1. Introduction and statement of results

For $\kappa \in \mathbb{Z}$, denote by $M^!_{2\kappa}$ the space of weight $2\kappa$ weakly holomorphic modular forms (i.e., those meromorphic modular forms whose only possible pole occurs at $i\infty$). Motivated by applications coming from the theory of $p$-adic modular forms, Guerzhoy [9] observed that even though the space $M^!_{2k}$ ($k \in \mathbb{N}$) is infinite dimensional and the Hecke operators $T_m$ (defined in (2.5)) increase the order of the pole, there is a meaningful Hecke theory on this space. Emulating the usual definition of Hecke eigenforms (i.e., for every $m \in \mathbb{N}$ there exists $\lambda_m \in \mathbb{C}$ such that $f|T_m = \lambda_m f$), Guerzhoy defined a weakly holomorphic Hecke eigenform to be any $f \in M^!_{2k}$ which satisfies

$$f|T_m \equiv \lambda_m f \pmod{J},$$

where $J$ is a specific subspace of $M^!_{2k}$ which is stable under the Hecke algebra. Here and throughout, for a vector space $S$ and a subspace $J \subset S$, for $f, g \in S$ the congruence $f \equiv g \pmod{J}$ means that $f - g \in J$. In [9], Guerzhoy chose $J = D^{2k-1}(M^!_{2-2k})$, where $D := \frac{1}{2\pi i} \frac{\partial}{\partial z}$, but it was later determined that $J = D^{2k-1}(S^!_{2-2k})$ is a better choice (cf. [5], [11]), where $S^!_{2\kappa}$ is the space of weak cusp forms of weight $2\kappa \in 2\mathbb{Z}$, i.e., those weakly holomorphic modular forms with vanishing constant term. Note that $D^{2k-1}$ and $T_m$ essentially commute (see [2,4]) and $S^!_{2-2k}$ is preserved by $T_m$, so $D^{2k-1}(S^!_{2-2k})|T_m \subseteq D^{2k-1}(S^!_{2-2k})$, and hence $D^{2k-1}(S^!_{2-2k})$ is stable under the Hecke operators. Thus a weakly holomorphic Hecke eigenform is any $f \in M^!_{2k} \setminus D^{2k-1}(S^!_{2-2k})$ for which there

Date: December 12, 2016.

The research of the first author is supported by the Alfried Krupp Prize for Young University Teachers of the Krupp foundation and the research leading to these results receives funding from the European Research Council under the European Union’s Seventh Framework Programme (FP/2007-2013) / ERC Grant agreement n. 335220 - AQSER. The research of the second author was supported by grants from the Research Grants Council of the Hong Kong SAR, China (project numbers HKU 27300314, 17302515, and 17316416).
exist $\lambda_m \in \mathbb{C}$ satisfying (for all $m \in \mathbb{N}$)

$$f|T_m \equiv \lambda_m f \pmod{D^{2k-1}(S_{2-2k}^t)}.$$  \hspace{1cm} (1.1)\]

Functions $f$ satisfying (1.1) are not eigenfunctions of $T_m$ in the usual sense. However a reasonable interpretation is to consider $[f] := f + D^{2k-1}(S_{2-2k}^t)$ as an element of the quotient space $M_{2k}/D^{2k-1}(S_{2-2k}^t)$. Functions $f$ satisfying (1.1) may then be viewed as eigenvectors $[f]$ of the Hecke operators in that factor space. As is usual, we exclude $[0] = D^{2k-1}(S_{2-2k}^t)$ in the definition of eigenvector.

