Journal of Combinatorics and Number Theory
Volume 1, Issue 1, pp. 75-88
ISSN 1942-5600
© 2009 Nova Science Publishers, Inc.

ON TWO CONJECTURES ABOUT MIXED SUMS OF SQUARES AND TRIANGULAR NUMBERS

Ben Kane*
Department of Mathematics, Radboud University,
Toernooiveld 1, 6525 Nijmegen, Netherlands

Abstract

We investigate mixed sums of triangular numbers and squares. We resolve a conjecture of Z.-W. Sun about representability of sums of this type by proving 6 of the 10 parts and giving counterexamples to the 4 other parts. We also show that the generalized Riemann hypothesis implies another conjecture of Z.-W. Sun about which explicit natural numbers may be represented.

Key Words: triangular numbers, odd Squares, sums of squares, quadratic forms, half-integral weight modular forms
2000 Mathematics Subject Classification: 11E25, 11E20, 11E45.

1. Introduction

Sums of squares and sums of triangular numbers have been studied extensively, going as far back as Fermat. Fermat asserted that every natural number was the sum of three triangular numbers, four squares, five pentagonal numbers, etc. Lagrange showed this claim for sums of four squares in 1770, while Gauss showed the result for three triangular numbers in 1796. The full assertion was later shown by Cauchy in 1813.

More recently, Sun [18] has considered mixed hybrid sums involving both triangular numbers and squares. That is, Sun has considered sums of the type

$$f_{a,b}(x,y) := a_1x_1^2 + \cdots + a_{m_1}x_{m_1}^2 + b_1T_{y_1} + \cdots + b_{m_2}T_{y_{m_2}},$$

where a_i and b_i are natural numbers and $T_n = n(n+1)/2$ is the n-th triangular number.

In [18], Sun investigates which sums with three terms represent every integer, so called *universal forms*, reducing the possible candidates to a short list which he then conjectured to be universal. Guo, Pan, and Sun [5] showed that at least all but one of these were indeed universal, while Sun and Oh [10] showed that every natural number could be written as a square plus an odd square plus a triangular number to complete the classification.

Theorem 1.1 (Sun and Oh, see [10]). *Every natural number has the form*

$$x^2 + 8T_y + T_z.$$

*E-mail address: bkane@science.ru.nl

Sun then conjectured the following (cf. [19]).

Conjecture 1.2 (Sun [19]). *Let m and n be any nonnegative integers. Then every sufficiently large natural number can be written in any of the following forms:*

$$2^m x^2 + 2^n y^2 + T_z \tag{1.1}$$
$$2^m x^2 + 2^n T_y + T_z \tag{1.2}$$
$$2^m T_x + 2^n T_y + T_z \tag{1.3}$$
$$x^2 + 2^n \cdot 3y^2 + T_z \tag{1.4}$$
$$x^2 + 2^n \cdot 3T_y + T_z \tag{1.5}$$
$$2^n \cdot 3x^2 + 2T_y + T_z \tag{1.6}$$
$$2^n \cdot 3T_x + 2T_y + T_z \tag{1.7}$$
$$2^n \cdot 5T_x + T_y + T_z \tag{1.8}$$
$$2T_x + 3T_y + 4T_z \tag{1.9}$$
$$2x^2 + 3y^2 + 2T_z. \tag{1.10}$$

We will see first that this conjecture does not hold in general. For formulas (1.2), (1.3), (1.7), and (1.8) we obtain explicit counterexamples to the conjecture.

Theorem 1.3.

$$x^2 + 16T_y + T_z$$

does not represent any natural number of the form $(p^2 - 17)/8$, where p is any prime congruent to 1 or 3 modulo 8, and is hence a counterexample to (1.2).

$$4T_x + 4T_y + T_z$$

and

$$8T_x + T_y + T_z$$

represent precisely the natural numbers not of the form $(a^2 - 9)/8$ and $(a^2 - 5)/4$, respectively, where a is any integer all of whose prime factors are congruent to 1 modulo 4. Hence both are counterexamples to (1.3).

$$192T_x + 2T_y + T_z$$

does not represent any natural number of the form $(3p^2 - 195)/8$ with p a prime congruent to 5 or 7 modulo 8, and hence it is a counterexample to (1.7).

$$160T_x + T_y + T_z$$

does not represent any natural number of the form $(5p^2 - 162)/8$ with p is a prime congruent to 5 or 7 modulo 8, and hence it is a counterexample to (1.8).

Remark 1.4. After submission, Sun has pointed out that the cases $4T_x + 4T_y + T_z$ and $8T_x + T_y + T_z$ are in fact implied in Oh and Sun's paper [10], as follows.

From Oh and Sun [10, Theorem 1.1(ii)],

$$\begin{aligned}&\{n \in \mathbb{Z}^+ : n \neq (2x+1)^2 + T_y + T_z, x, y, z \in \mathbb{Z}\}\\ &\quad = \{2T_m : m > 0, \text{ and all prime divisors of } 2m+1 \text{ are congruent to } 1 \pmod 4\}.\end{aligned}$$

It follows that a nonnegative integer n cannot be represented by $8T_x + T_y + T_z$ if and only if n has the form $2T_m - 1$ where $2m + 1$ has no prime divisors congruent to 3 modulo 4.

