Simultaneous s-cores and t-cores

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1 Introduction

A *partition* of $n \in \mathbb{N}$ is a finite sequence of positive integers $a_1, \ldots, a_k$ such that $a_1 \geq a_2 \geq \cdots \geq a_k$ and $\sum_{i=1}^{k} a_i = n$. We will call such a partition *standard*. This definition can be extended to include *partitions with parts of size zero*, by allowing a_i to be zero. We define $p(n)$ to be the number of partitions of n. There are many counting problems which use $p(n)$. For instance, if we have n computers, and we want to count the number of ways to assign n identical tasks to these computers, allowing multiple tasks to run on the same computer, then this is simply $p(n)$. The rate at which $p(n)$ grows with respect to n is surprising. The following asymptotic, due to Hardy and Ramanujan [6], is well known.

$$p\left(n\right) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}$$

For more information involving partition theory, see [1].

The *Ferrers-Young Diagram* of the partition $a_1, \ldots, a_k$ is a pictorial representation of a partition consisting of k rows of nodes with a_i nodes in the i^{th} row. For instance, consider partition P with $n = 10$, $a_1 = 5$, $a_2 = 3$, and $a_3 = 2$, we get the following Ferrers-Young Diagram:

$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 \\ 1 & \bullet & \bullet & \bullet & \bullet & \bullet \\ 2 & \bullet & \bullet & \bullet & & \\ 3 & \bullet & \bullet & & & \end{array}.$$

We will reference a node v with the ordered pair of $(row_v, column_v)$, indicating v's position in the diagram. We also define the *Ferrers-Young Diagram* *for a partition with parts of size zero* in the natural way (i.e. we will allow the Ferrers-Young diagram to have some empty rows).

In the Ferrers-Young Diagram, the number of nodes directly to the right of a node plus the number of nodes directly below the node plus one for the node itself forms the node's *hook number*. Partition P from the previous example has the following hook numbers:

$$\begin{matrix} 7 & 6 & 4 & 2 & 1 \\ 4 & 3 & 1 & & \\ 2 & 1 & & & \end{matrix}.$$

We define the hook number for nodes in the Ferrers-Young Diagrams of partitions with parts of size zero in the same manner, except that when calculating the first column's nodes' hook numbers, we shall replace "the number of nodes directly below the node" with "the number of rows below the node." If we add an empty row to our partition P, then we would have the following hook numbers:

$$\begin{matrix} 8 & 6 & 4 & 2 & 1 \\ 5 & 3 & 1 & & \\ 3 & 1 & & & \\ 0 & & & & \end{matrix}.$$

The *rim hook* of a node v is the set of nodes on the right-hand boundary of the Ferrers-Young diagram connecting the node at the right end of v's row to the node at the bottom of v's column. For instance, the rim hook for (2,1) in P's diagram is $\{(3,1),(3,2),(2,2),(2,3)\}$. The size of the rim hook set of a node is identical to the hook number of that node. The nodes in the rim hook for (2,1) are marked below by an X, and (2,1) is marked with a $\circ$.

$$\begin{matrix} & 1 & 2 & 3 & 4 & 5 \\ 1 & \bullet & \bullet & \bullet & \bullet & \bullet \\ 2 & \circ & X & X & & \\ 3 & X & X & & & \end{matrix}.$$

For a set A with $0 \in A$, we will define $A^{\times}$ to be the set $A \setminus \{0\}$. For some $t \in \mathbb{N}^{\times}$, and a partition p, we call p a *t-core* if its Ferrers-Young Diagram has no hook numbers which are a multiple of t. Counting the number of t-core partitions of n is useful in approximating $p(n)$[4].

The topic of this thesis is the study of the existence and structure of partitions that are simultaneously s and t cores, for some $s, t \in \mathbb{N}^{\times}$, which we shall denote as s/t-cores. This question was first addressed by Jackie Kohles, who considered s/t-cores for s and t relatively prime[9].

The Ferrers-Young Diagram of any 2-core must be triangular, since the length must differ by exactly 1 from each row to the next. Therefore, there is a 2-core partition of k if and only if k is a triangular number, a number of the form $\frac{i(i+1)}{2} = \sum_{j=1}^{i} j$. Furthermore, this 2-core partition of k is unique. While it is known that $\exists n \in \mathbb{N}^{\times}$ such that no 3-core of n exists, there is no known classification of which n have this property. Ono recently showed that for every $n \in \mathbb{N}^{\times}$, t-core partitions of n exist for every $t \geq 4$[5]. For t prime, this result was of independent interest in the Representation Theory of S_n; in particular, it turns out that a t-core partition of n gives us an object known as a defect zero t Brauer block[8]. In section 3, we shall define and explore this relationship between the representation theory of S_n and the theory of partitions.

Our investigation of this subject hinges on another visual representation of a partition, called an *abacus*. Given $t \in \mathbb{N}^{\times}$, a t-abacus has t columns numbered from 0 to $t-1$, and any number of rows starting with row 0. Some of the positions of this abacus are occupied by beads. Moving beads up and down in a column will play an important role in this representation, so we will consider an abacus with beads on strings that form the columns. A mapping from the natural numbers to positions on the abacus is defined as follows: the number $kt+j$, $0 \leq j \leq t-1$, corresponds to position (k, j) in the abacus. We shall henceforth refer to the number corresponding to a position in the abacus as the position's *structure number*. We will use the notation St_p to denote the set of structure numbers of the positions occupied by beads in the t-abacus representation of the partition p. If we are given a partition p, the t-abacus representation of p is the t-abacus with structure numbers equal to the first column hook numbers of the Ferrers-Young Diagram of p. Thus, St_p equals the set of first column hook numbers. If we do not allow the structure number 0, then there is an isomorphism between partitions and t-abacus representations. The 3-abacus representation of our partition

P would be as follows:

	0	1	2
0	$\vert$	$\vert$	$\bullet_2$
1	$\vert$	$\bullet_4$	$\vert$
2	$\vert$	$\bullet_7$	$\vert$
3	$\vert$	$\vert$	$\vert$

If we include the structure number 0, then we have an isomorphism between the set of all t-abacus representations and the Ferrers-Young Diagrams of partitions with parts of size zero. However, if we remove all of the empty rows in the Ferrer-Young Diagram with zero part sizes, then we get a standard Ferrers-Young Diagram. The action on an abacus corresponding to removing one of these rows is called a *rotation*. In a rotation, each structure number is reduced by one, and the bead in the (0,0) position is removed.

The relationship between t-cores and t-abacus representations will be useful in our investigation of t-cores. Therefore, we will use the following lemma from [8,3, and 11].

Lemma 1. *A partition is a t-core if and only if $\forall i, j \in \mathbb{N}$, a bead in the $(i+1, j)$ position of the t-abacus implies that there is a bead in the (i, j) position, and there is no bead in the $(0,0)$ position.*

Observing that the above abacus does not fulfill the necessary properties from this lemma gives us that P is not a 3-core.

Using an abacus argument, Jackie Kohles investigated s/t-cores with s and t relatively prime. In [9], she reached the following result:

Theorem 1. *(Kohles) When $s \in \mathbb{N}$ and $t \in \mathbb{N}$ are relatively prime, there are exactly $\frac{\binom{s+t}{s}}{s+t}$ s/t-cores.*

We now state the central results of this thesis. We begin by considering s/t-cores for s and t not relatively prime. A partition is a $gcd(s,t)$-core if and only if it has no hook numbers which are a multiple of $gcd(s,t)$. But in this case, it must not have any hook numbers which are a multiple of s or t. Therefore, it is an s/t-core. So it is only interesting to consider partitions which are s/t-cores and not $gcd(s,t)$-cores. Although there are only finitely many s/t-cores when $gcd(s,t) = 1$, we will show that

Theorem 2. *For distinct $s \in \mathbb{N}^\times$ and $t \in \mathbb{N}^\times$ such that $gcd(s,t) > 1$ there are infinitely many s/t-cores which are not $gcd(s,t)$-cores.*

We define $s' := \frac{s}{gcd(s,t)}$ and $t' := \frac{t}{gcd(s,t)}$ so that we may easily reference these values henceforth in the paper. Furthermore, we determine a complete $gcd(s,t)$-abacus description of these partitions given a similar (as of yet undetermined) representation of s'/t'-cores, based on the following definition: We say that the partition p is *placed in the i^{th} column of the k-abacus of q* if the row numbers of the beads in the i^{th} column of the k-abacus of q are identical to the structure numbers of p. For instance, the partition with structure numbers 2,4, and 6 (i.e. the partition $(4,3,2)$) is placed in the 1^{st} column of the following 3-abacus, marked here with $\circ$'s:

$$\begin{array}{cccc}
 & 0 & 1 & 2 \\
0 & | & | & \bullet \\
1 & | & | & | \\
2 & | & \circ & | \\
3 & | & | & | \\
4 & \bullet & \circ & | \\
5 & | & | & | \\
6 & \bullet & \circ & |
\end{array}$$

Theorem 3. *A partition p with no parts of size zero is an s/t-core with $gcd(s,t) > 1$ if and only if the partition placed in each column of the $gcd(s,t)$-abacus of p is an s'/t'-core partition, possibly with part sizes of zero. The partition placed in the 0^{th} column, however, cannot have parts of size zero.*

Our investigation of s/t-core partitions goes further than the above result, however. We add additional interesting properties to s/t-cores and giving a description of these partitions. One such property involves transposing the Ferrers-Young Diagram of a partition p about the diagonal, so that the first column becomes the first row and the first row becomes the first column. We call this new partition the *conjugate of partition p*. The following is the Ferrers-Young Diagram of the conjugate of P.

$$\begin{array}{cccc}
 & 1 & 2 & 3 \\
1 & \bullet & \bullet & \bullet \\
2 & \bullet & \bullet & \bullet \\
3 & \bullet & \bullet & \\
4 & \bullet & & \\
5 & \bullet & &
\end{array}.$$

If the conjugate of a partition p is exactly p, then we call p *self-conjugate*. One use of self-conjugate partitions is to determine the parity of the number of partitions of $n \in \mathbb{N}$ with a given property that is invariant with respect to conjugation. If the number of self-conjugate partitions with property T is even, then there must be an even number of partitions with property T, and otherwise there are an odd number of partitions with property T, since the other partitions are paired with their conjugates. A well known result, due to G.H. Hardy, gives a correspondence between self-conjugate partitions and partitions with distinct odd parts[7]. An alternate proof of this is included in our discussion of conjugation in section 6.

Theorem 4. *(Hardy) The number of self-conjugate partitions of an integer $n \in \mathbb{N}$ is identical to the number of partitions of n with distinct odd parts[7].*

As discussed earlier, we are interested in adding the property of self-conjugacy to our s/t-cores. We find the following description of s/t-cores that have the additional contstraint of self-conjugacy:

Theorem 5. *The $gcd(s,t)$-abacus representation for self-conjugate simultaneous s/t-cores is as follows:*

1) If $gcd(s,t)$ is even, then the partitions placed in columns $0, \ldots, \frac{gcd(s,t)}{2} - 1$ are arbitrary s'/t'-core partition. Furthermore, the partition placed in the $(gcd(s,t) - 1 - i)^{th}$ column is the conjugate of the partition placed in the i^{th} column.

2) If $gcd(s,t) = 2k+1$, then the partitions placed in columns $0, \ldots, k-1$ are arbitrary s'/t'-cores, the partition placed in the k^{th} column is a self conjugate s'/t'-core, and the partition placed in the $(2k-i)^{th}$ column is the conjugate of the partition placed in the i^{th} column.