Viewed in this light, the definition of weakly holomorphic Hecke eigenforms is perhaps not very enlightening at first glance. If one simply defines “eigenforms” as elements of some quotient space, then one can replace $D^{2k-1}(S_{2-2k}^t)$ with any space $J$ that is preserved under the action of the Hecke operators. It is hence natural to ask why $D^{2k-1}(S_{2-2k}^t)$ is the correct subspace. There are a few answers to this question. The initial perspective taken in [9] was a $p$-adic one, because for $f \in D^{2k-1}(S_{2-2k}^t)$ with integral coefficients the $(p^m n)$th coefficients become divisible by higher powers of $p$ as $m$ gets larger. Elements of $D^{2k-1}(S_{2-2k}^t)$ have vanishing period polynomials [5] and there is a recent cohomological interpretation as a space of coboundaries announced by Funke. Moreover certain regularized critical $L$-values are zero for elements in this space (see [4, Theorem 2.5]).

In this paper, we give another reason for the choice $J = D^{2k-1}(S_{2-2k}^t)$ by viewing these Hecke eigenforms in the framework of a regularized inner product $\langle \cdot, \cdot \rangle$, defined in [2] for arbitrary weakly holomorphic modular forms. Note that if $f \in M_{2k}$ is a classical holomorphic Hecke eigenform, then for each $m \in \mathbb{N}$ there exists $\lambda_m \in \mathbb{C}$, for which

$$\langle f|T_m, g \rangle = \langle \lambda_m f, g \rangle = \lambda_m \langle f, g \rangle$$ \hspace{1cm} (1.2)\]

for all $g \in M_{2k}$. Indeed, since the inner product is non-degenerate on $M_{2k}$ (see [15, Section 5]), (1.2) is equivalent to the usual definition of Hecke eigenforms on $M_{2k}$. Let

$$M_{2k}^! := \left\{ f \in M_{2k} : \langle f, g \rangle = 0 \text{ for all } g \in M_{2k} \right\}. \hspace{1cm} (1.3)$$

be the space of weakly holomorphic forms which are orthogonal to all of $M_{2k}^!$; from the point of view of the inner product (and (1.2) in particular), these functions are very naturally like zero.

The main point of this paper is to give another motivation of the definition in [2] by proving that the space $D^{2k-1}(S_{2-2k}^t)$ is the “degenerate part” of $M_{2k}^!$.
in the sense that it is orthogonal to all of \( M_{2k}^! \), from which one also conversely concludes that \( M_{2k}^{!,\perp} \) is a very large subspace.

**Theorem 1.1.** A function \( f \in M_{2k}^! \) is in \( M_{2k}^{!,\perp} \) if and only if \( f \in D^{2k-1}(S_{2-2k}) \).

As alluded to above, Theorem 1.1 yields another characterization of weakly holomorphic Hecke eigenforms.

**Corollary 1.2.** A function \( f \in M_{2k}^! \) is a weakly holomorphic Hecke eigenform if and only if for every \( m \in \mathbb{N} \) there exists \( \lambda_m \) satisfying (1.2) for all \( g \in M_{2k}^! \).

**Remark.** As shown in [2, Theorem 1.3], the inner product is Hermitian on \( M_{2k}^! \). Hence if \( f \) is a weakly holomorphic Hecke eigenform, then (1.2) implies that for all \( g \in M_{2k}^! \)

\[
\langle g, f | T_m \rangle = \overline{\lambda_m} \langle g, f \rangle.
\]

Moreover, by Lemma 2.1 below, the Hecke operators are also Hermitian. Thus, taking \( g = f \), we see that \( \lambda_m \in \mathbb{R} \) or \( \langle f, f \rangle = 0 \).

The paper is organized as follows. In Section 2, we introduce a family of non-holomorphic modular forms known as harmonic Maass forms which play an important role in the proof of Theorem 1.1 and then we explain how one can define regularized inner products. In Section 3, we prove Theorem 1.1 and Corollary 1.2.

## 2. Preliminaries

### 2.1. Harmonic Maass forms

We begin by defining harmonic Maass forms, which were first introduced by Bruinier–Funke [7].

