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SUMS OF TRIANGULAR NUMBERS AND t-CORE PARTITIONS

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Abstract

We prove a refinement of the t-core conjecture proven by Granville and Ono. We show that for every $n \geq g$ there are at least g partitions of n which are tg-core partitions but not g-core partitions unless $t = g = 2$ and $n = 4$ or $n = 10$. When investigating the case $t = g = 2$, we study the number of solutions to the equation $2\left(\binom{x+1}{2} + \binom{y+1}{2}\right) + \binom{z+1}{2} = n$ with $x, y, z \in \mathbb{Z}$.

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1. Introduction and Statement of Results

The theory of t-core partitions has played a role in a variety of areas. For example, when $t = p$ is prime they characterize p-modularly irreducible representations of the symmetric group S_n. Due to this connection, they have been studied by representation theorists such as in [8, 9, 14, 17]. They also played a role in establishing cranks in [6] and [5], which were used to show a combinatorial proof of Ramanujan's famous congruences for the partition function.

The *Ferrers-Young diagram* of the partition $\Lambda = (\lambda_1, \ldots, \lambda_d)$ of $n = \lambda_1 + \cdots + \lambda_d$ is formed by arranging n nodes in rows so that there are λ_i nodes in the i-th row. The *hook number* of a node is the number of nodes in the Ferrers-Young diagram to the right of the node plus the number of nodes below this node, plus one for the node itself.

Definition 1.1. *A t-core partition of n is a partition of n whose Ferrers-Young diagram has no hook numbers which are a multiple of t.*

Granville and Ono [7], using Lagrange's Four Square Theorem and the theory of modular forms, have shown that every nonnegative integer n may be partitioned by a t-core partition whenever $t \geq 4$ and otherwise the set of such n has measure zero. Using this result when $t = p$ is a prime, Granville and Ono completed the classification of simple groups with defect zero Brauer p-blocks. We shall show the following refinement of this theorem.

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Theorem 1.2. *Let n, t, g be positive integers such that $tg \geq 4$ and $t > 1$. Then there exists a tg-core partition of n which is not a g-core if and only if $n \geq g$. Moreover, there are at least g such partitions unless $g = t = 2$ and $n = 4$ or $n = 10$, in which case there is one.*

Remark 1.3. *This is best possible in the sense that every partition of $n < g$ is a g-core and when $n = g$ there are precisely g such partitions.*

Studying the case when $t = 2$ leads to an investigation of sums of triangular numbers. Gauss showed the following famous Eureka Theorem to determine representability by sums of triangular numbers.

Theorem 1.4 (Gauss)**.** *Every nonnegative integer n can be represented in the form*

$$\binom{x+1}{2} + \binom{y+1}{2} + \binom{z+1}{2}$$

with $x, y, z \in \mathbb{Z}$.

Let us next consider sums of the form

$$Q_k(x,y,z) := \binom{x+1}{2} + k\left(\binom{y+1}{2} + \binom{z+1}{2}\right).$$

We will see that a result similar to Theorem 1.4 holds for Q_2, which determines the number of 4-core partitions.

Theorem 1.5. *Let n be a nonnegative integer. Then the equation $Q_2(x,y,z) = n$ has more than one solution unless $n = 0, 1, 4$ (in which case it has exactly one), and it has more than two solutions unless $n = 0, 1, 2, 4, 10, 11, 16, 31$. Also, there are at least three 4-core partitions of n unless $n = 0, 1, 2, 4, 10, 11, 16, 31$.*

Remark 1.6. *Note that since 2 is not a triangular number $Q_k(x,y,z) = 2$ cannot have a solution for $k > 2$, and hence $k = 1$ and $k = 2$ are the only positive choices of k for which $Q_k(x,y,z) = n$ always has a solution.*

2. Sums of Triangular Numbers and 4-core Partitions

Investigating the case $t = g = 2$ quickly leads to determining the number of solutions to $Q_2(x,y,z) = n$.

