A Galois Connection Approach to Wei-Type Duality Theorems

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Abstract— In 1991, Wei proved a duality theorem that established an interesting connection between the generalized Hamming weights of a linear code and those of the dual code. Wei's duality theorem has since been extensively studied from different perspectives and extended to other settings. In this paper, we re-examine Wei's duality theorem and its various extensions, henceforth referred to as Wei-type duality theorems, from a new Galois connection perspective. Our approach is based on the observation that the generalized Hamming weights and the dimension/length profiles of a linear code form a Galois connection. The central result in this paper is a general Wei-type duality theorem for two Galois connections between finite subsets of $\mathbb{Z}$, from which all the known Wei-type duality theorems can be recovered. As corollaries of our central result, we prove new Wei-type duality theorems for w-demimatroids defined over finite sets and w-demi-polymatroids defined over modules with a composition series, which further allows us to unify and generalize all the known Wei-type duality theorems established for codes endowed with various metrics.

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1 Introduction

Let C be an (m, k) linear code over a field $\mathbb{F}$. For any $r \in \{0, 1, \ldots, k\}$, the r-th generalized Hamming weight (GHW) of C, denoted by $\mathbf{d}_r(C)$, is defined as

$$\mathbf{d}_r(C) = \min\{|\chi(D)| \mid D \text{ is an } (m, r) \text{ subcode of } C\}, \tag{1.1}$$

where for any $D \subseteq \mathbb{F}^m$,

$$\chi(D) = \{i \mid i \in \{1, \ldots, m\},\ \alpha_{(i)} \neq 0 \text{ for some } \alpha \in D\}$$

denotes the set of not-always-zero bit positions of D (see [36]). GHWs have been widely used to gauge the security performances of linear codes for secret sharing, secure network coding or distributed data storage. Relevant work for linear codes can be found in [5], [15], [25], [31], [36], which have been further extended to rank metric codes [24], [28].

Wei proved [36] a duality theorem that established an interesting connection between the GHWs of C and those of its dual code $C^{\perp}$. To state Wei's duality theorem more precisely, we first note that $C^{\perp}$ is an $(m, m-k)$-linear code and so the definition of GHWs applies to $C^{\perp}$ as well. Then, Wei's duality theorem states that the following two sets

$$\{\mathbf{d}_r(C) \mid r \in \{1, \ldots, k\}\},\ \{m + 1 - \mathbf{d}_r(C^{\perp}) \mid r \in \{1, \ldots, m-k\}\}, \tag{1.2}$$

form a partition of $\{1, \ldots, m\}$, and consequently, the GHWs of C and those of $C^{\perp}$ determine each other.

A closely related study can be found in Forney's 1994 paper [15]. More specifically, for any $l \in \{0, 1, \ldots, m\}$, the l-th dimension/length profile (DLP) of C, denoted by $\mathbf{K}_l(C)$, is defined as

$$\mathbf{K}_l(C) = \max\{\dim_{\mathbb{F}}(C \cap \delta(J)) \mid J \subseteq \{1, \ldots, m\},\ |J| = l\}, \tag{1.3}$$

where for any $J \subseteq \{1, \ldots, m\}$,

$$\delta(J) = \{\alpha \mid \alpha \in \mathbb{F}^m \ s.t.\ \forall\ i \in \{1, \ldots, m\} - J,\ \alpha_{(i)} = 0\} \tag{1.4}$$

denotes the set of all the elements in $\mathbb{F}^m$ whose positions outside of J are all zeros. Forney showed that for any $l \in \{0, 1, \ldots, m\}$, it holds that

$$\mathbf{K}_l(C^{\perp}) = \mathbf{K}_{m-l}(C) + l - k, \tag{1.5}$$

from which Wei's duality theorem can be recovered (see Theorems 2, 3 and 4 of [15] for more details).

Wei's duality theorem has since been generalized and extended in a number of directions. For codes over finite rings, Ashikhmin proved [2] a Wei-type duality theorem for linear codes over Galois rings. Horimoto and Shiromoto proved [20] a Wei-type duality theorem for linear codes over finite chain rings. For codes endowed with poset metric (see [9]), Wei-type duality theorems were proved by Barg and Purkayastha in [3] and by Moura and Firer in [29]. For rank metric codes (see [12], [16]), Wei-type duality theorems were proved by Ravagnani in [33], by Ducoat in [14] and by Martínez-Peñas and Matsumoto in [28], each in terms of a similar yet different version of generalized rank weights. A Wei-type duality theorem for sum-rank metric codes has been established by Martínez-Peñas in the recent paper [27].

Wei-type duality theorems were also proved for some combinatorial notions. In [6], Britz, Johnsen, Mayhew and Shiromoto proved two Wei-type duality theorems for demimatroids, and as consequences of these results, they further derived Wei-type theorems for matroids, graphs and transversals. In [17], Ghorpade and Johnsen proved Wei-type duality theorems for demi-polymatroids, a generalization of the q-analogue of a matroid defined over vector spaces (see [23], [34]), and as a consequence, they established a Wei-type duality theorem for rank metric code flags. A similar approach to [17] for demi-polymatroids was independently proposed by Britz, Mammomiti and Shiromoto in [7]. Recently, Panja, Pratihar and Hajatiana Randrianarisoa proved [32] a Wei-type duality theorem for sum-matroids, which is another generalization of the q-analogue of a matroid.

In this paper, we will re-examine Wei's original duality theorem and more generally Wei-type duality theorems from a new Galois connection perspective. Our starting point is the observation that the GHWs and DLPs of a linear code form a Galois connection, for which Wei's duality theorem holds; and moreover, Galois connections also naturally arise in similar or more general settings, where various analogues or generalizations of Wei's duality theorem hold.

In Section 2, we show that the GHWs and DLPs of a linear code form a Galois connection, and then we prove our main result Theorem 2.2, which is a Wei-type duality theorem for two Galois connections between finite subsets of $\mathbb{Z}$. This result is of central importance in the sense that it implies all the known Wei-type duality theorems, and moreover, it can be further applied to derive new Wei-type duality theorems, as detailed in later sections. In Section 3, as corollaries of Theorem 2.2, we establish Theorems 3.1 and 3.2, two bridging theorems that can facilitate the application of Theorem 2.2. Compared to Theorem 2.2, Theorems 3.1 and 3.2 can be used to recover known or derive new Wei-type duality theorems via simple substitution.

In Section 4, we prove Wei-type duality theorems for generalized weights and profiles of w-demimatroids defined over finite sets (Theorems 4.1 and 4.2). These results generalize the corresponding results for demimatroids ([5, 6, 26]). In Section 5, we prove a Wei-type duality theorem for generalized weights and profiles of w-demi-polymatroids defined over modules with a composition series (Theorem 5.1). Our result generalizes the corresponding results for (q, m)-demi-polymatroids defined over vector space over a field in [7, 17].

Sections 6 and 7 are devoted to generalized weights, profiles and Wei-type duality theorems for codes endowed with various metrics, more specifically, Gabidulin-Roth rank metric ([14], [16], [24]), poset metric ([9], [29]), Delsarte rank metric ([12], [18], [28], [29], [33]) and generalized Hamming weights with respect to rank of modules ([20]). Following the framework of [35], we consider codes over modules with a composition series. We show that generalized weights and profiles of such codes endowed with different metrics can be unified and rephrased in terms of the associated w-demimatroids or w-demi-polymatroids; and furthermore, dual w-demimatroids or w-demi-polymatroids are always associated to dual codes. Based on these observations, we prove Wei-type duality theorems for codes endowed with different metrics through a unified approach in Theorems 7.1–7.4, which respectively generalizes the corresponding existing Wei-type duality theorems.

2 Main Result

Throughout this paper, $\mathbb{N}$ will denote the set of all nonnegative integers, and the set of all positive integers will be denoted by $\mathbb{Z}^+$. For any $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, we will use $[a, b]$ to denote the set of all the integers between a and b, i.e., $[a, b] = \{i \mid i \in \mathbb{Z},\ a \leqslant i \leqslant b\}$. Note that if $a \geqslant b + 1$, then $[a, b] = \emptyset$.

2.1 Basics on Galois connections

In this subsection, we collect some basic facts on Galois connections between finite subsets of $\mathbb{Z}$ with respect to the order $\leqslant$. Here we note that the notion of Galois connection can be defined more generally for partially ordered sets (posets); see, e.g., [11, Definition 7.23] and [4, Page 124].