**Definition.** For \( \kappa \in \mathbb{Z} \), a harmonic Maass form of weight \( 2\kappa \) is a function \( F : \mathbb{H} \to \mathbb{C} \), which is real analytic on \( \mathbb{H} \) and satisfies the following conditions:

1. For every \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \), we have

\[
F \left( \frac{az + b}{cz + d} \right) = (cz + d)^{2\kappa} F(z).
\]

2. We have \( \Delta_{2\kappa}(F) = 0 \), where \( \Delta_{2\kappa} \) is the weight \( 2\kappa \) hyperbolic Laplace operator \( (z = x + iy) \)

\[
\Delta_{2\kappa} := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + 2\kappa iy \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).
\]

3. The function \( F \) grows at most linear exponentially at \( i\infty \).
A weight $2\kappa$ harmonic Maass form $F$ has a Fourier expansion of the type $(c_F(n, y) \in \mathbb{C})$

$$F(z) = \sum_{n \in \mathbb{Z}} c_F(n, y)e^{2\pi inz}.$$ 

(2.1)

If $F$ is weakly holomorphic, then $c_F(n, y) = c_F(n) \in \mathbb{C}$ is independent of $y$.

More generally, one may use condition (2) above to determine the dependence on $y$ and conclude that (2.1) naturally decomposes into holomorphic and non-holomorphic parts. Namely, for a harmonic Maass form $F$ of weight $2 - 2k < 0$,

$$F(z) = F^+(z) + F^-(z)$$

(2.2)

where, for some (unique) $c^\pm_F(n) \in \mathbb{C}$, we have

$$F^+(z) = \sum_{n \gg -\infty} c^+_F(n)e^{2\pi inz},$$

$$F^-(z) = c^-_F(0)y^{2k-1} + \sum_{n \ll \infty, n \neq 0} c^-_F(n)\Gamma(2k - 1, -4\pi ny)e^{2\pi inz}.$$

Here the incomplete gamma function is given by $\Gamma(\alpha, w) := \int_w^\infty e^{-t}t^{\alpha-1}dt$ (for $\text{Re}(\alpha) > 0$ and $w \in \mathbb{C}$). We call $F^+$ its holomorphic part of $F$ and $F^-$ the non-holomorphic part.

We let $H^\text{mg}_{2k}$ be the space of weight $2\kappa$ harmonic Maass forms. The operators $\xi^{2-2k} := 2iy^{2-2k}\frac{\partial}{\partial y}$ and $D^{2k-1}$, defined in the introduction, map $H^\text{mg}_{2-2k}$ to $\mathcal{M}_{2k}$, and we let $H_{2-2k}$ be the subspace of $H^\text{mg}_{2-2k}$ consisting of forms which map to cusp forms under $\xi^{2-2k}$. These differential operators also act naturally on the Fourier expansion (2.2). In particular, as collected in [3, Theorems 5.5 and 5.9],

$$\xi^{2-2k}(F(z)) = (2k - 1)c^-_F(0) - (4\pi)^{2k-1}\sum_{n \gg -\infty, n \neq 0} n^{2k-1}c^-_F(-n)e^{2\pi inz},$$

$$D^{2k-1}(F(z)) = -\frac{(2k - 1)!}{(4\pi)^{2k-1}}c^-_F(0) + \sum_{n \gg -\infty} n^{2k-1}c^+_F(n)e^{2\pi inz}.$$

Recall that the operators $D^{2k-1}$ and $\xi^{2-2k}$ are Hecke-equivariant (cf. [3, (7.4) and (7.5)]), i.e., for any harmonic Maass form $F$

$$\xi^{2-2k}(F|_{2-2k}T_m) = m^{1-2k}\xi^{2-2k}(F)|_{2k}T_m,$$

$$D^{2k-1}(F|_{2-2k}T_m) = m^{1-2k}D^{2k-1}(F)|_{2k}T_m.$$ 

(2.3) (2.4)
Here for $\kappa \in \mathbb{Z}$, the Hecke operators $T_m : H^{mg}_{2\kappa} \to H^{mg}_{2\kappa}$ is defined on the expansion (2.1) by

$$F(z)|_{2\kappa} T_m := \sum_{n \in \mathbb{Z}} \sum_{d \mid (m,n)} d^{2\kappa-1} c_F \left( \frac{mn}{d^2}, \frac{d^2 y}{m} \right) e^{2\pi i n z}.$$  \hfill (2.5)

