eta := \pm\begin{pmatrix} \frac{t+bu}{2} & cu \\ -au & \frac{t-bu}{2}\end{pmatrix},$$

where $\eta = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ and $t, u \in \mathbb{N}$ give the smallest solution to the Pell equation $t^2 - Du^2 = 4$. For $Q = [a, b, c]$, the subgroup Γ_η furthermore preserves the geodesic

$$S_Q := \left\{\tau \in \mathbb{H} : a\left|\tau\right|^2 + b\mathrm{Re}\left(\tau\right) + c = 0\right\}, \tag{3.3}$$

which is important in our study since the exceptional set E_D (defined in (1.5)) decomposes as $E_D = \bigcup_{Q\in\mathcal{Q}_D} S_Q$. These semi-circles have played an important role in the interrelation between integral and half-integral weight modular forms [21, 28].

Let $A \in \mathrm{SL}_2(\mathbb{R})$ satisfy $A\eta = \infty$ and $A\eta' = 0$. We note that one may choose

$$A = A_\eta := \pm\frac{1}{\sqrt{|\eta-\eta'|}}\begin{pmatrix} 1 & -\eta' \\ -\operatorname{sgn}\left(\eta-\eta'\right) & \operatorname{sgn}\left(\eta-\eta'\right)\eta\end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}). \tag{3.4}$$

Since g_η preserves the semi-circle S_Q, $A_\eta g_\eta A_\eta^{-1}$ is a scaling matrix $\left(\begin{smallmatrix}\zeta & 0\\ 0 & \zeta^{-1}\end{smallmatrix}\right)$ for some $\zeta \in \mathbb{R}$.

For $h_k(\tau) := \tau^{-k}$ (the constant term of the hyperbolic expansion of a modular form), we now define Petersson's classical hyperbolic Poincaré series [25]

$$P_{k,\eta}(\tau) := \sum_{\gamma\in\Gamma_\eta\backslash\Gamma_1} h_k\Big|_{2k} A\gamma(\tau), \tag{3.5}$$

which converges compactly for $k > 1$. By construction, $P_{k,\eta}$ satisfies weight $2k$ modularity and is holomorphic. Petersson proved that indeed $P_{k,\eta}$ is a cusp form and it was later shown that

$$P_{k,\eta} = \binom{2k-2}{k-1}\pi D^{\frac{1-k}{2}} f_{k,D,\mathcal{A}} \tag{3.6}$$

for $\mathcal{A} = [Q_0]$, where Q_0 has roots η, η' [18].

We move on to our construction of a weight $2-2k$ hyperbolic Poincaré series. Define

$$\varphi(v) := \int_0^v \sin(u)^{2k-2}\, du. \tag{3.7}$$

Noting that

$$\left|a\tau^2 + b\tau + c\right|^2 = Dy^2 + \left(a\left|\tau\right|^2 + bx + c\right)^2,$$

we see that

$$\arcsin\left(\frac{\sqrt{D}y}{|a\tau^2+b\tau+c|}\right) = \arctan\left|\frac{\sqrt{D}y}{a\left|\tau\right|^2+bx+c}\right|.$$

Therefore, using the fact that $\cos(\theta) \geq 0$ for $0 \leq \theta \leq \frac{\pi}{2}$, the change of variables $u = \sin(\theta)^2$ in the definition of the incomplete β-function yields (recall definition (1.3))

$$\psi\left(\frac{Dy^2}{\left|Q(\tau,1)\right|^2}\right) = \frac{1}{2}\beta\left(\frac{Dy^2}{|a\tau^2+b\tau+c|^2}; k-\frac{1}{2}, \frac{1}{2}\right) = \varphi\left(\arctan\left|\frac{\sqrt{D}y}{a\left|\tau\right|^2+bx+c}\right|\right), \tag{3.8}$$

where we understand the arctangent to be equal to $\frac{\pi}{2}$ if $a\left|\tau\right|^2 + bx + c = 0$.

Following our construction in the introduction, we set

$$\widehat{\varphi}(\tau) := \tau^{k-1}\operatorname{sgn}(x)\,\varphi\left(\arctan\left|\frac{y}{x}\right|\right). \tag{3.9}$$

We now define the weight $2-2k$ *locally harmonic hyperbolic Poincaré series* by

$$\mathcal{P}_{1-k,\eta}\left(\tau\right) := \sum_{\gamma\in\Gamma_\eta\backslash\Gamma_1} \widehat{\varphi}\Big|_{2-2k} A\gamma\left(\tau\right). \tag{3.10}$$

We show in Proposition 4.1 that $\mathcal{P}_{1-k,\eta}$ converges compactly for $k>1$.

We want to show that $\mathcal{P}_{1-k,\eta}$ and $\mathcal{F}_{1-k,D,\mathcal{A}}$ are connected in a way which is similar to the relation (3.6) between $P_{k,\eta}$ and $f_{k,D,\mathcal{A}}$. For a hyperbolic pair $\eta,\eta'\in\mathbb{R}$ with generator $g_\eta=\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ of Γ_η, chosen so that $\operatorname{sgn}\left(\gamma\right)=\operatorname{sgn}\left(\eta-\eta'\right)$, we define

$$Q_\eta(\tau,w) := \gamma\tau^2+\left(\delta-\alpha\right)\tau w-\beta w^2.$$

Conversely, for $Q=[a,b,c]\in\mathcal{Q}_D$, we choose the roots $\eta_Q=\frac{-b+\sqrt{D}}{2a}$, $\eta_Q'=\frac{-b-\sqrt{D}}{2a}$ and use the fact that $Q=Q_{\eta_Q}$ to obtain a correspondence. Note that $\operatorname{sgn}(\eta_Q-\eta_Q')=\operatorname{sgn}(a)$. We furthermore define $A_Q:=A_{\eta_Q}$, where A_η was defined in (3.4). For $Q\in\mathcal{Q}_D$, we denote the action of $\gamma\in\Gamma_1$ on Q by $Q\circ\gamma$. We first need to relate $A_\eta\gamma$ and A_Q.

Lemma 3.1. *For a hyperbolic pair η,η', $\gamma=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_1$, and $Q=Q_\eta\circ\gamma$, there exists a constant $r\in\mathbb{R}^+$ so that*

$$A_\eta\gamma=\begin{pmatrix}\sqrt{r} & 0\\ 0 & \frac{1}{\sqrt{r}}\end{pmatrix}A_Q \tag{3.11}$$

and hence in particular

$$\arg\left(A_\eta\gamma\tau\right)=\arg\left(A_Q\tau\right) \qquad \textit{and} \qquad \operatorname{sgn}\left(\operatorname{Re}\left(A_\eta\gamma\tau\right)\right)=\operatorname{sgn}\left(\operatorname{Re}\left(A_Q\tau\right)\right).$$

Moreover,

$$\tau\Big|_{-2}A_\eta\gamma\left(\tau\right)=\tau\Big|_{-2}A_Q\left(\tau\right)=\frac{-Q\left(\tau,1\right)}{\sqrt{D}}. \tag{3.12}$$

Proof. A direct calculation, using (3.4), yields

$$A_\eta\gamma\tau=\operatorname{sgn}(\eta-\eta')\frac{a-c\eta'}{a-c\eta}\left(\frac{\tau-\gamma^{-1}\eta'}{-\tau+\gamma^{-1}\eta}\right). \tag{3.13}$$

Denote $Q_\eta=[\alpha,\beta,\delta]$ and $Q=[a_Q,b_Q,c_Q]$ and recall that we have chosen Q_η (resp. η_Q) such that $\operatorname{sgn}\left(\alpha\right)=\operatorname{sgn}\left(\eta-\eta'\right)$ (resp. $\operatorname{sgn}(\eta_Q-\eta_Q')=\operatorname{sgn}(a_Q)$). Hence $\eta-\eta'=\frac{\sqrt{D}}{\alpha}$ and one now concludes the second identity of (3.12) after noting that

$$j\left(A_\eta,\tau\right)=\mp\frac{\operatorname{sgn}\left(\eta-\eta'\right)}{\sqrt{|\eta-\eta'|}}\left(\tau-\eta\right).$$

and applying (3.13) with $\eta=\eta_Q$ and $\gamma=I$. Since $Q=Q_\eta\circ\gamma$, γ sends the roots of Q to the roots of Q_η and hence either $\gamma^{-1}\eta=\eta_Q$ or $\gamma^{-1}\eta'=\eta_Q$. Since η_Q,η_Q' are ordered by $\operatorname{sgn}(\eta_Q-\eta_Q')=\operatorname{sgn}(a_Q)$, the identity $\gamma^{-1}\eta=\eta_Q$ is verified by

$$\begin{aligned}\operatorname{sgn}\left(a_Q\right)=\operatorname{sgn}\left(Q_\eta\left(a,c\right)\right)=\operatorname{sgn}\left(\alpha\right)\operatorname{sgn}\left(\left(a-c\eta\right)\left(a-c\eta'\right)\right)&\\ =\operatorname{sgn}\left(\frac{\eta-\eta'}{\left(a-c\eta\right)\left(a-c\eta'\right)}\right)=\operatorname{sgn}\left(\gamma^{-1}\eta-\gamma^{-1}\eta'\right).&\end{aligned}$$

Denoting $r:=\left|\frac{a-c\eta'}{a-c\eta}\right|$ and comparing (3.13) with the definition (3.4) of A_Q yields

$$A_\eta\gamma\tau=rA_Q\tau.$$

One concludes (3.11) from the fact that $A_\eta\gamma$ and A_Q both have determinant 1. Since τ is invariant by slashing with a scaling matrix in weight -2, the second identity of (3.12) follows, completing the proof. □

We now use Lemma 3.1 to show that under the natural correspondence between narrow classes $\mathcal{A} \subseteq \mathcal{Q}_D$ and hyperbolic pairs $\eta, \eta' \in \mathbb{R}$ given above, one has:

Lemma 3.2. *For every hyperbolic pair η, η' and $\mathcal{A} = [Q_\eta] \subseteq \mathcal{Q}_D$, one has*

$$\mathcal{P}_{1-k,\eta} = \binom{2k-2}{k-1} \pi D^{\frac{k}{2}} \mathcal{F}_{1-k,D,\mathcal{A}}.$$

Proof. By Lemma 3.1, (3.10) may be rewritten as

$$\mathcal{P}_{1-k,\eta}(\tau) = \frac{(-1)^{k-1}}{D^{\frac{k-1}{2}}} \sum_{Q \in \mathcal{A}} \operatorname{sgn}\left(\operatorname{Re}\left(A_Q\tau\right)\right) Q(\tau,1)^{k-1} \varphi\left(\arctan\left|\frac{\operatorname{Im}\left(A_Q\tau\right)}{\operatorname{Re}\left(A_Q\tau\right)}\right|\right). \tag{3.14}$$

We first note that $a \neq 0$ (since D is not a square, by assumption). From (3.13) with $\eta = \eta_Q$ and $\gamma = I$, one concludes

$$\operatorname{Re}\left(A_Q\tau\right) = -\frac{a|\tau|^2 + bx + c}{|a|\left|-\tau+\eta_Q\right|^2}, \qquad \operatorname{Im}\left(A_Q\tau\right) = \frac{y\sqrt{D}}{|a|\left|-\tau+\eta_Q\right|^2}. \tag{3.15}$$

This allows one to rewrite $\arctan\left|\frac{\operatorname{Im}(A_Q\tau)}{\operatorname{Re}(A_Q\tau)}\right|$. Using (3.8), it follows that (3.14) equals (3.2). □

4. Convergence of $\mathcal{F}_{1-k,D,\mathcal{A}}$

In this section we prove the convergence needed to show Theorem 1.1. We need the following simple property of $\arctan|z|$ for $z \in \mathbb{C}$:

$$\arctan|z| \leq \min\left\{|z|, \frac{\pi}{2}\right\}. \tag{4.1}$$

For a convergence estimate, we will also employ the following formula of Zagier ([30], Prop. 3). For a discriminant $0 < D = \Delta f^2$ with Δ a fundamental discriminant and $\operatorname{Re}(s) > 1$, one has

$$\sum_{a\in\mathbb{N}} \sum_{\substack{0\leq b<2a \\ b^2\equiv D \pmod{4a}}} a^{-s} = \frac{\zeta(s)}{\zeta(2s)} L_\Delta(s) \sum_{d|f} \mu(d)\chi_\Delta(d) d^{-s} \sigma_{1-2s}\left(\frac{f}{d}\right), \tag{4.2}$$

where $L_\Delta(s) := L(s,\chi_\Delta)$ is the Dirichlet L-series associated to the quadratic character $\chi_\Delta(n) := \left(\frac{\Delta}{n}\right)$, μ is the Möbius function, and $\sigma_s(n) := \sum_{d|n} d^s$.

Proposition 4.1. *For $k > 1$, $\mathcal{F}_{1-k,D,\mathcal{A}}$ converges compactly on $\mathbb{H}$.*

Proof. Assume that $\tau = x + iy$ is contained in a compact subset $\mathscr{C} \subset \mathbb{H}$. We note that although we unjustifiably reorder the summation multiple times before showing convergence, in the end we show that the resulting sum converges absolutely, hence validating the legality of this reordering.

Taking the absolute value of each term in (3.2) and extending the sum to all $Q \subset \mathcal{Q}_D$, we obtain (noting (3.8))

$$\frac{D^{\frac{1}{2}-k}}{\binom{2k-2}{k-1}\pi} \sum_{Q=[a,b,c]\in\mathcal{Q}_D} \left| Q(\tau,1)^{k-1} \varphi\left(\arctan\left|\frac{\sqrt{D}y}{a|\tau|^2+bx+c}\right|\right)\right|.$$

We may assume that $a>0$, since the case $a<0$ is treated by changing $Q \to -Q$. We next rewrite b as $b+2an$ with $0 \leq b < 2a$ and $n \in \mathbb{Z}$ and then split the sum into those summands with $|n|$ "large" and those with $|n|$ "small."

We first consider the case of large n, i.e., $|n| > 8\left(|\tau| + \sqrt{D}\right)$ and denote the corresponding sum by $\mathcal{G}_1$. One easily sees that

$$|Q(\tau,1)| \ll an^2, \tag{4.3}$$

where here and throughout the implied constant depends only on k unless otherwise noted. By estimating $|x| < |\tau| < \frac{|n|}{8}$ and $b < 2a$, one obtains (noting that $|n| > 8$)

$$\left|a|\tau|^2 + (b+2an)x + c\right| \geq |c| - |(b+2an)x| - a|\tau|^2 \geq |c| - \frac{a}{4}(|n|+1)|n| - \frac{an^2}{64} \geq |c| - \frac{19}{64}an^2.$$

However, $c = \frac{(b+2an)^2 - D}{4a}$, so that the bounds $|n| > 8$ and $D < \frac{n^2}{64}$ yield

$$|c| \geq a(|n|-1)^2 - \frac{n^2}{256a} \geq \frac{3}{4}an^2.$$

Therefore

$$\left|a|\tau|^2 + (b+2an)x + c\right| \gg an^2, \tag{4.4}$$

and hence by (4.1) one concludes

$$\arctan\left|\frac{\sqrt{D}y}{a|\tau|^2 + (b+2an)x + c}\right| \leq \left|\frac{\sqrt{D}y}{a|\tau|^2 + (b+2an)x + c}\right| \ll \frac{\sqrt{D}y}{an^2},$$

Using (3.7) and (3.8), one obtains the estimate

$$\int_0^{\arctan\left|\frac{\sqrt{D}y}{a|\tau|^2+bx+c}\right|} \sin(u)^{2k-2} du \ll \int_0^{\frac{\sqrt{D}y}{an^2}} |\sin(u)|^{2k-2}\, du.$$

Since $|\sin(u)| \leq u$ for $u \geq 0$, we conclude that

$$\int_0^{\frac{\sqrt{D}y}{an^2}} |\sin(u)|^{2k-2}\, du \leq \int_0^{\frac{\sqrt{D}y}{an^2}} u^{2k-2} du = \frac{1}{2k-1}\left(\frac{\sqrt{D}y}{an^2}\right)^{2k-1}. \tag{4.5}$$

Combining (4.3) and (4.5) and noting that all bounds are independent of b yields

$$\mathcal{G}_1(\tau) \ll y^{2k-1} D^{k-\frac{1}{2}} \sum_{a\in\mathbb{N}} \sum_{\substack{0\leq b<2a \\ b^2\equiv D \pmod{4a}}} a^{-k} \sum_{n>8\left(|\tau|+\sqrt{D}\right)} n^{-2k} \ll \left(\frac{y\sqrt{D}}{|\tau|+\sqrt{D}}\right)^{2k-1} \ll_{\mathscr{C},D} 1, \tag{4.6}$$

where we have estimated the inner sum against the corresponding integral and evaluated the outer two sums with (4.2). Since y (resp. $|\tau|$) may be bounded from above (resp. below) by a constant depending only on $\mathscr{C}$, it follows that $\mathcal{G}_1$ converges uniformly on $\mathscr{C}$.

We now move on to the case when $|n| \leq 8\left(|\tau| + \sqrt{D}\right)$ and denote the corresponding sum by $\mathcal{G}_2$. As in the case for n large, one easily estimates

$$|Q(\tau,1)| \ll a\left(|\tau| + \sqrt{D}\right)^2 \ll_{\mathscr{C},D} a. \tag{4.7}$$

We further split the sum over $a \in \mathbb{N}$. For $a > \frac{\sqrt{D}}{y}$ we have

$$\left|a|\tau|^2 + (b+2an)x + c\right| = \left|ay^2 + a\left(x+n+\frac{b}{2a}\right)^2 - \frac{D}{4a}\right| \gg ay^2. \tag{4.8}$$

Hence for the terms $a > \frac{\sqrt{D}}{y}$, we use (4.1) to obtain

$$\int_0^{\arctan\left|\frac{\sqrt{D}y}{a|\tau|^2+bx+c}\right|} \sin(u)^{2k-2}du \ll \int_0^{\frac{\sqrt{D}}{ay}} u^{2k-2}du = \frac{1}{2k-1}\left(\frac{\sqrt{D}}{ay}\right)^{2k-1}. \tag{4.9}$$

For $a \leq \frac{\sqrt{D}}{y}$ we simply note that by (4.1) we may trivially bound $\arctan\left|\frac{\sqrt{D}y}{a|\tau|^2+bx+c}\right| \leq \frac{\pi}{2}$ and, since $\sin(u) \geq 0$ for $0 \leq u \leq \pi$, we may trivially estimate the remaining terms by the constant

$$\int_0^{\frac{\pi}{2}} \sin(u)^{2k-2}du. \tag{4.10}$$

Bounding the sum over n trivially and using (4.7), (4.9), and (4.10) yields

$$\begin{aligned}\mathcal{G}_2(\tau) \ll{}& \left(|\tau|+\sqrt{D}\right)^{2k-1} \sum_{a\leq\frac{\sqrt{D}}{y}} \sum_{\substack{0\leq b<2a\\ b^2\equiv D \pmod{4a}}} a^{k-1}\\ &+ D^{k-\frac{1}{2}}\left(\frac{|\tau|+\sqrt{D}}{y}\right)^{2k-1} \sum_{a>\frac{\sqrt{D}}{y}} \sum_{\substack{0\leq b<2a\\ b^2\equiv D \pmod{4a}}} a^{-k} \ll \left(|\tau|+\sqrt{D}\right)^{2k-1}\frac{D^{\frac{k+1}{2}}}{y^{k+1}}.\end{aligned} \tag{4.11}$$

Here we have employed (4.2) for large a and used trivial estimates for all other sums, completing the proof. □

5. Values at exceptional points

In this section, we describe the behavior of $\mathcal{F}_{1-k,D,\mathcal{A}}$ along the circles of discontinuity E_D (defined in (1.5)). For each Q, S_Q (defined in (3.3)) partitions $\mathbb{H} \setminus S_Q$ into two open connected components (one "above" and one "below" S_Q), which, for $\varepsilon = \pm$, we denote by

$$\mathcal{C}_Q^{\varepsilon} := \left\{\tau \in \mathbb{H} : \varepsilon \operatorname{sgn}\left(\left|\tau - \frac{b}{2a}\right| - \frac{\sqrt{D}}{2|a|}\right) = 1\right\}. \tag{5.1}$$

For each $\tau \in \mathbb{H}$, we further define

$$\mathscr{B}_\tau = \mathscr{B}_{\tau,D} := \left\{Q \in \mathcal{Q}_D : \tau \in S_Q\right\}. \tag{5.2}$$

In order for the second condition in the definition of locally harmonic Maass forms to be meaningful, it is first necessary to show that the set E_D is nowhere dense in $\mathbb{H}$ and hence E_D partitions $\mathbb{H} \setminus E_D$ into (open) connected components.