By Sun [18, Theorem 1(iii)], and Oh-Sun [10, Theorem 2.1(ii)]

$$\begin{aligned}&\{n \in \mathbb{Z}^+ : n \neq (2x+1)^2 + (2y)^2 + T_z, x, y, z \in \mathbb{Z}\}\\ &\quad = \{T_m : m > 0, \text{ and all prime divisors of } 2m+1 \text{ are congruent to } 1 \pmod 4\}.\end{aligned}$$

Observe that

$$\begin{aligned} n &= 4T_x + 4T_y + T_z \text{ for some } x, y, z \in \mathbb{Z}\\ \iff 2n+2 &= (2x+1)^2 + (2y+1)^2 + 2T_z \text{ for some } x, y, z \in \mathbb{Z}\\ \iff n+1 &= (x+y+1)^2 + (x-y)^2 + T_z \text{ for some } x, y, z \in \mathbb{Z}\\ \iff n+1 &= (2u+1)^2 + (2v)^2 + T_z \text{ for some } u, v, z \in \mathbb{Z}.\end{aligned}$$

Thus, an integer n cannot be represented by $4T_x + 4T_y + T_z$ if and only if n has the form $T_m - 1$ with $2m + 1$ having no prime divisors congruent to 3 modulo 4. Note also that the "if" part is equivalent to Oh and Sun [10, Corollary 1.1(ii)].

Although such counterexamples to the conjecture exist, the nature of such counterexamples is tractable, and we will prove a revised version of the conjecture, resolving the conjecture conclusively in each case.

Theorem 1.5. *Let m and n be any nonnegative integers. Then for a sufficiently large natural number r, depending on n and m, the following equations hold*

1. $$2^m x^2 + 2^n y^2 + T_z = r.$$

2. $$2^m x^2 + 2^n T_y + T_z = r$$

   whenever $8r + 2^n + 1$ is not a square. This condition is empty when $n < 3$.

3. $$2^m T_x + 2^n T_y + T_z = r$$

   whenever $8r+2^n+2^m+1$ is not a square, or when $n = 0$ (or, symmetrically, $m = 0$) and $8r + 2^m + 2$ ($8r + 2^n + 2$, respectively) is twice a square.

4. $$x^2 + 2^n \cdot 3y^2 + T_z = r$$

5. $$x^2 + 2^n \cdot 3T_y + T_z = r$$

6.

$$2^n \cdot 3x^2 + 2T_y + T_z = r$$

7.

$$2^n \cdot 3T_x + 2T_y + T_z = r$$

whenever $8r + 3 \cdot 2^n + 3$ *is not* 3 *times a square.*

8.

$$2^n \cdot 5T_x + T_y + T_z = r$$

whenever $8r + 5 \cdot 2^n + 2$ *is not 10 times a square.*

9.

$$2T_x + 3T_y + 4T_z = r.$$

10.

$$2x^2 + 3y^2 + 2T_z = r.$$

Remark 1.6. This result is best possible in the sense that Theorem 1.3 gives forms which do not represent infinitely many natural numbers r such that $8r + c$ are in each of the exceptional square classes $t\mathbb{Z}^2$ listed in Theorem 1.5. In particular, $x^2 + 16T_y + T_z$ does not represent r if $8r + 17 = p^2$, whenever p is any prime congruent to 1 or 3 modulo 8, $4T_x + 4T_y + T_z$ and $8T_x + T_y + T_z$ do not represent r if $8r + 9 = a^2$ or $8r + 10 = 2a^2$, respectively, whenever all prime divisors of a are congruent to 1 modulo 4, $192T_x+2T_y+T_z$ and $160T_x + T_y + T_z$ do not represent r if $8r + 195 = 3p^2$ or $8r + 162 = 5p^2$, respectively, whenever p is a prime congruent to 5 or 7 modulo 8. It is important to note that for any fixed m, n our method will be sufficient to determine whether every sufficiently large integer is represented, or if there are infinitely many natural numbers of the form $8r + c = t\mathbb{Z}^2$ which are not represented.

Sun also makes several concrete observations based on computational evidence for many of these forms to determine what is "sufficiently large." However, our proof relies on a lower bound for the class numbers, and is hence ineffective, so that we cannot determine an explicit bound on r. Under the assumption of the Generalized Riemann Hypothesis (GRH), we can verify that the list given by Sun is complete. Sun also makes the following explicit conjecture (Conjecture 3 in [18]).

Conjecture 1.7. *Every natural number can be written in the form* $x^2 + 2y^2 + 3T_z$ *except* $r = 23$*, in the form* $x^2 + 5y^2 + 2T_z$ *except* $r = 19$*, in the form* $x^2 + 6y^2 + T_z$ *except* $r = 47$*, and in the form* $2x^2 + 4y^2 + T_z$ *except* $r = 20$*.*

Although our methods are not sufficient to completely resolve this conjecture, due to the ineffective nature of our bounds, we are able to obtain a partial result and a conditional proof of Conjecture 1.7, with the help of a computer, using the method of Ono and Soundararajan [12] which was used to (conditionally) determine the integers represented by $x^2+y^2+10z^2$.

Theorem 1.8. *Every sufficiently large natural number may be written in each of the forms given in Conjecture 1.7.*

Moreover, assuming GRH for Dirichlet L-functions and GRH for the L-functions of weight 2 new forms, Conjecture 1.7 holds.