We begin by recalling the definition of Galois connection between two finite subsets of $\mathbb{Z}$. Throughout this subsection, we let P and Q be two nonempty finite subsets of $\mathbb{Z}$.

Definition 2.1. *Given $\varphi : P \longrightarrow Q$, $\psi : Q \longrightarrow P$, (φ, ψ) is said to be a*

Galois connection between P and Q, if both φ and ψ preserve the order $\leqslant$ and for any $(a,b) \in P \times Q$, $a \leqslant \psi(b) \Longleftrightarrow \varphi(a) \leqslant b$.

The following basic facts on Galois connections will be used frequently.

Lemma 2.1. *For any Galois connection (φ, ψ) between P and Q, it holds that:*
(1) *For any $\lambda \in P$, $\varphi(\lambda) = \min\{b \mid b \in Q,\ \lambda \leqslant \psi(b)\}$;*
(2) *For any $\mu \in Q$, $\psi(\mu) = \max\{a \mid a \in P,\ \varphi(a) \leqslant \mu\}$;*
(3) *Let $d_0 = \min(Q)$. Then, $\varphi^{-1}[\{d_0\}] = \{a \mid a \in P,\ a \leqslant \psi(d_0)\}$;*
(4) *Let $d \in Q$ where $d \neq \min(Q)$, and let $v = \max\{b \mid b \in Q,\ b \leqslant d-1\}$. Then, for any $a \in P$, we have $d = \varphi(a) \Longleftrightarrow \psi(v) + 1 \leqslant a \leqslant \psi(d)$.*

Proof. (1) and (2) are special cases of [11, Proposition 7.31]. (3) and (4) follow from (1) and (2) via a routine verification. □

Remark 2.1. *We will show in Section 2.2 that the GHWs and DLPs of a linear code form a Galois connection. As a result, some relevant results on linear codes (e.g., [36, Corollary A] and [15, Section III]) can be reformulated in terms of Galois connection and thereby immediately follow from Lemma 2.1. Moreover, as detailed in later sections, Galois connections naturally arise in other settings, and some relevant results on rank metric codes (e.g., [33, Theorem 42]) or (q,m)-demi-polymatroids (e.g., [17, Lemma 10]) also follow from Lemma 2.1 after appropriate reformulation.*

We also need the following lemma, whose proof is straightforward and thus omitted.

Lemma 2.2. *For a nonempty (possibly infinite) set X, let $f : X \longrightarrow P$, $g : X \longrightarrow Q$ such that $\max(P) \in f[X]$, $\min(Q) \in g[X]$. Define $\varphi : P \longrightarrow Q$ as $\varphi(a) = \min\{g(u) \mid u \in X,\ a \leqslant f(u)\}$, and define $\psi : Q \longrightarrow P$ as $\psi(b) = \max\{f(u) \mid u \in X,\ g(u) \leqslant b\}$. Then, (φ, ψ) is a Galois connection between P and Q.*

2.2 Galois connections arising from GHWs and DLPs

In this subsection, we show that the GHWs and DLPs of a linear code form a Galois connection, which is a key observation underpinning our treatment of Wei's original duality theorem and more general Wei-type duality theorems.

Recall that in Section 1, for an (m,k) linear code C over a field $\mathbb{F}$, $\mathbf{d}_r(C)$ denotes its r-th GHW and $\mathbf{K}_l(C)$ its l-th DLP.

Theorem 2.1. *Define* $\varphi : [0,k] \longrightarrow [0,m]$ *as* $\varphi(r) = \mathbf{d}_r(C)$*, and define* $\psi : [0,m] \longrightarrow [0,k]$ *as* $\psi(l) = \mathbf{K}_l(C)$*. Then,* (φ,ψ) *is a Galois connection between* $[0,k]$ *and* $[0,m]$*.*

Proof. By [30, Theorem 2], for any $r \in [0,k]$, we have

$$\varphi(r) = \mathbf{d}_r(C) = \min\{|J| \mid J \subseteq [1,m],\ r \leqslant \dim_{\mathbb{F}}(C \cap \delta(J))\},$$

where for any $J \subseteq [1,m]$, $\delta(J)$ is defined as in (1.4). Moreover, it is straightforward to verify that for any $l \in [0,m]$, it holds that

$$\psi(l) = \mathbf{K}_l(C) = \max\{\dim_{\mathbb{F}}(C \cap \delta(J)) \mid J \subseteq [1,m],\ |J| \leqslant l\}.$$

Now applying Lemma 2.2 with $f : 2^{[1,m]} \longrightarrow [0,k]$ set to be $f(J) = \dim_{\mathbb{F}}(C \cap \delta(J))$, and $g : 2^{[1,m]} \longrightarrow [0,m]$ set to be $g(J) = |J|$, we conclude that (φ,ψ) is a Galois connection between $[0,k]$ and $[0,m]$. $\square$

2.3 The central theorem

In this subsection, we prove the main result of the paper, which is of central importance to this work in the sense that not only does it imply all the known Wei-type duality theorems, but also it can be used to derive new such theorems and applied to a variety of codes.

Throughout this subsection, we let $(k,m) \in \mathbb{N}^2$, $w \in \mathbb{Z}^+$.

Lemma 2.3. *Let* (φ,ψ) *be a Galois connection between* $[0,k]$ *and* $[0,m]$ *such that* $\psi(0) = 0$ *and* $\psi(l) - \psi(l-1) \leqslant w$ *for any* $l \in [1,m]$*. Define* $\eta : [0,m] \longrightarrow \mathbb{Z}$ *as* $\eta(l) = \psi(m-l) + wl - k$*. Then, we have:*
(1) $\eta(0) = 0$*,* $\eta(m) = wm - k$*;*
(2) *For any* $l \in [1,m]$*,* $0 \leqslant \eta(l) - \eta(l-1) \leqslant w$*;*
(3) *There exists* $\tau : [0, wm-k] \longrightarrow [0,m]$ *such that* (τ,η) *is a Galois connection between* $[0, wm-k]$ *and* $[0,m]$*. Moreover, for any* $u \in [0,k]$*,* $v \in [0, wm-k]$ *with* $\varphi(u) + \tau(v) = m+1$*, it holds true that* $u \not\equiv v + k \pmod{w}$*.*

Proof. (1) and (2) follow from straightforward computation, and we prove (3). Define $\tau : [0, wm-k] \longrightarrow [0,m]$ as $\tau(a) = \min\{b \mid b \in [0,m], a \leqslant \eta(b)\}$. Then, it can be readily verified that τ is well defined, and (τ,η) is a Galois connection between $[0, wm-k]$ and $[0,m]$. Now for $u \in [0,k]$, $v \in [0, wm-k]$ with $\varphi(u) + \tau(v) = m+1$, note that $\tau(v) = m+1-\varphi(u)$, $\varphi(u) \in [1,m]$ and applying (4) of Lemma 2.1 to (τ,η) and $(v, m+1-\varphi(u))$, we deduce that

$$\eta(m - \varphi(u)) + 1 \leqslant v \leqslant \eta(m + 1 - \varphi(u)),$$

which, together with the definition of η, implies that

$$\psi(\varphi(u)) + w(m - \varphi(u)) - k + 1 \leqslant v \leqslant \psi(\varphi(u) - 1) + w(m + 1 - \varphi(u)) - k.$$

Again, by Lemma 2.1, we have $u \leqslant \psi(\varphi(u))$ and $\psi(\varphi(u) - 1) \leqslant u - 1$. It then follows that

$$u + w(m - \varphi(u)) - k + 1 \leqslant v \leqslant u - 1 + w(m + 1 - \varphi(u)) - k,$$

which implies $1 \leqslant v + k - u - w(m - \varphi(u)) \leqslant w - 1$, and we deduce that $u \not\equiv v + k \pmod{w}$, completing the proof. $\square$

Remark 2.2. *Lemma 2.3 is largely inspired by [33, Theorem 37]. Similar results for rank metric code and for (q, m)-demi-polymatroids have been established in [28, Lemma 66] and [17, Theorem 13], respectively.*

The following proposition ought to be known, however we are not able to locate a reference and so we include its proof for completeness.