Lemma 5.1. *Suppose that $D > 0$ is a non-square discriminant. For every $\tau_0 = x_0 + iy_0 \in \mathbb{H}$, the following hold:*

1. *For all but finitely many $Q \in \mathcal{Q}_D$, we have that $\tau_0 \in \mathcal{C}_Q^+$. In particular, $\mathscr{B}_{\tau_0}$ is finite.*
2. *There exists a neighborhood N of τ_0 so that for every $[a,b,c] \notin \mathscr{B}_{\tau_0}$ and $\tau = x+iy \in N$,*

$$\operatorname{sgn}\left(a|\tau|^2 + bx + c\right) = \operatorname{sgn}\left(a|\tau_0|^2 + bx_0 + c\right) \neq 0.$$

Proof. (1) We define the open set

$$N_1 := \left\{ \tau = x + iy \in \mathbb{H} : |x - x_0| < 1, y > \frac{y_0}{2} \right\}.$$

If $|a| > \frac{\sqrt{D}}{y_0}$ and $\tau \in N_1$, then the inequality

$$\left| \tau - \frac{b}{2a} \right| \geq y > \frac{y_0}{2} > \frac{\sqrt{D}}{2|a|}$$

implies that $\tau \in \mathcal{C}_Q^+$. Moreover, for

$$|b| > 2|a| \max \left\{ |x_0 - 1|, |x_0 + 1| \right\} + \sqrt{D},$$

we have

$$\left| \tau - \frac{b}{2a} \right| > \left| \frac{2ax - b}{2a} \right| \geq \frac{|b| - 2|a||x|}{2|a|} > \frac{2|a| \left(\max \left\{ |x_0 - 1|, |x_0 + 1| \right\} - |x| \right) + \sqrt{D}}{2|a|}.$$

One immediately concludes that

$$N_1 \subseteq \mathcal{C}_Q^+ \tag{5.3}$$

for all but finitely many $Q \in \mathcal{Q}_D$. In particular, this proves the first statement.
(2) In order to prove the second statement, for $a, b, c \in \mathbb{Z}$, we define

$$N_{a,b,c} := \left\{ \tau = x + iy \in N_1 : \operatorname{sgn}\left(a|\tau|^2 + bx + c \right) = \operatorname{sgn}\left(a|\tau_0|^2 + bx_0 + c \right) \right\}.$$

We denote the intersection of these open sets by

$$N = N_Q := \bigcap_{[a,b,c] \in \mathcal{Q}_D \setminus \mathscr{B}_{\tau_0}} N_{a,b,c},$$

which we now prove is a neighborhood of τ_0 satisfying the second statement of the lemma. A short calculation shows that

$$\operatorname{sgn}\left(a|\tau|^2 + bx + c \right) = \operatorname{sgn}(a) \operatorname{sgn}\left(\left| \tau - \frac{b}{2a} \right| - \frac{\sqrt{D}}{2|a|} \right), \tag{5.4}$$

so that $N_{a,b,c} = N_1 \cap C_Q^{\varepsilon}$ with ε chosen such that $\tau_0 \in \mathcal{C}_Q^{\varepsilon}$. Hence by (5.3), we conclude that $N_{a,b,c} = N_1$ for all but finitely many $[a, b, c] \in \mathcal{Q}_D$. Therefore N is the intersection of finitely many $N_{a,b,c}$. Hence N is open and every $\tau \in N$ satisfies the conditions of the second statement, completing the proof. □

We are now ready to describe the value $\mathcal{F}_{1-k,D,\mathcal{A}}(\tau)$ whenever $\tau \in S_Q$ for some $Q \in \mathcal{Q}_D$.

Proposition 5.2. *If $\tau \in E_D$, then*

$$\mathcal{F}_{1-k,D,\mathcal{A}}(\tau) = \frac{1}{2} \lim_{w \to 0^+} \left(\mathcal{F}_{1-k,D,\mathcal{A}}(\tau + iw) + \mathcal{F}_{1-k,D,\mathcal{A}}(\tau - iw) \right).$$

Proof. We first split the sum (3.2) defining $\mathcal{F}_{1-k,D,\mathcal{A}}$ into $Q \in \mathscr{B}_\tau$ and $Q \notin \mathscr{B}_\tau$ (defined in (5.2)). Due to local uniform convergence, we may interchange the limit $w \to 0^+$ with the sum. Since

$\beta\left(t; k-\frac{1}{2}, \frac{1}{2}\right)$ is continuous as a function of $0 < t \leq 1$, one obtains

$$\frac{1}{2}\lim_{w\to 0^+}\left(\mathcal{F}_{1-k,D,\mathcal{A}}(\tau+iw)+\mathcal{F}_{1-k,D,\mathcal{A}}(\tau-iw)\right) \tag{5.5}$$
$$=\frac{(-1)^k D^{\frac{1}{2}-k}}{\binom{2k-2}{k-1}\pi}\sum_{Q=[a,b,c]\notin\mathscr{B}_\tau}\operatorname{sgn}\left(a|\tau|^2+bx+c\right)Q(\tau,1)^{k-1}\varphi\left(\arctan\left|\frac{\sqrt{D}y}{a|\tau|^2+bx+c}\right|\right)$$
$$+\frac{(-1)^k D^{\frac{1}{2}-k}}{2\pi\binom{2k-2}{k-1}}\sum_{\substack{Q=[a,b,c]\in\mathscr{B}_\tau\\ \varepsilon\in\{\pm\}}}\lim_{w\to 0^+}\Bigg(\operatorname{sgn}\left(a|\tau+\varepsilon iw|^2+bx+c\right)Q(\tau+\varepsilon iw,1)^{k-1}$$
$$\times\varphi\left(\arctan\left|\frac{\sqrt{D}(y+\varepsilon w)}{a|\tau+\varepsilon iw|^2+bx+c}\right|\right)\Bigg).$$

For each $Q=[a,b,c]\in\mathscr{B}_\tau$ and $0<w<y$, one concludes, since $\frac{b}{2a}$ is real, that

$$\left|\tau-iw-\frac{b}{2a}\right|-\frac{\sqrt{D}}{2|a|}<\left|\tau-\frac{b}{2a}\right|-\frac{\sqrt{D}}{2|a|}=0<\left|\tau+iw-\frac{b}{2a}\right|-\frac{\sqrt{D}}{2|a|}. \tag{5.6}$$

It follows from (5.4) that the $\pm$ terms on the right hand side of (5.5) have opposite signs. Since φ is continuous, one concludes that the sum over $Q\in\mathscr{B}_\tau$ vanishes, completing the proof. ☐

6. Action of ξ_{2-2k} and $\mathcal{D}^{2k-1}$

In this section, we determine the action of the operators ξ_{2-2k} and $\mathcal{D}^{2k-1}$ on $\mathcal{F}_{1-k,D,\mathcal{A}}$ (and $\mathcal{F}_{1-k,D}$). We prove the following proposition, which immediately implies Theorem 1.2.

Proposition 6.1. *Suppose that $k>1$, $D>0$ is a non-square discriminant, and $\mathcal{A}\subseteq\mathcal{Q}_D$ is a narrow class of binary quadratic forms. Then for every $\tau\in\mathbb{H}\setminus E_D$, the function $\mathcal{F}_{1-k,D}$ satisfies*

$$\begin{aligned}\xi_{2-2k}\left(\mathcal{F}_{1-k,D,\mathcal{A}}\right)(\tau) &= D^{\frac{1}{2}-k}f_{k,D,\mathcal{A}}(\tau),\\ \mathcal{D}^{2k-1}\left(\mathcal{F}_{1-k,D,\mathcal{A}}\right)(\tau) &= -D^{\frac{1}{2}-k}\frac{(2k-2)!}{(4\pi)^{2k-1}}f_{k,D,\mathcal{A}}(\tau).\end{aligned}$$

In particular, we have that

$$\Delta_{2-2k}\left(\mathcal{F}_{1-k,D,\mathcal{A}}\right)(\tau)=0. \tag{6.1}$$

Proof. Assume that $\tau\in\mathbb{H}\setminus E_D$. By Lemma 5.1, there is a neighborhood containing τ for which (3.2) is continuous and real differentiable. Inside this neighborhood, we use Lemma 3.2 to rewrite $\mathcal{F}_{1-k,D,\mathcal{A}}$ in terms of $\mathcal{P}_{1-k,\eta}$ for some hyperbolic pair η,η' and then act by ξ_{2-2k} and $\mathcal{D}^{2k-1}$ termwise on the expansion (3.10). However, the operator ξ_{2-2k} (resp. $\mathcal{D}^{2k-1}$) commutes with the group action of $\mathrm{SL}_2(\mathbb{R})$, so it suffices to compute the action of ξ_{2-2k} (resp. $\mathcal{D}^{2k-1}$) on $\widehat{\varphi}$ (defined in (3.9)). By Lemma 3.1 and (3.15), the assumption that $\tau\in\mathbb{H}\setminus E_D$ is equivalent to the restriction that $x\neq 0$ before slashing by $A\gamma$.

For $x\neq 0$, we use

$$\sin\left(\arctan\left|\frac{y}{x}\right|\right)=\frac{|y|}{\sqrt{x^2+y^2}} \tag{6.2}$$

to evaluate

$$\xi_{2-2k}\left(\widehat{\varphi}\right)(\tau)=iy^{2-2k}\operatorname{sgn}(x)\overline{\tau}^{k-1}\sin\left(\arctan\left|\frac{y}{x}\right|\right)^{2k-2}\left(-\frac{y\operatorname{sgn}(x)}{x^2+y^2}-i\frac{x\operatorname{sgn}(x)}{x^2+y^2}\right)=\tau^{-k}. \tag{6.3}$$

Using Lemma 3.2 and (3.6), on $\mathbb{H}\setminus E_D$ it follows that

$$\xi_{2-2k}\left(\mathcal{F}_{1-k,D,\mathcal{A}}\right)=\frac{D^{-\frac{k}{2}}}{\binom{2k-2}{k-1}\pi}\xi_{2-2k}\left(\mathcal{P}_{1-k,\eta}\right)=\frac{D^{-\frac{k}{2}}}{\binom{2k-2}{k-1}\pi}P_{k,\eta}=D^{\frac{1}{2}-k}f_{k,D,\mathcal{A}}.$$

Since $\xi_{2-2k}\left(\mathcal{F}_{1-k,D,\mathcal{A}}\right)$ is holomorphic in some neighborhood of τ, one immediately obtains (6.1) after using (2.1) to rewrite Δ_{2-2k}.

We next consider $\mathcal{D}^{2k-1}$. We first show that for $n\geq 0$ and $x\neq 0$ we have

$$(2\pi i)^n\mathcal{D}^n\left(\widehat{\varphi}\right)(\tau)=\frac{\Gamma(k)}{\Gamma(k-n)}\operatorname{sgn}(x)\tau^{k-1-n}\varphi\left(\arctan\left|\frac{y}{x}\right|\right)+\frac{P_n(x,y)}{\tau^n\overline{\tau}^{k-1}},\tag{6.4}$$

where $P_n(x,y)$ is the homogeneous polynomial of degree $2k-2$ defined inductively by $P_0(x,y):=0$ and

$$P_{n+1}(x,y):=\frac{-i}{2}\frac{\Gamma(k)}{\Gamma(k-n)}y^{2k-2}+\tau\frac{d}{d\tau}\left(P_n(x,y)\right)-nP_n(x,y)\tag{6.5}$$

for $n\geq 0$. The statement for $n=0$ is simply definition (3.9) of $\widehat{\varphi}$. We then use induction and apply (6.2) to establish (6.4) for $n\geq 0$.

In particular, for $n=2k-1$ the first term in (6.4) vanishes and thus we have

$$\mathcal{D}^{2k-1}\left(\widehat{\varphi}\right)(\tau)=\frac{P_{2k-1}(x,y)}{(2\pi i)^{2k-1}\tau^{2k-1}\overline{\tau}^{k-1}}.$$

However, in some neighborhood of τ, (6.1) implies that $\widehat{\varphi}$ is harmonic and hence $\mathcal{D}^{2k-1}\left(\widehat{\varphi}\right)$ is holomorphic. Thus

$$P_{2k-1}(x,y)=\overline{\tau}^{k-1}P(\tau)$$

for some polynomial $P\in\mathbb{C}[X]$. However, since $P_{2k-1}(x,y)$ is homogeneous of degree $2k-2$, it follows that

$$P_{2k-1}(x,y)=C\left|\tau\right|^{2k-2}=Cx^{2k-2}+O_y\left(x^{2k-3}\right)$$

for some constant $C\in\mathbb{C}$. In order to compute the constant, we note that, by (6.5), one easily inductively shows that for $n\geq 1$

$$P_{n+1}(x,y)=\frac{-i}{2}x^n\frac{d^n}{d\tau^n}\left(y^{2k-2}\right)+O_y\left(x^{n-1}\right).$$

We use this with $n=2k-2$ to obtain that

$$C=-\left(\frac{i}{2}\right)^{2k-1}(2k-2)!.$$

Hence it follows that

$$\mathcal{D}^{2k-1}\left(\widehat{\varphi}\right)(\tau)=-\frac{(2k-2)!}{(4\pi)^{2k-1}}\tau^{-k}.$$

Therefore, using Lemma 3.2 and (3.6) to rewrite $\mathcal{P}_{1-k,\eta}$ and $P_{k,\eta}$, we complete the proof with

$$\mathcal{D}^{2k-1}\left(\mathcal{F}_{1-k,D,\mathcal{A}}\right)(\tau)=-D^{\frac{1}{2}-k}\frac{(2k-2)!}{(4\pi)^{2k-1}}f_{k,D,\mathcal{A}}(\tau).$$

□

7. The expansion of $\mathcal{F}_{1-k,D,\mathcal{A}}$

In this section we investigate the "shape" of $\mathcal{F}_{1-k,D,\mathcal{A}}$. We are then able to prove that $\mathcal{F}_{1-k,D,\mathcal{A}}$ is a locally harmonic Maass form, completing the proof of Theorem 1.1. To describe the expansion of $\mathcal{F}_{1-k,D,\mathcal{A}}$, we first need some notation. Recall that for $\mathrm{Re}(s), \mathrm{Re}(w) > 0$, we have (for example, see (6.2.2) of [1])

$$\beta(s,w) := \beta(1;s,w) = \int_0^1 u^{s-1}(1-u)^{w-1}\,du = \frac{\Gamma(s)\Gamma(w)}{\Gamma(s+w)}. \tag{7.1}$$

In particular, by the duplication formula, one has

$$\beta\left(k-\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(k-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(k)} = \binom{2k-2}{k-1}2^{2-2k}\pi. \tag{7.2}$$

For $a > 0$, $b \in \mathbb{Z}$, and a narrow equivalence class $\mathcal{A} \subseteq \mathcal{Q}_D$, denote

$$r_{a,b}(\mathcal{A}) := \begin{cases} 1+(-1)^k & \text{if } \left[a,b,\frac{b^2-D}{4a}\right] \in \mathcal{A} \text{ and } \left[-a,-b,-\frac{b^2-D}{4a}\right] \in \mathcal{A},\\ 1 & \text{if } \left[a,b,\frac{b^2-D}{4a}\right] \in \mathcal{A} \text{ and } \left[-a,-b,-\frac{b^2-D}{4a}\right] \notin \mathcal{A},\\ (-1)^k & \text{if } \left[a,b,\frac{b^2-D}{4a}\right] \notin \mathcal{A} \text{ and } \left[-a,-b,-\frac{b^2-D}{4a}\right] \in \mathcal{A},\\ 0 & \text{otherwise.}\end{cases}$$

We define the constants

$$c_\infty(\mathcal{A}) := -\frac{1}{2^{2k-2}(2k-1)\binom{2k-2}{k-1}}\sum_{a\in\mathbb{N}} a^{-k} \sum_{\substack{b \pmod{2a}\\ b^2\equiv D \pmod{4a}}} r_{a,b}(\mathcal{A}), \tag{7.3}$$

$$c_\infty := -\frac{1}{2^{2k-2}(2k-1)\binom{2k-2}{k-1}}\frac{\zeta(k)}{\zeta(2k)}L_\Delta(k)\sum_{d\mid f}\mu(d)\chi_\Delta(d)d^{-k}\sigma_{1-2k}\left(\frac{f}{d}\right),$$

where $D = \Delta f^2$ and Δ is a fundamental discriminant. They play an important role in the expansions of $\mathcal{F}_{1-k,D,\mathcal{A}}$ and $\mathcal{F}_{1-k,D}$, respectively. For a connected component $\mathcal{C}$ of $\mathbb{H}\setminus E_D$, we also define

$$\mathcal{B}_\mathcal{C} = \mathcal{B}_{\mathcal{C},\mathcal{A}} := \left\{Q \in \mathcal{A} : \tau \in \mathcal{C}_Q^- \text{ for all } \tau \in \mathcal{C}\right\},$$

where $\mathcal{C}_Q^-$ was given in (5.1). The set $\mathcal{B}_\mathcal{C}$ consists of precisely those $Q \in \mathcal{A}$ for which S_Q (defined in (3.3)) circumscribes $\mathcal{C}$ and it is finite by Lemma 5.1. Furthermore, abusing notation, for $\alpha \in \mathbb{Q}\cup\{i\infty\}$, the (unique) connected component containing α on its boundary will be denoted by $\mathcal{C}_\alpha$. This connected component is unique because the set

$$\left\{\tau = x+iy \in \mathbb{H} : y > \frac{\sqrt{D}}{2}\right\} \subseteq \mathcal{C}_{i\infty}$$

and $\alpha = \gamma(i\infty)$ for some $\gamma \in \Gamma_1$. Before we state the theorem, we refer the reader back to the definitions of $f^*_{k,D,\mathcal{A}}$ and $\mathcal{E}_{f_{k,D,\mathcal{A}}}$, given in (1.6) and (1.7), respectively.