Given a partition p, we define the *t-core of p* to be the partition ascertained by successively removing rim hooks having length divisible by t until no node has a hook number which is a multiple of t. Thus, p is a t-core if and only if the t-core of p equals p. It is easy to verify that the 3-core of P is the depicted by the Ferrers-Young Diagram on the right:

$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 \\ 1 & \bullet & \bullet & \bullet & \bullet & \bullet \\ 2 & \bullet & \bullet & \bullet & & \\ 3 & \bullet & \bullet & & & \end{array} \longrightarrow \begin{array}{ccc} & 1 & 2 \\ 1 & \bullet & \bullet \\ 2 & \bullet & \\ 3 & \bullet & \end{array}.$$

It turns out that the k-core of a partition can easily be determined using the abacus representation. Removing a rim hook of a node that has a hook number which is a multiple of k in the Ferrers-Young diagram is equivalent to raising a bead in the k-abacus representation by one row[8]. It is easy to see that raising a bead by one row will change the overall partition size by k, the same size as the rim hook. To find the k-core, we begin by pushing all beads to the top. However, this may leave a bead in the (0,0) position. Thus, we then perform rotations until we have a standard partition. If the k-core of p is the empty partition, then we say that the partition has *empty k-core.* If the i^{th} column of the k-abacus of a partition has b_i beads, then this property is identical to having $b_0 \leq b_1 \leq \ldots b_{k-1} \leq b_0 - 1$, since pushing these beads up will give an empty partition. We call a partition which is an er-core with empty r-core an (e, r)-core. For example, the partition $1, 2, 3, 3, 3, 3, 3, 8, 9, 10$ of 45 is a $(2, 5)$-core. An investigation of the 10-abacus shows that it is a 10-core.

	0	1	2	3	4	5	6	7	8	9
0	|	•	|	•	|	•	•	•	•	•
2	|	|	|	|	|	•	|	•	|	•

Inspecting the 5-abacus representation of this partition reveals that it indeed has empty 5-core.

	0	1	2	3	4
0	|	•	|	•	|
1	•	•	•	•	•
2	|	|	|	|	|
3	•	|	•	|	•

A thorough investigation of (e, r)-cores can be found in [13]. In this thesis, we are intested in characterizing partitions that are simultaneously (e, r) and (e', r)-cores, which we shall denote $(e, r)/(e', r)$-cores. It is worth noting that these are merely $er/e'r$-cores with empty r-core. We find that the classification of $(e, r)/(e', r)$-cores is similar to the description given for $er/e'r$-cores. Thus, the structure of $(e, r)/(e', r)$-cores follows from theorem 3.

Corollary 1. *A partition p with no parts of size zero is an $(e, r)/(e', r)$-core with $e, e', r \in \mathbb{N}$, $r > 1$ if and only if the partition placed in each column of the r-abacus of p is an e/e'-core partition, possibly with parts of size zero, and if we consider b_i to be the number of beads in the i^{th} column of the r-abacus,* *then* $b_0 \leq b_1 \leq \cdots \leq b_{r-1} \leq b_0 - 1$. *The partition placed in the* 0^{th} *column, however, cannot have parts of size zero.*

The result given by corollary 1 gives us an interesting result about the number of $(e, r)/(e', r)$-cores with respect to e/e'-cores.

Corollary 2. *A bijection exists between* $(e, r)/(e', r)$*-cores and* e/e'*-cores. Additionally, if* $gcd(e, e') = 1$*, then there are* $\frac{\binom{e+e'}{e}}{e+e'}$ $(e, r)/(e', r)$*-cores.*

The generating function of a sequence $a_1, a_2, \ldots$ is denoted by $\sum_{i=1}^{\infty} a_i q^i$. When speaking about generation functions for partitions of with a particular property, we will say that the generating function for partitions of property Z is $\sum_{n=1}^{\infty} a_n q^n$, where a_n is the number of partitions of n with property Z. For instance, when a_n is the number of t-core partitions of n, a well known result by Garvan, Kim and Stanton [4] is that the generating function for t-cores is

$$\varphi_t(q) := \sum_{n=1}^{\infty} a_n q^n = \prod_{n=1}^{\infty} \frac{(1 - q^{nt})^t}{1 - q^n}.$$

We determine the generating functions for the properties discussed in this thesis. Due to the independence of the columns in the descriptions for the s/t-cores, our results about generating functions are centered around a lemma giving us the generating functions of partitions which have a given structure on each of their columns, but no dependence between the columns, which is proven in section 5. Consider the sets of partitions without parts of size zero $T_0, T_1, \ldots, T_{k-1}$. We are interested in finding the generating function for the set of partitions T such that for every partition in T the partition placed in the i^{th} column is a partition from the set T_i, except that the partition placed in every column but the 0^{th} column may have additional parts of size zero, and T contains all of partitions with this property. We can think of T as a sort of direct product of $T_0, T_1, \ldots, T_{k-1}$, due to the independence of the columns. Let $\delta_i(q)$ be the generating function for partitions in T_i.

Lemma 2. *If* $\delta_i(q)$ *is the generating function for partitions in* T_i*, then the generating function for* T *is*

$$\left(\prod_{i=0}^{k-1} \delta_i(q^k)\right) \cdot \prod_{i=1}^{\infty} \frac{(1 - q^{kn})^k}{1 - q^n}$$

Additionally, it should be noted that if each column has the same generating function $\psi(q)$, this becomes

$$(\psi(q^k))^k \cdot \prod_{n=1}^{\infty} \frac{(1-q^{kn})^k}{1-q^n}.$$

We determine the generating functions for the properties discussed in this thesis. Our generating functions are based on an unknown generating function for m/n-cores when m and n are relatively prime. We will use the symbol $\psi_{m,n}(q)$ to represent the generating function for m/n-cores. However, $\psi_{m,n}(q)$ is simply a polynomial when m and n are relatively prime, since there are only finitely many m/n-cores.

Theorem 6. *The generating function for s/t-core partitions is*

$$\psi_{s,t}(q) = \left(\psi_{s',t'}(q^{gcd(s,t)})\right)^{gcd(s,t)} \cdot \varphi_{gcd(s,t)}(q).$$

We also determine the generating function for simultaneous $(e,r)/(e,r')$-cores. It is interesting to note that this is identical to the generating function for $er/e'r$-cores without the multiplication by $\varphi_r(q)$. This seems to make sense due to the fact that we have "removed" the r-core part of this partition.

Theorem 7. *The generating function for $(e,r)/(e',r)$-cores is*

$$\left(\psi_{e,e'}(q^r)\right)^r$$

Adding self-conjugacy to our partitions gives a slightly more complicated generating function. For the remaining generating functions we will use $\gamma_k(q)$ to denote the generating function for self-conjugate k-cores and $\zeta_{m,n}(q)$ to represent the generating function for self-conjugate m/n-cores.

Adding the property of self-conjugacy to our s/t-cores gives the following generating function:

Theorem 8. *If $gcd(s,t) = 2k+1$ for some $k \in \mathbb{N}$, then the generating function for self-conjugate s/t-cores is*

$$\zeta_{s,t}(q) = \gamma_{gcd(s,t)}(q) \left(\psi_{s',t'}(q^{2gcd(s,t)})\right)^k \zeta_{s',t'}(q^{gcd(s,t)}).$$

If $gcd(s,t) = 2k$ for some $k \in \mathbb{N}$, then

$$\zeta_{s,t}(q) = \gamma_{gcd(s,t)}(q) \left(\psi_{s',t'}(q^{2gcd(s,t)})\right)^k.$$

The generating function for self-conjugate $(e,r)/(e',r)$-cores is related to $\zeta_{er,e'r}(q)$ in a very similar manner to the way that the generating function for $(e,r)/(e',r)$-cores is related to $er/e'r$-cores , except that this time we are dividing by $\gamma_r(q)$, instead of $\varphi_r(q)$.

Theorem 9. *If $r = 2k+1$ for some $k \in \mathbb{N}$, then the generating function for self-conjugate $(e,r)/(e',r)$-cores is*

$$\left(\psi_{e,e'}(q^{2r})\right)^k \zeta_{e,e'}(q^r)$$

If $r = 2k$ for some $k \in \mathbb{N}$, then the generating function for $(e,r)/(e',r)$-cores is

$$\left(\psi_{e,e'}(q^{2r})\right)^k$$

Although we were unable to determine the generating function for m/n-cores when m and n are relatively prime, we were interested in investigation the m/n-core with the largest number of rows in its Ferrers-Young Diagram. We refer to these as *maximum m/n-cores.* We show the following results about these partitions.

Theorem 10. *The partition P consisting exactly of the structure numbers which are not nonnegative linear combinations of m and n is an m/n-core. In addition, its Ferrers-Young Diagram has the most possible number of rows.*

Theorem 11. *Maximum m/n-cores are self-conjugate.*

Theorem 12. *For $m, n \in \mathbb{N}$, m and n relatively prime, Maximum m/n-cores partition the number*

$$\frac{(m^2-1)(n^2-1)}{24}.$$

It is left as an open problem that this is the largest number that an m/n-core can partition. If this is the case, this result gives a stopping point for the calculation of these generating functions. Additionally, this result will give us an interesting connection to representation theory which we will discuss in section 3.

The remainder of the thesis is organized in the following manner. In section 2, we will investigate interesting results from Partition Theory that will be of use in our characterization of s/t-cores. Our interest in section 3 will be to explore the relationship between t-cores and Representation Theory.

The first new results are given in section 4. Here we prove theorems 3 and 2, giving a characterization of s/t-cores, concluding that there are infinitely many s/t-cores which are not $gcd(s,t)$-cores when s and t are not relatively prime. We also characterize $(e,r)/(e',r)$-cores, proving corollary 1. In section 5, we investigate these characterizations further by determining the generating functions for s/t-cores and $(e,r)/(e',r)$-cores, proving theorems 6 and 7 and corollary 2. We also prove theorem 15, the extension of Ono's theorem (Theorem 14). This result shows the existence of an s/t-core, that is not a $gcd(s,t)$-core, of every $n > gcd(s,t)$ for s and t not relatively prime. To attain these results, we first prove lemma 2. To show the power of this lemma we proceed to give nice proofs of well known results as corollaries. We then explore conjugation of a partition in section 6. After determining the abacus structure of the conjugate of a partition, we determine a sufficient and necessary condition for self-conjugation based on the abacus representation. We give here a proof of theorem 4. We then characterize self-conjugate s/t-cores and $(e,r)/(e',r)$-cores in section 7, proving theorem 5. In section 8, we determine the generating functions for self-conjugate s/t-cores, proving theorem 8. We also prove the generating function for self-conjugate $(e,r)/(e',r) - cores$ given in theorem 9 in section 8. Finally, in section 9, we determine the properties of maximum s/t-cores with s and t relatively prime. We characterize these partitions in theorem 10, show that they are all self-conjugate in theorem 11, and end with a proof of the size of the maximum s/t-cores given in theorem 12. This result forms a nice connection to Representation Theory, which we will discuss in section 3.

2 A Useful Result from Partition Theory

Many of our arguments involve the abacus representation of a partition, so we find it useful to state here a theorem which will allow us to determine the size of a partition based on its abacus representation. A simple proof of the results can be realized by imagining adding one bead to the abacus at a time, and checking the change in size of the overall partition based on that bead being added. Let a partition Λ be given. Consider the k-abacus of this partition. James and Kerber in [8] obtain a natural correspondence between the partition Λ and a $(k+1)$-tuple of partitions

$$(\Lambda_{(k)}, \Lambda_0, \Lambda_1, \ldots, \Lambda_{k-1})$$

such that $\Lambda_{(k)}$ is the k-core of Λ and each Λ_i is the partition placed in the i^{th} column of the k-abacus of Λ. We will use the notation $|\Lambda|$ to denote the size of the number that is partitioned by Λ, Λ^C to denote the conjugate of Λ, and $HK(\Lambda)$ to denote the set of hook numbers of Λ.