Table 1. Equivalent Quadratic Forms

Mixed Sum f	Exceptional r	Quadratic Form Q	Congruence	Exceptional r'
$x^2+2y^2+3T_z$	$\{23\}$	$2x^2+3y^2+4z^2$	$3 \pmod 8$	$\{187\}$
$x^2+5y^2+2T_z$	$\{19\}$	$x^2+y^2+20z^2$	$1 \pmod 4$	$\{77\}$
$x^2+6y^2+T_z$	$\{47\}$	$x^2+2y^2+12z^2$	$1 \pmod 8$	$\{377\}$
$2x^2+4y^2+T_z$	$\{20\}$	$x^2+4y^2+32z^2$	$1 \pmod 8$	$\{161\}$

Remark 1.9. In light of Theorem 1.8, there is an elliptic curve E for each form such that any counterexample r to Conjecture 1.7 will give a (specific) discriminant D_r such that $L(\chi_{D_r}, s)$ has a Siegel zero or a (specific) discriminant D'_r such that the L-series of the D'_r-th quadratic twist of E contains a Siegel zero. Here D_r and D'_r vary linearly in r as a constant times $8r+c$ with c the constant in front of the term T_z

In our proof of Theorem 1.8, we also show that Conjecture 1.7 leads to the following equivalent statements.

Proposition 1.10. *In Table 1, the mixed sum f represents precisely every natural number other than the exceptional set of r if and only if the quadratic form Q represents every natural number in the given congruence class other than the exceptional set of r'.*

Our methods will be based upon the theory of (ternary) quadratic forms and half-integral weight modular forms. A good reference for quadratic forms is [6] and the survey paper of Schulze-Pillot [13], while a good reference for modular forms is [11].

2. Representations by Sufficiently Large Integers

In this section we will show our main results, Theorems 1.5 and 1.3. We will first show the main result and then show how the counterexamples arise naturally from our proof.

Proof of Theorem 1.5. Consider one of the forms of the conjecture written as

$$f_{a,b}(x,y) = a_1x_1^2 + \cdots + a_{m_1}x_{m_1}^2 + b_1T_{y_1} + \cdots + b_{m_2}T_{y_{m_2}}.$$

We first note that (extending the definition of triangular number to $T_{-x} = -x(-x+1)/2$ for symmetry), $f_{a,b}(x,y) = r$ if and only if

$$Q_{a,b}(x,y) := 8a_1x_1^2+\cdots+8a_{m_1}x_{m_1}^2+b_1(2y_1+1)^2+\cdots+b_{m_2}(2y_{m_2}+1)^2 = 8r+\sum_{i=1}^{m_2} b_i. \tag{2.1}$$

This is obtained simply by multiplying both sides of the equation by 8 and then adding $\sum_{i=1}^{m_2} b_i$ to both sides. Therefore, we will consider sums of the type $Q_{a,b}(x,y)$. Note that in each of our cases $Q = Q_{a,b}$ is a (ternary) quadratic form (which we shall denote Q' for the quadratic form) with the added condition that the b_i terms must be odd.

Consider the associated theta-series

$$\theta_Q := \sum_{x,y,z\in\mathbb{Z}} q^{Q(x,y,z)} = \sum_{r\in\mathbb{N}} a_Q(r)q^r,$$

where $a_Q(r)$ is the number of representations of r by Q. We may write, using inclusion/exclusion,

$$\theta_{Q(x_1,x_2,y_1)} = \theta_{Q'(x_1,x_2,y_1)} - \theta_{Q'(x_1,x_2,2y_1)}, \tag{2.2}$$

$$\theta_{Q(x_1,y_1,y_2)} = \theta_{Q'(x_1,y_1,y_2)} - \theta_{Q'(x_1,2y_1,y_2)} - \theta_{Q'(x_1,y_1,2y_2)} + \theta_{Q'(x_1,2y_1,2y_2)}, \tag{2.3}$$

and

$$\begin{aligned}\theta_{Q(y_1,y_2,y_3)} = \theta_{Q'(y_1,y_2,y_3)} &- \theta_{Q'(2y_1,y_2,y_3)} - \theta_{Q'(y_1,2y_2,y_3)} - \theta_{Q'(y_1,y_2,2y_3)} \\ &+ \theta_{Q'(2y_1,2y_2,y_3)} + \theta_{Q'(2y_1,y_2,2y_3)} + \theta_{Q'(y_1,2y_2,2y_3)} - \theta_{Q'(2y_1,2y_2,2y_3)}.\end{aligned} \tag{2.4}$$

Thus, θ_Q is a sum of finitely many modular forms (the theta-series of the above quadratic forms), and is thus itself a modular form.

The theta-series of a ternary quadratic form decomposes as follows:

$$\theta_{Q'} = (\theta_{Q'} - \theta_{\text{Spn}(Q')}) + (\theta_{\text{Spn}(Q')} - \theta_{\text{Gen}(Q')}) + \theta_{\text{Gen}(Q')},$$

where $\theta_{\text{Gen}(Q')}$ denotes the weighted average over the genus and $\theta_{\text{Spn}(Q')}$ denotes the weighted average over the spinor genus. Moreover, $(\theta_{Q'} - \theta_{\text{Spn}(Q')})$ is a cuspidal modular form whose Shimura lift is also cuspidal, $(\theta_{\text{Spn}(Q')} - \theta_{\text{Gen}(Q')})$ is a cuspidal modular form in the space of lifts of one dimensional theta-series, where only $t\mathbb{Z}^2$ coefficients are supported (all others are equal to zero) for finitely many squarefree integers t dividing the discriminant (cf. Schulze-Pillot [13], p. 7-9). We will call ta^2 a (primitive) *spinor exception* for Q' if ta^2 is not (primitively) represented by the spinor genus of Q', and we will call $t\mathbb{Z}^2$ a *spinor exceptional class* for Q' if t is not represented by one of the spinor genera in the genus of Q'. The r-th coefficient of the weighted average of the genus grows like a certain class number (see Jones [6]) when r has bounded divisibility by the *anisotropic primes* (primes p dividing twice the discriminant in which the number of representations does not grow locally), and hence the r-th coefficient grows like