Theorem 7.1. *Suppose that $k > 1$, $D > 0$ is a non-square discriminant, and $\mathcal{A} \subseteq \mathcal{Q}_D$ is a narrow equivalence class. Then, for every connected component $\mathcal{C}$ of $\mathbb{H}\setminus\bigcup_{Q\in\mathcal{A}}S_Q$, there exists a polynomial $P_{\mathcal{C},\mathcal{A}} \in \mathbb{C}[X]$ of degree at most $2k-2$ such that*

$$\mathcal{F}_{1-k,D,\mathcal{A}}(\tau) = D^{\frac{1}{2}-k}f^*_{k,D,\mathcal{A}}(\tau) - D^{\frac{1}{2}-k}\frac{(2k-2)!}{(4\pi)^{2k-1}}\mathcal{E}_{f_{k,D,\mathcal{A}}}(\tau) + P_{\mathcal{C},\mathcal{A}}(\tau) \tag{7.4}$$

for every $\tau \in \mathcal{C}$. This polynomial is explicitly given by

$$P_{\mathcal{C},\mathcal{A}}(\tau) = c_\infty(\mathcal{A}) - (-1)^k 2^{2-2k} D^{\frac{1}{2}-k} \sum_{Q=[a,b,c]\in\mathcal{B}_{\mathcal{C}}} \operatorname{sgn}(a) Q(\tau,1)^{k-1}. \tag{7.5}$$

Remark. In particular, for every $\tau \in \mathbb{H}$ with $y > \frac{\sqrt{D}}{2}$, $\mathcal{F}_{1-k,D,\mathcal{A}}$ has the Fourier expansion

$$\mathcal{F}_{1-k,D,\mathcal{A}}(\tau) = D^{\frac{1}{2}-k} f^*_{k,D,\mathcal{A}}(\tau) - D^{\frac{1}{2}-k}\frac{(2k-2)!}{(4\pi)^{2k-1}}\mathcal{E}_{f_{k,D,\mathcal{A}}}(\tau) + c_\infty(\mathcal{A}). \tag{7.6}$$

One now concludes Theorem 1.3 immediately by summing over all narrow classes $\mathcal{A} \subseteq \mathcal{Q}_D$.

Before proving Theorem 7.1, we note an immediate corollary which will prove useful in computing the periods of $f_{k,D}$.

Corollary 7.2. *Suppose that k is even. Then for every $\tau \in \mathcal{C}_0$,*

$$\mathcal{F}_{1-k,D}(\tau) = D^{\frac{1}{2}-k} f^*_{k,D}(\tau) - D^{\frac{1}{2}-k}\frac{(2k-2)!}{(4\pi)^{2k-1}}\mathcal{E}_{f_{k,D,\mathcal{A}}}(\tau) + P_{\mathcal{C}_0}(\tau),$$

where

$$P_{\mathcal{C}_0}(\tau) := c_\infty + 2^{3-2k} D^{\frac{1}{2}-k} \sum_{\substack{Q=[a,b,c]\in\mathcal{Q}_D\\ a<0<c}} Q(\tau,1)^{k-1}. \tag{7.7}$$

A key step in determining the constant term of (7.5) lies in computing the integral

$$\mathcal{I}_{a,D,k}(y) := \int_{-\infty}^{\infty} \left(a(w+iy)^2 - \frac{D}{4a}\right)^{k-1} \varphi\left(\arctan\left(\frac{\sqrt{D}y}{a(w^2+y^2) - \frac{D}{4a}}\right)\right) dw,$$

which is defined for $y > 0$, $a \in \mathbb{N}$, $k \in \mathbb{N}$, and $D > 0$ a non-square discriminant.

Lemma 7.3. *For $a \in \mathbb{N}$, D a non-square discriminant, and $k > 1$, we have*

$$\mathcal{I}_{a,D,k}(y) = (-1)^{k+1}\frac{D^{k-\frac{1}{2}}}{a^k 2^{2k-2}(2k-1)}\pi.$$

Due to the technical nature of the proof of Lemma 7.3, we first assume its statement and move its proof to the end of the section.

Proof of Theorem 7.1. Suppose that $\tau \in \mathcal{C}$. As described when defining f^* in (1.6), we have

$$\xi_{2-2k}\left(f^*_{k,D,\mathcal{A}}\right)(\tau) = f_{k,D,\mathcal{A}}(\tau), \tag{7.8}$$

$$\mathcal{D}^{2k-1}\left(f^*_{k,D,\mathcal{A}}\right)(\tau) = 0. \tag{7.9}$$

Since $\mathcal{D}(q^n) = nq^n$, one easily computes

$$\mathcal{D}^{2k-1}\left(\mathcal{E}_{f_{k,D,\mathcal{A}}}\right)(\tau) = f_{k,D,\mathcal{A}}(\tau), \tag{7.10}$$

where $\mathcal{E}_f$ ($f \in S_{2k}$) was defined in (1.7). Moreover, since $\mathcal{E}_{f_{k,D,\mathcal{A}}}$ is holomorphic,

$$\xi_{2-2k}\left(\mathcal{E}_{f_{k,D,\mathcal{A}}}\right)(\tau) = 0. \tag{7.11}$$

From (7.8), (7.11), and Proposition 6.1, it follows that

$$\xi_{2-2k}\left(\mathcal{F}_{1-k,D,\mathcal{A}} - D^{\frac{1}{2}-k} f^*_{k,D,\mathcal{A}} + D^{\frac{1}{2}-k}\frac{(2k-2)!}{(4\pi)^{2k-1}}\mathcal{E}_{f_{k,D,\mathcal{A}}}\right)(\tau) = 0,$$

and hence

$$P_{\mathcal{C},\mathcal{A}}(\tau) := \mathcal{F}_{1-k,D,\mathcal{A}}(\tau) - D^{\frac{1}{2}-k} f^{*}_{k,D,\mathcal{A}}(\tau) + D^{\frac{1}{2}-k}\frac{(2k-2)!}{(4\pi)^{2k-1}}\mathcal{E}_{f_{k,D,\mathcal{A}}}(\tau)$$

is holomorphic in $\mathcal{C}$. However, from (7.9), (7.10), and Proposition 6.1, we conclude that

$$\mathcal{D}^{2k-1}(P_{\mathcal{C},\mathcal{A}}) = 0.$$

It follows that $P_{\mathcal{C},\mathcal{A}}$ defines a polynomial of degree at most $2k-2$ inside $\mathcal{C}$, establishing (7.4).

We move on to the specific form of $P_{\mathcal{C},\mathcal{A}}$. Since $\mathcal{B}_{\mathcal{C}}$ is finite, we may prove the claim by induction on $\#\mathcal{B}_{\mathcal{C}}$. We begin with the case $\#\mathcal{B}_{\mathcal{C}} = 0$, which is precisely the case that $\mathcal{C} = \mathcal{C}_{i\infty}$. Note that for $\tau = x+iy$, the equation $a|\tau|^2 + bx + \frac{b^2-D}{4a} = 0$ gives the circle centered at $-\frac{b}{2a}$ of radius $\frac{\sqrt{D}}{2|a|} < \frac{\sqrt{D}}{2}$. Hence every $\tau \in \mathbb{H}$ with $\mathrm{Im}(\tau) > \frac{\sqrt{D}}{2}$ is in the same connected component $\mathcal{C}_{i\infty}$. It follows that $P_{\mathcal{C}_{i\infty},\mathcal{A}}$ is fixed under translations and hence is a constant which we now show agrees with $c_\infty(\mathcal{A})$.

For $y > \frac{\sqrt{D}}{2}$, we use Poisson summation on (3.2). One may restrict to $a > 0$ by the change of variables $a \to -a$ and $b \to -b$. Rewrite b as $b + 2an$ and note that

$$\begin{aligned} a|\tau|^2 + (b+2an)x + \frac{(b+2an)^2 - D}{4a} &= a|\tau+n|^2 + b(x+n) + \frac{b^2-D}{4a},\\ a\tau^2 + (b+2an)\tau + \frac{(b+2an)^2-D}{4a} &= a(\tau+n)^2 + b(\tau+n) + \frac{b^2-D}{4a}, \end{aligned}$$

and that the sgn term in (3.2) is always positive for $y > \frac{\sqrt{D}}{2}$. Hence (3.2) becomes

$$\mathcal{F}_{1-k,D,\mathcal{A}}(\tau) = \frac{(-1)^k D^{\frac{1}{2}-k}}{\binom{2k-2}{k-1}\pi}\sum_{a\in\mathbb{N}}\sum_{\substack{b \pmod{2a}\\ b^2\equiv D \pmod{4a}\\ Q=\left[a,b,\frac{b^2-D}{4a}\right]}} r_{a,b}(\mathcal{A})\sum_{n\in\mathbb{Z}} Q(\tau+n,1)^{k-1}$$
$$\times\varphi\left(\arctan\left|\frac{\sqrt{D}y}{a|\tau+n|^2 + b(x+n) + \frac{b^2-D}{4a}}\right|\right).$$

Applying Poisson summation to the inner sum and using the change of variables $w \to w - \frac{b}{2a} + iy$, the associated constant term becomes

$$\int_{-\infty+iy}^{\infty+iy} Q(w,1)^{k-1}\varphi\left(\arctan\left(\frac{\sqrt{D}y}{a|w|^2 + b\mathrm{Re}(w) + c}\right)\right)dw = \mathcal{I}_{a,D,k}(y).$$

We immediately conclude (7.6) by Lemma 7.3, establishing the case when $\mathcal{B}_{\mathcal{C}} = \emptyset$.

Next suppose that $\#\mathcal{B}_{\mathcal{C}} = n > 0$ and choose $Q_0 \in \mathcal{B}_{\mathcal{C}}$. Since two circles intersect at most twice and $\mathcal{B}_{\mathcal{C}}$ is finite by Lemma 5.1, it follows that there exists an (open) neighborhood N containing an arc along the geodesic S_{Q_0} (defined in (3.3)) which does not intersect any other geodesics S_Q for $Q \in \mathcal{Q}_D$. In other words, there exists $\tau_0 \in S_{Q_0}$ and a neighborhood N of τ_0 for which

$$N_1 := N \cap E_D \subset S_{Q_0}.$$

Thus N_1 is on the boundary of precisely two connected components, $\mathcal{C}$ and another connected component, which we denote $\mathcal{C}_1$. Then $\mathcal{C}_1$ contains those $\tau \in N$ for which $\tau = \tau_1 + iw$ for some $\tau_1 \in N_1$ and $w > 0$ and $\mathcal{C}$ contains those for which $\tau = \tau_1 - iw$. Our goal is to show (the analytic continuation of) identity (7.5) for every $\tau \in N_1$, hence concluding the result by the identity theorem. One sees immediately that $\mathcal{B}_{\mathcal{C}_1} \subsetneq \mathcal{B}_{\mathcal{C}}$, since $Q \notin \mathcal{B}_{\mathcal{C}_1}$. Hence by induction, we have

$$P_{\mathcal{C}_1,\mathcal{A}}(\tau) = c_\infty(\mathcal{A}) - (-1)^k 2^{2-2k} D^{\frac{1}{2}-k} \sum_{Q=[a,b,c]\in\mathcal{B}_{\mathcal{C}_1}} \operatorname{sgn}(a) Q(\tau,1)^{k-1}. \tag{7.12}$$

Since each summand in (7.4) is piecewise continuous, for $\tau_1 \in N_1$, we have

$$\lim_{w\to 0^+} \left(\mathcal{F}_{1-k,D,\mathcal{A}}(\tau - iw) - \mathcal{F}_{1-k,D,\mathcal{A}}(\tau + iw)\right) = P_{\mathcal{C},\mathcal{A}}(\tau) - P_{\mathcal{C}_1,\mathcal{A}}(\tau).$$

However, arguing as in (5.5) and (5.6), we may rewrite the limit to obtain, for every $\tau \in N_1$,

$$\begin{aligned} P_{\mathcal{C},\mathcal{A}}(\tau) - P_{\mathcal{C}_1,\mathcal{A}}(\tau) &= \lim_{w\to 0^+} \left(\mathcal{F}_{1-k,D,\mathcal{A}}(\tau - iw) - \mathcal{F}_{1-k,D,\mathcal{A}}(\tau + iw)\right) \\ &= -\frac{(-1)^k D^{\frac{1}{2}-k}}{\binom{2k-2}{k-1}\pi} \sum_{Q=[a,b,c]\in\mathscr{B}_{\tau,\mathcal{A}}} \operatorname{sgn}(a) Q(\tau,1)^{k-1} \beta\left(\frac{Dy^2}{|Q(\tau,1)|^2}; k-\frac{1}{2}, \frac{1}{2}\right), \end{aligned} \tag{7.13}$$

where $\mathscr{B}_{\tau,\mathcal{A}} := \{Q \in \mathcal{A} : \tau \in S_Q\}$. By the definition of N_1, we know that $\mathscr{B}_{\tau,\mathcal{A}} \subseteq \{Q_0, -Q_0\}$, because $S_Q = S_{\widetilde{Q}}$ if and only if $\widetilde{Q} = Q$ or $\widetilde{Q} = -Q$. Moreover, $|Q(\tau,1)|^2 = Dy^2$ for every $\tau \in N_1$. Since $\mathcal{B}_{\mathcal{C}} = \mathcal{B}_{\mathcal{C}_1} \cup (\{\pm Q_0\} \cap \mathcal{A})$, we may hence combine definition (7.1) of $\beta\left(k-\frac{1}{2}, \frac{1}{2}\right)$ with (7.13) and (7.12) to obtain (for every $\tau \in N_1$)

$$P_{\mathcal{C},\mathcal{A}}(\tau) = c_\infty(\mathcal{A}) - \frac{(-1)^k D^{\frac{1}{2}-k}}{\binom{2k-2}{k-1}\pi} \beta\left(k-\frac{1}{2}, \frac{1}{2}\right) \sum_{Q\in\mathcal{B}_{\mathcal{C}}} \operatorname{sgn}(a) Q(\tau,1)^{k-1}.$$

The result follows by (7.2). □

Proof of Corollary 7.2. The polynomial $P_{\mathcal{C}_0}$ is obtained by

$$P_{\mathcal{C}_0} = \sum_{\mathcal{A}} P_{\mathcal{C}_0,\mathcal{A}},$$

where the sum runs over all narrow classes of discriminant D. However, each $Q \in \mathcal{Q}_D$ is contained in precisely one narrow class $\mathcal{A}$, and hence, plugging in (7.5), one obtains

$$P_{\mathcal{C}_0}(\tau) = \sum_{\mathcal{A}} P_{\mathcal{C}_0,\mathcal{A}}(\tau) = \sum_{\mathcal{A}} c_\infty(\mathcal{A}) - 2^{2-2k} D^{\frac{1}{2}-k} \sum_{Q=[a,b,c]\in\bigcup_{\mathcal{A}}\mathcal{B}_{\mathcal{C}_0,\mathcal{A}}} \operatorname{sgn}(a) Q(\tau,1)^{k-1}.$$

Comparing (7.3) (with k even) and (4.2), we have

$$\sum_{\mathcal{A}} c_\infty(\mathcal{A}) = c_\infty,$$

and it remains to compute $\bigcup_{\mathcal{A}} \mathcal{B}_{\mathcal{C}_0,\mathcal{A}}$. This set consists of precisely those $Q = [a,b,c] \in \mathcal{Q}_D$ for which one root is positive and one root is negative, or in other words, $\operatorname{sgn}(ac) = -1$. By the change of variables $Q \to -Q$, we may assume that $a < 0 < c$. The corollary now follows. □

Proof of Lemma 7.3. We first set $\widetilde{y} := \frac{2a}{\sqrt{D}} y$ and make the change of variables $u = \frac{2a}{\sqrt{D}} w$, from which we obtain

$$\mathcal{I}_{a,D,k}(y) = \frac{D^{k-\frac{1}{2}}}{a^k 2^{2k-1}} \int_{-\infty}^{\infty} \left((u+i\widetilde{y})^2 - 1\right)^{k-1} \varphi\left(\arctan\left(\frac{2\widetilde{y}}{u^2+\widetilde{y}^2-1}\right)\right) du.$$

Now define

$$\mathcal{I}_k(\widetilde{y}) := \int_{-\infty}^{\infty} \left((u+i\widetilde{y})^2 - 1\right)^{k-1} \varphi\left(\arctan\left(\frac{2\widetilde{y}}{u^2+\widetilde{y}^2-1}\right)\right) du. \tag{7.14}$$

We next show that $\mathcal{I}_k(\widetilde{y})$ is independent of $\widetilde{y}>1$ (or equivalently $y>\frac{\sqrt{D}}{2a}$). Note that, for $a\in\mathbb{N}$ and $b \pmod{2a}$ ($b^2\equiv D \pmod{4a}$) fixed, either every $Q=[a,b,c]$ is an element of $\mathcal{A}$ or none of them are, because translations always give two equivalent quadratic forms. Recall that $\xi_{2-2k}\left(\mathcal{F}_{1-k,D,\mathcal{A}}\right)=f_{k,D,\mathcal{A}}$ and $D^{2k-1}\left(\mathcal{F}_{1-k,D,\mathcal{A}}\right)=cf_{k,D,\mathcal{A}}$, for some constant $c\in\mathbb{C}$, were shown termwise. Hence, arguing as before, but with a fixed, the polynomial in the connected component including $i\infty$ must be constant and hence we get independence of $y>\frac{\sqrt{D}}{2a}$, because no discontinuities exist for $y>\frac{\sqrt{D}}{2a}$. Thus, (7.14) is constant for $\widetilde{y}>1$. Since (7.14) is continuous for $\widetilde{y}>0$, (although only constant for $\widetilde{y}\geq 1$) for any $\widetilde{y}\geq 1$ we have that (7.14) agrees with

$$\lim_{\widetilde{y}\to 1^+}\mathcal{I}_k\left(\widetilde{y}\right)=\mathcal{I}_k\left(1\right)=\int_{-\infty}^{\infty}\left((u+i)^2-1\right)^{k-1}\varphi\left(\arctan\left(\frac{2}{u^2}\right)\right)du.$$

It hence suffices to prove

$$\mathcal{I}_k:=\mathcal{I}_k\left(1\right)=(-1)^{k-1}\frac{2\pi}{2k-1}. \tag{7.15}$$

We first expand

$$(u+i)^2-1=\left(u-\sqrt{2}\zeta_8^{-1}\right)\left(u-\sqrt{2}\zeta_8^{-3}\right), \tag{7.16}$$

where $\zeta_n:=e^{\frac{2\pi i}{n}}$. Now rewrite

$$\sin\left(u\right)^{2k-2}=-\left(-1\right)^k2^{2-2k}\sum_{m=0}^{2k-2}\binom{2k-2}{m}\left(-1\right)^m e^{i(2m-(2k-2))u}. \tag{7.17}$$

We may then explicitly integrate (7.17) as in definition (3.7) of φ, yielding

$$\varphi\left(v\right)=-\left(-1\right)^k2^{2-2k}\left(\binom{2k-2}{k-1}\left(-1\right)^{k-1}v-i\sum_{m\neq k-1}\frac{\binom{2k-2}{m}\left(-1\right)^m}{2m+2-2k}\left(e^{i(2m+2-2k)v}-1\right)\right).$$

We then use $e^{i\theta}=\cos\left(\theta\right)+i\sin\left(\theta\right)$ and (6.2) to expand

$$\begin{aligned}\varphi\left(\arctan\left(\frac{2}{u^2}\right)\right)&=\frac{1}{2^{2k-2}}\left(\binom{2k-2}{k-1}\arctan\left(\frac{2}{u^2}\right)+\left(-1\right)^k i\sum_{m\neq k-1}\frac{\binom{2k-2}{m}\left(-1\right)^m}{2m+2-2k}\right.\\&\quad\times\left.\left(\left(\cos\left(\arctan\left(\frac{2}{u^2}\right)\right)+i\sin\left(\arctan\left(\frac{2}{u^2}\right)\right)\right)^{2m+2-2k}-1\right)\right)\\&=\frac{1}{2^{2k-2}}\left(\binom{2k-2}{k-1}\arctan\left(\frac{2}{u^2}\right)+\left(-1\right)^k i\sum_{m\neq k-1}\frac{\binom{2k-2}{m}\left(-1\right)^m}{2m+2-2k}\left(\frac{u^2+2i}{u^2-2i}\right)^{m+1-k}\right),\end{aligned} \tag{7.18}$$

since the sum involving -1 vanishes. We now note that

$$f\left(z\right):=-i\left(1-\left(z+i\right)^2\right)^{k-1}\sum_{m\neq k-1}\frac{\binom{2k-2}{m}\left(-1\right)^m}{2m+2-2k}\left(\frac{z^2+2i}{z^2-2i}\right)^{m+1-k}$$

is a meromorphic function in z with no poles in the lower half plane (because the poles at $\sqrt{2}\zeta_8^{-1}$ and $\sqrt{2}\zeta_8^{-3}$ are cancelled by the zeros of order $k-1$ of $\left((z+i)^2-1\right)^{k-1}$ from (7.16)).