This correspondence has the properties that

$$(1)\ |\Lambda| = |\Lambda_{(k)}| + \sum_{i=0}^{k-1} k|\Lambda_i|$$

$$(2)\ \text{If } \ n \in Hk(\Lambda_{(k)}), \text{ then } \ k \nmid n$$

$$(3)\ Hk(\Lambda) = \left(\bigcup_{i=0}^{k-1} k \cdot Hk(\Lambda_i) + i \right) \cup Hk(\Lambda_{(k)}).$$

Furthermore, under this correspondence Λ^c corresponds with the $(k+1)$-tuple

$$(\Lambda^c_{(k)}, \Lambda^c_{k-1}, \Lambda^c_{k-2}, \ldots, \Lambda^c_0)$$

Thus we are placing the conjugate of Λ_i in the $(k-1-i)^{th}$ column, and the k-core is the conjugate of our previous k-core.

3 Representation Theory

Here we investigate the relationship between partition theory and the Representation Theory of the Symmetric Group. An *ordinary representation of* S_n is a homomorphism φ from S_n to $Gl_m(\mathbb{C})$ where $Gl_m(\mathbb{C})$ is the group of all invertible $m \times m$ matrices with complex coefficients. We call m the *dimension* of the representation φ.

Of course, the elements of $Gl_m(\mathbb{C})$ can be viewed as linear transformations on $\mathbb{C}^m$. We call a representation φ *irreducible* if there is no subspace $C \subset \mathbb{C}^m$ such that $\forall g \in S_n$, $\varphi(g)^>(C) \subset C$, where the notation $f^>(X)$ means the image of f over the set X. For example, if f is the function $f(x) := x^2$ and we have the set $X := \{3, 5, 13\}$, then $f^>(X) = \{9, 25, 169\}$. The *direct sum* of the representations $\varphi_0, \varphi_1, \ldots, \varphi_k$, $\bigoplus_{i=1}^k \varphi_i$ is the representation such that $(\bigoplus_{i=1}^k \varphi_i)(g) := \bigoplus_{i=1}^k(\varphi_i(g))$ where the direct sum for matrices is defined in the following way: If we have the matrices $X : V \longrightarrow V$ and $Y : W \longrightarrow W$, then $X \oplus Y$ is a matrix acting on 2-tuples of elements in V and W, such that for $v \in V$ and $w \in W$, $(X \oplus Y)(v, w) = (X(v), Y(w))$. Thus, the matrix $X \oplus Y$ is the block matrix

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$$

A represention is *completely reducible* if it can be written as a direct sum of irreducible representations. This means that under a change of basis for the matrices given by the representation is possible to write all of the matrices as a direct sum where for every matrix it is in the above form with the non-zero blocks forming an irreducible representation. We call these summands *constituents* of the representation.

Theorem. *(Maschke's Theorem) Every representation of a finite group having positive dimension is completely reducible.*

Modules play an important role in representation theory. A G-module, where G is a group with the identity element e, is a vector space V along with a group action for $g \in G,\ \mathbf{v} \in V,\ g\mathbf{v} : G \times V \longrightarrow V$ such that for all $g, h \in G,\ c, d \in \mathbb{C}$, and $\mathbf{v}, \mathbf{w} \in V$:

$$e\mathbf{v} = \mathbf{v},\ (gh)\mathbf{v} = g(h\mathbf{v}),\ \textit{and}\ g(c\mathbf{v} + d\mathbf{w}) = c(g\mathbf{v}) + d(g\mathbf{v})$$

W is a *submodule* of V if for every $\mathbf{w} \in W$ $g\mathbf{w} \in W$. V is an *irreducible module* if it has no non-trivial submodules. Consider, for a representation φ of G, $g\mathbf{v} = \varphi(g)\mathbf{v}$, where on the right hand side the operation is the normal multiplication. Since φ is a homomorphism, we find that φ fulfills all of the requirements for the group action on V[2]. Therefore V along with φ can be viewed as a G-module.

Two important examples of representations of S_n are the *regular representation* and the *defining representation.* The defining representation acting on $g \in S_n$ returns an $n \times n$ matrix with coefficients of 0 and 1, having a 1 in the (i, j) position if and only if $g(i) = j$. We call $\varphi(g)$ the *permutation matrix of g.* Consider an enumeration of the elements of S_n as $g_1, g_2, \ldots, g_{n!}$. If X is the matrix obtained by the regular representation operating on g, then X is an $n! \times n!$ matrix with 0 and 1 coefficients, where $X_{i,j} = 1$ if and only if $gg_i = g_j$. We will use R_{S_n} to represent the regular representation of S_n. This representation will help us later to determine the number of irreducible representations of S_n.

We will now proceed to given an overview of the rest of the section. Our goal is to establish a bijection between the partitions of n and the irreducible representations of S_n. We determine the existence of this bijection first, and then give a natural correspondence. The connection between the Representation Theory of S_n and partition theory is based on the cycle structure of permutations. We will therefore investigate the set of conjugacy classes, which are determined by the cycle structure. We will manipulate this set to find a basis for the set of $x \in \mathbb{C}S_n$ such that x commutes with every other element in $\mathbb{C}S_n$, where $\mathbb{C}S_n = \{\sum_{i=1}^{n!} c_i g_i : c_i \in \mathbb{C}, g_i \in S_n\}$. Along the way, we will develop the theory of characters, which will allow us to check if a representation is irreducible, and to break it into its irreducible constituents. A representation of S_n which contains every irreducible as a constituent is found, and it is determined that this representation indeed has $p(n)$ irreducibles. A nice correspondence is then determined between the irreducible representations and the partitions.

Consider the *conjugacy* classes of S_n. The conjugacy class of a permutation ρ is the set of all permutations of the form $\pi\rho\pi^{-1}$. Let $\pi, \psi \in S_n$ be given. If $\psi(i_1) = i_2$, then

$$\pi\psi\pi^{-1}(\pi(i_1)) = \pi(\psi(i_1)) = \pi(i_2).$$

Therefore, for every cycle in ψ, there is a corresponding cycle in $\pi\psi\pi^{-1}$ of the same length. Moreover, if ψ and ρ have the same cyle structure, then a permutation π can be constructed such that $\rho = \pi\psi\pi^{-1}$. Thus, the cycle structure completely defines the conjugacy classes. However, the cycle structure is merely a partition of the number n. This gives a nice correspondence between the conjugacy classes and partitions of n.

Since every representation can be broken into a direct sum of irreducible representations, then, under a certain basis, the corresponding matrix can be separated into a block matrix with each block along the diagonal being an irreducible representation, and all other blocks being zero. For example, if we have the direct sum $X^{(1)} \oplus X^{(2)}$, where $X^{(i)}$ is irreducible, then under a certain basis, we would have the following matrix.

$$\begin{pmatrix} X^{(1)} & 0 \\ 0 & X^{(2)} \end{pmatrix}$$

Moreover, since we can order the bases in any order for the matrix, we can write every representation in the form $\bigoplus_{i=1}^{n} m_i X^{(i)}$, where $X^{(i)}$ is irreducible, and $mX = \bigoplus_{i=1}^{m} X$.

If we have an action gv of a group G on a vector space V, and a corresponding action gw on a vector space W, then we call the linear transformation $\Phi : V \to W$ a G-homomorphism if $\Phi(gv) = g\Phi(v)$. If these vector spaces are modules, then consider the basis B for V and C for W. Choose $X(g)$ and $Y(g)$, the matrices of the representations corresponding to V and W with respect to these bases. We can write the matrix of Φ on these bases as T such that $TX(g) = Y(g)T$. It is clear that Φ is an isomorphism if and only if there is such a T which is invertible.

It is known that $ker(\Phi)$ is a subspace of the vector space V. Moreover, if we have $v \in ker(\Phi)$, then $\forall g \in \; G \; \Phi(gv) = g\Phi(v) = g0 = 0$, so $gv \in ker(\Phi)$. Therefore, $ker(\Phi)$ is a submodule of V, since it is closed under the action of G on V. A similar argument can be used to show that $im(\Phi)$ is a submodule of W. Since these are submodules, we know that if the modules are irreducible, then both of these submodules must either be the entire space or the trivial subspace $\{0\}$. This leads to the following result, due to Shur[12].

Theorem. (Shur's Lemma): If V and W are irreducible modules and $\Phi : V \longrightarrow W$ is a G-homomorphism, then either

1. Φ is an isomorphism, or
2. Φ is the zero map.

For a representation X, a useful object in algebra is the Commutant algebra

$$Com\ X := \{T : TX(g) = X(g)T \ \forall g \in G\},$$

The corresponding result to Shur's Lemma for matrices gives us a T_Φ which is either invertible or the zero matrix. Consider an irreducible representation $X(g)$. Let T be given such that $TX(g) = X(g)T \ \forall g \in G$. Then

$$(T - cI)X = TX - cX = XT - cX = XT - X(cI) = X(T - cI).$$

Therefore $T - cI$ must be invertible or the zero matrix for all $c \in \mathbb{C}$. Therefore, if c is an eigenvalue of the matrix T, then $(T - cI)$ is not invertible, so it must be the case that $T = cI$ for some $c \in \mathbb{C}$.

We also define the Endomorphism algebra:

$$End\ V := \{\varphi : V \to V \ : \varphi \ is \ a \ G - homomorphism\}.$$

We have already discussed the isomorphism between these two sets. We wish to use the commutant algebra to determine the number of distinct irreducibles which are contained in a representation. We would like to know what the commutant algebra looks like for different representations. Since we know that we can write a representation as $\bigoplus_{i=1}^{k} m_i X^{(i)}$, with $X^{(i)}$ irreducible and $X^{(1)} \neq X^{(2)}$, consider a representation $X = X^{(1)} \oplus X^{(2)}$. Then, if we have $T \in Com\ X$,

$$T = \begin{pmatrix} T_{(1,1)} & T_{(1,2)} \\ T_{(2,1)} & T_{(2,2)} \end{pmatrix}$$

Thus,

$$TX = \begin{pmatrix} T_{(1,1)}X^{(1)} & T_{(1,2)}X^{(2)} \\ T_{(2,1)}X^{(1)} & T_{(2,2)}X^{(2)} \end{pmatrix} = \begin{pmatrix} X^{(1)}T_{(1,1)} & X^{(1)}T_{(1,2)} \\ X^{(2)}T_{(2,1)} & X^{(2)}T_{(2,2)} \end{pmatrix} = XT$$

This gives us the set of equations

$$\begin{aligned} T_{(1,1)}X^{(1)} &= X^{(1)}T_{(1,1)} \\ T_{(1,2)}X^{(2)} &= X^{(1)}T_{(1,2)} \\ T_{(2,1)}X^{(1)} &= X^{(2)}T_{(2,1)} \\ T_{(2,2)}X^{(2)} &= X^{(2)}T_{(2,2)} \end{aligned}$$

Thus, since $X^{(i)}$ is irreducible, and $X^{(1)} \neq X^{(2)}$ we know that $T_{(1,1)} = c_1 I_{d_1}$ for some $c_1 \in \mathbb{C}$, $T_{(1,2)} = T_{(2,1)} = 0$, and $T_{(2,2)} = c_2 I_{d_2}$ for some $c_2 \in \mathbb{C}$, where $d_i = dim(X^{(i)})$. So $T = c_1 I_{d_1} \oplus c_2 I_{d_2}$ This can be shown more generally. For $X = \bigoplus_{i=1}^{k} X^{(i)}$, $T = \bigoplus_{i=1}^{k} c_i I_{d_i}$. In the case where we have m copies of the same irreducible $X^{(1)}$, then the matrix T fulfills the the equation

$$T_{(i,j)}X^{(1)} = X^{(1)}T_{(i,j)} \ \forall\ i, j \in \{1, 2\}$$

Thus, in this case, we get for some choice of $c_{(i,j)}$

$$T = \begin{pmatrix} c_{(1,1)}I_{d_1} & c_{(1,2)}I_{d_1} \\ c_{(2,1)}I_{d_1} & c_{(2,2)}I_{d_1} \end{pmatrix}$$

T can be defined using an operation on two matrices X and Y, called the *tensor product* of X and Y, where

$$X \otimes Y = \begin{pmatrix} X_{(1,1)}Y & X_{(1,2)}Y & \cdots \\ X_{(2,1)}Y & X_{(2,2)}Y & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

So T can be written as $C \otimes I_{d_1}$. In general, for a representation $X = \sum_{i=1}^{k} m_i X^{(i)}$, we have that $Com\ X = \{\bigoplus_{i=1}^{k}(Y_{m_i} \otimes I_{d_i})\ :\ Y_{m_i} \in\ M_{m_i}$, where M_{m_i} is the set of all $m_i \times m_i$ matrices.