Proof of Theorem 1.5. It is well known that the generating function for t-cores is

$$\psi_t(q) := \prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{1-q^n} \tag{2.1}$$

and clearly satisfies

$$\psi_{ab}(q) = \psi_a(q^b)^b \psi_b(q). \tag{2.2}$$

Thus, the generating function for 4-cores may be written as

$$\psi_2(q^2)^2 \psi_2(q).$$

However, it is well known that $\psi_2(q) = \sum_{n\geq 0} q^{\binom{n+1}{2}}$, so the number of 4-core partitions of n is precisely the number of solutions to $Q_2(x, y, z) = n$. Observing this, Ono has shown in [11] that the number of 4-core partitions of n is precisely equal to the number of solutions to $8n + 5 = x^2 + 2y^2 + 2z^2$. Using the fact that $x^2 + 2y^2 + 2z^2$ is a genus 1 quadratic form, Ono and Sze [13] showed that the number of solutions is exactly

$$\frac{1}{2} \sum_{\substack{D|8n+5 \\ \frac{8n+5}{D}=u^2}} h(-4D). \tag{2.3}$$

Here $h(-D)$ is the class number of $\mathbf{O}_{-D}$, the size of the group of fractional ideals of the order $\mathbf{O}_{-D}$ modulo the principal ideals. We are interested in when (2.3) is 1 or 2. Due to the factor of $\frac{1}{2}$, this corresponds to the classification of orders with class number 1, 2, 3, and 4. Given the class number of $\mathbf{O}_{-D}$ for a fundamental discriminant $-D$, Dirichlet's class number formula gives the class number for $\mathbf{O}_{-Dc^2}$ explicitly, so it suffices to determine the imaginary quadratic fields with class number less than 5. Using Baker [3, 4] and Stark's [15, 16] independent solutions to Gauss's class number one and class number 2 problems, (2.3) equals 1 only when $n = 0, 1,$ or 4. By determining an effective lower bound for the class number, Oesterlé [10] solved the class number 3 problem, and Arno [1] solved the class number 4 problem, showing in our case that there are two solutions precisely when $n = 2, 10, 11, 16,$ or 31. □

Remark 2.1. *Let an integer N be given. Using Oesterlé's bound [10]*

$$h(-d) > \frac{1}{7000} \prod_{p|d}^{*} \left(1 - \frac{\lfloor 2\sqrt{p} \rfloor}{p+1}\right) \ln d,$$

where the $*$ indicates that the largest prime divisor is not included, or the known lists for imaginary quadratic fields with class number less than $2N$ when N is small enough, one can extend this argument to give a complete list of n for which (2.3) is less than N, or, equivalently, there are less than N 4-core partitions of n.

3. Partitions which are tg-core but not g-core

We will use the theory of modular forms, Theorem 1.4, Theorem 1.5, and the generating function for partitions which are simultaneously s-cores and t-cores (denoted here as s/t-cores) to show Theorem 1.2.

Proof of Theorem 1.2. We first note that for $n < g$ every partition of n is clearly a g-core since every hook number is less than g. This concludes the only if portion of the statement.

Assume that $g \geq 4$. Here we will have the stronger result that there are always at least g tg/sg-core partitions of n which are not g-cores for any $s > 1$ and $t > 1$. In [2] it was established that the generating function for partitions which are $sg/(tg)$-cores is given by

$$(\psi_{s,t}(q^g))^g \cdot \psi_g(q), \tag{3.1}$$

where $\psi_{s,t}$ is the generating function for partitions which are s/t-cores and $\psi_g(q) =: \sum_{n\geq 0} a_n q^n$ is the generating function for g-core partitions. In [2], the statement of equation (3.1) is restricted to the case where s and t are relatively prime, but the result holds more generally by using equation (2.2). From (3.1), the generating function for partitions which are sg/tg-core but not g-core is

$$[(\psi_{s,t}(q^g))^g - 1] \cdot \psi_g(q). \tag{3.2}$$

We know from Granville and Ono [7] that $a_n > 0$ for every n. Noting that the unique partition (1) of 1 is always an s/t-core, equation (3.2) equals

$$\sum_{n\geq 0} b_n q^n := \left[gq^g + O(q^{2g})\right] \cdot \sum_{n\geq 0} a_n q^n,$$

where the coefficients in $O(q^{2g})$ are all nonnegative. Hence for every $n \geq 0$, $b_{n+g} \geq g a_n \geq g$. This establishes the result for $g \geq 4$.