In order to evaluate $\mathcal{I}_k$, for $R>0$ we let C_R denote the path from $-R$ to R followed by the semi-circle in the lower half plane from R to $-R$. Define

$$g^{\pm}(z) := \frac{i}{2}\log\left(\frac{z-\sqrt{2}\zeta_8^{\pm 1}}{z-\sqrt{2}\zeta_8^{\pm 3}}\right),$$

where $\log(z)$ is the principal branch. One easily checks that the branch cuts for $g^{\pm}$ are the the lines connecting $\zeta_8^{\pm 1}$ and $\zeta_8^{\pm 3}$ and the branch cuts for $\log\left(\frac{z^2-2i}{z^2+2i}\right)$ are those lines radially from the point 0 to $\sqrt{2}\zeta_8^{2j-1}$ $(1\le j\le 4)$. Hence the sum of the logarithms equals the logarithm of the product for every $z\in C_R$ by the identity theorem (since they agree when the parameter is real). Therefore, for all $z\in C_R$, we have (see (4.4.31) of [1])

$$g^{+}(z)-g^{-}(z)=\frac{i}{2}\log\left(\frac{z^2-2i}{z^2+2i}\right)=\operatorname{arccot}\left(\frac{z^2}{2}\right)=\arctan\left(\frac{2}{z^2}\right).$$

We may henceforth interchange between the original definition of $\varphi\left(\operatorname{arccot}\left(\frac{z^2}{2}\right)\right)$ and that involving logarithms (in particular, in (7.18)). We hence evaluate

$$\int_{C_R}\left(f(z)+2^{2-2k}\binom{2k-2}{k-1}\left((z+i)^2-1\right)^{k-1}\left(g^{+}(z)-g^{-}(z)\right)\right)dz.$$

Using (6.2), for those z on the semi-circle, one easily obtains

$$\left|\left((z+i)^2-1\right)^{k-1}\varphi\left(\operatorname{arccot}\left(\frac{z^2}{2}\right)\right)\right|\ll R^{-2k}\to 0.$$

Hence the integral along the semi-circle vanishes for $R\to\infty$. Therefore

$$\mathcal{I}_k=\lim_{R\to\infty}\int_{C_R}\left(f(z)+2^{2-2k}\binom{2k-2}{k-1}\left((z+i)^2-1\right)^{k-1}\left(g^{+}(z)-g^{-}(z)\right)\right)dz.$$

Since $f(z)$ and $\left((z+i)^2-1\right)^{k-1}g^{+}(z)$ are holomorphic in the lower half plane, the Residue Theorem yields

$$\int_{C_R}\left(f(z)+2^{2-2k}\binom{2k-2}{k-1}\left((z+i)^2-1\right)^{k-1}g^{+}(z)\right)dz=0.$$

Using integration by parts, one obtains

$$\begin{aligned}\text{(7.19)}\quad \int_{C_R}\left((z+i)^2-1\right)^{k-1}g^{-}(z)\,dz&=\frac{i}{2}\int_{C_R}\left((z+i)^2-1\right)^{k-1}\log\left(\frac{z-\sqrt{2}\zeta_8^{-1}}{z-\sqrt{2}\zeta_8^{-3}}\right)dz\\ &=-\frac{i}{2}\int_{C_R}\left(\int_0^z\left((u+i)^2-1\right)^{k-1}du\right)\left(\frac{1}{z-\sqrt{2}\zeta_8^{-1}}-\frac{1}{z-\sqrt{2}\zeta_8^{-3}}\right)dz.\end{aligned}$$

Applying the Residue Theorem to (7.19) (noting simple poles and a minus sign from taking the integral clockwise) and recalling the identity (7.1), we obtain

$$\begin{aligned}\mathcal{I}_k&=2^{2-2k}\pi\binom{2k-2}{k-1}\int_{\sqrt{2}\zeta_8^{-3}}^{\sqrt{2}\zeta_8^{-1}}\left((u+i)^2-1\right)^{k-1}du\\ &=2\pi(-1)^{k-1}\binom{2k-2}{k-1}\int_0^1(u(1-u))^{k-1}du=2\pi(-1)^{k-1}\binom{2k-2}{k-1}\beta(k,k)=\frac{2\pi(-1)^{k-1}}{2k-1},\end{aligned}$$

where $u\to 2u+\sqrt{2}\zeta_8^{-3}$ in the second identity. This is the desired equality (7.15). □

We are finally ready to prove Theorem 1.1. By taking linear combinations of the $\mathcal{F}_{1-k,D,\mathcal{A}}$, it suffices to show the following.

Theorem 7.4. *For $k>1$, D a non-square discriminant, and $\mathcal{A}\subset\mathcal{Q}_D$ a narrow class, the function $\mathcal{F}_{1-k,D,\mathcal{A}}$ is a weight $2-2k$ locally harmonic Maass form with exceptional set E_D.*

Proof. Suppose that $\gamma_1\in\Gamma_1$. By Lemma 3.2, we may choose a hyperbolic pair η,η' so that

$$\mathcal{F}_{1-k,D,\mathcal{A}}\Big|_{2-2k}\gamma_1=\frac{D^{-\frac{k}{2}}}{\binom{2k-2}{k-1}\pi}\mathcal{P}_{1-k,\eta}\Big|_{2-2k}\gamma_1=\frac{D^{-\frac{k}{2}}}{\binom{2k-2}{k-1}\pi}\sum_{\gamma\in\Gamma_\eta\backslash\Gamma_1}\widehat{\varphi}\Big|_{2-2k}A\gamma\gamma_1.$$

Due to the absolute convergence proven in Proposition 4.1, we may rearrange the sum, from which we conclude weight $2-2k$ modularity. The local harmonicity of $\mathcal{F}_{1-k,D,\mathcal{A}}$ was shown in (6.1). Condition 3 is precisely Proposition 5.2. The functions $\mathcal{E}_{f_{k,D,\mathcal{A}}}$ and $f^*_{k,D,\mathcal{A}}$ decay towards $i\infty$. Thus, using (7.5) with $\mathcal{C}=\mathcal{C}_{i\infty}$, (7.4) implies that $\mathcal{F}_{1-k,D,\mathcal{A}}$ is bounded towards $i\infty$. □

8. Relations to period polynomials

The main goal of this section is to use Corollary 7.2 to supply a new proof of Theorem 1.4, i.e., the fact that the even periods of $f_{k,D}$ are rational. We begin by giving a formal definition of periods and period polynomials. For $f\in S_{2k}$ and $0\le n\le 2k-2$, the *n-th period of f* is defined by (see Section 1.1 of [23])

$$r_n(f):=\int_0^\infty f(it)\,t^n dt=n!\,(2\pi)^{-n-1}L(f,n+1),\tag{8.1}$$

where $L(f,s)$ is the L-series associated to f. These can be nicely packaged into a *period polynomial*

$$r(f;X):=\int_0^{i\infty}f(z)\,(X-z)^{2k-2}dz=\sum_{n=0}^{2k-2}i^{1-n}\binom{2k-2}{n}r_n(f)\,X^{2k-2-n}$$

and we denote the even part of the period polynomial by

$$r^+(f;X):=\sum_{\substack{0\le n\le 2k-2\\ n\text{ even}}}(-1)^{\frac{n}{2}}\binom{2k-2}{n}r_n(f)\,X^{2k-2-n}.$$

We note that a theory of period polynomials for weakly holomorphic modular forms has also been developed (see [3]).

We now describe how the polynomials $P_{\mathcal{C},\mathcal{A}}$ in Theorem 7.1 are related to period polynomials. We note that while neither $f^*_{k,D,\mathcal{A}}$ nor $\mathcal{E}_{f_{k,D,\mathcal{A}}}$ satisfy modularity, up to the constant term they are the non-holomorphic and holomorphic parts of certain harmonic weak Maass forms, respectively. This follows because the operator ξ_{2-2k} is surjective by work of Bruinier and Funke [9] and $\mathcal{D}^{2k-1}$ is surjective by work of Bruinier, Ono, and Rhoades [11]. For $\gamma\in\Gamma_1$, $f^*_{k,D,\mathcal{A}}$ and $\mathcal{E}_{f_{k,D,\mathcal{A}}}$ satisfy

$$f^*_{k,D,\mathcal{A}}\Big|_{2-2k}\gamma(\tau)=f^*_{k,D,\mathcal{A}}+r_\gamma(\tau),\tag{8.2}$$

$$\mathcal{E}_{f_{k,D,\mathcal{A}}}\Big|_{2-2k}\gamma(\tau)=\mathcal{E}_{f_{k,D,\mathcal{A}}}+R_\gamma(\tau)\tag{8.3}$$

for certain period polynomials r_γ and R_γ (each is of degree at most $2k-2$). However, it is known that there exists $C\in\mathbb{C}$ such that

$$-\frac{(2k-2)!}{(4\pi)^{2k-1}}R_\gamma(\tau)=r^c_\gamma(\tau)+C\left(j(\gamma,\tau)^{2k-2}-1\right),\tag{8.4}$$

where $P^c \in \mathbb{C}[X]$ is the polynomial whose coefficients are the complex conjugates of the coefficients of $P \in \mathbb{C}[X]$ [3, 19]. The following proposition relates the period polynomials to the polynomials $P_{\mathcal{C},\mathcal{A}}$ from the previous section.

Proposition 8.1. *Suppose that $D > 0$ is a non-square discriminant, $\mathcal{A} \subseteq \mathcal{Q}_D$ is a narrow class, $\mathcal{C}$ is a connected component of $\mathbb{H} \setminus E_D$, $\tau \in \mathcal{C}$, and $\gamma \in \Gamma_1$. Then*

$$P_{\mathcal{C},\mathcal{A}}(\tau) = D^{\frac{1}{2}-k} r_\gamma(\tau) - D^{\frac{1}{2}-k}\frac{(2k-2)!}{(4\pi)^{2k-1}} R_\gamma(\tau) + P_{\gamma\mathcal{C},\mathcal{A}}(\gamma\tau)\, j(\gamma,\tau)^{2k-2}.$$

In particular, if $\gamma\mathcal{C} = \mathcal{C}_{i\infty}$, then

$$P_{\mathcal{C},\mathcal{A}}(\tau) = D^{\frac{1}{2}-k} r_\gamma(\tau) - D^{\frac{1}{2}-k}\frac{(2k-2)!}{(4\pi)^{2k-1}} R_\gamma(\tau) + c_\infty(\mathcal{A})\, j(\gamma,\tau)^{2k-2}. \tag{8.5}$$

Proof. By the modularity of $\mathcal{F}_{1-k,D,\mathcal{A}}$, we have

$$0 = \mathcal{F}_{1-k,D,\mathcal{A}}\Big|_{2-2k}\gamma(\tau) - \mathcal{F}_{1-k,D,\mathcal{A}}(\tau).$$

However, plugging in (7.4) and definitions (8.2) and (8.3) of the period polynomials, this becomes

$$0 = D^{\frac{1}{2}-k} r_\gamma(\tau) - D^{\frac{1}{2}-k}\frac{(2k-2)!}{(4\pi)^{2k-1}} R_\gamma(\tau) + P_{\gamma\mathcal{C},\mathcal{A}}(\gamma\tau)\, j(\gamma,\tau)^{2k-2} - P_{\mathcal{C},\mathcal{A}}(\tau).$$

This yields the first statement of the proposition. The second statement simply follows from the fact that $P_{\mathcal{C}_{i\infty},\mathcal{A}} = c_\infty(\mathcal{A})$ by (7.5). ☐

Proof of Theorem 1.4. In order to get information about the even periods, we first show that

$$r(f_{k,D};\tau) - r^c(f_{k,D};\tau) = 2ir^+(f_{k,D};\tau). \tag{8.6}$$

To see this, note that $f_{k,D}(iy)$ is real because the change of variables $b \to -b$ yields

$$\sum_{Q=[a,b,c]\in\mathcal{Q}_D} (-a+iyb+c)^{-k} = \overline{\sum_{Q=[a,b,c]\in\mathcal{Q}_D} (-a+iyb+c)^{-k}}.$$

The integral (8.1) defining $r_n(f)$ is hence also real, from which (8.6) follows.

Plugging $\gamma = S$ into (8.5) and summing over all narrow classes, we obtain

$$P_{\mathcal{C}_0}(\tau) = D^{\frac{1}{2}-k} r_S(\tau) - D^{\frac{1}{2}-k}\frac{(2k-2)!}{(4\pi)^{k-1}} R_S(\tau) + c_\infty\tau^{2k-2}, \tag{8.7}$$

where $P_{\mathcal{C}_0}$ was defined in (7.7). However, it can be proven (see (1.13) of [3]) that

$$R_S(\tau) = -\frac{(2\pi i)^{2k-1}}{(2k-2)!} r(f_{k,D};\tau).$$

Hence by (8.4) and (8.6), we may rewrite (8.7) as

$$\begin{aligned} P_{\mathcal{C}_0}(\tau) &= -2^{1-2k} i D^{\frac{1}{2}-k}\left(-r^c(f_{k,D};\tau) + r(f_{k,D};\tau)\right) + C\left(\tau^{2k-2}-1\right) + c_\infty\tau^{2k-2} \\ &= 2^{2-2k} D^{\frac{1}{2}-k} r^+(f_{k,D};\tau) + C\left(\tau^{2k-2}-1\right) + c_\infty\tau^{2k-2} \end{aligned}$$

for some constant C. We now use Corollary 7.2 to rewrite the left hand side, obtaining

$$c_\infty + 2^{3-2k} D^{\frac{1}{2}-k} \sum_{\substack{Q=[a,b,c]\in\mathcal{Q}_D \\ a<0<c}} Q(\tau,1)^{k-1} = 2^{2-2k} D^{\frac{1}{2}-k} r^+(f_{k,D};\tau) + C\left(\tau^{k-2}-1\right) + c_\infty\tau^{k-2}.$$

Rearranging yields (1.8), completing the proof. ☐

Remark. We note that the above method may also be applied to reprove the rationality of the even periods of $f_{k,D,\mathcal{A}} + f_{k,D,-\mathcal{A}}$ (cf. Theorem 5 of [23]). Note that a symmetrization is made here so that a statement similar to (8.6) holds. Without this symmetrization, one would only obtain rationality for the imaginary part of the periods of $f_{k,D,\mathcal{A}}$.

9. Hecke operators

In this section, we investigate the action of the Hecke operators on $\mathcal{F}_{1-k,D}$, proving Theorem 1.5. For a prime p, recall that the weight $2-2k$ Hecke operator T_p acts on a translation invariant function $f : \mathbb{H} \to \mathbb{C}$ by

$$f\Big|_{2-2k} T_p(\tau) := p^{1-2k} f(p\tau) + p^{-1} \sum_{r \pmod p} f\left(\frac{\tau + r}{p}\right). \tag{9.1}$$

In order to prove Theorem 1.5, we first compute the action of T_p on the intermediary function

$$\mathcal{G}_{1-k,D}(\tau) := \frac{D^{\frac{1-k}{2}}}{\binom{2k-2}{k-1}\pi} \sum_{Q=[a,b,c]\in\mathcal{Q}'_D} \operatorname{sgn}\left(a|\tau|^2 + bx + c\right) Q(\tau,1)^{k-1} \psi\left(\frac{Dy^2}{|Q(\tau,1)|^2}\right),$$

where $\mathcal{Q}'_D$ denotes the set of primitive $Q = [a,b,c] \in \mathcal{Q}_D$ (i.e., those with $(a,b,c) = 1$).

Proof of Theorem 1.5. We first prove that

$$\mathcal{G}_{1-k,D}\Big|_{2-2k} T_p = \begin{cases} p^{-k}\mathcal{G}_{1-k,Dp^2} + p^{-k}\left(1 + \left(\frac{D}{p}\right)\right)\mathcal{G}_{1-k,D} & \text{if } p^2 \nmid D, \\ p^{-k}\mathcal{G}_{1-k,Dp^2} + p^{-k}\left(p - \left(\frac{D/p^2}{p}\right)\right)\mathcal{G}_{1-k,\frac{D}{p^2}} & \text{if } p^2 \mid D. \end{cases} \tag{9.2}$$

We define the multiset

$$\mathcal{B} := \left\{\left[ap^2, bp, c\right], \left[a, bp + 2ar, ar^2 + bpr + cp^2\right] : 0 \le r \le p-1,\ a > 0,\ [a,b,c] \in \mathcal{Q}'_D\right\}$$

and for $g \in \mathbb{N}$, we define the set

$$\mathcal{B}(g) := \left\{[\mathsf{A},\mathsf{B},\mathsf{C}] \in \mathcal{Q}_{Dp^2} : (\mathsf{A},\mathsf{B},\mathsf{C}) = g\right\}.$$

We first note that all $Q \in \mathcal{B}$ have discriminant Dp^2. A direct calculation yields

$$\mathcal{G}_{1-k,D}\Big|_{2-2k} T_p(\tau) = \sum_{Q\in\mathcal{B}} \operatorname{sgn}\left(a|\tau|^2 + bx + c\right) Q(\tau,1)^{k-1} \varphi\left(\arctan\left|\frac{\sqrt{D}y}{a|\tau|^2+bx+c}\right|\right).$$

In determining the action of the Hecke operators on the classical hyperbolic Poincaré series, Parson [24] determined precisely how many choices of primitive $[a,b,c] \in \mathcal{Q}_D$ yield a representation of each $[\mathsf{A},\mathsf{B},\mathsf{C}] \in \mathcal{B}(g)$ with $g \in \left\{1, p, p^2\right\}$. Then (9.2) follows from this enumeration and the fact that each summand in (1.4) is homogeneous of degree $k-1$ in the variables a, b, c.