The *center* of a ring S, $Z_S := \{x \in S : xy = yx\ \forall y \in S\}$. For example, consider $C_{m_i} \in Z_{M_{m_i}}$, where M_{m_i} is the set of all $m_i \times m_i$ matrices. But if we then consider the matrices which give the i^{th} row with right multiplication, and i^{th} column with left multiplication (i.e. the matrix with all zeros except a one in the (i,i) position), then it is obvious that since these matrices are equivalent, that all entries off of the diagonal must be zero. Moreover, if we use the matrices which swap two rows with right multiplication and columns with left multiplication (i.e. the matrix with ones in the (i,j) and (j,i) positions and zeros otherwise), then we will get that the value along the diagonal is a constant. Therefore, $C_{m_i} = c_i I_{m_i}$ for some $c_i \in \mathbb{C}$.

Since $Z_{Com\ X}$ will be used to form an isomorphism to conjugacy classes, we would like to classify $Z_{Com\ X}$. Let $C \in Z_{Com\ X}$, $T \in Com\ X$ be given. Choose C_{m_i}, Y_{m_i} such that $C = \bigoplus_{i=1}^{k}(C_{m_i} \otimes I_{d_i})$ and $T = \bigoplus_{i=1}^{k}(Y_{m_i} \otimes I_{d_i})$. Using the definitions of $\oplus$ and $\otimes$, it can easily be shown that $(A \oplus B)(C \oplus D) = AC \oplus BD$ and $(A \otimes B)(C \otimes D) = AC \otimes BD$ for all square matrices A, B, C, D such that $dim(A) = dim(C)$ and $dim(B) = dim(D)$. Using these result, it is easy to show that, since $C \in Z_{Com\ X}$,

$$\bigoplus_{i=1}^{k}(Y_{m_i} C_{m_i} \otimes I_{d_i}) = TC = CT = \bigoplus_{i=1}^{k}(C_{m_i} Y_{m_i} \otimes I_{d_i})$$

However, this can only occur when $C_{m_i} Y_{m_i} = Y_{m_i} C_{m_i}\ \forall Y_{m_i} \in\ M_{m_i}$. But then $C_{m_i} \in Z_{M_{m_i}}$, and we already know that all members of this set are merely a constant times the identity. So $C = \bigoplus_{i=1}^{k} c_i I_{m_i} \otimes I_{d_i}$. Thus $dim(Z_{Com\ X}) = k$, since we can choose any values for the c_i's. Remembering that there is an isomorphism from $Com\ X$ to $End\ V$ gives us that $dim(Z_{End\ V}) = k$.

A useful tool for determining distinctions between representations is the *character* of the representation. The character of $X(g)$ is the *trace* of the matrix $X(g)$, where the trace is the sum of the diagonal elements. The trace is invariant under change of basis, so $tr(XYX^{-1}) = tr(Y)$. We may think of the character of a representation as a vector of size $|G|$, where the value for g_i of the vector is the trace of $X(g_i)$.

Under the same group, two characters have the same dimension, $|G|$, so we can therefore consider the normal *inner product* on two characters χ and

ψ for representations of G,

$$\langle \chi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}$$

Moreover, if an orthonormal basis is chosen for V, then we can choose a matrix representation Y for the representation with character ψ such that $Y(g^{-1}) = Y(g)^{-1} = \overline{Y(g)}^T$, and thus $\overline{\psi(g)} = tr(\overline{Y(g)}) = tr(Y(g^{-1})^T) = tr(Y(g^{-1})) = \psi(g^{-1})$.

Theorem 13. *Two irreducible representations X with character χ and Y with character ψ are identical if and only if $\langle \chi, \psi \rangle = 1$, and otherwise, $\langle \chi, \psi \rangle = 0$ [12].*

Since the trace of a direct sum is the sum of the traces, it is easy to see that if we have the representation $X := \bigoplus_{i=1}^k m_i X^{(i)}$ then $\chi = \sum_{i=1}^k m_i \chi^{(i)}$, where χ is the character for the representation X, and $\chi^{(i)}$ is the character for $X^{(i)}$. Moreover,

$$\left\langle \chi, \chi^{(j)} \right\rangle = \left\langle \sum_{i=1}^k m_i \chi^{(i)}, \chi^{(j)} \right\rangle = \sum_{i=1}^k m_i \left\langle \chi^{(i)}, \chi^{(j)} \right\rangle = m_j.$$

We are now able to show the importance of the regular representation. Since the regular representation has only 0's and 1's, the character will be the number of ones along the diagonal of the matrix, or the number of fixed points under left multiplication. However, if $gh = h$ for some $h \in S_n$, then $g = ghh^{-1} = hh^{-1} = e$, so the trace of the matrix for the the regular representation $X(g)$ is 0 if $g \neq e$, and it is $|S_n|$ otherwise. Indeed, since the homomorphism must map the identity to the identity matrix, we must have for any representation that the trace is $|S_n|$. Consider writing the regular representation as $\bigoplus_{i=1}^k m_i X^{(i)}$. We can assume that all of the irreducibles are mentioned here, but that only finitely many m_i's are nonzero, since we have a finite dimensional representation. From above, we have that the number of copies of $X^{(i)}$, $m_i =< \chi, \chi^{(i)} >= \frac{1}{|S_n|} \sum_{g \in S_n} \chi(g) \chi^{(i)}(g^{-1})$. However, we have shown that $\chi(g) = 0$ if $g \neq e$, and $\chi(e) = |S_n|$. Thus, we have $m_i = \frac{1}{|S_n|} |S_n| \chi^{(i)}(e) = dim(X^{(i)})$. Therefore, we have that every irreducible is included in the regular representation and the number of times each irreducible occurs is identical to its dimension.

It is now sufficient to show that $k = p(n)$, to get the isomorphism that we desire. Consider the module $\mathbb{C}(S_n)$, the module with elements in S_n and constants from $\mathbb{C}$. It is all of the the sums of the form $\sum_{i=0}^{n!} c_i g_i$. It has already been determined that $dim(Z_{End\ \mathbb{C}(S_n)}) = dim(Z_{Com\ X}) = k$. Consider $\theta_v \in End\ \mathbb{C}(S_n)$, $\theta_v(w) := wv$. Then an isomorphism between $End\mathbb{C}(S_n)$ and $\mathbb{C}(S_n)$ exists. Consider the mapping $\theta(v) := \theta_v$. However, $ker(\theta) = \{v \in \mathbb{C}(S_n) : \theta(v) = 0\} = \{v \in \mathbb{C}(S_n) : \theta_v(g) = 0\ \forall g \in S_n\} \subset \{v \in \mathbb{C}(S_n) : \theta_v(e) = 0\} = \{v \in \mathbb{C}(S_n) : 0 = ve = v\} = \{0\}$. Moreover, let $\Theta \in End\ \mathbb{C}(S_n)$ be given. Choose $v \in \mathbb{C}(S_n)$ such that $\Theta(e) = v$. Then $\Theta(g) = \Theta(ge) = g\Theta(e) = gv = \theta_v(g)$. So the mapping is also surjective. Also, $\Theta(vw) = \theta_{vw} = \theta_w \theta_v$. So Θ forms an anti-isomorphism between $\mathbb{C}(S_n)$ and $End\ \mathbb{C}(S_n)$. So if we have $v \in Z_{\mathbb{C}(S_n)}$, $w \in \mathbb{C}(S_n)$, then

$$\theta_v \theta_w = \Theta(wv) = \Theta(vw) = \theta_w \theta_v.$$

So $\theta_v \in Z_{End\mathbb{C}(S_n)}$. Also, if we have $\theta_v \in Z_{End\mathbb{C}(S_n)}$ and $\theta_w \in End\mathbb{C}(S_n)$, then

$$\Theta(wv) = \theta_v \theta_w = \theta_w \theta_v = \Theta(vw) \implies wv = vw.$$

Thus we the centers of $End\mathbb{C}(S_n)$ and $\mathbb{C}(S_n)$ are isomorphic. So $k = dim(Z_{End\mathbb{C}(S_n)}) = dim(Z_{\mathbb{C}(S_n)})$. Let $z = \sum_{i=1}^{j} c_i g_i \in Z_{\mathbb{C}(S_n)}$ be given. Let $h \in S_n$ be given. Since $z \in Z_{\mathbb{C}(S_n)}$, $\sum_{i=1}^{j} c_i g_i = z = hzh^{-1} = \sum_{i=1}^{j} c_i h g_i h^{-1}$. But then, as we cycle through all of the $h \in S_n$, we get that every element of g_i's conjugacy class, each with coefficient equal to c_i. Thus, if we have m Conjugacy classes $K_1, \ldots, K_m$, then we can consider $z_i := \sum_{g \in K_i} g$, and rewrite z as $z = \sum_{i=1}^{m} c_i z_i$. Moreover, it has been shown that these z_i form a basis for the space[12], and thus $k = m$. Therefore, an isomorphism has been determined between the number of irreducible representations and the number of conjugacy classes of S_n, which we already determined was the same as the number of partitions of n.