Proposition 2.1. *Suppose that (φ, ψ) is a Galois connection between $[0, k]$ and $[0, m]$. Then, the following three statements are equivalent to each other:*

(1) *$\psi(l) - \psi(l - 1) \leqslant w$ for any $l \in [1, m]$;*

(2) *$|\varphi^{-1}[\{l\}]| \leqslant w$ for any $l \in [1, m]$;*

(3) *$\varphi(r) + 1 \leqslant \max\{\varphi(r + w), 1\}$ for any $r \in [0, k - w]$;*

Moreover, if $\psi(0) = 0$, then $\varphi(a) \in [1, m]$ for any $a \in [1, k]$, and **(1)**–**(3)** *are equivalent to the following*

(4) *$\varphi(r) + 1 \leqslant \varphi(r + w)$ for any $r \in [0, k - w]$.*

Proof. By (4) of Lemma 2.1, for any $l \in [1, m]$, we have $|\varphi^{-1}[\{l\}]| = \psi(l) - \psi(l - 1)$, which immediately implies (1) $\Longleftrightarrow$ (2).

Next we show (1) $\Longleftrightarrow$ (3).

(1) $\Longrightarrow$ (3) Consider $r \in [0, k - w]$. If $\varphi(r + w) \leqslant 0$, then $\varphi(r) + 1 \leqslant \varphi(r + w) + 1 \leqslant 1$, which implies (3) as desired. In the following we assume $1 \leqslant \varphi(r+w)$. By (1), we have $\psi(\varphi(r+w)) - \psi(\varphi(r+w) - 1) \leqslant w$, and by (1) of Lemma 2.1, $r + w \leqslant \psi(\varphi(r + w))$, which implies that $r \leqslant \psi(\varphi(r+w) - 1)$. Now, applying Definition 2.1 to $(r, \varphi(r+w) - 1)$, we have $\varphi(r) \leqslant \varphi(r+w) - 1$, which again implies (3).

(3) $\Longrightarrow$ (1) Consider $l \in [1, m]$. If $\psi(l) \leqslant w$, then note that $0 \leqslant \psi(l - 1)$, we deduce that $\psi(l) - \psi(l - 1) \leqslant w$, which implies (1). In the following we assume $\psi(l) \geqslant w$, therefore $\psi(l) - w \in [0, k - w]$, which, together with (3), implies that $\varphi(\psi(l) - w) + 1 \leqslant \max\{\varphi(\psi(l)), 1\}$. By (2) of Lemma 2.1, we have $\varphi(\psi(l)) \leqslant l$, also note that $1 \leqslant l$, we deduce that $\varphi(\psi(l) - w) + 1 \leqslant l$,

and hence $\varphi(\psi(l)-w) \leqslant l-1$. Applying Definition 2.1 to $(\psi(l)-w, l-1)$, we have $\psi(l)-w \leqslant \psi(l-1)$, which again implies (1).

The remainder of the proposition follows from (3) of Lemma 2.1 and the proven fact that (1) $\Longleftrightarrow$ (3). $\square$

We also need the following lemma, whose proof is straightforward and thus omitted.

Lemma 2.4. *Assume $wm \geqslant k$. For any $\gamma \in \mathbb{Z}$, define $\mathcal{U}_{(\gamma)} = \{u \mid u \in [1,k], u \equiv \gamma + k \pmod{w}\}$, $\mathcal{V}_{(\gamma)} = \{v \mid v \in [1, wm-k], v \equiv \gamma \pmod{w}\}$. Then, we have $|\mathcal{U}_{(\gamma)}| + |\mathcal{V}_{(\gamma)}| = m$.*

We are now ready to prove the main result of this paper.

Theorem 2.2. *Let (φ, ψ) be a Galois connection between $[0,k]$ and $[0,m]$ such that $\psi(0) = 0$ and for any $l \in [1,m]$, $\psi(l) - \psi(l-1) \leqslant w$. Let (τ, η) be a Galois connection between $[0, wm-k]$ and $[0,m]$. For any $\gamma \in \mathbb{Z}$, define $\mathcal{A}_{(\gamma)} = \{\varphi(u) \mid u \in [1,k],\ u \equiv \gamma + k \pmod{w}\}$, $\mathcal{B}_{(\gamma)} = \{m+1-\tau(v) \mid v \in [1, wm-k],\ v \equiv \gamma \pmod{w}\}$. Then, the following four statements are equivalent to each other:*
(1) *For any $l \in [0,m]$, $\eta(l) = \psi(m-l) + wl - k$;*
(2) *$\eta(0) = 0$, $\eta(l) - \eta(l-1) \leqslant w$ for any $l \in [1,m]$, and for any $(u,v) \in [1,k] \times [1, wm-k]$, $\varphi(u) + \tau(v) = m+1 \Longrightarrow u \not\equiv\ v + k \pmod{w}$;*
(3) *For any $\gamma \in \mathbb{Z}$, $\mathcal{A}_{(\gamma)} \cap \mathcal{B}_{(\gamma)} = \emptyset$, $\mathcal{A}_{(\gamma)} \cup \mathcal{B}_{(\gamma)} = [1,m]$;*
(4) *For any $\gamma \in \mathbb{Z}$, $\mathcal{A}_{(\gamma)} \cup \mathcal{B}_{(\gamma)} = [1,m]$.*

Proof. As detailed below, the proof consists of the following 5 steps. Throughout the proof, for any $\gamma \in \mathbb{Z}$, we define $\mathcal{U}_{(\gamma)} = \{u \mid u \in [1,k],\ u \equiv \gamma + k \pmod{w}\}$, $\mathcal{V}_{(\gamma)} = \{v \mid v \in [1, wm-k],\ v \equiv \gamma \pmod{w}\}$.
(**1**) $\Longrightarrow$ (**2**) This follows from Lemma 2.3.
(**2**) $\Longrightarrow$ (**3**) Fix $\gamma \in \mathbb{Z}$. Note that for any $u \in \mathcal{U}_{(\gamma)}$ and $v \in \mathcal{V}_{(\gamma)}$, it holds true that $u \equiv\ v + k \pmod{w}$, which, by (2), implies that $\varphi(u) \neq m+1-\tau(v)$ and furthermore $\mathcal{A}_{(\gamma)} \cap \mathcal{B}_{(\gamma)} = \emptyset$. By Proposition 2.1, $\varphi(a) \in [1,m]$ for any $a \in [1,k]$, and $\varphi(r)+1 \leqslant \varphi(r+w)$ for any $r \in [0, k-w]$, which imply that $\mathcal{A}_{(\gamma)} \subseteq [1,m]$ and $|\mathcal{A}_{(\gamma)}| = |\mathcal{U}_{(\gamma)}|$. Again by Proposition 2.1, $\tau(a) \in [1,m]$ for any $a \in [1, wm-k]$, and $\tau(r)+1 \leqslant \tau(r+w)$ for any $r \in [0, w(m-1)-k]$, which imply that $\mathcal{B}_{(\gamma)} \subseteq [1,m]$ and $|\mathcal{B}_{(\gamma)}| = |\mathcal{V}_{(\gamma)}|$. It then follows from Lemma 2.4 that $|\mathcal{U}_{(\gamma)}| + |\mathcal{V}_{(\gamma)}| = m$ and therefore $|\mathcal{A}_{(\gamma)}| + |\mathcal{B}_{(\gamma)}| = m$, which, together with $\mathcal{A}_{(\gamma)} \cap \mathcal{B}_{(\gamma)} = \emptyset$, implies that $|\mathcal{A}_{(\gamma)} \cup \mathcal{B}_{(\gamma)}| = m$. Finally, using the fact $\mathcal{A}_{(\gamma)} \cup \mathcal{B}_{(\gamma)} \subseteq [1,m]$, we deduce that $\mathcal{A}_{(\gamma)} \cup \mathcal{B}_{(\gamma)} = [1,m]$, as desired.
(**3**) $\Longrightarrow$ (**4**) This is trivial.