Denote $D = \Delta f^2$ with Δ a fundamental discriminant. We make use of the identity

$$\mathcal{F}_{1-k,D} = D^{-\frac{k}{2}} \sum_{g\mid f} \mathcal{G}_{1-k,\Delta g^2}$$

and apply (9.2) to $\mathcal{G}_{1-k,\Delta g^2}$. This yields

$$\begin{aligned}(9.3)\quad \mathcal{F}_{1-k,D}\Big|_{2-2k}T_p &= D^{-\frac{k}{2}}\sum_{g^2|D}\mathcal{G}_{1-k,\Delta g^2}\Big|_{2-2k}T_p\\ &= \left(Dp^2\right)^{-\frac{k}{2}}\sum_{g|f,\,p\nmid g}\left(\mathcal{G}_{1-k,\Delta(gp)^2}+\left(1+\left(\frac{\Delta g^2}{p}\right)\right)\mathcal{G}_{1-k,\Delta g^2}\right)\\ &\quad+\left(Dp^2\right)^{-\frac{k}{2}}\sum_{g|f,\,p|g}\left(\mathcal{G}_{1-k,\Delta(gp)^2}+\left(p-\left(\frac{\Delta(g/p)^2}{p}\right)\right)\mathcal{G}_{1-k,\Delta\left(\frac{g}{p}\right)^2}\right).\end{aligned}$$

We next combine

$$\sum_{g|f,\,p\nmid g}\left(\mathcal{G}_{1-k,\Delta(gp)^2}+\mathcal{G}_{1-k,\Delta g^2}\right)+\sum_{p|g|f}\mathcal{G}_{1-k,\Delta(gp)^2}=\sum_{g|fp}\mathcal{G}_{1-k,\Delta g^2}=\left(Dp^2\right)^{\frac{k}{2}}\mathcal{F}_{1-k,Dp^2}$$

and

$$\sum_{g|f,\,p|g}\mathcal{G}_{1-k,\Delta\left(\frac{g}{p}\right)^2}=D^{\frac{k}{2}}p^{-k}\mathcal{F}_{1-k,\frac{D}{p^2}}$$

to rewrite the right hand side of (9.3) as

$$\mathcal{F}_{1-k,Dp^2}+p^{1-2k}\mathcal{F}_{1-k,\frac{D}{p^2}}+p^{-k}D^{-\frac{k}{2}}\left(\sum_{g|f,\,p\nmid g}\left(\frac{\Delta g^2}{p}\right)\mathcal{G}_{1-k,\Delta g^2}-\sum_{g|f,\,p|g}\left(\frac{\Delta(g/p)^2}{p}\right)\mathcal{G}_{1-k,\Delta\left(\frac{g}{p}\right)^2}\right).$$

If $p\nmid f$, then (1.9) follows by noting that $\left(\frac{\Delta f^2}{p}\right)=\left(\frac{\Delta g^2}{p}\right)$ for every $g\mid f$. If $p\mid f$, then we note that $\left(\frac{\Delta(g/p)^2}{p}\right)=0$ unless $p\|g$. In this case, the two remaining sums cancel by making the change of variables $g\to gp$ in the last sum. Hence when $p\mid f$ one obtains

$$\mathcal{F}_{1-k,D}\Big|_{2-2k}T_p=\mathcal{F}_{1-k,Dp^2}+p^{1-2k}\mathcal{F}_{1-k,\frac{D}{p^2}},$$

from which (1.9) follows because $\left(\frac{D}{p}\right)=0$. This completes the proof.

□

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THETA LIFTS AND LOCAL MAASS FORMS

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Abstract. The first two authors and Kohnen have recently introduced a new class of modular objects called locally harmonic Maass forms, which are annihilated almost everywhere by the hyperbolic Laplacian operator. In this paper, we realize these locally harmonic Maass forms as theta lifts of harmonic weak Maass forms. Using the theory of theta lifts, we then construct examples of (non-harmonic) local Maass forms, which are instead eigenfunctions of the hyperbolic Laplacian almost everywhere.

1. Introduction and statement of results

In [7], a new class of modular objects was introduced. These functions, known as locally harmonic Maass forms, satisfy negative weight modularity and are annihilated almost everywhere by the hyperbolic Laplacian (see Section 2 for the relevant definitions), mirroring harmonic weak Maass forms. Recent interest in harmonic weak Maass forms initiated with their systematic treatment by Bruinier and Funke [13]. Following their appearance in the theory of mock theta functions due to Zwegers [34], it has been shown that harmonic weak Maass forms have applications ranging from partition theory (for example [2, 4, 6, 9, 11]) and Zagier's duality [33] relating "modular objects" of different weights (for example [10]) to derivatives of L-functions (for example [14, 15]). They also arise in mathematical physics, as recently evidenced in Eguchi, Ooguri, and Tachikawa's [16] investigation of moonshine for the largest Mathieu group M_{24}. The main difference between locally harmonic Maass forms and harmonic weak Maass forms is that there are certain geodesics along which locally harmonic weak Maass forms are not necessarily real analytic and may even exhibit discontinuities.

In this paper, we realize the locally harmonic Maass forms studied in [7] as theta lifts of harmonic weak Maass forms. Theta lifts form connections between different types of modular objects and the regularization of Harvey–Moore [19] and Borcherds [3] allow one to extend their definitions to previously divergent theta integrals. In particular, the Shimura lift [26] was realized as a theta lift by Niwa [24]. Borcherds [3] later placed this into the framework of a larger family of theta lifts. Following his work, theta lifts have more recently appeared in a variety of applications including generalized Kac–Moody algebras [18] and the arithmetic of Shimura varieties [15]. To expound upon one example, Katok and Sarnak [21] used theta lifts to relate the central value of the L-series of a Maass cusp form to the Fourier coefficients of corresponding Maass cusp forms under the Shimura lift. This extended a famous result of Waldspurger [30] proving that the central value of the L-function of an integral weight Hecke eigenform is proportional to the square of a coefficient of its half-integral weight counterpart under the Shintani lift. Tunnell [28] later exploited this link to express the central value of the L-function of an elliptic curve in terms of the coefficients of a theta function associated to a

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ternary quadratic form. Tunnell's Theorem gives a solution to the ancient congruent number problem (conditional on the Birch and Swinnerton-Dyer conjecture).

Following Bruinier's [12] application of Borcherds lifts to harmonic weak Maass forms, Bruinier and Funke [13] extended theta lifts to harmonic weak Maass forms. Due to the theory built around theta lifts, one may naturally extend the definition of locally harmonic Maass forms to include local Maass forms, i.e., functions with the above properties of locally harmonic Maass forms except that instead of being annihilated by the hyperbolic Laplacian, they are eigenfunctions. The locally harmonic Maass forms investigated in [7] have a natural connection to the Shimura [26] and Shintani [27] lifts, which we next describe.

For $k \in 2\mathbb{N}$ and a discriminant $D > 0$, Zagier [31] defined the functions

$$f_{k,D}(z) := \frac{D^{k-\frac{1}{2}}}{\binom{2k-2}{k-1}\pi} \sum_{Q \in \mathcal{Q}_D} Q(z,1)^{-k}, \tag{1.1}$$

where $\mathcal{Q}_D$ denotes the set of binary quadratic forms of discriminant $D \in \mathbb{Z}$. Zagier showed that $f_{k,D} \in S_{2k}$, the space of weight $2k$ cusp forms for $\mathrm{SL}_2(\mathbb{Z})$ and it was later noticed that the $f_{k,D}$ could be naturally realized as (linear combinations of) hyperbolic Poincaré series defined by Petersson [25]. The functions $f_{k,D}$ reappeared in the (holomorphic) kernel function for the Shimura and Shintani lifts

$$\Omega(z,\tau) := \sum_{0<D\equiv 0,1 \pmod 4} f_{k,D}(z) e^{2\pi i D\tau}$$

between S_{2k} and $S^{+}_{k+\frac{1}{2}}$ (Kohnen's plus space of weight $k+\frac{1}{2}$ modular forms), which was defined by Kohnen and Zagier [23]. For $g \in S^{+}_{k+\frac{1}{2}}$, the Petersson inner product $\langle g, \Omega(-\overline{z}, \cdot)\rangle$ equals $(-1)^{k/2}2^{2-3k}$ times the Shimura lift of g. Kohnen and Zagier used Ω to explicitly compute the constant of proportionality in Waldspurger's result, in turn proving nonnegativity of the central L-values of Hecke eigenforms.

As indicated above, the functions $f_{k,D}$ may be interpreted in terms of theta lifts. To describe this, we define Shintani's [27] non-holomorphic kernel function. Throughout we write $\tau = u + iv \in \mathbb{H}$, $z = x + iy \in \mathbb{H}$, and denote for $Q = [a, b, c] \in \mathcal{Q}_D$

$$Q_z := \frac{1}{y}\left(a|z|^2 + bx + c\right).$$

Using this notation, Shintani's theta function projected into Kohnen's plus space equals

$$\Theta(z,\tau) := y^{-2k} v^{\frac{1}{2}} \sum_{\substack{D\in\mathbb{Z}\\ Q\in\mathcal{Q}_D}} Q(z,1)^k e^{-4\pi Q_z^2 v} e^{2\pi i D\tau}. \tag{1.2}$$

The function $\Theta(-\overline{z}, \tau)$ transforms like a modular form of weight $k + \frac{1}{2}$ in τ and weight $2k$ in z (see Proposition 3.2 (1)). Integrating the D-th weight $k + \frac{1}{2}$ (holomorphic) Poincaré series against Θ yields $f_{k,D}$. One can use Borcherds's [3] aforementioned regularized version $\langle f, g\rangle^{\mathrm{reg}}$ of the Petersson inner product (see Section 2 for a definition) to extend the utility of the Shimura lift (realized as Niwa's [24] theta lift) to weak Maass forms. To be more precise, for a weight $k + \frac{1}{2}$ weak Maass form H with eigenvalue

$$\lambda_s := \left(s - \frac{k}{2} - \frac{1}{4}\right)\left(1 - s - \frac{k}{2} - \frac{1}{4}\right)$$

under the hyperbolic Laplacian $\Delta_{k+\frac{1}{2}}$, we define the theta lift

$$\Phi_k(H)(z) := \langle H, \Theta(z, \cdot)\rangle^{\mathrm{reg}}.$$

By choosing an appropriate input, this lift leads to the natural generalization

$$f_{k,s,D}(z) := \sum_{Q \in \mathcal{Q}_D} Q(z,1)^{-k} \varphi_s\left(\frac{Dy^2}{|Q(z,1)|^2}\right) \tag{1.3}$$

of $f_{k,D}$. Here, for $0 < w \leq 1$ and $\mathrm{Re}(s) \geq \frac{k}{2} + \frac{1}{4}$, using the usual ${}_2F_1$ notation for Gauss's hypergeometric function, we define

$$\varphi_s(w) := \frac{\Gamma\left(s + \frac{k}{2} - \frac{1}{4}\right) D^{\frac{k}{2}+\frac{1}{4}}}{6\Gamma(2s)\,(4\pi)^{\frac{k}{2}-\frac{1}{4}}} w^{s-\frac{k}{2}-\frac{1}{4}} {}_2F_1\left(s + \frac{k}{2} - \frac{1}{4}, s - \frac{k}{2} - \frac{1}{4}; 2s; w\right),$$

which is easily seen to be a constant when $s = \frac{k}{2} + \frac{1}{4}$. Note that for $\mathrm{Re}(s) > \frac{k}{2} + \frac{1}{4}$, the Euler integral representation of the ${}_2F_1$ (see (4.3)) yields

$$\varphi_s(w) = \frac{\Gamma\left(s + \frac{k}{2} - \frac{1}{4}\right) D^{\frac{k}{2}+\frac{1}{4}} w^{s-\frac{k}{2}-\frac{1}{4}}}{6\Gamma\left(s + \frac{k}{2} + \frac{1}{4}\right) \Gamma\left(s - \frac{k}{2} - \frac{1}{4}\right) (4\pi)^{\frac{k}{2}-\frac{1}{4}}} \int_0^1 (1-t)^{s+\frac{k}{2}-\frac{3}{4}} t^{s-\frac{k}{2}-\frac{5}{4}} (1-wt)^{-s-\frac{k}{2}+\frac{1}{4}}\, dt.$$

In order to obtain the functions $f_{k,s,D}$, we apply the theta lift Φ_k to the D-th Poincaré series $P_{k+\frac{1}{2},s,D}$ (see (2.12)) of weight $k + \frac{1}{2}$ with eigenvalue λ_s under $\Delta_{k+\frac{1}{2}}$ in Kohnen's plus space. In the special case that $s = \frac{k}{2} + \frac{1}{4}$, this Poincaré series is precisely the classical cuspidal Poincaré series and $f_{k,\frac{k}{2}+\frac{1}{4},D}$ is essentially $f_{k,D}$ because $\varphi_{\frac{k}{2}+\frac{1}{4}}$ is a constant. We next show that in general the functions $f_{k,s,D}$ are local Maass forms with exceptional set given by the closed geodesics

$$E_D := \left\{ z = x + iy \in \mathbb{H} : \exists a, b, c \in \mathbb{Z},\ b^2 - 4ac = D,\ a|z|^2 + bx + c = 0 \right\}. \tag{1.4}$$

Theorem 1.1. *Suppose that $s \in \mathbb{C}$ satisfies $\mathrm{Re}(s) \geq \frac{k}{2} + \frac{1}{4}$ and $D > 0$ is a discriminant. Then the following hold.*

(1) *The function $f_{k,s,D}$ is a local Maass form of weight $2k$ and eigenvalue $4\lambda_s$ under Δ_{2k} with exceptional set E_D. Moreover,*

$$f_{k,\frac{k}{2}+\frac{1}{4},D} = \frac{2^{2k-3}}{3(2k-1)} (4\pi D)^{\frac{3}{4}-\frac{k}{2}} f_{k,D}, \tag{1.5}$$

which is a cusp form.

(2) *The theta lift Φ_k maps weight $k + \frac{1}{2}$ weak Maass forms with eigenvalue λ_s under $\Delta_{k+\frac{1}{2}}$ to weight $2k$ local Maass forms with eigenvalue $4\lambda_s$ under Δ_{2k}. In particular, the image of the D-th Poincaré series under the theta lift Φ_k equals*

$$\Phi_k\left(P_{k+\frac{1}{2},s,D}\right) = f_{k,s,D}.$$

Remark. The function $f_{k,s,D}$ is continuous for every $\mathrm{Re}(s) \geq \frac{k}{2} + \frac{1}{4}$, but whenever $\lambda_s \neq 0$ there exist points along E_D along which $f_{k,s,D}$ is not differentiable. In particular, one should note the astonishing fact that while the functions are not differentiable for $\lambda_s \neq 0$, the case $\lambda_s = 0$ yields a (holomorphic) cusp form by (1.5).

We now investigate the general properties of the theta lift. Let T_p and T_p^2 denote the Hecke operators of integral and half-integral weight, respectively (see (2.3) and (2.4)). We next show that the theta lift commutes with the Hecke operators.

Theorem 1.2.

(1) *For every weight $k + \frac{1}{2}$ weak Maass form H with eigenvalue λ_s with $\mathrm{Re}(s) \geq \frac{k}{2} + \frac{1}{4}$*

$$\Phi_k(H)\Big|_{2k} T_p = \Phi_k\left(H\Big|_{k+\frac{1}{2}} T_{p^2}\right).$$

(2) *If Re*$(s) \geq \frac{k}{2}+\frac{1}{4}$ *and* $s \neq \frac{k}{2}+\frac{1}{4}$, *then the lift* Φ_k *is injective on the space of weak Maass forms with eigenvalue* λ_s *under* $\Delta_{k+\frac{1}{2}}$.

We next describe a theta lift which parallels the construction of Shintani [27] and Niwa [24] in negative weight. Define the following theta function

$$\Theta^*(z,\tau) := v^k \sum_{\substack{D\in\mathbb{Z}\\ Q\in\mathcal{Q}_D}} Q_z Q(z,1)^{k-1} e^{-\frac{4\pi|Q(z,1)|^2 v}{y^2}} e^{-2\pi i D\tau}. \tag{1.6}$$

The function Θ^* transforms like a modular form of weight $\frac{3}{2}-k$ in τ and weight $2-2k$ in z (see Proposition 3.2 (2)). Similar to the positive weight case, for a weak Maass form H of weight $\frac{3}{2}-k$, we define the theta lift by

$$\Phi^*_{1-k}(H)(z) := \langle H, \Theta^*(-\bar{z},\cdot)\rangle^{\mathrm{reg}}.$$

Since the space of weak Maass forms is spanned by the Poincaré series $P_{\frac{3}{2}-k,s,D}$ (defined in (2.12)), it suffices to consider their image under the theta lifting. This leads to the definition

$$\mathcal{F}_{1-k,s,D}(z) := \sum_{Q\in\mathcal{Q}_D} \operatorname{sgn}(Q_z)\, Q(z,1)^{k-1} \varphi^*_s\left(\frac{Dy^2}{|Q(z,1)|^2}\right), \tag{1.7}$$

where, for $0 < w \leq 1$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s) \geq \frac{k}{2}-\frac{3}{4}$, we define

$$\varphi^*_s(w) := \frac{\Gamma\left(s+\frac{k}{2}-\frac{1}{4}\right)(4\pi D)^{\frac{3}{4}-\frac{k}{2}}}{12\sqrt{\pi}\Gamma(2s)} w^{\frac{k}{2}-\frac{3}{4}+s} {}_2F_1\left(s-\frac{k}{2}+\frac{1}{4}, s+\frac{k}{2}-\frac{3}{4}; 2s; w\right).$$

The Euler integral representation (4.3) again implies that

$$\varphi^*_s(w) = \frac{\Gamma\left(s+\frac{k}{2}-\frac{1}{4}\right)(4\pi D)^{\frac{3}{4}-\frac{k}{2}}}{12\sqrt{\pi}\Gamma\left(s+\frac{k}{2}-\frac{3}{4}\right)\Gamma\left(s-\frac{k}{2}+\frac{3}{4}\right)} w^{\frac{k}{2}-\frac{3}{4}+s} \int_0^1 t^{s+\frac{k}{2}-\frac{7}{4}}(1-t)^{s-\frac{k}{2}-\frac{1}{4}}(1-wt)^{-s+\frac{k}{2}-\frac{1}{4}}\, dt.$$

In the special case that $s=\frac{k}{2}+\frac{1}{4}$, a change of variables yields the locally harmonic Maass form

$$\mathcal{F}_{1-k,D}(z) := \frac{1}{12\psi(1)}(4\pi D)^{\frac{3}{4}-\frac{k}{2}} \sum_{Q\in\mathcal{Q}_D} \operatorname{sgn}(Q_z)\, Q(z,1)^{k-1}\, \psi\left(\frac{Dy^2}{|Q(z,1)|^2}\right),$$

investigated in [7]. Here

$$\psi(v) := \frac{1}{2}\beta\left(v; k-\frac{1}{2}, \frac{1}{2}\right)$$

is a special value of the incomplete β-function, which is defined for $r,s\in\mathbb{C}$ satisfying $\mathrm{Re}(r)$, $\mathrm{Re}(s) > 0$ by $\beta(v;s,r) := \int_0^v u^{s-1}(1-u)^{r-1}\,du$. In [7], the first two authors and Kohnen introduced the functions $\mathcal{F}_{1-k,D}$ and showed that they transform like weight $2-2k$ modular forms and are locally harmonic in every neighborhood of $\mathbb{H}$ which does not intersect E_D. More generally, the functions $\mathcal{F}_{1-k,s,D}$ are local Maass forms with exceptional set E_D.