It remains to determine the combinatorial correspondence between the irreducibles and partitions. The argument involves ordering the partitions in such a way that the representation for the i^{th} element in the ordering contains exactly one irreducible constituent that does not occur in any of the preceding representations. This correspondence is based on the *lexicographical* ordering of partitions. In this ordering, a partition $a_0, \ldots, a_m$ of n is greater than the partition $b_0, \ldots, b_k$ if and only if

$$for\ some\ i \in \mathbb{N}\ a_i > b_i\ and\ a_j = b_j \forall\ j \in \mathbb{N},\ j < i.$$

It is clear that this ordering is a total ordering of the partitions. We also need to define the *induced representation* of a representation ψ. If $H \subset G$ is given, then the induced representation of ψ from H to G is determined in the following manner: if $e = g_1, \ldots, g_{|G/H|} = g_{\frac{|G|}{|H|}}$ are each a representative from distinct (left) cosets of H in G, then the induced representation, as given in [2] is:

$$\psi \uparrow_H^G (g) := \begin{pmatrix} \psi(e^{-1}ge) & \psi(e^{-1}gg_2) & \cdots & \psi(e^{-1}gg_{|G/H|}) \\ \psi(g_2^{-1}ge) & \psi(g_2^{-1}gg_2) & \cdots & \psi(g_2^{-1}gg_{|G/H|}) \\ \vdots & \vdots & \ddots & \vdots \\ \psi(g_{|G/H|}^{-1}ge) & \psi(g_{|G/H|}^{-1}gg_2) & \cdots & \psi(g_{|G/H|}^{-1}gg_{|G/H|}) \end{pmatrix}$$

where $\psi(g) = 0$ if $g \notin H$

Let partition $\lambda = \lambda_1, \ldots, \lambda_k$ of n be given. Consider the following subset of S_n:

$$S_\lambda := \bigoplus_{i=1}^{k} S_{\{(\sum_{j=1}^{i-1} \lambda_j)+1, \ldots, (\sum_{j=1}^{i} \lambda_j)+1\}}$$

where, for a set A, S_A is the set of all permutations of the elements of A. So S_λ is simply the set of permutations that permute the first λ_1 elements in any way, the next λ_2 elements in any way, independently, etc. For example, if we had the partition $\lambda = (5, 3, 2)$, then

$$S_\lambda = S_{\{1,2,3,4,5\}} \oplus S_{\{6,7,8\}} \oplus S_{\{9,10\}}.$$

This is clearly a subset of S_{10}. It is easy to see that $S_{[\mathbf{n}]} = S_n$, and $S_\lambda \simeq \bigoplus_{i=1}^{k} S_{\lambda_i}$. For $\lambda = (1, 1, \ldots, 1)$, we clearly have $S_\lambda = e$, the identity element. It turns out that if we induce the identity representation from S_λ to S_n then we will be able to determine the irreducibles.

For $\lambda = (n)$, the partition of n with only n,X $H = G$, so the induced representation of the identity representation is the identity representation. Order the partitions in *reverse* lexicographic order, giving each a superscript corresponding to its position in the ordering. Remember that each representation is a direct sum of irreducible representations. It turns out that if we look at the induced representations $I \uparrow_{S_\lambda}^{S_n}$ for λ in our ordering, then we will get exactly one new irreducible representation going from $\lambda^{(i)}$ to $\lambda^{(i+1)}$[12]. We will call this irreducible representation $\psi_{\lambda^{(i+1)}}$. It turns out that if we consider the sign representation A, where $A(\pi) = sgn(\pi)$, then, abusing the

$\cap$ notation, where $\psi \cap \varphi = \{x : x \textit{ irreducible}, < \psi, x > \neq 0, < \varphi, x > \neq 0\}$ (i.e. the irreducible constituents that ψ and φ have in common), we have, with λ^C denoting the conjugate of λ, the following equality given in [8]:

$$\psi_(\lambda) = I \uparrow_{S_\lambda}^{S_n} \cap A \uparrow_{S_{\lambda^C}}^{S_n}$$

Since we know that there are the same number of partitions as irreducible representations, we know that we have found all of the irreducible representations in this construction. We will now show the preceding process to determine the irreducible representations of S_3. We only need to give the induced representation from $S_{(2,1)}$, since $I \uparrow_{S_{(3)}}^{S_3} = I$ and $I \uparrow_{S_{(1,1,1)}}^{S_3} = R_{S_3}$. Here we use e to represent the identity element. We consider representatives e, (23), and (13) of the cosets. Since these are all their own inverses, we will not write the inverse when determining the induced representations. We will also use ψ to represent the representation $I_{S_{(2,1)}}$ that is 0 for everything outside of $S_{(2,1)}$.

Example 1.

$$I\uparrow_{S_{(2,1)}}^{S_3}(e) = \begin{pmatrix} \psi(eee) & \psi(ee(23)) & \psi(ee(13)) \\ \psi((23)ee) & \psi((23)e(23)) & \psi((23)e(13)) \\ \psi((13)ee) & \psi((13)e(23)) & \psi((13)e(13)) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I\uparrow_{S_{(2,1)}}^{S_3}((12)) = \begin{pmatrix} \psi(e(12)e) & \psi(e(12)(23)) & \psi(e(12)(13)) \\ \psi((23)(12)e) & \psi((23)(12)(23)) & \psi((23)(12)(13)) \\ \psi((13)(12)e) & \psi((13)(12)(23)) & \psi((13)(12)(13)) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$I\uparrow_{S_{(2,1)}}^{S_3}((13)) = \begin{pmatrix} \psi(e(13)e) & \psi(e(13)(23)) & \psi(e(13)(13)) \\ \psi((23)(13)e) & \psi((23)(13)(23)) & \psi((23)(13)(13)) \\ \psi((13)(13)e) & \psi((13)(13)(23)) & \psi((13)(13)(13)) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$I\uparrow_{S_{(2,1)}}^{S_3}((23)) = \begin{pmatrix} \psi(e(23)e) & \psi(e(23)(23)) & \psi(e(23)(13)) \\ \psi((23)(23)e) & \psi((23)(23)(23)) & \psi((23)(23)(13)) \\ \psi((13)(23)e) & \psi((13)(23)(23)) & \psi((13)(23)(13)) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I\uparrow_{S_{(2,1)}}^{S_3}((123)) = \begin{pmatrix} \psi(e(123)e) & \psi(e(123)(23)) & \psi(e(123)(13)) \\ \psi((23)(123)e) & \psi((23)(123)(23)) & \psi((23)(123)(13)) \\ \psi((13)(123)e) & \psi((13)(123)(23)) & \psi((13)(123)(13)) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$I\uparrow_{S_{(2,1)}}^{S_3}((132)) = \begin{pmatrix} \psi(e(132)e) & \psi(e(132)(23)) & \psi(e(132)(13)) \\ \psi((23)(132)e) & \psi((23)(132)(23)) & \psi((23)(132)(13)) \\ \psi((13)(132)e) & \psi((13)(132)(23)) & \psi((13)(132)(13)) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

We will try now to determine the constituents of this representation. Consider $V := \left\{ \begin{pmatrix} c \\ c \\ c \end{pmatrix} : c \in \mathbb{C} \right\}$ and $W := \left\{ \begin{pmatrix} c_1 \\ c_2 \\ -c_1 - c_2 \end{pmatrix} : c_1, c_2 \in \mathbb{C} \right\}$. It is easy to check that for every $g \in S_3$, $\mathbf{v} \in V$, $I \uparrow_{S_{(2,1)}}^{S_3} (g)\mathbf{v} \in V$ and $\forall \mathbf{w} \in W, I \uparrow_{S_{(2,1)}}^{S_3} (g)\mathbf{w} \in W$. So V and W are submodules of $\mathbb{C}^3$. Consider the (orthonormal) basis $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ of V, and the basis $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ of W. We know that there are corresponding representations φ_V and φ_W for the submodules V and W, respectively, which are also representations of S_3. Additionally, we know that $I \uparrow_{S_{(2,1)}}^{S_3} = \varphi_V \oplus \varphi_W$. There must also be exactly one more irreducible in this representation than the number of irreducibles in $I \uparrow_{S_{(3)}}^{S_3} = I$, since it is second in the reverse lexicographic ordering $(3), (2,1), (1,1,1)$. Therefore, as it is clear that $\varphi_V = I$, since it is a one-dimensional representation, we have that φ_W is irreducible, and it is unique from I_{S_3} and A_{S_3}. Therefore, once we do a change of basis to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ we get the following matrices for the representation

φ:

$$\begin{aligned}
\varphi(e) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\varphi((12)) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
\varphi((13)) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{3}{2} & \frac{1}{2} \end{pmatrix} \\
\varphi((23)) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{3}{2} & \frac{1}{2} \end{pmatrix} \\
\varphi((123)) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{3}{2} & -\frac{1}{2} \end{pmatrix} \\
\varphi((132)) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{3}{2} & -\frac{1}{2} \end{pmatrix}
\end{aligned}$$

Under this basis, It is obvious that φ_V is the identity representation. We get the following matrices for φ_W.

$$\begin{aligned}
\varphi_W(e) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\varphi_W((12)) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
\varphi_W((13)) &= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} \\
\varphi_W((23)) &= \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} \\
\varphi_W((123)) &= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} \end{pmatrix} \\
\varphi_W((132)) &= \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}
\end{aligned}$$

Therefore, we have determined the unknown irreducible representation of S_3 of dimension 2. We now have a complete classification of the irreducible representations of S_3. This concludes our example.

There is a change of bases such that $\forall g \in S_n$, the entries of the $\varphi(g)$ matrix are all integers[8]. This allows us to consider all of the entries of $\varphi(g)$ $(modp)$. We call one of these representations *p-modularly irreducible* if it is irreducible with respect to $Gl((\mathbb{Z}_p, \cdot))$, in the same way that we determined irreducibility above. Since we know that these p-modular representations are direct summands of p-modularly irreducible representations, we can construct a graph, called the *Brauer Graph*, where, for irreducible representations φ, ψ, an edge exists between $\varphi(modp)$ and $\psi(modp)$ iff they have an p-modularly irreducible constituent in common. This breaks up the irreducible (in the non-modular sense) representations into equivalence classes. Each equivalence class consists of a connected subgraph which is isolated from the rest of the graph. These equivalence classes are called *p-blocks*. If a p-block contains only one element, then the representation corresponding to this p-block is said to have *defect 0 p-block*. It was found that the partitions corresponding to these irreducible representations were exactly the p-core partitions[8]. We have already noted that a partition theory result proved by Ono in [5] showed

Theorem 14. *$\forall t \in \mathbb{N}^{\times}$, $t \geq 4$, $n \geq 1$, $\exists$ a t-core partition of n.*

Therefore, this means that there is a defect 0 p-block representation of S_n for every prime $p > 4$. This brings an interesting question about simultaneous p/q-cores, p, q prime. It would be interesting to find out for which $n \in \mathbb{N}$ there exists p/q-core partitions, since this would tell us when there is a representation with defect 0 p-block and defect 0 q-block. It is particularly interesting since there are only finitely many of these.

In our investigation of this question, we found that

Theorem 15. *$\forall k \geq 4$, $m, n \in \mathbb{N}$, $gcd(m, n) = 1$, $j \in \mathbb{N}$, $j \geq k$, there exists a mk/nk-core partition of j which is not a k-core partition.*

Unfortunately, since mk, nk are not prime, this result does not give us any information in representation theory, since $(\mathbb{Z}_{mk}, \cdot)$ and $(\mathbb{Z}_{nk}, \cdot)$ do not form groups.

An interesting result which connects the research involved in this thesis with representation theory is listed below.

Theorem 16. *With $gcd(m, n) = 1$, the size of maximum m/n-cores is $\frac{(m^2-1)(n^2-1)}{24}$.*

This theorem implies that, specifically when m and n are prime, the largest $k \in \mathbb{N}$ such that there is a partition of k that has both a defect zero m block and a defect zero n block is when $k = \frac{(m^2-1)(n^2-1)}{24}$.

4 The General Structure of s/t-cores with $gcd(s,t) > 1$

We begin with a mapping that determines the correspondence between positions of beads in the k-abacus for a partition P with beads in the kn-abacus for P. We will use this mapping to determine a mapping from the $gcd(s,t)$-abacus of P to both the s-abacus and t-abacus of P. This will allow us to determine the s/t-core property while investigating only the $gcd(s,t)$-abacus.