$(\mathbf{4}) \Longrightarrow (\mathbf{3})$ Fix $\gamma \in \mathbb{Z}$. By Lemma 2.4, it holds true that $|\mathcal{U}_{(\gamma)}|+|\mathcal{V}_{(\gamma)}| = m$. Then, from the facts $|\mathcal{A}_{(\gamma)}| \leqslant |\mathcal{U}_{(\gamma)}|$, $|\mathcal{B}_{(\gamma)}| \leqslant |\mathcal{V}_{(\gamma)}|$, $\mathcal{A}_{(\gamma)} \cup \mathcal{B}_{(\gamma)} = [1, m]$, we deduce that $|\mathcal{A}_{(\gamma)}|+|\mathcal{B}_{(\gamma)}| \leqslant |\mathcal{A}_{(\gamma)} \cup \mathcal{B}_{(\gamma)}|$, which implies that $\mathcal{A}_{(\gamma)} \cap \mathcal{B}_{(\gamma)} = \emptyset$, and so (3) follows.
$(\mathbf{3}) \Longrightarrow (\mathbf{1})$ By Lemma 2.3, there exists uniquely a Galois connection (ξ, ζ) between $[0, wm-k]$ and $[0, m]$ such that $\zeta(l) = \psi(m-l) + wl - k$ for any $l \in [0, m]$, and moreover, it follows from Lemma 2.3 and Proposition 2.1 that for any $r \in [0, w(m-1)-k]$,

$$\xi(r) + 1 \leqslant \xi(r + w). \tag{2.1}$$

Now for a fixed, yet arbitrary $\gamma \in \mathbb{Z}$, define $\mathcal{L}_{(\gamma)} = \{m+1-\xi(v) \mid v \in \mathcal{V}_{(\gamma)}\}$. Then, using a parallel argument as in the step of (2) $\Longrightarrow$ (3) (with (τ, η) replaced by (ξ, ζ)), we deduce that $\mathcal{A}_{(\gamma)} \cap \mathcal{L}_{(\gamma)} = \emptyset$, $\mathcal{A}_{(\gamma)} \cup \mathcal{L}_{(\gamma)} = [1, m]$, which, together with $\mathcal{A}_{(\gamma)} \cap \mathcal{B}_{(\gamma)} = \emptyset$ and $\mathcal{A}_{(\gamma)} \cup \mathcal{B}_{(\gamma)} = [1, m]$, imply that $\mathcal{L}_{(\gamma)} = \mathcal{B}_{(\gamma)}$ and immediately,

$$f[\mathcal{V}_{(\gamma)}] = \mathcal{B}_{(\gamma)} = \mathcal{L}_{(\gamma)} = g[\mathcal{V}_{(\gamma)}],$$

where $f : \mathcal{V}_{(\gamma)} \longrightarrow \mathbb{Z}$ is defined as $f(v) = m + 1 - \tau(v)$ and $g : \mathcal{V}_{(\gamma)} \longrightarrow \mathbb{Z}$ is defined as $g(v) = m + 1 - \xi(v)$. Since both τ and ξ preserves the order $\leqslant$, it holds true that for any $c, d \in \mathcal{V}_{(\gamma)}$ with $c \leqslant d$, we have $f(c) \geqslant f(d)$, $g(c) \geqslant g(d)$. Noting that by (2.2), g is injective, we infer that $f = g$, which implies that $\tau(v) = \xi(v)$ for any $v \in [1, wm-k]$ with $v \equiv \gamma \pmod{w}$. It follows from the arbitrariness of γ that $\tau(v) = \xi(v)$ for any $v \in [1, wm-k]$. By (3) of Lemma 2.1, $\tau(0) = \xi(0) = 0$, which implies that $\tau = \xi$. Since (τ, η) and (ξ, ζ) are Galois connections between $[0, wm-k]$ and $[0, m]$, we apply (2) of Lemma 2.1 to reach $\eta = \zeta$, and then (1) follows. $\square$

Remark 2.3. *With the help of Theorem 2.1, Theorem 2.2 can be used to recover Wei's original duality theorem. More specifically, set $w = 1$; and as in Theorem 2.1, set φ, ψ to be the GHWs and DLPs of C, respectively; and moreover, set τ, η to be those of $C^{\perp}$, respectively. Then, by Theorem 2.1, (φ, ψ) is a Galois connection between $[0, k]$ and $[0, m]$, and (τ, η) a Galois connection between $[0, m-k]$ and $[0, m]$. It can then be verified that (1) of Theorem 2.2 boils down to Forney's result in (1.5), and (3) of Theorem 2.2 boils down to Wei's result in (1.2), thereby recovering Wei's duality theorem.*

Wei's duality theorem has been extensively studied and extended to other codes or even settings beyond coding theory. To the best of our knowledge, all Wei-type duality theorems, previously known in the literature or newly established in this paper, are special cases of Theorem 2.2 with the two pairs of

Galois connections appropriately set. In this sense, Theorem 2.2 reveals the essence of Wei-type duality theorems from a Galois connection perspective.

3 Bridging theorems

In this section, to facilitate the application of Theorem 2.2, we establish two bridging theorems that can be used to recover known or derive new Wei-type duality theorems via simple substitutions.

3.1 The first bridging theorem

Throughout this subsection, we let Y be a nonempty set, $m \in \mathbb{N}$, and $g : Y \longrightarrow [0, m]$ be a surjective map. Also fix $w \in \mathbb{Z}^+$, $k \in \mathbb{N}$, and choose $f : Y \longrightarrow [0, k]$ such that the following four conditions hold:

$$\forall\ y \in Y,\ g(y) = 0 \Longrightarrow f(y) = 0; \tag{3.1}$$

$$\forall\ y \in Y,\ wg(y) - f(y) \leqslant wm - k; \tag{3.2}$$

$$\forall\ u \in Y\ s.t.\ g(u) \leqslant m-1,\ \exists\ v \in Y\ s.t.\ g(v) = g(u)+1,\ f(u) \leqslant f(v); \tag{3.3}$$

$$\forall\ v \in Y\ s.t.\ g(v) \geqslant 1,\ \exists\ u \in Y\ s.t.\ g(u) = g(v)-1,\ f(v)-f(u) \leqslant w. \tag{3.4}$$

Now, let X be a nonempty set, and $\sigma : X \longrightarrow Y$ be a surjective map. Define $\mu : X \longrightarrow [0, m]$ as $\mu(t) = m - g(\sigma(t))$, and define $h : X \longrightarrow \mathbb{Z}$ as $h(t) = f(\sigma(t)) + w\mu(t) - k$.

The following lemma, whose proof is straightforward and thus omitted, lists some elementary properties of the functions defined as above.

Lemma 3.1. **(1)** *For any $y \in Y$ with $g(y) = m$, it holds true that $f(y) = k$.*
(2) *For any $v \in Y$, we have $f(v) \leqslant wg(v)$. In particular, $k \leqslant wm$.*
(3) *$\mu : X \longrightarrow [0, m]$ is a surjective map.*
(4) *$h[X] \subseteq [0, wm - k]$, and moreover, for any $t \in X$ with $\mu(t) = m$, it holds that $h(t) = wm - k$.*
(5) *For any $c \in X$ with $\mu(c) \leqslant m - 1$, there exists $d \in X$ such that $\mu(d) = \mu(c) + 1$, $h(c) \leqslant h(d)$.*

Finally, we define $\varphi : [0, k] \longrightarrow [0, m]$ as

$$\varphi(a) = \min\{g(u) \mid u \in Y,\ a \leqslant f(u)\}, \tag{3.5}$$

and define $\psi : [0, m] \longrightarrow [0, k]$ as

$$\psi(b) = \max\{f(u) \mid u \in Y,\ g(u) \leqslant b\}. \tag{3.6}$$

Moreover, define $\tau : [0, wm - k] \longrightarrow [0, m]$ as

$$\tau(a) = \min\{\mu(t) \mid t \in X,\ a \leqslant h(t)\}, \tag{3.7}$$

and define $\eta : [0, m] \longrightarrow [0, wm - k]$ as

$$\eta(b) = \max\{h(t) \mid t \in X,\ \mu(t) \leqslant b\}. \tag{3.8}$$

Proposition 3.1. *φ, ψ, τ, η are well defined. Moreover, we have:*
(1) *(φ, ψ) is a Galois connection between $[0, k]$ and $[0, m]$;*
(2) *For any $b \in [0, m]$, $\psi(b) = \max\{f(u) \mid u \in Y,\ g(u) = b\}$;*
(3) *$\psi(0) = 0$, and for any $l \in [1, m]$, we have $\psi(l) - \psi(l-1) \leqslant w$;*
(4) *For any $a \in [1, k]$, $\varphi(a) \in [1, m]$, and for any $r \in [0, k - w]$, we have $\varphi(r) + 1 \leqslant \varphi(r + w)$;*
(5) *(τ, η) is a Galois connection between $[0, wm - k]$ and $[0, m]$;*
(6) *For any $b \in [0, m]$, $\eta(b) = \max\{h(t) \mid t \in X,\ \mu(t) = b\}$.*

Proof. Since $g : Y \longrightarrow [0, m]$ is surjective and $k \in f[Y]$ (by (1) of Lemma 3.1), φ and ψ are well defined. By (3) and (4) of Lemma 3.1, we have $wm - k \in h[X]$, and hence τ and η are well defined. To finish the proof, we infer that (1) and (5) follow from Lemma 2.2, (2) follows from (3.3), (3) follows from (3.1) and (3.4), (4) follow from (3) and Proposition 2.1, and (6) follows from (5) of Lemma 3.1. □

We are now ready to present and prove the first bridging theorem in this section.