Theorem 1.3. *Suppose that* k *is even,* $D>0$ *is a discriminant, and* $s\in\mathbb{C}$ *satisfies Re*$(s) \geq \frac{k}{2}-\frac{3}{4}$. *Then the following hold.*

(1) *The function* $\mathcal{F}_{1-k,s,D}$ *is a local Maass form of weight* $2-2k$ *with eigenvalue* $4\lambda_s$ *under* Δ_{2-2k} *and exceptional set* E_D.

(2) *The theta lift Φ^*_{1-k} maps weight $\frac{3}{2}-k$ weak Maass forms with eigenvalue λ_s under $\Delta_{\frac{3}{2}-k}$ to weight $2-2k$ local Maass forms with eigenvalue $4\lambda_s$ under Δ_{2-2k}. In particular, the image of $P_{\frac{3}{2}-k,s,D}$ under the theta lift is*

$$\Phi^*_{1-k}\left(P_{\frac{3}{2}-k,s,D}\right) = \mathcal{F}_{1-k,s,D}. \tag{1.8}$$

Remarks.

(1) The functions $\mathcal{F}_{1-k,s,D}$ are never continuous. That is to say, for every s and D satisfying the conditions of Theorem 1.3, there exist points along E_D for which $\mathcal{F}_{1-k,s,D}$ exhibits discontinuities.
(2) Although $\mathcal{F}_{1-k,D}$ is never continuous, one may add a piecewise polynomial function to obtain a real analytic function. The polynomial in question is related to the period polynomial of $f_{k,D}$ and was thoroughly investigated in [7].
(3) In the omitted case $k=1$ and $\lambda_s=0$, Hövel [20] has constructed locally harmonic Maass forms via a theta lift. The relationship with the Shimura and Shintani lifts as well as its geometric interpretation were further investigated there.
(4) The regularized theta lifts considered here should also have a geometric interpretation. One expects that their images represent cohomology classes of geodesic cycles as currents.

We again turn to the general properties of this theta lift. In particular, it also commutes with the Hecke operators.

Theorem 1.4. *Suppose that $s\in\mathbb{C}$ satisfies $Re(s)\geq \frac{k}{2}-\frac{3}{4}$. The following hold.*

(1) *For every weak Maass form H of weight $\frac{3}{2}-k$ in Kohnen's plus space with eigenvalue λ_s under $\Delta_{\frac{3}{2}-k}$, one has*

$$\Phi^*_{1-k}(H)\Big|_{2-2k} T_p = \Phi^*_{1-k}\left(H\Big|_{\frac{3}{2}-k} T_{p^2}\right). \tag{1.9}$$

(2) *The lift Φ^*_{1-k} is injective on the space of weak Maass forms with eigenvalue λ_s under $\Delta_{\frac{3}{2}-k}$.*

Remark. In [7], it was shown that the functions $\mathcal{F}_{1-k,D}$ satisfy relations under the Hecke operators which seemed to imply a natural connection to weight $\frac{3}{2}-k$ objects. This is explained by the relation (1.9) between integral and half-integral weight Hecke operators.

The images $\mathcal{F}_{1-k,s,D}$ and $f_{k,s,D}$ under the two theta lifts considered in this paper are related through the antiholomorphic differential operator $\xi_\kappa := 2iy^\kappa\overline{\frac{\partial}{\partial\overline{z}}}$.

Theorem 1.5. *Suppose that $k>0$ is an even integer, D is a positive discriminant, and $s\in\mathbb{C}$ satisfies $Re(s)\geq\frac{k}{2}+\frac{1}{4}$.*

(1) *For every $z\notin E_D$, we have that*

$$\xi_{2-2k}\left(\mathcal{F}_{1-k,s,D}(z)\right) = 2\left(\overline{s}-\frac{3}{4}+\frac{k}{2}\right) f_{k,\overline{s},D}(z). \tag{1.10}$$

(2) *For $z\notin E_D$, we have that*

$$\xi_{2k}\left(f_{k,s,D}(z)\right) = 2\left(\overline{s}-\frac{k}{2}-\frac{1}{4}\right)\mathcal{F}_{1-k,\overline{s},D}(z). \tag{1.11}$$

Theorem 1.5 states that for $s \geq \frac{k}{2} + \frac{1}{4}$ the following commutative diagram holds:

$$\begin{array}{ccc}
P_{\frac{3}{2}-k,s,D} & \xrightarrow{\Phi_{1-k}^{*}} & \mathcal{F}_{1-k,s,D} \\
\Big\downarrow {\scriptstyle \xi_{\frac{3}{2}-k}} & & \Big\downarrow {\scriptstyle \xi_{2-2k}} \\
\left(\overline{s} - \frac{3}{4} + \frac{k}{2}\right) P_{k+\frac{1}{2},\overline{s},D} & \xrightarrow{2\Phi_k} & 2\left(\overline{s} - \frac{3}{4} + \frac{k}{2}\right) f_{k,\overline{s},D} \\
\Big\downarrow {\scriptstyle \xi_{k+\frac{1}{2}}} & & \Big\downarrow {\scriptstyle \xi_{2k}} \\
-\lambda_s P_{\frac{3}{2}-k,s,D} & \xrightarrow{4\Phi_{1-k}^{*}} & -4\lambda_s \mathcal{F}_{1-k,s,D}
\end{array}$$

Denote the d-th Shimura [26] lift by $\mathscr{S}_d$ and $P_{\kappa,D} := P_{\kappa,\frac{k}{2}+\frac{1}{4},D}$. In the special case that $s = \frac{k}{2} + \frac{1}{4}$ (see Corollary 9 of [23] for the constant multiple of $\mathscr{S}_1$), the diagram becomes the following:

$$\begin{array}{ccc}
P_{\frac{3}{2}-k,D} & \xrightarrow{\Phi_{1-k}^{*}} & \mathcal{F}_{1-k,D} \\
\Big\downarrow {\scriptstyle \xi_{\frac{3}{2}-k}} & & \Big\downarrow {\scriptstyle \xi_{2-2k}} \\
\left(k - \frac{1}{2}\right) P_{k+\frac{1}{2},D} & \xrightarrow[3^{-1}2^{-k}\mathscr{S}_1]{2\Phi_k} & \frac{2^{2k-3}}{3} \left(4\pi D\right)^{\frac{3}{4}-\frac{k}{2}} f_{k,D}
\end{array}$$

Remarks.

(1) The above diagram extends work of Bruinier and Funke [13] and Hövel [20] in the case of $O(2,1)$ to higher weight.

(2) By applying (6.1) (used to obtain (1.10)) to s-derivatives of weak Maass forms, one could also obtain links between modular objects known as sesquiharmonic forms [5]. These functions map to weakly holomorphic modular forms under the hyperbolic Laplacian.

The paper is organized as follows. In Section 2, we recall the theory of weak Maass forms and give a formal definition of local Maass forms. Section 2.1 is devoted to the properties of the regularized inner product. The modularity properties of the theta functions are enunciated in Section 3, where we derive a number of interrelations between the theta functions through differential operators. The image of Φ_k (Theorem 1.1 (2)) is determined in Section 4, while Section 5 is devoted to the image of Φ_{1-k}^{*} (Theorem 1.3 (2)) and the injectivity of the lift (Theorem 1.4 (2)). In Section 6, Theorem 1.5 is established and the relationship between $f_{k,s,D}$ and $\mathcal{F}_{1-k,s,D}$ is then used to conclude Theorems 1.1 (1) and 1.3 (1). Finally, Section 7 concludes the paper with a discussion of the Hecke operators and the injectivity of Φ_k (Theorems 1.2 and 1.4 (1)).

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2. Basic facts on weak and local Maass forms

In this section, we recall the basic definitions necessary to describe the modular objects and the theta lifts used in this paper. We first define the regularized inner product used in the definitions of Φ and Φ^*. In order to understand the relationship between lifts in different spaces, we then define the Hecke operators, which act formally on any translation invariant function. We then recall Kohnen's plus space and weak Maass forms, upon which we apply our theta lifts. The next subsection is devoted to constructing Poincaré series which span these spaces of weak Maass forms. Following this, we give the definition of local Maass forms, which are the focus of this paper.

Thoughout this section, $\kappa \in \frac{1}{2}\mathbb{Z}$ and we set $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ whenever $\kappa \in \mathbb{Z}$, while $\Gamma := \Gamma_0(4)$ if $\kappa \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$.

2.1. Regularized inner products and Hecke operators.

For $T > 0$, denote the truncated fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ by

$$\mathcal{F}_T := \left\{ \tau \in \mathbb{H} : |u| \leq \frac{1}{2}, |\tau| \geq 1, v \leq T \right\}. \tag{2.1}$$

For a finite index subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ we further define

$$\mathcal{F}_T(\Gamma) := \bigcup_{\gamma \in \Gamma \setminus \mathrm{SL}_2(\mathbb{Z})} \gamma \mathcal{F}_T.$$

In particular, we set $\mathcal{F}_T(4) := \mathcal{F}_T(\Gamma_0(4))$. For two functions G and H satisfying weight κ modularity for the group Γ, we define, whenever the limit exists, the regularized inner product

$$\langle G, H \rangle^{\mathrm{reg}} := \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]} \lim_{T \to \infty} \int_{\mathcal{F}_T(\Gamma)} G(\tau) \overline{H(\tau)} v^{\kappa} \frac{dudv}{v^2}.$$

We use the following lemma, which follows by standard arguments using Stokes's Theorem.

Lemma 2.1. *Suppose that F, $G : \mathbb{H} \to \mathbb{C}$ are real analytic functions that satisfy $F|_{2-\kappa}\gamma = F$ and $G|_{\kappa}\gamma = G$ for all $\gamma \in \Gamma$. Then*

$$\int_{\mathcal{F}_T(\Gamma)} \xi_{2-\kappa}\left(F(\tau)\right) \overline{G(\tau)}\, v^{\kappa-2}\, du\, dv + \int_{\mathcal{F}_T(\Gamma)} \xi_{\kappa}\left(G(\tau)\right) \overline{F(\tau)}\, v^{-\kappa}\, du\, dv = -\int_{\partial \mathcal{F}_T(\Gamma)} \overline{F(\tau) G(\tau)}\, d\overline{\tau}. \tag{2.2}$$

A number of important operators are Hermitian with respect to the regularized inner product. One such class of operators is the *Hecke operators*. Suppose that F is a function satisfying weight κ modularity and write its Fourier expansion as

$$F(\tau) = \sum_{n \in \mathbb{Z}} a_v(n) e^{2\pi i n u}.$$

If $\kappa \in \mathbb{Z}$ (resp. $\kappa \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$), then for a prime p, the Hecke operator T_p (resp. T_{p^2}) is defined by

$$F\Big|_{\kappa} T_p(\tau) := \sum_{n \in \mathbb{Z}} \left(a_v(pn) + p^{\kappa-1} a_v\left(\frac{n}{p}\right) \right) e^{2\pi i n u}, \tag{2.3}$$

$$F\Big|_{\kappa} T_{p^2}(\tau) := \sum_{n \in \mathbb{Z}} \left(a_v\left(p^2 n\right) + p^{\kappa-\frac{3}{2}} \left(\frac{(-1)^{\kappa-\frac{1}{2}} n}{p} \right) a_v(n) + p^{2\kappa-2} a_v\left(\frac{n}{p^2}\right) \right) e^{2\pi i n u}. \tag{2.4}$$

We apply the regularized inner product to (half-integral weight) weak Maass forms, which we define in the following subsection.

2.2. **Weak Maass forms.** When $\kappa \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}$, we are interested in weight κ real analytic modular forms on Γ in Kohnen's plus space. This means that the Fourier expansions are supported on the coefficients n satisfying $(-1)^{\kappa-\frac{1}{2}}n \equiv 0, 1 \pmod 4$. We use pr to denote the projection operator (see Section 2.3 of [22]) into Kohnen's plus space. It is useful to recall that if F is modular in Kohnen's plus space for Γ, then its Fourier expansions at the cusps 0 and $\frac{1}{2}$ are determined by the expansion at $i\infty$ (see [22] for a proof in the holomorphic case). Like the Hecke operators, the projection operator pr is Hermitian with respect to the regularized inner product, i.e.,

$$\left\langle G \middle| \mathrm{pr}, H \right\rangle^{\mathrm{reg}} = \left\langle G, H \middle| \mathrm{pr} \right\rangle^{\mathrm{reg}}. \tag{2.5}$$

The real analytic modular forms of particular interest for this paper are weak Maass forms. A good background reference for weak Maass forms is [13]. Recall that we write $\tau = u + iv$ throughout. For $\kappa \in \frac{1}{2}\mathbb{Z}$, the weight κ *hyperbolic Laplacian* is defined by

$$\Delta_\kappa := \Delta_{\kappa,\tau} := -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + i\kappa v\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

It is related to the operator $\xi_\kappa = \xi_{\kappa,\tau} := 2iv^\kappa \overline{\frac{\partial}{\partial\overline{\tau}}}$ through

$$\Delta_\kappa = -\xi_{2-\kappa} \circ \xi_\kappa.$$

In order to define weak Maass forms, we require

$$\mathcal{M}_{\kappa,s}(t) := |t|^{-\frac{\kappa}{2}} M_{\frac{\kappa}{2}\operatorname{sgn}(t),\, s-\frac{1}{2}}(|t|), \tag{2.6}$$

where $M_{\mu,s-\frac{1}{2}}$ is the usual M-Whittaker function. For $\mathrm{Re}(s \pm \mu) > 0$ and $v > 0$, we have the integral representation

$$M_{\mu,s-\frac{1}{2}}(v) = v^s e^{\frac{v}{2}} \frac{\Gamma(2s)}{\Gamma(s+\mu)\Gamma(s-\mu)} \int_0^1 t^{s+\mu-1}(1-t)^{s-\mu-1}e^{-vt}dt. \tag{2.7}$$

In the special case that $\mu = s$, we have

$$M_{\mu,s-\frac{1}{2}}(v) = e^{-\frac{v}{2}}v^s. \tag{2.8}$$

Furthermore, as $v \to \infty$, the Whittaker function satisfies the following asymptotic behavior for $\mu \neq s$:

$$M_{\mu,s-\frac{1}{2}}(v) \sim \frac{\Gamma(2s)}{\Gamma(s-\mu)} e^{\frac{v}{2}} v^{-\mu}. \tag{2.9}$$

We move on to the definition of weak Maass forms. For $s \in \mathbb{C}$ a *weak Maass form* $F : \mathbb{H} \to \mathbb{C}$ of weight κ for Γ with eigenvalue $\lambda = \left(s - \frac{\kappa}{2}\right)\left(1 - s - \frac{\kappa}{2}\right)$ is a real analytic function satisfying:

(1) For every $\gamma \in \Gamma$, one has $F|_\kappa\gamma = F$, where $|_\kappa$ denotes the usual weight κ slash-operator.
(2) One has $\Delta_\kappa(F) = \lambda F$.
(3) There exist $a_1, \ldots, a_N \in \mathbb{C}$ for which

$$F(\tau) - \sum_{m=1}^{N} a_m \mathcal{M}_{\kappa,s}(4\pi \operatorname{sgn}(\kappa) m v)\, e^{2\pi i m \operatorname{sgn}(\kappa) u} = O\left(v^{1-\mathrm{Re}(s)-\frac{\kappa}{2}}\right).$$

There are analogous conditions at the other cusps of Γ.

2.3. **Poincaré series.** One builds explicit examples of weak Maass forms by constructing Poincaré series [17]. For $m \in \mathbb{Z} \setminus \{0\}$, the function

$$\psi_{m,\kappa}(s;\tau) := (4\pi|m|)^{\frac{\kappa}{2}}\,\Gamma(2s)^{-1}\mathcal{M}_{\kappa,s}(4\pi m v)\,e^{2\pi i m u}$$

is an eigenfunction for Δ_κ with eigenvalue $\left(s-\frac{\kappa}{2}\right)\left(1-s-\frac{\kappa}{2}\right)$. Thus, one concludes that for $\mathrm{Re}(s) > 1$ the Poincaré series

$$P_{\kappa,s,\Gamma,m}(\tau) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \psi_{\mathrm{sgn}(\kappa)m,\kappa}(s;\tau)\Big|_\kappa \gamma, \tag{2.10}$$

where $\Gamma_\infty := \{\pm\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right) : n \in \mathbb{Z}\}$, is also an eigenfunction under Δ_κ with the same eigenvalue. Moreover, the space of weight κ weak Maass forms with this eigenvalue is spanned by such Poincaré series. The Poincaré series satisfies the growth condition

$$P_{\kappa,s,\Gamma,m}(\tau) - \psi_{\mathrm{sgn}(\kappa)m,\kappa}(s;\tau) = O\left(v^{1-\mathrm{Re}(s)-\frac{\kappa}{2}}\right). \tag{2.11}$$

In the case that $\kappa \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, we then project the Poincaré series into Kohnen's plus space, defining

$$P_{\kappa,s,m} := P_{\kappa,s,\Gamma_0(4),m}\Big|\,\mathrm{pr}\,. \tag{2.12}$$

In the special cases that $s = 1-\frac{\kappa}{2}$ or $s = \frac{\kappa}{2}$, the resulting Poincaré series is harmonic. For $D \neq 0$, the positive and negative weight Poincaré series are related to each other via

$$\xi_\kappa\left(P_{\kappa,s,D}\right) = \left(\overline{s} - \frac{\kappa}{2}\right) P_{2-\kappa,\overline{s},D}. \tag{2.13}$$

2.4. **Local Maass forms.** Mirroring the definition of weak Maass forms, for $\kappa \in 2\mathbb{Z}$, $\lambda \in \mathbb{C}$, and a measure zero set E, we call a function $\mathcal{F}$ a weight κ *local Maass form* with eigenvalue λ and exceptional set E if $\mathcal{F}$ satisfies the following:

(1) For every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, one has $\mathcal{F}|_\kappa\gamma = \mathcal{F}$
(2) For every $\tau \notin E$ there exists a neighborhood around τ for which $\mathcal{F}$ is real analytic and
$$\Delta_\kappa(\mathcal{F})(\tau) = \lambda\mathcal{F}(\tau).$$
(3) For $\tau \in E$ one has
$$\mathcal{F}(\tau) = \frac{1}{2}\lim_{r\to 0^+}\left(\mathcal{F}(\tau+ir) + \mathcal{F}(\tau-ir)\right).$$
(4) The function $\mathcal{F}$ exhibits at most polynomial growth as $v \to \infty$.

Examples of locally harmonic Maass forms (those with eigenvalue 0) are given in [7] as "quadratic form Poincaré series." In this paper, we give further examples of local Maass forms via theta lifts.

3. Indefinite theta functions

In this section we collect several important properties of the theta functions (1.2) and (1.6). The modularity properties of these indefinite theta functions follow by a result of Vignéras [29]. To state these, we define the Euler operator $E := \sum_{i=1}^n w_i \frac{\partial}{\partial w_i}$. As usual, we denote the Gram matrix associated to a nondegenerate quadratic form q on $\mathbb{R}^n$ by A. The *Laplacian* associated to q is then defined by $\Delta := \langle \frac{\partial}{\partial w}, A^{-1}\frac{\partial}{\partial w}\rangle$. Here $\langle\cdot,\cdot\rangle$ denotes the usual inner product on $\mathbb{R}^n$.