Lemma 3. *For a given partition p, the following mapping defines the correspondence of bead positions in the k-abacus representation of p to bead positions in the kn-abacus.*

$$(an+c,b) \leftrightarrow (a,ck+b),\ 0 \le c < n,\ 0 \le b < k$$

Proof. Consider a bead in the $(an+c,b)$ position of the k-abacus. The structure number corresponding to this bead is

$$k(an+c)+b = kan+kc+b = a(kn)+(ck+b).$$

Thus, the structure number corresponds to a bead in the $(a,ck+b)$ position of the kn-abacus. □

We now determine a $gcd(s,t)$-abacus classification of s/t-cores based on the structure of s'/t'-cores:

Theorem 3. *A standard partition p is an s/t-core with $s,t \in \mathbb{N},\ gcd(s,t) > 1$ if and only if the partition placed in each column of the $gcd(s,t)$-abacus of p is an s'/t'-core partition, possibly with part sizes of zero. The partition placed in the 0^{th} column, however, cannot have part sizes of zero.*

Proof. Consider first the restrictions on a s'/t'-core partition from lemma 1. Let a standard s'/t'-core partition p be given. Assume that $n \in St_p$. Since the partition is an s'-core, we must also have

$$n - is' \in St_p,\ \forall i \in \mathbb{N} : n - is' > 0$$

as each column in its s'-abacus must be filled in from the top. Since it is also a t'-core, we thus have

$$n - is' - jt' \in St_p\ \forall i,j \in \mathbb{N} : n - is' - jt' > 0.$$

Since p is a standard partition, $0 \notin St_p$. It follows that

$$\{is' + jt'\} \cap St_p = \emptyset$$

Consider the mapping from the $gcd(s,t)$-abacus to the s-abacus, from lemma 3:

$$(as' + c, i) \leftrightarrow (a, c \cdot gcd(s,t) + i),\ 0 \leq c < s,\ 0 \leq i < gcd(s,t).$$

An equivalent mapping is made between the $gcd(s,t)$ and the t-abacus. Since the image under this mapping of the k^{th} column of the $gcd(s,t)$-abacus goes to the set of columns $\{l : l \equiv k(mod\ gcd(s,t))\}$ of the s-abacus, the abacus mapping can be decomposed into independent column mappings.

Consider the k^{th} column of the $gcd(s,t)$ abacus. Let the bead in the $(as' + c)^{th}$ row of the k^{th} column be filled. This maps to the a^{th} row, $(gcd(s,t) \cdot c + k)^{th}$ column of the s-abacus. Then, the bead in the

$$(as' + c - is')^{th} = ((a - i)s' + c)^{th}$$

row of the k^{th} column will map to the $(a - i)^{th}$ row, $(gcd(s,t) \cdot c + k)^{th}$ column of the s-abacus. Since, in the s-abacus, this bead is in the same column as the bead mapped from the $(as' + c)^{th}$ row, k^{th} column of the $gcd(s,t)$-abacus, but has a smaller row number, and because we have an s-core, we must have this bead filled in. The argument that the position in the k^{th} column, $(as' - jt' + c)^{th}$ row must be filled in is identical. Therefore, the partition placed in the k^{th} column of p has the s'/t' restrictions from lemma 1, and therefore must be an s'/t'-core. Furthermore, since p cannot have zero part sizes, the partition placed in the 0^{th} column is cannot have zero part sizes. This concludes the proof. □

It is easy to see from this classification that there are infinitely many s/t-cores which are not $gcd(s,t)$-cores when s and t are not relatively prime.

Theorem 2. *if $s \neq t$, $gcd(s,t) > 1$, there are infinitely many simultaneous s/t-cores which are not $gcd(s,t)$-cores.*

Proof. Consider the partition λ such that $\lambda_1 = 1$, of 1. This partition is trivially an s'/t'-core. Now consider the partition which has beads only in the first column of the $gcd(s,t)$-abacus, and λ placed in the first column, adding any number of parts of size zero to λ. We know that the first column must exist, since $gcd(s,t) > 1$. This is an s/t-core by Theorem 3, and it is not a $gcd(s,t)$-core because the beads in the first column are not all pushed up. □

We find a similar description for $(e,r)/(e',r)$-core partitions.

Corollary 1. *A standard partition p is an $(e,r)/(e',r)$-core with $e, e', r \in \mathbb{N}$, $r > 1$ if and only if the partition placed in each column of the r-abacus of p is an e/e'-core partition, possibly with parts of size zero, and if we consider b_i to be the number of beads in the i^{th} column of the r-abacus, then $b_0 \leq b_1 \leq \cdots \leq b_{r-1} \leq b_0 - 1$. The partition placed in the 0^{th} column, however, cannot have parts of size zero.*

Proof. It is clear that an $(e,r)/(e',r)$-core is simply an $er/e'r$-core with empty r-core. Therefore, since all $(e,r)/(e',r)$-cores are $er/e'r$-cores, they must all have the properties listed in theorem 3. Additionally, if we take the r-core of a partition, it is clear that this will be the empty partition if and only if the r-abacus has $b_0 \leq b_1 \leq \cdots \leq b_{r-1} \leq b_0 - 1$. □

5 Generating Functions based on the columns of a k-abacus

Consider sets of standard partitions $T_0, T_1, \ldots, T_{k-1}$. We are interested in finding the generating function for the set of partitions T consisting of all partitions p such that the partition placed in the i^{th} column of the k-abacus of p is a partition from the set T_i with some number of parts of size zero added. We can think of T as a sort of direct product of $T_0, T_1, \ldots, T_{k-1}$, due to the independence of the columns. Let $\varphi_i(q)$ be the generating function for partitions in T_i.

Lemma 2. *If $\varphi_i(q)$ is the generating function for partitions in T_i, then the generating function for T is*

$$\left(\prod_{i=0}^{k-1} \varphi_i(q^k)\right) \cdot \prod_{i=1}^{\infty} \frac{(1-q^{kn})^k}{1-q^n}$$

Additionally, it should be noted that if each column has the same generating function $\psi(q)$, this becomes

$$(\psi(q^k))^k \cdot \prod_{n=1}^{\infty} \frac{(1-q^{kn})^k}{1-q^n}.$$

Proof. Let $(p_0, \ldots, p_{k-1}, t)$ be given such that $p_i \in T_i$ and $|p_i| = \lambda_i \; for i = 0, \ldots, k-1$, and t is a k-core partition such that $|t| = \lambda_k$. Consider the partition P_1 with p_i placed in the i^{th} column of the k-abacus. Choose the column j of P_1 with the most beads. Consider the partition P_2 which is identical to P_1 except that zero part sizes are added to the partition placed in every column such that they all have the same number of beads as j. Since we may add zero part sizes to each of the columns, it is clear that $P_2 \in T$. From [8], as indicated in section 2, we know that P_2 partitions the number

$$k \cdot \left(\sum_{i=0}^{k-1} \lambda_i \right)$$

since k-coring the partition will give us a k-core of size 0. It is important to note that this result is independent of the column in which the p_i's are placed. We know that since t is a k-core, we can completely determine t based on the number of beads placed in each column, t_i. Now consider the partition P which is identical to the P_2, except that the the partition placed in the i^{th} column of P has an addition t_i zero part sizes. It is trivial to see that $P \in T$. P will be a partition of size

$$k \cdot \left(\sum_{i=0}^{k-1} \lambda_i \right) + \lambda_k.$$

This mapping is obviously injective. To show surjectivity, we will reverse the mapping in the following way: Take $P \in T$. Consider t, the k-core of P, with t_i beads in the i^{th} column of its k-abacus. Add zero part sizes to P until the partition placed in each column has at least t_i zero part sizes. This partition is still identical to P, but now allowing zero part sizes. Now we may reverse the mapping given above, to get the correct $n+1$-tuple. Therefore, we have an isomorphism.

Therefore, we have defined an isomorphism between $(k+1)$-tuples of k partitions of the desired structures and a k-core, to partitions in T, with the corresponding size being completely determined by the $(k+1)$-tuple. Note that the generating function for a k-core is $\prod_{i=1}^{\infty} \frac{(1-q^{kn})^k}{1-q^n}$, as given by Garvan, Stanton, and Kim in [4]. Since we may choose any combination of partitions in our $(k+1)$-tuple, we get the generating function

$$\left(\prod_{i=0}^{k-1} \varphi_i(q^k)\right) \cdot \prod_{i=1}^{\infty} \frac{(1-q^{kn})^k}{1-q^n}.$$

□

In addition to the assumptions made for lemma 2, we will assume that the partitions also have empty k-core.

Lemma 4. *The generating function for partitions in T, as described in Lemma 2, with empty k-core is*

$$\prod_{i=0}^{k-1} \varphi_i(q^k)$$

Additionally, it should be noted that if each column has the same generating function $\psi(q)$, this becomes

$$(\psi(q^k))^k$$

Proof. Since the r-core entry of the $(k+1)$-tuple used in Lemma 2 will always be 0, the proof follows the proof of Lemma 2 without the final step of adding an arbitrary k-core structure. □

We can use this to show that

Corollary 2. *An isomorphism exists between $(m,k)/(n,k)$-cores and m/n-cores. Additionally, if $(m,n)=1$, then there are $\frac{\binom{m+n}{m}}{m+n}$ $(m,k)/(n,k)$-cores.*

Proof. We may obtain an isomorphism from the mapping taken in the proof of Lemma 2, ignoring the k-core part. The second statement follows directly from the results that Kohles had involving the number of m/n-cores[9].. □

To show the strength of the method proven in lemmas 2 and 4 we give alternate proofs herein for the following well known results.

Corollary 3. *The generating function for (e, r)-cores is*

$$\left(\prod_{n=1}^{\infty} \frac{(1-q^{nre})^e}{1-q^{nr}}\right)^r.$$

Proof. It can be shown using the mapping given in lemma 3 that a partition is an (e, r)-core if and only if the partition placed every column is an e-core, with the overall partition having empty r-core. We know that the generating function for e-cores is

$$\prod_{n=1}^{\infty} \frac{(1-q^{ne})^e}{1-q^n}.$$

Using lemma 4, replacing q with q^r and multiplying these identical generating functions together gives us the desired generating function. ☐

Corollary 4. *The generating function for km-cores is*

$$\prod_{n=1}^{\infty} \left(\frac{1-q^{kmn}}{1-q^{kn}}\right)^k \cdot \frac{\left(1-q^{kn}\right)^k}{1-q^n} = \frac{1-q^{kmn}}{=} \prod_{n=1}^{\infty} \frac{\left(1-q^{kmn}\right)^{km}}{1-q^n}$$

Proof. It is clear that the a partition is a km-core if and only if the partition placed in each column of the k-abacus is an m-core. Therefore, from Lemma 2, the desired generating function is attained. ☐

Theorem 15. *$\forall k \geq 4$, $m, n \in \mathbb{N}$, $gcd(m, n) = 1$, $j \in \mathbb{N}$, $j \geq k$, there exists a mk/nk-core partition of j which is not a k-core partition.*

Proof. Consider the sets $T_0 = \{(1)\}, T_i = \emptyset \forall i \in \mathbb{N},\ 0 < i < k$. We would like to determine the generating function for the set T of partitions which have a choice from T_i in the i^{th} column as described in lemma 2. It is easy to see that the generating functions are $\varphi_0 = 1 + q$ and $\varphi_i = 1$. Then, from lemma 2, we have that the generating function is

$$(1+q^k) \cdot \prod_{i=1}^{\infty} \frac{(1-q^{kn})^k}{1-q^n}.$$

However, if we remove all of the k-cores, then we get the generating function

$$(1+q^k) \cdot \prod_{i=1}^{\infty} \frac{(1-q^{kn})^k}{1-q^n} - \prod_{i=1}^{\infty} \frac{(1-q^{kn})^k}{1-q^n} = q^k \cdot \prod_{i=1}^{\infty} \frac{(1-q^{kn})^k}{1-q^n}.$$

This gives us the result that if there is a k-core partition of $n \in \mathbb{N}$ then there is a mk/nk-core partition of $n+k$ which is not a k-core. Using theorem 14, proven by Ono, we know that for every $n \in \mathbb{N}^{\times}$, since $k \geq 4$ there is a k-core partition of n. Thus we know that $\forall n > k$ we have an mk/nk-core partition that is not a k-core. Additionally, the partition with the single structure number k, attained by not adding any k-core part to the partition forced by the T_i's, is not a k core, but it is an mk/nk-core, and has size k. Thus we have shown the desired result. □

We now use the developed method to prove the generating functions for the partitions which we have investigated thus far.