Theorem 3.1. **(1)** *For any $l \in [0, m]$, $\eta(l) = \psi(m - l) + wl - k$.*
(2) *For any $\gamma \in \mathbb{Z}$, define $\mathcal{A}_{(\gamma)} = \{\varphi(u) \mid u \in [1, k],\ u \equiv \gamma + k \pmod{w}\}$, $\mathcal{B}_{(\gamma)} = \{m + 1 - \tau(v) \mid v \in [1, wm - k],\ v \equiv \gamma \pmod{w}\}$. Then, we have $\mathcal{A}_{(\gamma)} \cap \mathcal{B}_{(\gamma)} = \emptyset$, $\mathcal{A}_{(\gamma)} \cup \mathcal{B}_{(\gamma)} = [1, m]$.*

Proof. For any $l \in [0, m]$, noticing that $\sigma : X \longrightarrow Y$ is surjective, we conclude that $\{\sigma(t) \mid t \in X,\ \mu(t) = l\} = \{u \mid u \in Y,\ g(u) = m - l\}$. Thus, by (2) and (6) of Proposition 3.1, we have

$$\begin{aligned}
\eta(l) &= \max\{f(\sigma(t)) + w\mu(t) - k \mid t \in X,\ \mu(t) = l\} \\
&= \max\{f(\sigma(t)) + wl - k \mid t \in X,\ \mu(t) = l\} \\
&= \max\{f(\sigma(t)) \mid t \in X,\ \mu(t) = l\} + wl - k \\
&= \max\{f(u) \mid u \in Y,\ g(u) = m - l\} + wl - k \\
&= \psi(m - l) + wl - k,
\end{aligned}$$

which completes the proof of (1). Now with (1) and (1), (3), (5) of Proposition 3.1, (2) is an immediate consequence of Theorem 2.2. □

Remark 3.1. *To summarize, we start with the tuple* $(Y, m, g, w, k, f, X, \sigma)$*, from which* (μ, h) *is determined. Then, using* (g, f) *and* (μ, h)*, we define two Galois connections* (φ, ψ) *and* (τ, η) *via (3.5)–(3.8), which leads to a Wei-type duality theorem as in Theorem 3.1.*

3.2 The second bridging theorem

In this subsection, we focus on a special case of Theorem 3.1. To be specific, we begin with the following definition.

Definition 3.1. *Let* $(Y, \preccurlyeq)$ *be a poset with the least element* 0_Y *and the greatest element* π_Y*, and let* $g : Y \longrightarrow \mathbb{N}$*. Then, the tuple* $((Y, \preccurlyeq), g)$ *is said to be an abundance, if the following three conditions hold:*

$$\forall\ (x, y) \in Y \times Y,\ x \preccurlyeq y \Longrightarrow g(x) \leqslant g(y); \tag{3.9}$$

$$\forall\ u \in Y\ s.t.\ g(u) \leqslant g(\pi_Y) - 1, \exists\ v \in Y\ s.t.\ (u \preccurlyeq v,\ g(v) = g(u) + 1); \tag{3.10}$$

$$\forall\ v \in Y\ s.t.\ g(v) \geqslant 1, \exists\ u \in Y\ s.t.\ (u \preccurlyeq v,\ g(u) = g(v) - 1). \tag{3.11}$$

Remark 3.2. *Definition 3.1 is inspired by [21. Proposition 1.1], where it is shown that any partial order over a finite set has an abundance of ideals, and we will dwell on this particular case later in Section 4.*

Next, we will establish the second bridging theorem as a corollary of Theorem 3.1. To this end, we adopt the notations in Definition 3.1, and assume $((Y, \preccurlyeq), g)$ is an abundance with $m = g(\pi_Y)$. It can be readily verified that $g(0_Y) = 0$, $g[Y] = [0, m]$, and therefore g is a surjective map from Y to $[0, m]$. Now, we choose $w \in \mathbb{Z}^+$ and $f : Y \longrightarrow \mathbb{Z}$ such that the following two conditions hold:

(1) $f(0_Y) = 0$;

(2) For any $(x, y) \in Y \times Y$ with $x \preccurlyeq y$, $0 \leqslant f(y) - f(x) \leqslant w(g(y) - g(x))$.

Let $k = f(\pi_Y)$. Then, we have $f[Y] \subseteq [0, k]$. Moreover, it can be readily verified that the conditions (3.1)–(3.4) hold for g, w, k and f. Now let X be a set, and let $\sigma : X \longrightarrow Y$ be a surjective map. By now, we have the tuple $(Y, m, g, w, k, f, X, \sigma)$ to which Theorem 3.1 can be applied.

Theorem 3.2. *Define* $\mu : X \longrightarrow [0, m]$ *as* $\mu(t) = m - g(\sigma(t))$*,* $h : X \longrightarrow \mathbb{Z}$ *as* $h(t) = f(\sigma(t)) + w\mu(t) - k$*, and define* φ*,* ψ*,* τ*,* η *exactly in the way as in (3.5)–(3.8). Then, the conclusions of Theorem 3.1 hold for* (φ, ψ) *and* (τ, η)*.*

As an application of Theorem 3.2, in the following example, we derive a Wei-type duality theorem for the q-analogue of a matroid (q-matroid). Here we consider q-matroids defined for any complemented modular lattice of finite length (see [23, Section 10], [11, Definitions 4.4 and 4.13]), which include both matroids and q-matroids defined over finite dimensional vector spaces over a field as special cases (see [30], [23]).

Example 3.1. *Let $(Y, \wedge, \vee, \preccurlyeq)$ be a complemented modular lattice of finite length with the least element 0_Y and the greatest element π_Y. For any $v \in Y$, let $\mathrm{len}\,(v)$ denote the length of the sublattice $\{u \mid u \in Y,\ u \preccurlyeq v\}$ (see [11, Definition 2.37]). Apparently, len is a map from Y to $\mathbb{N}$, and furthermore, it is straightforward to verify that $((Y, \preccurlyeq), \mathrm{len}\,)$ is an abundance. We let $m = \mathrm{len}\,(\pi_Y)$.*

Let $\rho : Y \longrightarrow \mathbb{Z}$ such that $((Y, \preccurlyeq), \rho)$ is a q-matroid, i.e., the following three conditions hold:

(1) *For any $v \in Y$, $0 \leqslant \rho(v) \leqslant \mathrm{len}\,(v)$;*

(2) *For any $(x, y) \in Y \times Y$ with $x \preccurlyeq y$, it holds that $\rho(x) \leqslant \rho(y)$;*

(3) *For any $(x, y) \in Y \times Y$, $\rho(x \wedge y) + \rho(x \vee y) \leqslant \rho(x) + \rho(y)$.*

Note that (1)–(3) imply that $\rho(0_Y) = 0$, and for any $(x, y) \in Y \times Y$ with $x \preccurlyeq y$, it holds true that $0 \leqslant \rho(y) - \rho(x) \leqslant \mathrm{len}\,(y) - \mathrm{len}\,(x)$. We let $k = \rho(\pi_Y)$.

Let $\sigma : Y \longrightarrow Y$ be any bijective map such that

$$\forall\ (x, y) \in Y \times Y,\ \sigma(x) \preccurlyeq \sigma(y) \Longleftrightarrow y \preccurlyeq x. \tag{3.12}$$

Now, we have the tuple $(Y, m, \mathrm{len}\,, 1, k, \rho, Y, \sigma)$ to which Theorem 3.2 can be applied.

Define $\mu : Y \longrightarrow [0, m]$ as $\mu(v) = m - \mathrm{len}\,(\sigma(v))$. Then, by (3.12), we deduce that $\mu = \mathrm{len}\,$. Define $\theta : Y \longrightarrow \mathbb{Z}$ as $\theta(v) = \rho(\sigma(v)) + \mathrm{len}\,(v) - k$, and define φ, ψ, τ, η exactly in the way as in (3.5)–(3.8). It can be readily verified that $((Y, \preccurlyeq), \theta)$ is also a q-matroid. Hence, an application of Theorem 3.2 yields a Wei-type duality theorem for q-matroid. This result generalizes the Wei-type duality theorems for matroids [6, Theorem 1], for sum-matroids [32, Theorem 12] and for sum-rank metric codes [27, Theorem 2].

4 Wei-type duality theorems for w-demimatroids

Throughout this section, we let E be a finite set with $|E| = m$.