Theorem 3.1 (Vignéras). *Suppose that $n \in \mathbb{N}$, q is a nondegenerate quadratic form on $\mathbb{R}^n$, $L \subset \mathbb{R}^n$ is a lattice on which q takes integer values, and $p : \mathbb{R}^n \to \mathbb{C}$ is a function satisfying the following conditions:*

(i) *The function* $f(w) := p(w)e^{-2\pi q(w)}$ *times any polynomial of degree at most 2 and all partial derivatives of* f *of order at most 2 are elements of* $L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$.
(ii) *For some* $\lambda \in \mathbb{Z}$, *the function* p *satisfies*

$$\left(E - \frac{\Delta}{4\pi}\right) p = \lambda p.$$

Then the indefinite theta function

$$v^{-\frac{\lambda}{2}} \sum_{w \in L} p\left(w\sqrt{v}\right) e^{2\pi i q(w)\tau}$$

is modular of weight $\lambda + \frac{n}{2}$ *for* $\Gamma_0(N)$ *and character* $\chi \cdot \chi_{-4}^{\lambda}$, *where* N *and* χ *are the level and character of* q *and* χ_{-4} *is the unique primitive Dirichlet character of conductor* 4.

Remark. Note that the definition of the character given in Vignéras [29] differs to that given by Shimura [26] by a factor of χ_{-4}^{λ}. We adopt Shimura's notation here.

Applying Theorem 3.1 to Θ and Θ^* yields their modularity properties (see [8] for details).

Proposition 3.2.
(1) *The function* $\Theta(-\overline{z}, \tau)$ *transforms like a modular form of weight* $k + \frac{1}{2}$ *in Kohnen's plus space on* $\Gamma_0(4)$ *in* τ *and weight* $2k$ *on* $\mathrm{SL}_2(\mathbb{Z})$ *in* z.
(2) *The function* Θ^* *transforms like a modular form of weight* $\frac{3}{2} - k$ *in Kohnen's plus space on* $\Gamma_0(4)$ *in* τ *and weight* $2 - 2k$ *on* $\mathrm{SL}_2(\mathbb{Z})$ *in* z.

The following lemma is the key relation needed to establish a link between the functions $f_{k,s,D}$ and $\mathcal{F}_{1-k,s,D}$. The correspondence is formed through a relation betwen the respective differential operators in τ and z on Θ and Θ^*, mirroring an important connection formed in [13].

Lemma 3.3. *For every integer* $k \geq 1$, *one has*

$$\xi_{k+\frac{1}{2},\tau}\left(\Theta(z,\tau)\right) = -iy^{2-2k}\frac{\partial}{\partial z}\Theta^*(-\overline{z},\tau), \tag{3.1}$$

$$\xi_{\frac{3}{2}-k,\tau}\left(\Theta^*(-\overline{z},\tau)\right) = -iy^{2k}\frac{\partial}{\partial z}\Theta(z,\tau). \tag{3.2}$$

Proof: We first prove (3.1). We compute that $\frac{\partial}{\partial z}\Theta^*(-\overline{z},\tau)$ equals

$$\sum_{\substack{D\in\mathbb{Z}\\ Q\in\mathcal{Q}_D}} Q(-\overline{z},1)^{k-1} e^{-\frac{4\pi v}{y^2}|Q(-\overline{z},1)|^2} e^{-2\pi i D\tau}\left(\frac{\partial}{\partial z}Q_{-\overline{z}} - 4\pi Q_{-\overline{z}} v \frac{\partial}{\partial z}\left(\frac{|Q(-\overline{z},1)|^2}{y^2}\right)\right).$$

We then use

$$|Q(z,1)|^2 = Q_z^2 y^2 + D y^2 \tag{3.3}$$

and

$$y^2\frac{\partial}{\partial z}Q_{-\overline{z}} = \frac{i}{2}Q(-\overline{z},1)$$

to obtain

$$-iy^{2-2k}\frac{\partial}{\partial z}\Theta^*(-\overline{z},\tau) = \frac{1}{2}y^{-2k}v^k \sum_{\substack{D\in\mathbb{Z}\\ Q\in\mathcal{Q}_D}} Q(-\overline{z},1)^k e^{-4\pi Q_{-\overline{z}}^2 v} e^{-2\pi i D\overline{\tau}}\left(1 - 8\pi Q_{-\overline{z}}^2 v\right).$$

We similarly compute the action of $\xi_{k+\frac{1}{2},\tau}$ on Θ. A straightforward calculation yields

$$\xi_{k+\frac{1}{2},\tau}\left(\Theta(z,\tau)\right) = \frac{1}{2}y^{-2k}v^k \sum_{\substack{D\in\mathbb{Z}\\ Q\in\mathcal{Q}_D}} Q(\overline{z},1)^k e^{-4\pi Q_z^2 v} e^{-2\pi i D\overline{\tau}}\left(1 - 8\pi Q_z^2 v\right).$$

Equation (3.1) now follows immediately by the change of variables $Q = [a, b, c] \to [a, -b, c] =: \widetilde{Q} \in \mathcal{Q}_D$, noting that

$$\widetilde{Q}\left(\overline{z}, 1\right) = Q\left(-\overline{z}, 1\right) \qquad \text{and} \qquad \widetilde{Q}_z = Q_{-\overline{z}}. \tag{3.4}$$

We move on to proving (3.2). Since $Q_z \in \mathbb{R}$, a direct calculation, mirroring the proof of (3.1) and using (3.4), yields

$$\xi_{\frac{3}{2}-k,\tau}\left(\Theta^*(-\overline{z},\tau)\right) = v^{\frac{1}{2}} \sum_{\substack{D\in\mathbb{Z}\\ Q\in\mathcal{Q}_D}} Q_z Q(z,1)^{k-1} e^{-4\pi Q_z^2 v} e^{2\pi i D\tau} \left(k - \frac{4\pi v}{y^2}\left|Q(z,1)\right|^2\right).$$

We next obtain (3.2) by showing that $-iy^{2k}\frac{\partial}{\partial z}\Theta\left(z,\tau\right)$ equals

$$- iv^{\frac{1}{2}} y^2 \sum_{\substack{D\in\mathbb{Z}\\ Q\in\mathcal{Q}_D}} Q(z,1)^{k-1} e^{-4\pi Q_z^2 v} e^{2\pi i D\tau} \left(k\frac{\partial}{\partial z}\left(y^{-2}Q\left(z,1\right)\right) - 8\pi Q_z y^{-2} Q\left(z,1\right) v \frac{\partial}{\partial z} Q_z\right)$$

$$= v^{\frac{1}{2}} \sum_{\substack{D\in\mathbb{Z}\\ Q\in\mathcal{Q}_D}} Q_z Q(z,1)^{k-1} e^{-4\pi Q_z^2 v} e^{2\pi i D\tau} \left(k - \frac{4\pi v}{y^2}\left|Q(z,1)\right|^2\right),$$

where in the last line we have used

$$y^2 \frac{\partial}{\partial z} Q_z = \frac{i}{2} Q(\overline{z},1) \qquad \text{and} \qquad y^2\frac{\partial}{\partial z}\left(y^{-2}Q\left(z,1\right)\right) = iQ_z.$$

□

The following lemma relates the regularized inner products in positive and negative weight through the ξ-operator.

Lemma 3.4. *Suppose that $D > 0$ is a discriminant and $z \notin E_D$. Then for every s with $Re(s) \geq \frac{k}{2} + \frac{1}{4}$ one has*

$$\left\langle \xi_{k+\frac{1}{2}}\left(P_{k+\frac{1}{2},s,D}\right), \Theta^*\left(-\overline{z},\cdot\right)\right\rangle^{reg} = -\overline{\left\langle P_{k+\frac{1}{2},s,D}, \xi_{\frac{3}{2}-k}\left(\Theta^*\left(-\overline{z},\cdot\right)\right)\right\rangle^{reg}} \tag{3.5}$$

and

$$\left\langle \xi_{\frac{3}{2}-k}\left(P_{\frac{3}{2}-k,s,D}\right), \Theta(z,\cdot)\right\rangle^{reg} = -\overline{\left\langle P_{\frac{3}{2}-k,s,D}, \xi_{k+\frac{1}{2}}\left(\Theta(z,\cdot)\right)\right\rangle^{reg}}. \tag{3.6}$$

Proof: Note that all of the regularized integrals exist, as will be shown in the proofs of Theorem 1.1 (2) and 1.3 (2). We begin with the proof of (3.5) and abbreviate $P := P_{k+\frac{1}{2},s,D}$. By Lemma 2.1, we have

$$\left\langle \xi_{k+\frac{1}{2}}\left(P\right), \Theta^*\left(-\overline{z},\cdot\right)\right\rangle^{\mathrm{reg}} + \overline{\left\langle P, \xi_{\frac{3}{2}-k}\left(\Theta^*\left(-\overline{z},\cdot\right)\right)\right\rangle^{\mathrm{reg}}} = -\frac{1}{6}\lim_{T\to\infty}\int_{\partial\mathcal{F}_T(4)} \overline{P(\tau)\Theta^*\left(-\overline{z},\tau\right)}d\overline{\tau},$$

provided that the limit exists. Hence our goal is to show that the limit on the right hand side is zero. A standard argument reduces this claim to showing that

$$\lim_{T\to\infty}\int_0^1 P(u+iT)\Theta^*\left(-\overline{z},u+iT\right)du = 0 \tag{3.7}$$

as well as vanishing of similar integrals around the other cusps of $\Gamma_0(4)$. However, since both P and Θ^* are in Kohnen's plus space, the vanishing of the corresponding integrals at the other cusps may be reduced to showing that (3.7) vanishes.

In order to prove (3.7), we first recall the growth condition (2.11) and note that $\Theta^*(-\overline{z}, u+iT)$ decays exponentially as $T \to \infty$. Indeed, using (3.3), one can show that for fixed $z \in \mathbb{H}$ the quadratic form

$$Q^*(a,b,c) := D - \frac{2|Q(z,1)|^2}{y^2} = -D + 2Q_z^2$$

is positive definite on the lattice of all binary quadratic forms $Q = [a,b,c] \in \mathcal{Q}_D$. After evaluating the integral over u, one reduces (3.7) to showing that

$$\lim_{T\to\infty} R_T = 0,$$

where

$$R_T := \mathcal{M}_{k+\frac{1}{2},s}(4\pi DT)\, T^k \sum_{Q\in\mathcal{Q}_D} Q_{-\overline{z}} Q(-\overline{z},1)^{k-1} e^{-\frac{4\pi|Q(-\overline{z},1)|^2 T}{y^2}} e^{2\pi DT}.$$

However, the asymptotic behavior for the Whittaker function coming from (2.8) and (2.9) yields

$$\mathcal{M}_{k+\frac{1}{2},s}(4\pi DT) \ll_{k,s,D} e^{2\pi DT} T^{-k-\frac{1}{2}}.$$

Using (3.3), we may hence bound

$$R_T \ll_{k,s,D} T^{-\frac{1}{2}} \sum_{Q\in\mathcal{Q}_D} Q_{-\overline{z}} Q(-\overline{z},1)^{k-1} e^{-4\pi Q_{-\overline{z}}^2 T}.$$

Since $z \notin E_D$ (and hence $-\overline{z} \notin E_D$), $Q_{-\overline{z}}^2 > 0$ for every $Q \in \mathcal{Q}_D$ and hence R_T exhibits exponential decay as $T \to \infty$. This concludes (3.7), yielding (3.5). The proof of (3.6) follows analogously.

□

4. Image of the theta lift Φ_k

In this section, we introduce a spectral parameter in the classical Shintani lift.

Proof of Theorem 1.1 (2): In order to compute the regularized inner product, we use a method of Zagier [32]. He defined a regularization which he used for functions which grow at most polynomially, but the method may be extended to the functions of interest here, as we now describe. We first define

$$\mathbb{H}_T := \bigcup_{\gamma\in\mathrm{SL}_2(\mathbb{Z})} \gamma\mathcal{F}_T = \bigcup_{\gamma\in\Gamma_0(4)} \gamma\mathcal{F}_T(4).$$

We first use (2.5) together with the fact that $\Theta = \Theta|\,\mathrm{pr}$ to compute

$$\left\langle P_{k+\frac{1}{2},s,D}, \Theta(z,\cdot)\right\rangle^{\mathrm{reg}} = \left\langle P_{k+\frac{1}{2},s,\Gamma_0(4),D}\middle|\,\mathrm{pr}, \Theta(z,\cdot)\right\rangle^{\mathrm{reg}} = \left\langle P_{k+\frac{1}{2},s,\Gamma_0(4),D}, \Theta(z,\cdot)\right\rangle^{\mathrm{reg}}. \tag{4.1}$$

Then the usual unfolding argument yields

$$\left\langle P_{k+\frac{1}{2},s,D}, \Theta(z,\cdot)\right\rangle^{\mathrm{reg}} = \frac{1}{6}\lim_{T\to\infty}\int_{\Gamma_\infty\backslash\mathbb{H}_T} \psi_{D,k+\frac{1}{2}}(s;\tau)\,\overline{\Theta(z,\tau)} v^{k+\frac{1}{2}} \frac{du dv}{v^2}.$$

We now rewrite

$$\mathbb{H}_T = \left\{\tau\in\mathbb{H}\middle|\mathrm{Im}(\tau)\le T\right\} \setminus \bigcup_{\substack{c\ge1\\ a\in\mathbb{Z}\\ (a,c)=1}} S_{\frac{a}{c}}(T),$$

where $S_{\frac{a}{c}}(T)$ is the disc of radius $\frac{1}{2c^2T}$ tangent to the real axis at $\frac{a}{c}$. Hence, we have

$$\left\langle P_{k+\frac{1}{2},s,D}, \Theta(z,\cdot)\right\rangle^{\mathrm{reg}} = \lim_{T\to\infty}\left(I_1(T) + I_2(T)\right),$$

where

$$I_1(T) := \frac{1}{6}\int_0^T \int_0^1 \psi_{D,k+\frac{1}{2}}(s;\tau)\,\overline{\Theta(z,\tau)}v^{k+\frac{1}{2}}\frac{dudv}{v^2},$$
$$I_2(T) := -\frac{1}{6}\sum_{c\geq 1}\sum_{\substack{a \pmod c\\ (a,c)=1}} \int_{S_{\frac{a}{c}}(T)} \psi_{D,k+\frac{1}{2}}(s;\tau)\,\overline{\Theta(z,\tau)}v^{k+\frac{1}{2}}\frac{dudv}{v^2}.$$

We first consider $I_1(T)$. Evaluating the integral over u and using (3.3), we obtain

$$\text{(4.2)}\quad \lim_{T\to\infty} I_1(T) = \frac{(4\pi D)^{\frac{k}{2}+\frac{1}{4}}}{6y^{2k}\Gamma(2s)}\sum_{Q\in Q_D} Q(\overline{z},1)^k \int_0^\infty e^{-2\pi D v}\mathcal{M}_{k+\frac{1}{2},s}(4\pi D v)e^{-4\pi Q_z^2 v}v^{k-1}dv$$
$$= \frac{(4\pi D)^{\frac{1}{4}-\frac{k}{2}}}{6y^{2k}\Gamma(2s)}\sum_{Q\in\mathcal{Q}_D} Q(\overline{z},1)^k\, I\left(\frac{Dy^2}{|Q(\tau,1)|^2}\right),$$

where for $0<w<1$ we define

$$I(w) := \int_0^\infty \mathcal{M}_{k+\frac{1}{2},s}(v)e^{\frac{v}{2}}e^{-vw^{-1}}v^{k-1}dv.$$

In the case that $s\neq \frac{k}{2}+\frac{1}{4}$, we insert the definition (2.6) of $\mathcal{M}_{k+\frac{1}{2},s}(v)$ and then substitute the integral representation (2.7) of the M-Whittaker function when $\mathrm{Re}(s)>\frac{k}{2}$. The change of variables $t\to 1-t$ yields

$$I(w) = \frac{\Gamma(2s)}{\Gamma\left(s+\frac{k}{2}+\frac{1}{4}\right)\Gamma\left(s-\frac{k}{2}-\frac{1}{4}\right)}\int_0^1 (1-t)^{s+\frac{k}{2}-\frac{3}{4}}t^{s-\frac{k}{2}-\frac{5}{4}}\int_0^\infty v^{s+\frac{k}{2}-\frac{5}{4}}e^{-v\left(w^{-1}-t\right)}dvdt$$
$$= \frac{\Gamma(2s)\Gamma\left(s+\frac{k}{2}-\frac{1}{4}\right)w^{s+\frac{k}{2}-\frac{1}{4}}}{\Gamma\left(s+\frac{k}{2}+\frac{1}{4}\right)\Gamma\left(s-\frac{k}{2}-\frac{1}{4}\right)}\int_0^1 (1-t)^{s+\frac{k}{2}-\frac{3}{4}}t^{s-\frac{k}{2}-\frac{5}{4}}(1-wt)^{-\frac{k}{2}-s+\frac{1}{4}}dt.$$

We then rewrite the integral using the Euler integral representation for ${}_2F_1$ (see (15.3.1) in [1]), given for $\mathrm{Re}(C)>\mathrm{Re}(B)>0$ and $|w|<1$ by

$$\text{(4.3)}\quad {}_2F_1(A,B;C;w) = \frac{\Gamma(C)}{\Gamma(B)\Gamma(C-B)}\int_0^1 t^{B-1}(1-t)^{C-B-1}(1-wt)^{-A}dt.$$

Thus

$$I(w) = \Gamma\left(s+\frac{k}{2}-\frac{1}{4}\right)w^{s+\frac{k}{2}-\frac{1}{4}}{}_2F_1\left(s+\frac{k}{2}-\frac{1}{4}, s-\frac{k}{2}-\frac{1}{4}; 2s; w\right).$$

Inserting this into (4.2) shows that $\lim_{T\to\infty} I_1(T) = f_{k,s,D}$.

For $s=\frac{k}{2}+\frac{1}{4}$, inserting (2.8) into (4.2) yields

$$\text{(4.4)}\quad \lim_{T\to\infty} I_1(T) = \frac{D^{\frac{k}{2}+\frac{1}{4}}\Gamma(k)}{6(4\pi)^{\frac{k}{2}-\frac{1}{4}}\Gamma\left(k+\frac{1}{2}\right)}\sum_{Q\in\mathcal{Q}_D} Q(z,1)^{-k} = f_{k,\frac{k}{2}+\frac{1}{4},D}(z).$$

To conclude (1.8), it remains to show that $I_2(T)$ vanishes as $T\to\infty$. We first assume that $4\mid c$ and choose $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_0(4)$. A direct calculation shows that

$$\gamma S_{\frac{a}{c}} = \left\{\tau\in\mathbb{H}\middle| v\geq T\right\}.$$

Hence, the change of variables $\tau \to \gamma\tau$, together with the modularity of Θ in Proposition 3.2, yields

$$I_2(T) = -\frac{1}{6}\int_T^\infty \int_{-\infty}^\infty \overline{\Theta\left(-\overline{z}, \tau\right)} \left(c\overline{\tau} + d\right)^{k+\frac{1}{2}} \psi_{D,k+\frac{1}{2}}\left(s; \gamma\tau\right) \operatorname{Im}\left(\gamma\tau\right)^{k+\frac{1}{2}} \frac{dudv}{v^2}.$$

Using the facts that $\operatorname{Im}\left(\gamma\tau\right) = \frac{v}{|c\tau+d|^2}$ and Θ is translation invariant, the integral may be rewritten as

$$-\frac{1}{6}\int_T^\infty \int_0^1 \overline{\Theta\left(-\overline{z}, \tau\right)} \sum_{n=-\infty}^{\infty} \psi_{D,k+\frac{1}{2}}\left(s; \tau\right)\Big|_{k+\frac{1}{2}} \gamma \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} v^{k+\frac{1}{2}} \frac{dudv}{v^2}.$$

Taking the sum over all a, c with $4 \mid c > 0$, the inner sum precisely evaluates as

$$P_{k+\frac{1}{2},s,\Gamma_0(4),D} - \psi_{D,k+\frac{1}{2}}\left(s; \tau\right).$$

Comparing the polynomial growth in (2.11) with the exponential decay of $\Theta\left(-\overline{z}, \tau\right)$ towards $i\infty$, one concludes that the limit $T \to \infty$ vanishes. A similar argument shows that the contribution to $I_2(T)$ coming from $4 \nmid c$ also vanishes as $T \to \infty$. This yields (1.8).