$$a_{\theta_{\text{Gen}(Q')}}(r) \gg r^{1/2-\epsilon},$$

whenever r is locally represented, by Siegel's (ineffective) lower bound for the class numbers [16]. Since the Shimura lift of $(\theta_{Q'} - \theta_{\text{Spn}(Q')})$ is cuspidal, Duke's bound [2] gives

$$a_{\theta_{Q'}-\theta_{\text{Spn}(Q')}}(r) \ll r^{3/7+\epsilon},$$

as observed by Duke and Schulze-Pillot [3]. Therefore, outside of the coefficients which are supported by $(\theta_{\text{Spn}(Q')} - \theta_{\text{Gen}(Q')})$ or $(\theta_{\text{Spn}(Q'(x,y,2z))} - \theta_{\text{Gen}(Q'(x,y,2z))})$, Equation (2.2) gives in that case

$$a_Q(r) = a_{\theta_{\text{Gen}(Q'(x,y,z))}}(r) - a_{\theta_{\text{Gen}(Q'(x,y,2z))}}(r) + O(r^{3/7+\epsilon}).$$

We now investigate the difference

$$a_{\theta_{\text{Gen}(Q'(x,y,z))}}(r) - a_{\theta_{\text{Gen}(Q'(x,y,2z))}}(r).$$

Using Siegel's averaging formula, the coefficients of these forms are given by the product of the limit of the number of solutions modulo a prime power p^m divided by p^m. Since these forms are equivalent p-adically for all primes $p \neq 2$, it follows that $a_{\theta_{\text{Gen}(Q'(x,y,2z))}}(r) = c_r a_{\theta_{\text{Gen}(Q'(x,y,z))}}(r)$, where c_r is a constant which only depends on r modulo a fixed power of 2. Clearly $c_r \leq 1$, since the number of representations of r with z arbitrary is less than or equal to the number of representations with z even. We then note that modulo a fixed power of 2 this difference of local densities is equal to the number of solutions with z odd. Moreover, Siegel's averaging theorem [17] shows that r is represented by one of the forms in the genus if and only if it is locally represented at all of the primes. Thus, $c_r = 1$ if and only if r is not locally represented by Q' with z odd.

Taking c_r' to be the weighted sum of the above c_r from the inclusion/exclusion, the same argument as above shows that Equations (2.3) and (2.4) also give

$$a_Q(r) = (1 - c_r')a_{\theta_{\text{Gen}(Q'(x,y,z))}}(r) + O(r^{3/7+\epsilon})$$

for coefficients not supported by $\theta_{\text{Spn}(Q'')} - \theta_{\text{Gen}(Q'')}$ for any Q'' occuring in the inclusion/exclusion, where $1 - c_r' = 0$ if and only if r is not locally represented.

Thus, any sufficiently large integer which is locally represented by Q and has bounded divisibility by the anisotropic primes, other than (possibly) spinor exceptional square classes $t\mathbb{Z}^2$, with t a squarefree divisor of twice the discriminant of Q', are represented globally by Q.

We will now proceed to show that each of the forms (1.1), (1.2), (1.3), (1.4), (1.5), (1.6), (1.7), (1.8), (1.9), and (1.10) give a form $Q_{a,b}$ which locally represents every integer of the form

$$8r + \sum_{i=1}^{m_2} b_i$$

from Equation (2.1). We will then show which possible exceptional square classes may occur in each case.

We first note that the anisotropic primes must divide twice the discriminant and hence in each case these can only be 2, 3, or 5. The prime 2 can be ignored, since the congruence conditions modulo 8 of the integers we would like to represent by $Q_{a,b}$ in each of our examples automatically implies bounded divisibility at 2. For those cases where 3 and 5 occur in the discriminant we use the fact that 2 is invertible p-adically so that we only need to check the local conditions at 3 and 5 for $n = 0$ or $n = 1$ and $m = 0$ or $m = 1$, verifying in each case that 3 and 5 are not anisotropic. Therefore we only need to check p-adically at each prime for existence of a solution.

It is well known that a solution exists for primes p relatively prime to the discriminant, so we only need to consider primes which divide the discriminant. For (1.1), (1.2), and (1.3), the discriminant is a power of 2, so we only need to consider solutions modulo a sufficiently large power of 2. Checking a fixed power of 2 and applying Hensel's lemma shows that the 2-adic conditions are indeed satisfied in each case.

Therefore, other than spinor exceptional square classes, we have the desired result for these three types of forms. Since the discriminants of each of these forms are a power of 2, the only possible spinor exceptional classes are $\mathbb{Z}^2$ and $2\mathbb{Z}^2$, so $t = 1$ or $t = 2$.

For forms of type (1.1) we note that $8r + 1$ is never 2 times a square and for a square $(2a + 1)^2 = 8r + 1$ we have the explicit solution $x = y = 0$ and $z = a$, so we obtain the desired statement.