We begin with the definition of w-demimatroid.

Definition 4.1. *For any $f : 2^E \longrightarrow \mathbb{Z}$ and $w \in \mathbb{Z}^+$, (E, f) is said to be a w-demimatroid if the following two conditions hold:*
(1) *$f(\emptyset) = 0$;*
(2) *For any $A, B \subseteq E$ with $A \subseteq B$, $0 \leqslant f(B) - f(A) \leqslant w(|B| - |A|)$.*

Remark 4.1. *The notion of a w-demimatroid generalizes that of the well known demimatroid [5, 6] as a demimatroid is a de facto 1-demimatroid.*

The proof of the following proposition is straightforward and hence omitted.

Proposition 4.1. *For $w \in \mathbb{Z}^+$ and a w-demimatroid (E, f), define $h : 2^E \longrightarrow \mathbb{Z}$ as $h(A) = f(E - A) + w|A| - f(E)$. Then, (E, h) is a w-demimatroid with $h(E) = wm - f(E)$, and moreover, for any $A \subseteq E$, it holds that $f(A) = h(E - A) + w|A| - h(E)$.*

Remark 4.2. *Similar to the demimatroid case ([5, 6, 26]), (E, h) derived in Proposition 4.1 can be regarded as the dual w-demimatroid of (E, f). Proposition 4.1 implies that (E, f) is also the dual w-demimatroid of (E, h); in other words, the dual operation for w-demimatroid is an involution ([6, Page 5]).*

In the following, we will proceed to define generalized weights and profiles for a w-demimatroid.

First of all, we consider a special case of Definition 3.1. For $\mathcal{C} \subseteq 2^E$ such that $\emptyset \in \mathcal{C}$, $E \in \mathcal{C}$, we define $g : \mathcal{C} \longrightarrow \mathbb{N}$ as $g(A) = |A|$. Then, by Definition 3.1, $((\mathcal{C}, \subseteq), g)$ is an abundance, if the following two conditions hold:

(1) For any $A \in \mathcal{C}$, $A \subsetneqq E$, there exists $B \in \mathcal{C}$ with $A \subseteq B$, $|B| = |A|+1$;

(2) For any $B \in \mathcal{C}$, $B \neq \emptyset$, there exists $A \in \mathcal{C}$ with $A \subseteq B$, $|A| = |B|-1$.
For convenience, throughout this section, we will say $\mathcal{C}$ is an *abundance* provided that $\mathcal{C}$ satisfied the above (1) and (2), i.e., $((\mathcal{C}, \subseteq), g)$ is an abundance. If $\mathcal{C}$ is an abundance, then it is straightforward to verify that $\mathcal{D} \triangleq \{E - A \mid A \in \mathcal{C}\}$ is also an abundance.

Now we define generalized weights and profiles for w-demimatroids. Throughout the rest of this section, we let w be a fixed positive integer.

Definition 4.2. *Let (E, f) be a w-demimatroid with $k = f(E)$, and let $\mathcal{C} \subseteq 2^E$ be an abundance. Then, for any $a \in [0, k]$, the a-th generalized weight of $(f, \mathcal{C})$, denoted by $\mathbf{d}_a(f, \mathcal{C})$, is defined as $\min\{|B| \mid B \in \mathcal{C},\ a \leqslant f(B)\}$, and for any $b \in [0, m]$, the b-th profile of $(f, \mathcal{C})$, denoted by $\mathbf{K}_b(f, \mathcal{C})$, is defined as $\max\{f(B) \mid B \in \mathcal{C},\ |B| = b\}$.*

Now we derive a Wei-type duality theorem for w-demimatroids. Let (E, f) be a w-demimatroid with $k = f(E)$, and define $h : 2^E \longrightarrow \mathbb{Z}$ as $h(B) = f(E - B) + w|B| - k$.

Let $\mathcal{C} \subseteq 2^E$ be an abundance, and let $\mathcal{D} = \{E - A \mid A \in \mathcal{C}\}$. Define $g : \mathcal{C} \longrightarrow \mathbb{Z}$ as $g(A) = |A|$. Then, $((\mathcal{C}, \subseteq), g)$ is an abundance with $g(E) = m$. Let $f_{\mathcal{C}} = f \mid_{\mathcal{C}}$. Then, $f_{\mathcal{C}}(E) = k$, and by Definition 4.1, $f_{\mathcal{C}}$ satisfies the following two conditions:

(1) $f_{\mathcal{C}}(\emptyset) = 0$;

(2) For any $A, B \in \mathcal{C}$ with $A \subseteq B$, $0 \leqslant f_{\mathcal{C}}(B) - f_{\mathcal{C}}(A) \leqslant w(g(B) - g(A))$.

Define $\sigma : \mathcal{D} \longrightarrow \mathcal{C}$ as $\sigma(B) = E - B$. Obviously, $\sigma : \mathcal{D} \longrightarrow \mathcal{C}$ is bijective. By now, we have the tuple $(\mathcal{C}, m, g, w, k, f_{\mathcal{C}}, \mathcal{D}, \sigma)$ that is needed for applying Theorem 3.2.

Now, define $\mu : \mathcal{D} \longrightarrow \mathbb{Z}$ as $\mu(B) = m - g(\sigma(B))$. Then, it can be readily verified that for any $B \in \mathcal{D}$, we have $\mu(B) = |B|$, $f_{\mathcal{C}}(\sigma(B)) + w \cdot \mu(B) - k = h(B)$. Define φ, ψ, τ, η exactly in the way as in (3.5)–(3.8). Then, by Definition 4.1, for any $a \in [0, k]$, $b \in [0, m]$, we have $\varphi(a) = \mathbf{d}_a(f, \mathcal{C})$, $\psi(b) = \mathbf{K}_b(f, \mathcal{C})$. Similarly, for any $a \in [0, wm - k]$, $b \in [0, m]$, it holds that $\tau(a) = \mathbf{d}_a(h, \mathcal{D})$, $\eta(b) = \mathbf{K}_b(h, \mathcal{D})$.

The above discussions, together with Theorem 3.2, have implied the following Wei-type duality theorem for generalized weights and profiles of $(f, \mathcal{C})$ and $(h, \mathcal{D})$.

Theorem 4.1. **(1)** *For any $l \in [0, m]$, $\mathbf{K}_l(h, \mathcal{D}) = \mathbf{K}_{m-l}(f, \mathcal{C}) + wl - k$.*
(2) *For any $\gamma \in \mathbb{Z}$, let $\mathcal{A}_{(\gamma)} = \{\mathbf{d}_u(f, \mathcal{C}) \mid u \in [1, k],\ u \equiv \gamma + k \pmod{w}\}$, $\mathcal{B}_{(\gamma)} = \{m + 1 - \mathbf{d}_v(h, \mathcal{D}) \mid v \in [1, wm - k],\ v \equiv \gamma \pmod{w}\}$. Then, we have $\mathcal{A}_{(\gamma)} \cap \mathcal{B}_{(\gamma)} = \emptyset$, $\mathcal{A}_{(\gamma)} \cup \mathcal{B}_{(\gamma)} = [1, m]$.*

From now on, we focus our attention to posets over E, from which the aforementioned abundance subset $\mathcal{C} \subseteq 2^E$ arise naturally, as detailed below.

Let $\mathbf{P} = (E, \preccurlyeq_{\mathbf{P}})$ be a poset. For any $B \subseteq E$, recall that B is an ideal of $\mathbf{P}$, if for any $v \in B$ and $u \in E$, $u \preccurlyeq_{\mathbf{P}} v$ implies $u \in B$. Let $\mathcal{I}(\mathbf{P})$ denote the set of all ideals of $\mathbf{P}$. Then, by [29, Proposition 7] or [21, Proposition 1.1], $\mathcal{I}(\mathbf{P})$ is an abundance. Also consider the dual poset of $\mathbf{P}$, denoted by $\overline{\mathbf{P}} = (E, \preccurlyeq_{\overline{\mathbf{P}}})$, where for any $(u, v) \in E \times E$, $u \preccurlyeq_{\overline{\mathbf{P}}} v \Longleftrightarrow v \preccurlyeq_{\mathbf{P}} u$. Then, [20, Lemma 1.2] implies that $\mathcal{I}(\overline{\mathbf{P}}) = \{E - B \mid B \in \mathcal{I}(\mathbf{P})\}$.