Theorem 6. *If $\varphi_k(q)$ is the generating function for k-core partitions, and $\psi_{m,n}(q)$ is the generating function for simultaneous m/n-core partitions with $gcd(m,n) = 1$, then the generating function for simultaneous mk/nk-core partitions is*

$$\left(\psi_{m,n}(q^k)\right)^k \cdot \varphi_k(q).$$

Proof. Since we have shown that a partition is an mk/nk-core if and only if the partition placed in each column is an m/n-core from theorem 3, this follows immediately from Lemma 2. □

When $m = 2$ and $n = 3$ there is a nice structure to the generating function, which is depicted in the following example:

Example 2. There is only one simultaneous 2/3-core partition, namely, the single partition of 1. Therefore, the generating function for 2/3-cores is $1+q$. Thus the generating function for $2k/3k$-cores is

$$(1+q^k)^k \cdot x \prod_{i=1}^{\infty} \frac{(1-q^{kn})^k}{1-q^n} = (\sum_{i=0}^{k} (\binom{k}{i} q^k)) \cdot \prod_{i=1}^{\infty} \frac{(1-q^{kn})^k}{1-q^n}$$

Theorem 7. *if $\psi_{e,e'}(q)$ is the generating function for simultaneous e/e' cores, then the generating function for simultaneous $(e,r)/(e',r)$ cores is*

$$\left(\psi_{e,e'}(q^r)\right)^r.$$

Proof. We know that a partition is an $(e,r)/(e',r)$-core if and only if it is an $er/e'r$-core with empty r-core. This occurs if and only if the partition placed in each column is an e/e'-core and the overall partition has empty r-core. Therefore, this follows immediately from Lemma 4. □

6 Conjugation

Let a partition Λ be given. Let n be the largest structure number for Λ.

Theorem 17. *The partition Λ^c, the conjugate of Λ, is determined in the following manner. Pair structure numbers as follows:*

$$(n, 0), (n-1, 1), \ldots, (n-i, i), \cdots, (n - \lfloor\frac{n}{2}\rfloor, \lfloor\frac{n}{2}\rfloor).$$

Then $n - i \in St_\Lambda$ iff $i \notin St_{\Lambda^c}$ $\forall i \in \mathbb{N} : i < n$

Proof. Consider the Ferrers-Young Diagram of Λ. Let m be the number of nodes in the $(i+1)^{st}$ column and k be the difference between the number of nodes in the i^{th} and $(i+1)^{st}$ columns. Then the difference between the i^{th} and $(i+1)^{st}$ largest structure numbers of Λ^c will be $k+1$. These columns will differ by k iff the $\{m+1, m+2, \ldots, m+k\}^{th}$ rows are all of length i, the m^{th} row has more than i nodes, and the $(m+k+1)^{th}$ row has less than i nodes. The structure numbers corresponding to these rows in Λ will differ by 1. It is trivial to show that the largest structure number of Λ will also be the largest structure number of Λ^c. Therefore, starting at the first column of the Ferrers-Young Diagram, corresponding to the structure number n, we may choose k such that $n-k-1$ is the next structure number of Λ^c, and Λ has structure numbers $\{1, 2, \ldots, k\}$, corresponding to equal part sizes of 1 in the Ferrers-Young Diagram. Additionally, since the $(k+1)^{th}$ row does not have the same part size, Λ will not have the structure number $k+1$. Thus, if we continue this process for all of the columns of the Ferrers-Young Diagram, we find that $n - i \in St_{\Lambda^c}$ iff Λ does not have a structure number i. ☐

It can easily be observed that taking this operation modulo an integer will result in an operation which will only depend on the columns. Thus, we got the following result involving conjugating t-cores

Corollary 5. *If we have a t-core with t-abacus column lengths*

$$(0, C_1, C_2, \ldots C_k, \ldots, C_{t-1}),$$

where $C_k \geq C_i$, $\forall i \in \mathbb{N}$, $i < t$ and $k \geq i \, \forall i : C_k = C_i$, then the lengths of Λ^c are

$$(0, C_k - C_{k-1}, C_k - C_{k-2}, \ldots, C_k - C_1, C_k, C_k - C_{t-1} - 1, C_k - C_{t-2} - 1, \ldots, C_k - C_{k+1} - 1)$$

Proof. Theorem 17 gives the process which, taken modulo t, will give us this result. □

Furthermore, we determine the properties necessary and condition for self-conjugacy of a partition.

Corollary 6. *Λ is self-conjugate iff for every pair of structure numbers $(n-i, i)$, Λ contains exactly one of the structure numbers.*

Proof. Since the conjugate partition Λ^c has the structure number i iff Λ does not have the structure number $n-i$, this follows directly from Theorem 17. □

Fix $0 \leq k < t$, $k \equiv n\,(mod t)$ so that the k^{th} column of the normalized t-abacus will serve as our "pivot" column for the conjugation operation.

Corollary 7. *A t-core partition is self conjugate iff for every pair of columns of the t-abacus $(i, k-i)$ such that $0 \leq i \leq \lfloor \frac{k}{2} \rfloor$, $C_i + C_{k-i} = C_k$. Additionally, for every pair of columns of the t-abacus $(k+i, t-1-i)$, $C_{k+i} + C_{t-1-i} = C_k - 1$.*

Proof. This follows directly from Corollary 5. □

The above characterization of self-conjugate partitions in corollary 7 leads itself to a nice alternate proof of the generating function for all self conjugate partitions.

Theorem 4. *The generating function for all self conjugate partitions is identical to the generating function for partitioning n with distinct odd integers, namely*

$$\prod_{n=0}^{\infty}(1 + q^{2n+1}).$$

Proof. Let a self conjugate partition Λ be given. Choose the largest structure number $2k+1$ of Λ. Based on Theorem 17, $\forall i \in \{0, \ldots, k\}$, Λ has exactly one of the structure numbers $(i, 2k+1-i)$. Consider generating the partition Λ by entering structure numbers one at a time, starting with $2k+1$, then 1 or $2k+1-1$, then 2 or $2k+1-2$, etc. When a structure number is added, it adds a part size of its structure number minus the number of structure numbers less than itself. It also takes away 1 from the part sizes of all structure numbers greater than itself. Therefore, it adds its structure number minus the number of structure numbers currently in the partition to the overall partition size. Since there are i structure numbers in the partition when we choose to add either the structure number i or $2k+1-i$, we either add $i-i=0$ or $2k+1-i-i=2(k-i)+1$ to the overall partition size. Therefore, since we are adding only distinct odd integers, we have exactly the number of ways to partition n with distinct odd integers. This gives us the desired generating function $\prod_{n=0}^{\infty}(1+q^{2n+1})$. □

7 General Structure Theorem for Self-Conjugate s/t-cores With $gcd(s,t) > 1$

Theorem 5. *The general structure for self-conjugate simultaneous s/t-cores is as follows:*

1) If $gcd(s,t)$ is even, place $\frac{gcd(s,t)}{2}$ simultaneous s'/t'-cores in columns $0, \ldots, \frac{gcd(s,t)}{2}-1$. Next, place the conjugate of each i^{th} column in the $(gcd(s,t)-i)^{th}$ column. Fill in a $gcd(s,t)$-core part at the top of each column.

2) If $gcd(s,t) = 2k+1$, place k simultaneous s'/t'-cores in columns $0, \ldots, k-1$. Next, place a self-conjugate s'/t'-core partition in the k^{th} column. Place the conjugate partitions of each i^{th} column in the $(gcd(s,t)-i)^{th}$ column. Finally, add a $gcd(s,t)$-core structure at the top of each column.

Proof. Due to the isomorphism from a partition to the partitions in the columns of its k-abacus, determined by James and Kerber, and the isomorphism between a $(k+1)$-tuple representing this partition, and its conjugate, we get the following isomorphism between a $(k+1)$-tuple for a partition and its conjugate:

$$(p_k, p_0, p_1, \ldots, p_{k-1}) \mapsto (p_k^c, p_{k-1}^c, \ldots, p_0^c)$$

Let a self-conjugate, simultaneous s/t-core partition Λ corresponding to $(p_{(gcd(s,t))}, p_0, p_1, \ldots, p_{gcd(s,t)-1})$, with $p_{(gcd(s,t))}$ a $gcd(s,t)$-core be given. We know from the general structure of simultaneous s/t-cores that each p_i, $i < gcd(s,t)$, must be a simultaneous s'/t'-core. Since the conjugate of Λ corresponds to $(p_k^c, p_{k-1}^c, \ldots, p_0^c)$, and Λ is self-conjugate, we must have $p_i = p_{k-1-i}^c \, \forall i < gcd(s,t)$. In the case where $gcd(s,t)$ is odd, then the k^{th} column is paired with itself, and thus it must be self-conjugate. This proves the theorem. □

8 Generating Function for Self-Conjugate s/t-cores With $gcd(s,t) > 1$

Let the generating function $\gamma(q)$ of self-conjugate $gcd(s,t)$-cores be given. Let the generating function $\psi_{s',t'}(q)$ of simultaneous s'/t'-cores be given. Let the generating function $\zeta_{s',t'}(q)$ of self-conjugate simultaneous s'/t'-cores be given.

Theorem 8. *If $gcd(s,t) = 2k+1$ for some $k \in \mathbb{N}$, then the generating function for self-conjugate s/t-cores is*

$$\gamma(q)\left(\psi_{s',t'}(q^{2gcd(s,t)})\right)^k \zeta_{s',t'}(q^{gcd(s,t)}).$$

If $gcd(s,t) = 2k$ for some $k \in \mathbb{N}$, then the generating function for self-conjugate s/t-cores is

$$\gamma(q)\left(\psi_{s',t'}(q^{2gcd(s,t)})\right)^k.$$

Proof. Suppose $gcd(s,t) = 2k$ for some $k \in \mathbb{N}$. Let simultaneous s'/t'-core partitions $p_0, \ldots, p_{k-1}$ of sizes $\lambda_0, \ldots, \lambda_{k-1}$ be given. Let a self-conjugate $gcd(s,t)$-core partition $p_{(k)}$ of size λ_k be given. In a similar manner to the proof of Lemma 2, place the beads corresponding to p_i, $i < k$ into the i^{th} column of the $gcd(s,t)$-abacus. Place beads corresponding to p_i^c, $i < k$, into the $(2k-i)^{th}$ column. Then fill in from the top until all columns have the same number of beads. Next, add beads to each column until the partition $p_{(k)}$ can be obtained by $gcd(s,t)$-coring. This corresponds to the $(2k+1)$-tuple

$$(p_{(k)}, p_0, p_1, \ldots, p_{k-1}, p_{k-1}^c, \ldots, p_0^c).$$

From [9], the size for this partition is

$$(\sum_{i=0}^{k-1} gcd(s,t) \cdot 2\lambda_i) + \lambda_k.$$

The number of choices for positions of the λ_i's are the multinomial coefficients. Therefore, we obtain the generating function

$$\gamma(q)\left(\psi_{s',t'}(q^{2gcd(s,t)})\right)^k.$$

Suppose $gcd(s,t) = 2k+1$ for some $k \in \mathbb{N}$. The proof follows Theorem 5 by a similar argument. □

The generating function for self-conjugate $(e,r)/(e',r)$-cores is related to the generating function for self-conjugate $er/e'r$-cores in a similar manner to the relationship found earlier for these partitions without the added self-conjugacy constraint. Let $\psi_{e,e'}(q)$ be the generating function for simultaneous e/e'-cores. Let $\zeta_{e,e'}(q)$ be the generating function for self-conjugate simultaneous e/e'-cores.