$\square$

5. Image of the theta lift Φ^*_{1-k}

We next compute the image of Φ^*_{1-k} with the method from Section 4.

Proof of Theorem 1.3 (2): Following the argument in the proof of Theorem 1.1 (2), we may reduce the theorem to evaluating

$$(5.1)\quad \frac{1}{6}\lim_{T\to\infty}\int_0^T \int_0^1 \psi_{D,\frac{3}{2}-k}\left(s; \tau\right) \overline{\Theta^*\left(-\overline{z}, \tau\right)} v^{\frac{3}{2}-k} \frac{dudv}{v^2} \\ - \frac{1}{6}\lim_{T\to\infty}\sum_{c\geq 1}\sum_{\substack{a \pmod{c} \\ (a,c)=1}} \int_{S_{\frac{a}{c}}(T)} \psi_{D,\frac{3}{2}-k}\left(s; \tau\right) \overline{\Theta^*\left(-\overline{z}, \tau\right)} v^{\frac{3}{2}-k} \frac{dudv}{v^2}.$$

Using the argument from before, the second summand vanishes. We use (3.3) to rewrite the exponential in the theta series as

$$\left(b^2 - 4ac\right) u + iv \left(2Q_{-\overline{z}}^2 + \left(b^2 - 4ac\right)\right).$$

Therefore, evaluating the integral over u and then making the change of variables $Q \to \widetilde{Q}$ (as defined before (3.4)), it suffices to compute

$$(5.2)\quad \frac{1}{6}(4\pi D)^{\frac{1}{4}-\frac{k}{2}}\Gamma(2s)^{-1}\sum_{Q\in\mathcal{Q}_D} Q_z Q(z,1)^{k-1}\mathcal{I}\left(\frac{Dy^2}{|Q(z,1)|^2}\right),$$

where

$$\mathcal{I}(w) := \int_0^\infty \mathcal{M}_{\frac{3}{2}-k,s}(-v)\, e^{\frac{v}{2}}\, v^{-\frac{1}{2}}\, e^{-vw^{-1}} dv.$$

Inserting the definition (2.6) of $\mathcal{M}_{\frac{3}{2}-k,s}$ and the integral representation (2.7) of the M-Whittaker function, we evaluate

(5.3)

$$\mathcal{I}(w) = \frac{\Gamma(2s)}{\Gamma\left(s-\frac{k}{2}+\frac{3}{4}\right)\Gamma\left(s+\frac{k}{2}-\frac{3}{4}\right)}\int_0^1 t^{s+\frac{k}{2}-\frac{7}{4}}(1-t)^{s-\frac{k}{2}-\frac{1}{4}}\int_0^\infty v^{s+\frac{k}{2}-\frac{5}{4}}e^{-v\left(t-1+w^{-1}\right)}dvdt$$
$$= \frac{\Gamma(2s)\Gamma\left(s+\frac{k}{2}-\frac{1}{4}\right)}{\Gamma\left(s-\frac{k}{2}+\frac{3}{4}\right)\Gamma\left(s+\frac{k}{2}-\frac{3}{4}\right)}w^{s+\frac{k}{2}-\frac{1}{4}}\int_0^1(1-t)^{s+\frac{k}{2}-\frac{7}{4}}t^{s-\frac{k}{2}-\frac{1}{4}}\left(1-wt\right)^{-s-\frac{k}{2}+\frac{1}{4}}dt.$$

We again employ the Euler integral representation (4.3) to show that

$$\mathcal{I}(w) = \Gamma\left(s+\frac{k}{2}-\frac{1}{4}\right)w^{s+\frac{k}{2}-\frac{1}{4}}{}_2F_1\left(s+\frac{k}{2}-\frac{1}{4}, s-\frac{k}{2}+\frac{3}{4}; 2s; w\right).$$

We then rewrite the hypergeometric function by using the Euler transform

$${}_2F_1\left(A,B;C;w\right) = (1-w)^{C-A-B}\,{}_2F_1\left(C-A,C-B;C;w\right)$$

to yield

$$\mathcal{I}(w) = \Gamma\left(s+\frac{k}{2}-\frac{1}{4}\right)(1-w)^{-\frac{1}{2}}w^{s+\frac{k}{2}-\frac{1}{4}}{}_2F_1\left(s-\frac{k}{2}+\frac{1}{4}, s+\frac{k}{2}-\frac{3}{4}; 2s; w\right).$$

Finally, we conclude that (5.2) equals (1.8) by using (3.3) to rewrite $|Q_z|$ in terms of $\frac{Dy^2}{|Q(z,1)|^2}$. □

We next establish the injectivity of the lift.

Proof of Theorem 1.4 (2): Since the Poincaré series $P_{\frac{3}{2}-k,s,D}$ span the space of weak Maass forms and are linearly independent (which can be seen by comparing their principal parts), it is enough to show that the functions $\mathcal{F}_{1-k,s,D}$ are linearly independent. This follows by proving that any linear combination

$$\mathcal{F} := \sum_{j=1}^n a_j\mathcal{F}_{1-k,s,D_j}$$

with a_j not all zero exhibits discontinuities and is hence nonzero. Comparing the sets E_{D_j} of geodesics defined in (1.4) implies the result. □

6. Relation between positive and negative weight local Maass forms

In this section we relate $f_{k,s,D}$ and $\mathcal{F}_{1-k,s,D}$.

Proof of Theorem 1.5: We prove (1.10) by establishing that for $P := P_{\frac{3}{2}-k,s,D}$ and $z\notin E_D$, one has

$$\xi_{2-2k}\left(\Phi^*_{1-k}\left(P\right)(z)\right) = 2\Phi_k\left(\xi_{\frac{3}{2}-k}(P)\right)(z). \tag{6.1}$$

We first use (3.6) and then (3.1) to obtain for $z\notin E_D$

(6.2)
$$\Phi_k\left(\xi_{\frac{3}{2}-k}(P)\right)(z) = \left\langle\xi_{\frac{3}{2}-k}\left(P\right),\Theta\left(z,\cdot\right)\right\rangle^{\mathrm{reg}} = -\overline{\left\langle P,\xi_{k+\frac{1}{2}}\left(\Theta\left(z,\cdot\right)\right)\right\rangle^{\mathrm{reg}}}$$
$$= -\overline{\left\langle P,-iy^{2-2k}\frac{\partial}{\partial z}\Theta^*\left(-\overline{z},\cdot\right)\right\rangle^{\mathrm{reg}}} = \frac{iy^{2-2k}}{6}\frac{\partial}{\partial z}\int_{\mathcal{F}_0(4)}^{\mathrm{reg}}\overline{P(\tau)}\Theta^*\left(-\overline{z},\tau\right)v^{\frac{3}{2}-k}\frac{dudv}{v^2}$$
$$= iy^{2-2k}\frac{\partial}{\partial z}\overline{\left\langle P,\Theta^*\left(-\overline{z},\cdot\right)\right\rangle^{\mathrm{reg}}}.$$

Since

$$\xi_\kappa\left(G(z)\right)=2iy^\kappa\frac{\partial}{\partial z}\overline{G(z)}, \tag{6.3}$$

we conclude (6.1) from (6.2).

We now apply Theorem 1.3 (2), (6.1), (2.13), and finally Theorem 1.1 (2) to yield

$$\begin{aligned}\xi_{2-2k}\left(\mathcal{F}_{1-k,s,D}(z)\right)=\xi_{2-2k}\left(\Phi^*_{1-k}\left(P_{\frac{3}{2}-k,s,D}\right)(z)\right)&=2\Phi_k\left(\xi_{\frac{3}{2}-k}\left(P_{\frac{3}{2}-k,s,D}\right)\right)(z)\\&=2\left(\overline{s}-\frac{3}{4}+\frac{k}{2}\right)\Phi_k\left(P_{k+\frac{1}{2},\overline{s},D}\right)(z)=2\left(\overline{s}-\frac{3}{4}+\frac{k}{2}\right)f_{k,\overline{s},D}(z).\end{aligned}$$

This concludes the proof of (1.10).

We next prove (1.11). Denoting $P:=P_{k+\frac{1}{2},s,D}$, we use (3.5) to conclude that for $z\notin E_D$

$$\Phi^*_{1-k}\left(\xi_{k+\frac{1}{2}}(P)\right)(z)=\left\langle\xi_{k+\frac{1}{2}}(P),\Theta^*(-\overline{z},\cdot)\right\rangle^{\mathrm{reg}}=-\overline{\left\langle P,\xi_{\frac{3}{2}-k}\left(\Theta^*(-\overline{z},\cdot)\right)\right\rangle^{\mathrm{reg}}}. \tag{6.4}$$

We then employ (3.2) and (6.3) to obtain

$$\Phi^*_{1-k}\left(\xi_{k+\frac{1}{2}}(P)\right)(z)=iy^{2k}\frac{\partial}{\partial z}\overline{\langle P,\Theta(z,\cdot)\rangle^{\mathrm{reg}}}=\frac{1}{2}\xi_{2k}\left(\Phi_k(P)(z)\right). \tag{6.5}$$

Combining this with Theorem 1.3 (2), (2.13), and Theorem 1.1 (2) yields

$$\begin{aligned}\left(\overline{s}-\frac{k}{2}-\frac{1}{4}\right)\mathcal{F}_{1-k,\overline{s},D}(z)&=\left(\overline{s}-\frac{k}{2}-\frac{1}{4}\right)\Phi^*_{1-k}\left(P_{\frac{3}{2}-k,\overline{s},D}\right)(z)\\&=\Phi^*_{1-k}\left(\xi_{k+\frac{1}{2}}\left(P_{k+\frac{1}{2},s,D}\right)\right)(z)=\frac{1}{2}\xi_{2k}\left(\Phi_k\left(P_{k+\frac{1}{2},s,D}\right)(z)\right)=\frac{1}{2}\xi_{2k}\left(f_{k,s,D}(z)\right).\end{aligned}$$

□

We are now ready to prove Theorem 1.1 (1) and Theorem 1.3 (1).

Proof of Theorem 1.1 (1): Note that

$$\overline{\Theta(z,\tau)}=\Theta(-\overline{z},-\overline{\tau}).$$

Hence $f_{k,s,D}$ is modular of weight $2k$ by Proposition 3.2.

The functions $f_{k,s,D}$ are continuous since for $\mathrm{Re}(C)>\mathrm{Re}(A+B)$, the hypergeometric function ${}_2F_1(A,B;C;w)$, and hence $\varphi_s(w)$, is continuous for $w\leq 1$. This implies condition (3).

For $z\notin E_D$, (1.11) and (1.10) imply that

$$\Delta_{2k}\left(f_{k,s,D}(z)\right)=-\xi_{2-2k}\left(\xi_{2k}\left(f_{k,s,D}(z)\right)\right)=4\lambda_s f_{k,s,D}(z).$$

A straightforward calculation shows that $f_{k,s,D}(z)$ grows at most polynomially as $y\to\infty$.

Finally, one uses (4.4) and the duplication formula for the Γ-function to conclude (1.5).

□

Remark. The non-differentiability of $f_{k,s,D}$ follows by using (1.11) and then proving that the functions $\mathcal{F}_{1-k,s,D}$ are not continuous. Computational evidence indicates that $f_{k,s,D}(z)$ decays exponentially as $y\to\infty$.

Proof of Theorem 1.3 (1): Noting that

$$\overline{\Theta^*(-\overline{z},\tau)}=\Theta^*(z,-\overline{\tau}),$$

Proposition 3.2 implies that $\mathcal{F}_{1-k,s,D}$ is modular of weight $2-2k$.

The proof that $\mathcal{F}_{1-k,s,D}$ is an eigenfunction under Δ_{2-2k} with eigenvalue $4\lambda_s$ follows by (1.10) and (1.11) precisely as in the proof of Theorem 1.1 (1).

In order to show condition (3) in the definition of local Maass forms, we first note that $\varphi_s^*(w)$ is continuous for $0 < w \leq 1$. The locally uniform convergence of the sum allows us to pull the limit $r \to 0^+$ of $\mathcal{F}_{1-k,s,D}(z \pm ir)$ into each term. Define

$$\mathscr{B}_z := \left\{ Q \in \mathcal{Q}_D \middle| Q_z = 0 \right\}.$$

By Lemma 5.1 of [7], there are only finitely many $Q \in \mathscr{B}_z$. Note that

$$\operatorname{sgn}(Q_z) = \operatorname{sgn}(Q_{z \pm ir})$$

for r sufficiently small and $Q \notin \mathscr{B}_z$, while for $Q \in \mathscr{B}_z$ one has

$$\operatorname{sgn}(Q_{z+ir}) = -\operatorname{sgn}(Q_{z-ir}).$$

Hence, since the terms of $\mathcal{F}_{1-k,s,D}(z)$ with $Q \in \mathscr{B}_z$ vanish,

$$\frac{1}{2} \lim_{r \to 0^+} \left(\mathcal{F}_{1-k,s,D}(z+ir) + \mathcal{F}_{1-k,s,D}(z-ir) \right) = \sum_{Q \notin \mathscr{B}_z} \operatorname{sgn}(Q_z)\, Q(z,1)^{k-1} \varphi_s^* \left(\frac{Dy^2}{|Q(z,1)|^2} \right)$$
$$= \mathcal{F}_{1-k,s,D}(z).$$

A direct calculation shows that $\mathcal{F}_{1-k,s,D}(z)$ grows at most polynomially as $y \to \infty$. □

Remark. To show that $\mathcal{F}_{1-k,s,D}$ exhibits discontinuities along the set E_D, one computes

$$\lim_{r \to 0^+} \left(\mathcal{F}_{1-k,s,D}(z+ir) - \mathcal{F}_{1-k,s,D}(z-ir) \right)$$

similarly as in the proof of Theorem 1.3 (1). It is shown to be nonzero by using Gauss's summation formula to conclude that $\varphi_s^*(1) \neq 0$.

If D is not a square and $\operatorname{Re}(s) \geq \frac{k}{2} + \frac{1}{4}$, then computational evidence indicates that $\mathcal{F}_{1-k,s,D}$ is bounded as $y \to \infty$.

7. Hecke operators

In this section, we consider the action of the Hecke operators on the theta lifts.

Proof of Theorem 1.4 (1): Since the Poincaré series span the space of weight $\frac{3}{2} - k$ weak Maass forms, it suffices to compute the action of the Hecke operators on Poincaré series. As in the proof of Theorem 1.5 of [7], one can show that

$$\mathcal{F}_{1-k,s,D}\Big|_{2-2k} T_p = \mathcal{F}_{1-k,s,Dp^2} + p^{-k} \left(\frac{D}{p} \right) \mathcal{F}_{1-k,s,D} + p^{1-2k} \mathcal{F}_{1-k,s,\frac{D}{p^2}}.$$

Hence by Theorem 1.3 (1), equation (1.9) follows by the easily verified identity

$$P_{\frac{3}{2}-k,s,D}\Big|_{\frac{3}{2}-k} T_{p^2} = P_{\frac{3}{2}-k,s,Dp^2} + p^{-k} \left(\frac{D}{p} \right) P_{\frac{3}{2}-k,s,D} + p^{1-2k} P_{\frac{3}{2}-k,s,\frac{D}{p^2}}.$$

□

We now move on to the positive weight case.

Proof of Theorem 1.2: We first prove Theorem 1.2 (1). Let $\mathcal{H}$ be a weight $2k$ local Maass form with exceptional set E_D which is continuous everywhere. Since continuity is preserved by the Hecke operators, one easily checks that $\mathcal{H}|_{2k}T_p$ is a local Maass form. To determine the exceptional set for $\mathcal{H}$, recall that the weight $2k$ Hecke operator may be written as

$$\mathcal{H}\Big|_{2k} T_p(\tau) = p^{2k-1} \mathcal{H}(p\tau) + p^{-1} \sum_{r \pmod{p}} \mathcal{H}\left(\frac{\tau + r}{p} \right).$$

By computing the image of E_D under $\tau \to p\tau$ and $\tau \to \frac{\tau+r}{p}$, one concludes that $\mathcal{H}|_{2k}T_p$ has exceptional set $E := E_{Dp^2} \supset E_D$. Hence it suffices to prove the statement for $z \notin E$.

Suppose that H is a weak Maass form of weight $k+\frac{1}{2}$ with eigenvalue λ_s under $\Delta_{k+\frac{1}{2}}$. Since $\xi_{\frac{3}{2}-k}$ surjects onto the space of weak Maass forms of weight $k+\frac{1}{2}$ with eigenvalue λ_s (see [13]), we may choose such a weight $\frac{3}{2}-k$ weak Maass form G such that $\xi_{\frac{3}{2}-k}(G)=H$. But then by (1.9), (6.1), and the fact that the Hecke operators commute with ξ_{2-2k}, for $z\notin E$, we have that

$$\Phi_k(H)\Big|_{2k}T_p(z)=\frac{1}{2}\xi_{2-2k}\left(\Phi_{1-k}^*(G)\right)\Big|_{2k}T_p(z)=\frac{1}{2}\xi_{2-2k}\left(\Phi_{1-k}^*\left(G\Big|_{\frac{3}{2}-k}T_{p^2}\right)(z)\right).$$

We now use (6.5) and the fact that $\xi_{\frac{3}{2}-k}$ commutes with the Hecke operators to obtain

$$\xi_{2-2k}\left(\Phi_{1-k}^*\left(G\Big|_{\frac{3}{2}-k}T_{p^2}\right)(z)\right)=2\Phi_k\left(\xi_{\frac{3}{2}-k}\left(G\Big|_{\frac{3}{2}-k}T_{p^2}\right)\right)(z)=2\Phi_k\left(H\Big|_{k+\frac{1}{2}}T_{p^2}\right)(z),$$

as desired.

We move on to Theorem 1.2 (2). Assume that $\Phi_k(F)\equiv 0$ for a weak Maass form F with eigenvalue $\lambda_s\neq 0$. Writing $G:=-(4\lambda_s)^{-1}\xi_{k+\frac{1}{2}}(F)$, by (6.1) we have that

$$0=\xi_{2-2k}\left(\Phi_{1-k}^*(G)\right).$$

Since $\Phi_{1-k}^*(G)$ is an eigenfunction under Δ_{2-2k} with eigenvalue $4\lambda_s\neq 0$, we have

$$0=-\left(4\lambda_s\right)^{-1}\xi_{2k}\left(\xi_{2-2k}\left(\Phi_{1-k}^*(G)\right)\right)=\Phi_{1-k}^*(G).$$

Since Φ_{1-k}^* is injective, we conclude that $G\equiv 0$. However,

$$\xi_{\frac{3}{2}-k}(G)=F,$$

and hence $F\equiv 0$. □

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