For forms of type (1.2), $8r + 2^n + 1$ is even if and only if $n = 0$ (and in that case 2 $\pmod 8$). But in this case we have the solution $x = 0$ and $y = z = a$ for $2(2a + 1)^2$, so $t = 2$ does not appear in our analysis. When $n < 3$, $8r + 2^n + 1$ is not 1 or 0 mod 8, so squares do not appear in our analysis when $n < 3$. The case $t = 1$ gives the condition of the theorem.

For forms of type (1.3) $8r + 2^n + 2^m + 1$ is even if and only if $m > 0$ and $n = 0$ (up to symmetry), so we only need to consider twice a square when $n = 0$. The case $t = 1$ gives the other condition of the theorem.

Forms of type (1.4), (1.5), (1.6), (1.7), (1.9) and (1.10) give Q' with discriminant a power of two times 3. Therefore, for these forms we need to check the local condition at 2 and at 3. The 2-adic argument for forms of type (1.4), (1.5), (1.6), (1.7) follow exactly as above for the previous 3 types of forms, using Hensel's Lemma. For the 3-adic argument we only need to show that there is a solution modulo 9 and then use Hensel's Lemma. Since 2 is invertible modulo a 3 power, we only need to consider the cases $n = 0$ or $n = 1$. A simple check shows that the local conditions are satisfied in this case. The local conditions for the forms (1.9) and (1.10) follow directly by direct calculation.

Therefore, the result holds outside of the spinor exceptional square classes for these forms. For (1.9) and (1.10) the genus only has one spinor genus so there are no spinor exceptional square classes, and these follow immediately. For all others, the only possible spinor exceptional classes are $\mathbb{Z}^2$, $2\mathbb{Z}^2$, $3\mathbb{Z}^2$, and $6\mathbb{Z}^2$. For (1.4), $8r + 1 = ta^2$ only has a solution modulo 8 if $t_1 = 1$, but in this case we have the solution $x = y = 0$ and $z = a$. For forms of type (1.5) we have

$$8r + 3 \cdot 2^n + 1 \equiv \begin{cases} 1 & \text{if } n \geq 3 \\ 5 & \text{if } n = 2 \\ 7 & \text{if } n = 1 \\ 4 & \text{if } n = 0 \end{cases} \pmod 8$$

Hence we only need to consider the spinor exceptional class with $t = 1$ for $n \geq 3$ and $t = 2$ for $n = 1$. A quick check shows that $2\mathbb{Z}^2$ is not a spinor exceptional class for $n = 1$ since the genus equals the spinor genus in this case. Therefore only the case $t = 1$ is possible. However, Schulze Pillot [14] gives necessary and sufficient conditions p-adically for t to be a spinor exception. We will only need here the necessary condition 3-adically (which is due to Kneser [9]). Earnest, Hsia, and Hung have given an easy determination of when these conditions are satisfied [4]. They show that the necessary condition implies that if p is ramified in $\mathbb{Q}(\sqrt{-td})$ then

$$L_p \cong b_1 x^2 + b_2 3^r y^2 + b_3 3^s z^2$$

with b_i being p-adic units and $0 < r < s$. However, we have 3 ramified in $\mathbb{Q}(\sqrt{-3 \cdot 2^{n+3} t})$ whenever 3 does not divide t, and $r = 0$ in our case, so it follows that 1 cannot be a spinor exception for $8x^2 + 2^n 3y^2 + z^2$ for any n. But our sum (2.3) only contains quadratic forms of this type, and the result follows.

For (1.6), the congruence $8r + 3 \equiv 3 \pmod 8$ implies that only $t = 3$ may occur, but $x = 0$, $y = z = a$ gives a solution to $3a^2$. For (1.7) the congruence condition modulo 8 implies that only $t = 6$ is possible for $n = 0$, $t = 1$ for $n = 1$, and $t = 3$ is possible for

$n \geq 3$. A quick check for $n = 0$ and $n = 1$ show that these spinor exceptions do not occur, and we are left with the remaining condition.

Finally we show the result for forms of type (1.8). In this case, the discriminant is a power of 2 times 5, and the local conditions are shown as above. The only possible spinor exceptional classes are those with $t = 1$, $t = 2$, $t = 5$, and $t = 10$. We again look at the congruence conditions modulo 8 to determine that the only possible spinor exceptions equal to $8r + 5 \cdot 2^n + 2$ are twice a square or 10 times a square when $n \geq 3$. As in the case of (1.5) we then argue 5-adically to show that 5 must be a divisor of t, so the case $t = 2$ cannot occur.

□

We will now show that our counterexamples to the original conjecture are of the exceptional type arising from spinor exceptional square classes in the associated quadratic form, as evidenced in the above proof.

Proof of Theorem 1.3. In light of Theorem 1.5, for each of the counterexamples we will show that the associated form Q does not represent ta^2 for infinitely many integers a, with $t\mathbb{Z}^2$ the possible spinor exceptional square class which occurs as a condition in the given statement.