Now consider $g < 4$ and $t \geq 4$. Using equation (2.2) and subtracting $\psi_g(q)$, the generating function for tg-cores which are not g-cores is

$$[(\psi_t(q^g))^g - 1] \cdot \psi_g(q).$$

Since every integer n is partitioned by a t-core, it follows that the generating function may be written as

$$\left[\sum_{n\geq 1} g q^{gn} + c_n q^{gn}\right] \sum_{n\geq 0} a_n q^n = \sum_{n'\geq 0, m\geq 1} g a_{n'} q^{n'+gm} + a_{n'} c_m q^{n'+gm},$$

where $c_n \geq 0$ and a_n is the number of g-core partitions of n. Let an integer $n \geq g$ be given. Then $n = n' + gm$ for some $0 \leq n' < g$ and $m \geq 1$. Clearly $a_{n'} > 0$ because every partition is a g-core, so that we know there exist at least g such tg-core partitions of n which are not g-cores, establishing the result when $t \geq 4$. The only remaining cases are $t = g = 2$, $t = g = 3$, $t = 2$ and $g = 3$, and, finally, $t = 3$ and $g = 2$.

For $t = g = 2$, the generating function for 4-cores which are not 2-cores is the number of solutions to $Q_2(x, y, z) = n$ minus the number of solutions to $n = \binom{x+1}{2}$, so that we have the desired bound by Theorem 1.5, since $n = 2, 4, 11, 16,$ and 31 are not triangular numbers.

When $tg = 6$, we use (2.2) to write the generating function for 6-cores as

$$(\psi_2(q^3))^3 \psi_3(q) =: \sum_{n\geq 0} a_n q^n.$$

We first check $n \leq 10$ by hand for $g = 2$ and $g = 3$. Noting that the n-th coefficient of $\psi_2^3(q)$ is the number of ways that n may be written as the sum of 3 triangular numbers, Theorem 1.4 gives

$$(\psi_2(q^3))^3 = \sum_{n\geq 0} (1 + e_n) q^{3n},$$

where $e_n \geq 0$. For $g = 3$, we obtain the generating function

$$\left(\sum_{n\geq 1}(1+e_n)q^{3n}\right)\cdot \psi_3(q)$$

for 6-cores which are not 3-cores, so that the result follows for $n > 10$ from the fact that there is a 3-core for $0, 1, 2, 4, 5, 6, 8, 9, 10$. For $g = 2$, we note that it sufficies to show that $a_n \geq 3$. Since there is a 3-core of $0, 1, 2, 4, 5, 6, 8, 9, 10$, it follows that for $a_n \geq 3$ when $n \geq 10$.

The only remaining case is $t = g = 3$. We again check $n \leq 10$ by hand. The generating function for 9-cores is given as above by

$$\psi_3(q^3)^3\psi_3(q). \tag{3.3}$$

As observed by Ono in [12], $\psi_3(q^3)^3$ is a weight 3 Eisenstein series on $\Gamma_0(3)$ with Dirichlet character $\varepsilon(n) = \left(\frac{n}{3}\right)$, so it can be expanded as

$$\psi_3(q^3)^3 = \sum_{n\geq 1}\sigma_{2,\varepsilon}(n)q^{3(n-1)},$$

where

$$\sigma_{2,\varepsilon}(n) := \sum_{D|n}\varepsilon\left(\frac{n}{D}\right)D^2.$$

It is easy to check that $\sigma_{2,\varepsilon}(n) > 0$ for every n. Therefore, as above, subtracting $\psi_3(q)$ from (3.3) gives

$$\left(\sum_{n\geq 2}\sigma_{2,\varepsilon}(n)q^{3(n-1)}\right)\psi_3(q).$$

Hence, the result follows analogously to the other cases for $n > 10$, since there is a 3-core partition of $0, 1, 2, 4, 5, 6, 8, 9$, and 10. □

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