Remark 4.3. *There exists $\mathcal{C} \subseteq 2^E$ such that $\mathcal{C}$ is an abundance, yet for any poset $\mathbf{P} = (E, \preccurlyeq_{\mathbf{P}})$, $\mathcal{C} \neq \mathcal{I}(\mathbf{P})$. As an example, consider $E = \{1, 2, 3\}$, $\mathcal{C} = \{\emptyset, \{1\}, \{2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.*

Now, as a special case of Definition 4.2, we give the following Definition.

Definition 4.3. *Let (E, f) be a w-demimatroid with $k = f(E)$, and suppose that $\mathbf{P} = (E, \preccurlyeq_{\mathbf{P}})$ is a poset. Then, for any $a \in [0, k]$, the a-th generalized weight of $(f, \mathbf{P})$, denoted by $\mathbf{d}_a(f, \mathbf{P})$, is defined as $\mathbf{d}_a(f, \mathcal{I}(\mathbf{P}))$, and for any $b \in [0, m]$, the b-th profile of $(f, \mathbf{P})$, denoted by $\mathbf{K}_b(f, \mathbf{P})$, is defined as $\mathbf{K}_b(f, \mathcal{I}(\mathbf{P}))$.*

Remark 4.4. *Let $\mathbf{P} = (E, \preccurlyeq_{\mathbf{P}})$ be a poset, and for any $B \subseteq E$, let $\langle B \rangle_{\mathbf{P}} = \{u \mid u \in E, \ \exists \ v \in B \ s.t. \ u \preccurlyeq_{\mathbf{P}} v\}$ denote the ideal generated by B. Then, for any $a \in [0, k]$, $\mathbf{d}_a(f, \mathbf{P})$ can be alternatively defined as $\min\{|\langle B \rangle_{\mathbf{P}}| \mid B \subseteq E, \ a \leqslant f(B)\}$, and for any $b \in [0, m]$, $\mathbf{K}_b(f, \mathbf{P})$ can be alternatively defined as $\max\{f(B) \mid B \subseteq E, \ |\langle B \rangle_{\mathbf{P}}| \leqslant b\}$.*

Now let (E, f) be a w-demimatroid with $k = f(E)$, and define $h : 2^E \longrightarrow \mathbb{Z}$ as $h(B) = f(E - B) + w|B| - k$. For a poset $\mathbf{P} = (E, \preccurlyeq_{\mathbf{P}})$ and its dual poset $\overline{\mathbf{P}} = (E, \preccurlyeq_{\overline{\mathbf{P}}})$, a combination of Theorem 4.1, Definition 4.3 and the known fact $\mathcal{I}(\overline{\mathbf{P}}) = \{E - B \mid B \in \mathcal{I}(\mathbf{P})\}$ yields the following Wei-type duality theorem for generalized weights and profiles of $(f, \mathbf{P})$ and $(h, \overline{\mathbf{P}})$.

Theorem 4.2. **(1)** *For any $l \in [0, m]$, $\mathbf{K}_l(h, \overline{\mathbf{P}}) = \mathbf{K}_{m-l}(f, \mathbf{P}) + wl - k$;*
(2) *For any $\gamma \in \mathbb{Z}$, define $\mathcal{A}_{(\gamma)} = \{\mathbf{d}_u(f, \mathbf{P}) \mid u \in [1, k], \ u \equiv \gamma + k \pmod{w}\}$, $\mathcal{B}_{(\gamma)} = \{m + 1 - \mathbf{d}_v(h, \overline{\mathbf{P}}) \mid v \in [1, wm - k], \ v \equiv \gamma \pmod{w}\}$. Then, we have $\mathcal{A}_{(\gamma)} \cap \mathcal{B}_{(\gamma)} = \emptyset$, $\mathcal{A}_{(\gamma)} \cup \mathcal{B}_{(\gamma)} = [1, m]$.*

Remark 4.5. *As in Theorem 4.2, with w set to be 1, we recover the Wei-type duality theorem for demimatroids [6, Theorem 6]. If we let $\preccurlyeq_{\mathbf{P}}$ be the trivial (anti-chain) partial order on E, Theorem 4.2 becomes an analogue of the Wei-type duality theorem for (q, m)-demimatroids [7, Theorem 4].*

It is well known in coding theory that the generalized weights and profiles of linear codes over a division ring can be recast as those of their associated demimatroids ([5, 6]). We will show in Sections 6 and 7 that similar relations hold for codes over ring modules with a poset metric and their associated w-demimatroids. In particular, the Wei-type duality theorem for w-demimatroids (Theorem 4.2) will lead to a Wei-type duality theorem for codes endowed with a poset metric (see Section 7.2).

5 Wei-type duality theorems for w-demi-polymatroids

As combinatorial notions arising from rank metric codes ([18, 19, 33]), demi-polymatroids defined over vector spaces over a field have been studied extensively in [7] and [17]. In this section, we define and study w-demi-polymatroids over modules, with a focus on those with a composition series (see, e.g., [1, Chapter 3], [10, Section 13], [35, Section 4]).

Throughout this section, we let R and S be rings, which we always assume to be associative with multiplicative unit.

Recall that for any left R-module M, M is said to be *simple* if and only if $M \neq \{0\}$ and M has no left R-submodules other than $\{0\}$ and M. A *composition series* of M is a chain of left R-submodules

$$\{0\} = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_{t-1} \subseteq L_t = M,$$

where $t \in \mathbb{N}$ and for any $i \in [1, t]$, L_i/L_{i-1} is a simple left R-module. For any left R-module M, $\mathcal{P}(M)$ will denote the set of all left R-submodules of M. We note that all the above notions parallelly apply to right S-modules.

Now we present a notion that is a slightly modified version of [35, Section 4, Paragraph 4], which generalizes dimension function of vector spaces over a field.

Let Ω be a nonempty collection of simple left R-modules. For any left R-module M with a composition series $\{0\} = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_t = M$, we define

$$\rho_\Omega(M) = |\{i \mid i \in [1, t], \exists\, W \in \Omega \ s.t.\ L_i/L_{i-1} \cong W \text{ as left } R\text{-modules}\}|. \tag{5.1}$$

Similarly, let Δ be a nonempty collection of simple right S-modules. Then, for any right S-module N with a composition series $\{0\} = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_{p-1} \subseteq H_p = N$, we define

$$\lambda_\Delta(N) = |\{i \mid i \in [1, p], \exists\, Z \in \Delta \ s.t.\ H_i/H_{i-1} \cong Z \text{ as right S-modules}\}|. \tag{5.2}$$

By the Jordan-Hölder theorem ([10, Theorem 13.7]), both $\rho_\Omega(M)$ and $\lambda_\Delta(N)$ are independent of the choice of the composition series, and hence well-defined. We will show in Section 5.2 that ρ_Ω and λ_Δ are closely related in suitable settings, which will lead to Wei-type duality theorems.

Remark 5.1. *If R is a field or division ring, which implies that the only simple left R-module up to isomorphism is ${}_RR$, then for any finite dimensional left R-module M, $\rho_\Omega(M)$ becomes the dimension of ${}_RM$, i.e., $\rho_\Omega(M) = \dim_R(M)$. Therefore the notion ρ_Ω naturally extends the dimension function of vector spaces over a field. More generally, let R be an arbitrary ring, as long as Ω contains all the simple left R-modules up to isomorphism, $\rho_\Omega(M)$ then becomes the length of a composition series of M. We note that similar properties hold for λ_Δ in a parallel fashion.*

5.1 Generalized weights and profiles of w-demi-polymatroids

In this subsection, we will focus on left R-modules, since all the discussions remain valid for right S-modules parallelly.

Throughout this subsection, we let Ω be a nonempty collection of simple left R-modules, and let M be a left R-module with a composition series.

As detailed in the following Lemma, which is a consequence of [35, Proposition 4.1], ρ_Ω maintains some basic properties of the dimension function of vector spaces over a field.

Lemma 5.1. **(1)** *For any* $U, V \in \mathcal{P}(M)$ *with* $U \subseteq V$, $\rho_\Omega(V) = \rho_\Omega(U) + \rho_\Omega(V/U)$.
(2) *Define* $g : \mathcal{P}(M) \longrightarrow \mathbb{N}$ *as* $g(V) = \rho_\Omega(V)$. *Then,* $((\mathcal{P}(M), \subseteq), g)$ *is an abundance.*
(3) *For any* $X, Y \in \mathcal{P}(M)$, $\rho_\Omega(X + Y) + \rho_\Omega(X \cap Y) = \rho_\Omega(X) + \rho_\Omega(Y)$.
(4) *For* $s \in \mathbb{Z}^+$, $(L_{(1)}, \ldots, L_{(s)}) \in \mathcal{P}(M)^s$ *such that* $L_{(s)} \subseteq \cdots \subseteq L_{(1)}$, *and* $X, Y \in \mathcal{P}(M)$ *with* $X \subseteq Y$, *it holds that*

$$0 \leqslant \left(\sum_{i=1}^{s} (-1)^{i-1} \rho_\Omega(L_{(i)} \cap Y)\right) - \left(\sum_{i=1}^{s} (-1)^{i-1} \rho_\Omega(L_{(i)} \cap X)\right) \leqslant \rho_\Omega(Y) - \rho_\Omega(X).$$

Now we define w-demi-polymatroids over modules.