We will first show the case for $4T_x + 4T_y + T_z$. The associated form $Q'(x, y, z) := 4x^2 + 4y^2 + z^2$ is genus 1. Local conditions (modulo 8) show that the difference of sums to obtain $8r+9$ is $Q(x, y, z) = Q'(x, y, z) - Q'(x, 2y, z)$, since otherwise the local conditions are not satisfied. However, $Q''(x, y, z) := 4x^2 + 16y^2 + z^2$ is spinor genus 1. Therefore, $\theta_{Q'} - \theta_{\text{Spn}(Q')} = 0$, $\theta_{\text{Gen}(Q')} - \theta_{\text{Spn}(Q')} = 0$, and $\theta_{Q''} - \theta_{\text{Spn}(Q'')} = 0$, so that, calculating the constant in front of $\theta_{\text{Gen}(Q')}$ exactly,

$$\theta_Q = c\theta_{\text{Gen}(Q')} - (\theta_{\text{Spn}(Q'')} - \theta_{\text{Gen}(Q'')}),$$

so that $a_Q(8r + 9) = 2a_{Q'''}(8r + 9)$, where $Q''' = 4x^2 + 4y^2 + 5z^2 + 4xz$ is the (unique) representative of the other spinor genus in the genus of Q''. Therefore, r will be represented if and only if $8r + 9$ is represented by Q''', which is spinor genus 1 (and satisfies local conditions), and hence represents every integer of this type except for the spinor exceptions. However 1 is a spinor exception for Q''', so it follows from the work of Schulze-Pillot [14] that if p is a prime which splits in $K = \mathbb{Q}(\sqrt{-16}) = \mathbb{Q}(i)$ then Q''' will not represent p^2, and hence neither will Q. To determine completely the integers not represented by Q''', one may then follow Schulze-Pillot [14] to see that the integers r not represented are precisely those for which $8r + 9$ is a square which has divisors that all split in $\mathbb{Q}(i)$, which occurs if and only if every prime divisor of $8r + 9$ is congruent to 1 modulo 4.

For $8T_x+T_y+T_z$, we similarly have that $8x^2+y^2+z^2$ is genus 1 and that $32x^2+y^2+z^2$ is spinor regular with $2\mathbb{Z}^2$ a spinor exceptional square class. Again $K = \mathbb{Q}(\sqrt{-8 \cdot 2}) = \mathbb{Q}(i)$. Therefore, the integers not represented by $8T_x + T_y + T_z$ are precisely those r for which $8r + 10$ is twice a square a where all of the divisors of a are congruent to 1 modulo 4. We include this case as a second counterexample to (1.3) to show that both conditions which we have in the theorem are necessary.

For $x^2 + 16T_y + T_z$, the inclusion/exclusion sum gives $Q'(x, y, z) - Q'(x, 2y, z)$ with $Q' = 8x^2 + 16y^2 + z^2$, a genus 1 form. Since $\mathbb{Z}^2$ is a spinor exceptional square class for

$32x^2 + 16y^2 + z^2$ we argue as above to obtain $8r + 17$ is not represented by Q if and only if $8r + 17$ is a square with all divisors split in $\mathbb{Q}(\sqrt{-2})$.

For the other cases, $t\mathbb{Z}^2$ will be a spinor exceptional square class for both $Q'(x, y, z)$ and $Q'(2x, y, z)$ which is represented by both $Q'(x, y, z)$ and $Q'(2x, y, z)$. Therefore, for any prime p which is inert in $K := \mathbb{Q}(\sqrt{-td})$ (Here d is the discriminant of the form, which is the same up to a square for both quadratic forms), then tp^2 will not be represented primitively by the spinor genus of $Q'(x, y, z)$ or $Q'(2x, y, z)$, and hence not by these forms (see [15], page 352).

For $192T_x + 2T_y + T_z$ we have the spinor exceptional class from $t = 3$. Clearly, $Q(x, y, z)$ does not represent $t = 3$, because the odd condition dictates that the smallest integer represented by Q is $192 + 2 + 1 = 195$. Therefore, the number of representations of $t = 3$ by $Q'(x, y, z)$ and $Q'(2x, y, z)$ are equal. Fix an arbitrary prime p inert in $K = \mathbb{Q}(\sqrt{-2})$. Since there are no primitive representations of tp^2 by $Q'(x, y, z)$ and $Q'(2x, y, z)$, it follows that the number of representations of tp^2 by $Q'(x, y, z)$ equals the number of representation of tp^2 by $Q'(2x, y, z)$. For $160T_x + T_y + T_z$, we have $t = 10$ and $K = \mathbb{Q}(i)$, and the argument follows as above. □

3. GRH and Mixed Sums

In [8], the author considers sums of the type $f_{a,b}$ where $a = 0$. Using the decomposition given in Equation (2.4), an algorithm is shown to determine, conditional upon GRH, which integers are represented by $f_{0,b}$. This is based upon an algorithm described by Ono and Soundararajan [12] to determine the integers represented by the particular form $x^2 + y^2 + 10z^2$, as generalized to more general forms by the author in [7]. We will briefly explain the theory behind the algorithm and then use the algorithm to conclude Theorem 1.8.

We start by decomposing the associated quadratic form Q' as described in the previous section, namely

$$\theta_{Q'} = (\theta_{Q'} - \theta_{\text{Spn}(Q')}) + (\theta_{\text{Spn}(Q')} - \theta_{\text{Gen}(Q')}) + \theta_{\text{Gen}(Q')}.$$

For $(\theta_{\text{Spn}(Q')} - \theta_{\text{Gen}(Q')})$, we use the results of Schulze-Pillot [14] to determine all (primitive) spinor exceptions. Outside of these finitely many square classes, we have a cuspidal weight $3/2$ modular form $g := (\theta_{Q'} - \theta_{\text{Spn}(Q')})$ whose Shimura lift is cuspidal plus $E_{Q'} := \theta_{\text{Gen}(Q')}$. Since the Shimura lift of g is cuspidal and the Hecke operators commute with the Shimura lift, we may further decompose g into

$$g = \sum_{i=1}^{m} b_i g_i,$$

where g_i are a fixed set of weight $3/2$ eigenforms which each lift to weight 2 normalized eigenforms G_i ($a_{G_i}(1) = 1$) under our choice of Shimura lift. One then uses the following result of Waldspurger [20].