\subsection{Regularized inner products.}

For two cusp forms $f, g \in S_{2k}$, Petersson’s classical inner product is defined by

$$\langle f, g \rangle := \int_{\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} f(z) \overline{g(z)} y^{2k} \frac{dx dy}{y^2}. \hfill (2.6)$$

This was extended to an inner product over all of $M^!_{2k}$ in a series of steps. The first such attempt to do so appears to be by Petersson himself [14], which was later rediscovered and extended by Harvey–Moore [10] and Borcherds [1] as we describe below. Setting

$$\mathcal{F}_T := \left\{ z \in \mathbb{H} : |z| \geq 1, y \leq T, -\frac{1}{2} \leq x \leq \frac{1}{2} \right\},$$

the first regularized inner product is defined by

$$\langle f, g \rangle := \lim_{T \to \infty} \int_{\mathcal{F}_T} f(z) \overline{g(z)} y^{2k} \frac{dx dy}{y^2}. \hfill (2.7)$$

Borcherds extended this regularization by multiplying the integrand by $y^{-s}$ and taking the constant term of the analytic continuation around $s = 0$ of the analytic continuation in $s$. The regularization (2.7) coincides with (2.6) whenever (2.6) converges. Unfortunately, the regularization (2.7) and even Borcherds’s extension also do not always exist. In particular, one cannot use them to define a meaningful norm for weakly holomorphic forms. This problem was overcome to obtain a regularized inner product for arbitrary $f, g \in M^!_{2k}$ by Diamantis, Ehlen, and the first author [2]. The idea is simple. One multiplies the integrands with a function that forces convergence and then analytically continues.

For this, observe that for $\text{Re}(w) \gg 0$, the integral

$$I(f, g; w, s) := \int_{\mathcal{F}} f(z) \overline{g(z)} y^{2k-s} e^{-wy} \frac{dx dy}{y^2}$$

converges and is analytic. For every $\varphi \in (\pi/2, 3\pi/2) \setminus \{\pi\}$ it has an analytic continuation $I_\varphi(f, g; w, s)$ to $U_\varphi \times \mathbb{C}$ with $U_\varphi \subset \mathbb{C}$ a certain open set. Then
define
\[ \langle f, g \rangle_\varphi := \text{CT}_{s=0} I_\varphi(f, g; 0, s) - i \sum_{n>0} c_f(-n) c_g(-n) \text{Im} \left( E_{2-2k}(-4\pi n) \right), \]
where \( \text{CT}_{s=0} F(s) \) denotes the constant term of an analytic function \( F \), \( c_\varphi \) are defined as in (2.1), and \( E_{r,\varphi} \) is the generalized exponential integral (see [12, (8.19.3)]) defined with branch cut on the ray \( \{ xe^{i\varphi} : x \in \mathbb{R}^+ \} \). In [2] it is shown that for \( f, g \in M_{2k} \), \( \langle f, g \rangle \) exists and is independent of the choice of \( \varphi \). Furthermore, it equals the regularization (2.7) whenever (2.7) exists.

The following lemma describes that the inner product from [2] commutes with the Hecke algebra.

**Lemma 2.1.** For \( k \in \mathbb{N} \), the Hecke operators are Hermitian on \( M_{2k} \).

**Proof:** We split the proof into three cases. First we assume that \( f \in S_{2k} \) and \( g \in M_{2k} \), then we use fact that the inner product is Hermitian (see [2, Theorem 1.3]) for the case that \( f \in M_{2k} \) and \( g \in S_{2k} \), and finally we consider \( f = g = E_{2k} \); these cover all cases by linearity.