Theorem 9. *If $r = 2k+1$ for some $k \in \mathbb{N}$, then the generating function for self-conjugate $(e,r)/(e',r)$-cores is*

$$\left(\psi_{e,e'}(q^{2r})\right)^k \zeta_{e,e'}(q^r)$$

If $r = 2k$ for some $k \in \mathbb{N}$, then the generating function for $(e,r)/(e',r)$-cores is

$$\left(\psi_{e,e'}(q^{2r})\right)^k$$

Proof. This follows immediately from Lemma 4 and Theorem 8. □

9 Maximum simultaneous m/n-cores with $gcd(m,n) = 1$

Since our generating functions depend on the (polynomial) generating functions for m/n-cores with $gcd(m,n) = 1$, we are interested in finding the largest $k \in \mathbb{N}$ for which an m/n-core partition exists. Additionally, these objects are of interest in Representation Theory, as indicated in section 3. We investigate here the m/n-core with the most number of rows in the Ferrers-Young Diagram. We shall call these *maximum m/n-cores*. We know that no m/n-core can have structure numbers which are positive linear combinations of m and n with non-negative coefficients, so we consider the partition with all of the structure numbers which are not positive linear combinations of m and n. We will show that this is always an m/n-core, and has the most number of rows. For example, if $m = 5$ and $n = 7$, we have the following partition, represented in its 5-abacus:

	0	1	2	3	4
0	$\vert$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
1	$\vert$	$\bullet$	$\vert$	$\bullet$	$\bullet$
2	$\vert$	$\bullet$	$\vert$	$\bullet$	$\vert$
3	$\vert$	$\bullet$	$\vert$	$\bullet$	$\vert$
4	$\vert$	$\vert$	$\vert$	$\bullet$	$\vert$

We show further that the size of a partition of this type is $\frac{(m^2-1)(n^2-1)}{24}$ and that these partitions are also self conjugate. It will be useful to use the notation $LC_{m,n}$ to denote the set of all nonnegative linear combinations of m and n.

Theorem 10. *The partition P consisting exactly of the structure numbers which are not nonnegative linear combinations of m and n is an m/n-core. In addition, its Ferrers-Young Diagram has the most possible number of rows.*

Proof. We know that P is an m-core if and only if $\forall k > m,\ k \in St_P \implies k-m \in St_P$ and $0 \notin St_P$. We already know $0 \notin St_P$. Let $k > m,\ k \in St_P$ be given. Then k is not a linear combination of m and n. Thus, if $k-m \in LC_{m,n}$, then, as $m \in LC_{m,n}, k = (k-m) + m \in LC_{m,n}$. So $k - m \in St_P$. The same argument can be used to show that P is an n-core. Therefore P is an m/n-core. We also know that P has exactly one row in its Ferrers-Young diagram for every structure number. If we add any structure number, then this number must be a linear combination of m and n, but then this would require $0 \in St_P$, so we cannot add any new structure numbers. ☐

Lemma 5. $St_P = \{b - i :\ i \in LC_{m,n} \cap [0, b]\}$*, where b is the largest $i \in \mathbb{N}$: $i \notin\ LC_{m,n}$.*

Proof. We will denote $LC_{m,n} \cap [0, b]$ by $LC^b_{m,n}$. We know that since m and n are relatively prime that we can make find a positive linear combination of every number greater than $mn - m - n = (m - 1)(n - 1) - 1$. If we reorder the columns in the m-abacus of P, from the shortest column to the longest column, then the i^{th} column would have $\lfloor \frac{in}{m} \rfloor$ beads in it, since the lines for the m-core and n-core parts allow at most this many beads to be in that row. Since m and n are relatively prime, $\frac{in}{m} - \lfloor \frac{in}{m} \rfloor$ will take on every possible remainder from 1 to $m-1$, in the following sum for the size of St_P:

$$\begin{aligned}\sum_{i=1}^{m-1}\lfloor\frac{in}{m}\rfloor &= \sum_{i=1}^{m-1}\frac{in}{m}\rfloor - \frac{in}{m} + \frac{in}{m} \\ &= \sum_{i=1}^{m-1} -\frac{i}{m} + \frac{in}{m} \\ &= \frac{(m-1)(n-1)}{2}\end{aligned}$$

Therefore, since $St_P \cap LC^b_{m,n} = \emptyset$ and $St_P \cup LC^b_{m,n} = [0,b]$ $|St_P| = |LC^b_{m,n}|$. It is obvious that $|\{b-i: \ i \in LC^b_{m,n}\}| = |LC^b_{m,n}|$ and we have already shown in the proof of theorem 10 that $St_P \supset \{b-i: \ i \in LC^b_{m,n}\}$. Therefore, it follows that $St_P = \{b-i: \ i \in LC^b_{m,n}\}$. □

Theorem 11. *Maximum m/n-cores are self-conjugate.*

Proof. Let the maximum m/n-core partition P be given. From Corollary 6, we know that a partition is self-conjugate iff $\forall i \in \mathbb{N}, \ i \leq b, \ i \in St_P \Leftrightarrow b-i \notin St_P$. Let $i \in LC_{m,n}$ be given. Then $b-i \notin LC_{m,n}$, since $b-i, i \in LC_{m,n} \implies b-i+i = b \in LC_{m,n}$. Let $k \in St_P$ be given. Then, since, from Lemma 5, $St_P = \{b-i: \ i \in LC^b_{m,n}\}$ there is $j \in LC_{m,n}$ such that $k = b-j$, so $b-k = j \notin St_P$. □

We would now like to show the size of these partitions:

Theorem 12. *The size of maximum m/n-cores is $\frac{(m^2-1)(n^2-1)}{24}$.*

Proof. Consider the partition with exactly the structure numbers from 1 to b. This would represent a partition of $mn-m-n$. Consider removing all of the elements of ${LC^b_{m,n}}^{\times}$. After removing all of these structure numbers, we will get the maximum m/n-core. To calculate the size of this partition, we consider the change in the number which is partitioned by incrementally removing the positive linear combination structure numbers. If the structure number k from partition p of y is removed from a partition, then the corresponding row in the Ferrers-Young Diagram has k- (# rows with lower structure numbers) nodes. However, if the rows above it keep the same structure number, then we must add another node to each of these rows to compensate for the missing row beneath them. For notational ease, we will define for a true/false function f $T_f := \{x \in St_P : f(x)\}$. Therefore, the overall effect is that the new partition p' partitions $y - (k - \#T_{<k} + \#T_{>k}) =$

$$y - k + \#T_{\neq k}.$$

So removing all of the linear combinations from the partition will give

$$b - \sum_{x \in \ {LC^b_{m,n}}^\times} x + \sum_{i=1}^{|{LC^b_{m,n}}^\times|} b - i.$$

In order to calculate $\sum_{i=1}^{|{LC^b_{m,n}}^\times|} b - i$ we remember from Lemma 5 $St_P = \{b - i : i \in LC^b_{m,n}\}$. Therefore there is an isomorphism between $LC^b_{m,n}$ and St_P, $\varphi : {LC^b_{m,n}}^\times \mapsto St_P$, $\varphi(i) = b - i$. Moreover, $LC^b_{m,n} \cup St_P = [0, b]$ and $LC^b_{m,n} \cap St_P = \emptyset$, so, as we have already removed 0, we have $|{LC^b_{m,n}}^\times| = \frac{b+1}{2} - 1 = \frac{(m-1)(n-1)}{2} - 1$. Thus

$$b + \sum_{i=1}^{|{LC^b_{m,n}}^\times|} b - i = b + \sum_{i=1}^{\frac{(m-1)(n-1)}{2}} b - i \quad = \frac{(m-1)(n-1)}{8}(-2 + 3(m-1)(n-1))$$

To calculate the first sum, a slightly more complicated procedure is necessary. Consider the disjoint decomposition of $LC^b_{m,n} = \bigcup_{i=1}^{n-2}\{im + jn : i, j \in \mathbb{N},\ im + jn < b\}$. Then we have that

$$\sum_{x \in \ {LC^b_{m,n}}^\times} x = \sum_{x \in \bigcup_{i=1}^{m-2}\{in+jm : i,j \in \mathbb{N},\ in+jm<b\}} = \sum_{i=1}^{n-2} \sum_{\{j \in \mathbb{N} : in+jm<b\}} in + jm$$

For each i, choose $j \in \mathbb{N}$ such that for $r_i := b - (in + jm),\ 0 \le r_i < n$ Since m and n are relatively prime, we will attain exactly every value for r_i from 1 to $n - 1$. Consider the step function f_i which is equal to $in + jm$ on the interval $in + (j-1)m$ to $in + jm$. Then, if you take the integral of this function from $in - m$ to $b - r_i$ and then divide by m, the width of each interval of the step function, you will get the desired value. To integrate this step function, we estimate the function by $f(x) := x$, and comute the error. It is clear to see that the difference of the integrals is the area of a right triangle of height m and width m. The area of this triangle is $\frac{m^2}{2}$. Therefore, the error for the entire sum of the integrals is $\frac{m^2}{2}(\#\ triangles) = \frac{m^2}{2}(|LC_{m,n}^{b}{}^{\times}| + 1)$, since we have one triangle for each linear combination, and an addition triangle for $i = 0, j = 0$. We have already calculated $|LC_{m,n}^{b}{}^{\times}|$, however. So we have $\frac{(m-1)(n-1)}{2}$ such triangles. Therefore, using the equalities $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$ and $\sum_{i=0}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$ gives us

$$\begin{aligned}
\sum_{x\in\ LC_{m,n}^{b}{}^{\times}} x &= \frac{1}{m}\left(\left(\sum_{i=0}^{m-2}\int_{in-m}^{b-r_i} x\right) + \frac{m^2(m-1)(n-1)}{4}\right) \\
&= \frac{1}{2m}\left(\sum_{i=0}^{m-2}(b-r_i)^2 - (in-m)^2\right) + \frac{m(m-1)(n-1)}{4} \\
&= \frac{1}{2m}\left(\sum_{i=0}^{m-2}(b-(i+1))^2 - (in-m)^2\right) + \frac{m(m-1)(n-1)}{4} \\
&= \frac{(m-1)(n-1)}{12}(4mn - 5n - 5m + 1)
\end{aligned}$$

So, this gives us the overall answer of

$$\begin{aligned}
& b + \sum_{i=1}^{|LC_{m,n}^{b}{}^{\times}|}(b-i) - \sum_{x\in\ LC_{m,n}^{b}{}^{\times}} x \\
&= \frac{(m-1)(n-1)}{8}(-2 + 3(m-1)(n-1)) - \frac{(m-1)(n-1)}{12}(4mn - 5n - 5m + 1) \\
&= \frac{(m-1)(n-1)}{24}(-6 + 9(m-1)(n-1) - 8mn + 10n + 10m - 2) \\
&= \frac{(m-1)(n-1)}{24}(mn + m + n + 1) \\
&= \frac{(m-1)(n-1)(m+1)(n+1)}{24} \\
&= \frac{(m^2-1)(n^2-1)}{24}
\end{aligned}$$

which is the size of the partition which we set out to prove.

□

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