Theorem (Waldspurger)**.** *Let a weight $3/2$ Hecke eigenform g_i of level N with Nebentypus χ whose Shimura lift is G_i. Then if $r_1/r_2 \in \mathbb{Q}_p^{x^2}$ for every $p \mid N$,*

$$a_{g_i}^2(r_1)L(G_i, \left(\frac{-1}{\cdot}\right)\chi^{-1}\chi_{r_2}, 1)\chi\left(\frac{r_2}{r_1}\right)r_2^{1/2} = a_{g_i}^2(r_2)L(G_i, \left(\frac{-1}{\cdot}\right)\chi^{-1}\chi_{r_1}, 1)r_1^{1/2},$$

where $L(G_i,\chi',s)$ is the L-series of G_i twisted by the character χ'.

We then find a representative r_2 modulo squares in $\mathbb{Q}_p$ so that the coefficient $a_{g_i}(r_2)\neq 0$ (if one exists). If we define

$$c_i := \frac{a_{g_i}^2(r_2)}{r_2^{1/2}L(G_i,\left(\frac{-1}{\cdot}\right)\chi^{-1}\chi_{r_2},1)}$$

then for each r_1 equivalent to r_2 modulo squares, we have

$$a_{g_i}^2(r_1)=\frac{c_i}{\chi(r_1/r_2)}L(G_i,\chi'_{r_1},1)r_1^{1/2}.$$

Now we note that to obtain the theta series for Q we are taking the sums and differences of finitely many of these theta series for quadratic forms Q'', and in each case $E_{Q''}=c_{Q''}E_{Q'}$, where $c_{Q''}$ is some constant which only depends modulo squares 2-adically. But, as shown above, $\sum_{Q''}c_{Q''}>0$ whenever the integer is represented locally with the appropriate odd conditions. Thus, taking the sum of each of these from the appropriate Equation (2.2), (2.3), or (2.4) we have, for integers represented locally,

$$a_Q(r)=ca_E(r)+\sum_{i=1}^{m}b_i\sqrt{\frac{c_i}{\chi(r/r_2)}L(G_i,\chi'_r,1)r^{1/2}}.$$

For r square free, the coefficients $a_E(r)$ are certain class numbers, so Dirichlet's class number formula (cf. [1]) allows us to write

$$a_E(r)=c'L(\chi''_r,1)r^{1/2},$$

where $L(\chi''_r,s)$ is the L-series of the appropriate character χ''_r. We then simply note that r is not represented by Q if and only if $a_Q(r)=0$, and then rearrange and divide by $L(\chi'',1)$, bounding the ratios $L(G_i,\chi'_r,1)/L(\chi'',1)^2$ using the bounds in [7].

We now use this algorithm to solve the conjecture assuming GRH.

Proof of Theorem 1.8. We first note that in each of these cases we have $Q(x,y,z)=Q'(x,y,z)-Q'(x,y,2z)$ with $Q'(x,y,z)=8ax^2+8by^2+cT_z$ for some a,b, with $c=1,2$ or 3. Thus there are no solutions to $8r+c=Q'(x,y,2z)\pmod 8$, so $Q(x,y,z)=Q'(x,y,z)$. We check the local conditions for Q' and note that Q' does not have c as a spinor exception (the only one possible because of the congruence conditions modulo 8) in each case (one can merely check trivially that it represents c). Therefore, every sufficiently large integer is represented by the form.

We now proceed to show Proposition 1.10 and then use our algorithm given above to determine the integers represented in each case. In each case, the resulting form is genus two and thus will decompose as $E+g$, with E the weighted average among the genus and g a Hecke eigenform (hence, since g has rational coefficients, its lift G will be the L-series of an elliptic curve). We then in each case use an argument similar to that given in [7] to determine (unconditionally) that all of the non squarefree integers must be represented by the form.

For $r = x^2+5y^2+2T_z$, we get $Q'(x,y,z) = 8x^2+40y^2+2z^2$. But $Q'(x,y,z) = 8r+2$ if and only if

$$4x^2 + 20y^2 + z^2 = 4r + 1.$$

However, solutions to $x_2^2 + 20y_2^2 + z^2 = 8r + 1$ can only exist if x is even (up to symmetry of x and z), so we have half the number of solutions to

$$x^2 + 20y^2 + z^2 = 4r + 1,$$

giving the assertion of Proposition 1.10. We then use our algorithm to show the integers (not) represented by $x^2+20y^2+z^2$ and check which are 1 modulo 4. This form is genus 2, so the theta series decomposes as $E+g$ with g a Hecke eigenform. Using the algorithm in [8], all squarefree integers 1 mod 4 greater than 12288 (we get three different bounds depending on the value of $\left(\frac{-20}{4r+1}\right)$, and take the largest one) are represented by $x^2 + 20y^2 + z^2$. A quick computer check then verifies that the only squarefree integer smaller than 10^8 which is 1 modulo 4 and not represented by $x^2 + 20y^2 + z^2$ is 77.