Following [3, (1.15)], one defines, for \( f, g \in M_{2k} \),
\[ \{ f, g \}_0 := \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{c_f(-n)c_g(n)}{n^{2k-1}}. \]

The ratio \( c_g(n)/n^{2k-1} \) can also be interpreted as the \( n \)th coefficient of the holomorphic part of any harmonic Maass form \( G \) for which \( D_{2k-1}(G) = g \). This yields a connection with the Bruinier–Funke pairing (cf. [7] for the original definition restricted to certain subspaces)
\[ \{ f, G \} := \sum_{n \in \mathbb{Z}} c_f(-n)c^+_G(n) \]

between \( f \in M_{2k} \) and \( G \in H_{2-2k}^{mg} \).

We first assume that \( f \in S_{2k} \) and \( g \in M_{2k} \). In this case
\[ \{ f, g \}_0 = \{ f, G \}. \] (2.8)

This is related to the inner product by [2, Theorem 4.1]; namely,
\[ \{ f, G \} = \langle f, \xi_{2-2k}(G) \rangle. \] (2.9)

Setting \( h := \xi_{2-2k}(G) \), we combine (2.9) with (2.8) and the fact that the Hecke operators are Hermitian with respect to \( \langle \cdot, \cdot \rangle_0 \) to obtain
\[ \langle f|_{2k} T_m, h \rangle = \{ f|_{2k} T_m, G \} = \{ f|_{2k} T_m, g \}_0 = \{ f, g|_{2k} T_m \}_0. \]
Next, by (2.4), we have
\[ D^{2k-1}(G|_{2-2k} T_m) = m^{1-2k} D^{2k-1}(G)|_{2k} T_m = m^{1-2k} g|_{2k} T_m. \]
Plugging this into (2.8) and using (2.9), yields
\[ \{f, g|_{2k} T_m\}_0 = m^{2k-1} \{f, G|_{2-2k} T_m\} = m^{2k-1} \langle f, \xi_{2-2k} (G|_{2-2k} T_m) \rangle. \]
We finally use (2.3) to obtain
\[ m^{2k-1} \langle f, \xi_{2-2k} (G|_{2-2k} T_m) \rangle = \langle f, h|_{2k} T_m \rangle. \]
Altogether, we have therefore shown that
\[ \langle f|_{2k} T_m, h \rangle = \langle f, h|_{2k} T_m \rangle. \quad (2.10) \]
Since \( \xi_{2-2k} \) is surjective on \( M'_{2k} \), we obtain the claim for arbitrary \( f \in S'_{2k} \) and \( h \in M'_{2k} \).

Next suppose that \( f \in M'_{2k} \) and \( g \in S'_{2k} \). Since the inner product is Hermitian (see [2, Theorem 1.3]), by (2.10) we have
\[ \langle f|_{2k} T_m, g \rangle = \langle g, f|_{2k} T_m \rangle = \langle g|_{2k} T_m, f \rangle = \langle f, g|_{2k} T_m \rangle. \]
Finally, for \( f = E_{2k} = g \), we obtain the result directly from the fact that both \( f \) and \( g \) are Hecke eigenforms with the same real eigenvalues.

3. Degeneracy and Proofs of Theorem 1.1 and Corollary 1.2

In this section, we show that the space \( M'_{2k} \) defined in (1.3) coincides with the space \( D^{2k-1}(S'_{2-2k}) \) which appears in the definition of Hecke eigenforms for weakly holomorphic modular forms. In [2 Corollary 4.5], the space \( M'_{2k} \), defined in (1.3), was explicitly determined. Theorem 1.1 rewrites this in a form which is useful for Hecke eigenforms. In order to prove Theorem 1.1, we also require the following useful lemma about the flipping operator
\[ \mathfrak{F}_\kappa (F(z)) := -\frac{y^{-\kappa}}{(-\kappa)!} R_{-\kappa} (F(z)). \]
The following lemma follows from the calculation in the proof of [8, Theorem 1.1] and [8, Remark 7]; it may be found in this form in [3 Proposition 5.14].

**Lemma 3.1.** If \( F \in H^\text{mg}_{2-2k} \), then the operator \( \tilde{\xi}_{2-2k} \) satisfies
\[ \xi_{2-2k}(\tilde{\xi}_{2-2k}(F)) = \left(\frac{4\pi}{2k - 2}\right)^{2k-1} D^{2k-1}(F), \quad (3.1) \]
\[ D^{2k-1}(\tilde{\mathcal{F}}_{2-2k}(F)) = \frac{(2k-2)!}{(4\pi)^{2k-1}} \xi_{2-2k}(F). \]  

(3.2)

Furthermore, \( \tilde{\mathcal{F}}_{2-2k} \circ \tilde{\mathcal{F}}_{2-2k}(F) = F \).

We now use Lemma 3.1 to prove Theorem 1.1.

**Proof of Theorem 1.1:** By [2, Corollary 4.5], \( f \in M_{2k}^! \) is orthogonal to \( M_{2k}^! \) if and only if there exists \( F \in H_{2-2k}^{mg} \) for which \( F^+ = 0 \) and \( \xi_{2-2k}(F) = f \). By (3.2), we have

\[ f = \xi_{2-2k}(F) = \frac{(4\pi)^{2k-1}}{(2k-2)!} D^{2k-1}(\tilde{\mathcal{F}}_{2-2k}(F)). \]  

(3.3)

We conclude that \( \tilde{\mathcal{F}}_{2-2k}(F) \in M_{2-2k}^! \) because this is the kernel of \( \xi_{2-2k} \). Thus by (3.3), we have \( f \in D^{2k-1}(S_{2-2k}^!) \). Moreover, [6] displayed formula before Lemma 3.1 states that (note that in this paper the flipping operator is renormalized)

\[ c_{\tilde{\mathcal{F}}_{2-2k}(F)}^+(0) = -c_{\mathcal{F}}^+(0) = 0. \]

Thus \( F^+ = 0 \) is equivalent to \( f \in D^{2k-1}(S_{2-2k}^!) \), which is the claim. \( \square \)

We conclude the paper by using Theorem 1.1 to show that one may obtain an alternative characterization of weakly holomorphic Hecke eigenforms by requiring (1.2) to hold for all \( g \in M_{2k}^! \) instead of (1.1).

**Proof of Corollary 1.2:** By Theorem 1.1 (1.1) is equivalent to

\[ f|T_m \equiv \lambda_m f \pmod{M_{2k}^{1,\perp}}. \]

Hence if \( f \) is a weakly holomorphic Hecke eigenform, then \( f|T_m = \lambda_m f + h \) for some \( h \in M_{2k}^{1,\perp} \). Thus for every \( g \in M_{2k}^! \), linearity of the regularized inner product implies that

\[ \langle f|T_m, g \rangle = \langle \lambda_m f + h, g \rangle = \lambda_m \langle f, g \rangle + \langle h, g \rangle = \lambda_m \langle f, g \rangle. \]

We conclude that there exists \( \lambda_m \) satisfying (1.2) for every \( g \in M_{2k}^! \).

Conversely, assume that (1.2) holds for all \( g \in M_{2k}^! \) and set

\[ h := f|T_m - \lambda_m f. \]
Linearity of the regularized inner product together with (1.2) then implies that
\[ \langle h, g \rangle = \langle f | T_m - \lambda_m f, g \rangle = \langle f | T_m, g \rangle - \lambda_m \langle f, g \rangle = 0. \]
This implies that \( h \in M^{1,\perp}_{2k} = D^{2k-1}(S_{2-2k}^1), \) where the identity follows by Theorem 1.1. Thus
\[ f | T_m - \lambda_m f \in D^{2k-1}(S_{2-2k}^1), \]
or in other words, (1.1) holds. □

**References**


Mathematical Institute, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany
E-mail address: kbringma@math.uni-koeln.de

Department of Mathematics, University of Hong Kong, Pokfulam, Hong Kong
E-mail address: bkane@maths.hku.hk