For $x^2 + 2y^2 + 3T_z$, we need to find solutions to $8r + 3 = 8x^2 + 16y^2 + 3z^2$. However, any solution to $2x_2^2 + 4y_2^2 + 3z^2 = 8r + 3$ must have x and y even, so this is equivalent to $2x^2 + 4y^2 + 3z^2 = 8r + 3$ and we get the assertion of Proposition 1.10. The form is genus 2, so the theta series decomposes as $E + g$ with g a Hecke eigenform. We then use the algorithm in [8] to show that every squarefree integer which is 3 modulo 8, relatively prime to 3 and greater than 1.89×10^9 is represented by $2x^2 + 4y^2 + 3z^2$, while those which are not relatively prime to 3 and greater than 21291 are represented. We then do a quick check by computer to verify that every natural number less than 5×10^{10}, other than 187, which is congruent to 3 modulo 8 is represented by $2x^2 + 4y^2 + 3z^2$.

Next we consider $x^2 + 6y^2 + T_z$. In this case we have solutions to $8r + 1 = 8x^2 + 48y^2 + z^2$. But the number of solutions to $2x^2 + 12y^2 + z^2 = 8r + 1$ equals the number of solutions to $8r+1 = 8x^2+48y^2+z^2$, so r is represented if and only if $8r+1$ is represented by $2x^2 + 12y^2 + z^2$, verifying the statement in Proposition 1.10. As above, our algorithm shows the result for every natural number less than 1.6×10^8. We then check by computer to verify that every natural number less than 2×10^9, other than 377, which is congruent to 1 modulo 8 is represented by $2x^2 + 12y^2 + z^2$.

Finally, for $2x^2 + 4y^2 + T_z$, we need to find solutions to $8r + 1 = 16x^2 + 32y^2 + z^2$. Similarly to above, the number of solutions to $8r+1 = 4x^2+32y^2+z^2$ equals the number of solutions of $8r+1 = 16x^2+32y^2+z^2$, verifying the statement in Proposition 1.10. Our algorithm shows the result for every natural number greater than 5.2×10^8. We then check by computer to verify that every natural number less than 5×10^{10}, other than 161, which is congruent to 1 modulo 8 is represented by $4x^2 + 32y^2 + z^2$. □

Acknowledgements. The author would like to thank Prof. T. H. Yang for bringing the problem to his attention, and would also like to thank Prof. Z.-W. Sun and the anonymous referee for helpful comments about the exposition of the paper.

References

[1] H. Davenport, *Multiplicative Number Theory*, Springer, New York, 1980.

[2] W. Duke, *Hyperbolic distribution problems and half-integral weight Maass forms*, Invent. Math. **92**(1988), 73–90.

[3] W. Duke, R. Schulze-Pillot, *Representations of integers by positive ternary quadratic forms and equidistribution of lattice points on ellipsoids*, Invent. Math. **99**(1990), 49–57.

[4] A. G. Earnest, J. S. Hsia, and D. Hung, *Primitive representations by spinor genera of ternary quadratic forms*, J. London Math. Soc. (2)**50**(1994), 222–230.

[5] S. Guo, H. Pan and Z.-W. Sun, *Mixed sums of squares and triangular numbers (II)*, Integers **7**(2007), # A56, 5pp (electronic).

[6] B. Jones, *The arithmetic theory of quadratic forms*, Carcus Monograph Series, No. 10, Math. Assoc. Amer., Buffalo, New York, 1950, 212 pp.

[7] B. Kane, *Representations of integers by ternary quadratic forms*, Int. J. Number Theory, to appear.

[8] B. Kane, *Representing sets with sums of triangular numbers*, preprint, 2008.

[9] M. Kneser, *Darstellungsmasse indefiniter quadratischer formen*, Math. Z. **11**(1961), 188–194.

[10] B.-K. Oh and Z.-W. Sun, *Mixed sums of squares and triangular numbers (III)*, J. Number Theory **129**(2009), 964–969.

[11] K. Ono, *Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-series,* Amer. Math. Soc., Providence, R.I., 2003.

[12] K. Ono and K. Soundararajan, *Ramanujan's ternary quadratic form*, Invent. Math. **130**(1997), 415–454.

[13] R. Schulze-Pillot, *Representations by integral quadratic forms - a survey*, in: Algebraic and Arithmetic Theory of Quadratic Forms, pp. 303–321, Contemp. Math., 344, Amer. Math. Soc., Providence, R.I., 2004.

[14] R. Schulze-Pillot, *Darstellung durch spinorgeschlechter ternärer quadratischer formen*, J. Number Theory **12**(1980), 529–540.

[15] R. Schulze-Pillot, *Exceptional integers for genera of integral ternary positive definite quadratic forms*, Duke Math. J. **102**(2000), 351–357.

[16] C. Siegel, *Über die klassenzahl quadratischer zahlkörper*, Acta Arith. **1**(1936), 83–86.

[17] C. Siegel, *Über die analytische theorie der quadratischen formen*, Ann. Math. **36**(1935), 527–606.

[18] Z.-W. Sun, *Mixed sums of squares and triangular numbers*, Acta. Arith. **127**(2007), 103-113.

[19] Z.-W. Sun, *A message to number theory mailing list*, April 27, 2008. http://listserv.nodak.edu/cgi-bin/wa.exe?A2=ind0804&L=nmbrthry&T=0&P=1670.

[20] J.-L. Waldspurger, *Sur les coefficients de Fourier des formes modulaires de poids demi-entier*, J. Math. Pures Appl. **60**(1981), 375–484.