Definition 5.1. *For any* $f : \mathcal{P}(M) \longrightarrow \mathbb{Z}$ *and* $w \in \mathbb{Z}^+$, (M, f, Ω) *is said to be a* w*-demi-polymatroid if the following two conditions hold:*
(1) $f(\{0\}) = 0$;
(2) *For any* $X, Y \in \mathcal{P}(M)$ *with* $X \subseteq Y$, $0 \leqslant f(Y) - f(X) \leqslant w(\rho_\Omega(Y) - \rho_\Omega(X))$.

Set R as a field in Definition 5.1. Then, as in Remark 5.1, $\rho_\Omega = \dim_R$, and we recover the definition of w-demi-polymatroids defined over vector spaces over a field (see [17, Definition 33], [7, Section 3]).

Now we define generalized weights and profiles for w-demi-polymatroids.

Definition 5.2. *Let* $f : \mathcal{P}(M) \longrightarrow \mathbb{Z}$ *and* $w \in \mathbb{Z}^+$ *such that* (M, f, Ω) *is a* w*-demi-polymatroid with* $f(M) = k$, *and let* $\Phi \subseteq \mathcal{P}(M)$ *such that* $\{0\} \in \Phi$, $M \in \Phi$ *and* $((\Phi, \subseteq), (\rho_\Omega) \mid_\Phi)$ *is an abundance. For any* $a \in [0, k]$, *the* a*-th generalized weight of* $((M, f, \Omega), \Phi)$, *denoted by* $\mathbf{d}_a((M, f, \Omega), \Phi)$, *is defined as* $\min\{\rho_\Omega(W) \mid W \in \Phi,\ a \leqslant f(W)\}$, *and for any* $b \in [0, \rho_\Omega(M)]$, *the* b*-th profile of* $((M, f, \Omega), \Phi)$, *denoted by* $\mathbf{K}_b((M, f, \Omega), \Phi)$, *is defined as* $\max\{f(W) \mid W \in \Phi,\ \rho_\Omega(W) = b\}$.

Adopting the notations in Definition 5.2, an application of Proposition 3.1 yields the following corollary.

Corollary 5.1. **(1)** *Define* $\varphi : [0,k] \longrightarrow [0,\rho_\Omega(M)]$ *as* $\varphi(a) = \mathbf{d}_a((M,f,\Omega),\Phi)$, *and define* $\psi : [0,\rho_\Omega(M)] \longrightarrow [0,k]$ *as* $\psi(b) = \mathbf{K}_b((M,f,\Omega),\Phi)$. *Then,* (φ,ψ) *is a Galois connection between* $[0,k]$ *and* $[0,\rho_\Omega(M)]$.
(2) $\mathbf{K}_0((M,f,\Omega),\Phi) = 0$, $\mathbf{K}_l((M,f,\Omega),\Phi) - \mathbf{K}_{l-1}((M,f,\Omega),\Phi) \leqslant w$ *for any* $l \in [1,\rho_\Omega(M)]$.
(3) *For any* $r \in [0,k-w]$, $\mathbf{d}_r((M,f,\Omega),\Phi) + 1 \leqslant \mathbf{d}_{r+w}((M,f,\Omega),\Phi)$.

5.2 Wei-type duality theorems

A necessary step for establishing Wei-type duality theorems for w-demi-polymatroids is to extend the notion of dual space of vector space over a field. To this end, we will use non-degenerated bilinear maps defined for modules (see [1, Theorem 30.1] and [10, Theorem 58.8]), which appears to be suitable for deriving Wei-type duality theorems.

Throughout this subsection, we let U be an R-S bimodule such that:

$$\text{For any simple left } R\text{-module } X, \operatorname{Hom}_R(X,U) \text{ is a simple right } S\text{-module;} \tag{5.3}$$

$$\text{For any simple right } S\text{-module } Y, \operatorname{Hom}_S(Y,U) \text{ is a simple left } R\text{-module.} \tag{5.4}$$

Let M be a left R-module with a composition series, and let N be a right S-module. Fix a *bilinear map* $\varpi : M \times N \longrightarrow U$, i.e., for any $a,b \in M$, $c,d \in N$, $r \in R$, $s \in S$, it holds that:

(1) $\varpi(a+b,c) = \varpi(a,c) + \varpi(b,c)$;

(2) $\varpi(a,c+d) = \varpi(a,c) + \varpi(a,d)$;

(3) $\varpi(ra,cs) = r\varpi(a,c)s$.

For any $C \subseteq M$, define $C^\perp = \{y \mid y \in N,\ \varpi(x,y) = 0 \text{ for all } x \in C\}$, and for any $D \subseteq N$, define ${}^\perp D = \{x \mid x \in M,\ \varpi(x,y) = 0 \text{ for all } y \in D\}$. We further assume ϖ is *non-degenerated*, i.e., $M^\perp = \{0\}$, ${}^\perp N = \{0\}$.

Example 5.1. *Let* $\mathbb{F}$ *be a field, and set* $R = S = U = \mathbb{F}$. *It can be readily verified that* $\mathbb{F}$ *satisfies (5.3) and (5.4). Now we fix a nonempty finite set* E, *and set* $M = N = \mathbb{F}^E$. *Furthermore, let* ϖ *denote the standard inner product of* $\mathbb{F}^E$, *i.e., for any* $\alpha,\beta \in \mathbb{F}^E$, $\varpi(\alpha,\beta) = \sum_{e\in E} \alpha_{(e)} \cdot \beta_{(e)}$. *Then,* ϖ *is non-degenerated. Now for any* $\mathbb{F}$*-subspace* $C \subseteq \mathbb{F}^E$, $C^\perp$ *is exactly the dual space of* C, *or alternatively in coding theoretic terms,* C *is a linear code and* $C^\perp$ *is its dual code.*

More generally, let A *be a quasi-Frobenius ring, i.e.,* A *is left and right Artinian and is injective as a left* A*-module (e.g.,* A *set to be a division ring, a Galois ring or a finite chain ring, see [2, 5, 20]), and set* $R = S = U = A$. *Then,* A *satisfies (5.3) and (5.4) (see [10, Section 58]). Now we proceed* *exactly the same as in the previous paragraph. And with $\mathbb{F}$ replaced by A, all the discussions in the previous paragraph remain valid.*

The following lemma, which is a consequence of [1, Theorem 30.1], is essential for deriving Wei-type duality theorems (also see [35, Equations (45) and (47)]).

Lemma 5.2. *N has a composition series. Moreover, we have:*
(1) *For any $C \in \mathcal{P}(M)$, $D \in \mathcal{P}(N)$, it holds that ${}^{\perp}\left(C^{\perp}\right) = C$, $\left({}^{\perp}D\right)^{\perp} = D$;*
(2) *Assume Ω is a nonempty collection of simple left R-modules, and define $\Delta = \{\mathrm{Hom}_R(X, U) \mid X \in \Omega\}$. Then, Δ is a collection of simple right S-modules by (5.3), and moreover, for any $D \in \mathcal{P}(N)$, it holds true that $\lambda_{\Delta}(D) = \rho_{\Omega}(M) - \rho_{\Omega}({}^{\perp}D)$.*

We are ready to establish Wei-type duality theorems for w-demi-polymatroids. Throughout the rest of this subsection, we let Ω be a nonempty collection of simple left R-modules, and let $\Delta = \{\mathrm{Hom}_R(X, U) \mid X \in \Omega\}$.

First, by Lemma 5.2, we have the following analogue of Proposition 4.1.

Proposition 5.1. **(1)** *For $f : \mathcal{P}(M) \longrightarrow \mathbb{Z}$ and $w \in \mathbb{Z}^+$ such that (M, f, Ω) is a w-demi-polymatroid with $f(M) = k$, define $h : \mathcal{P}(N) \longrightarrow \mathbb{Z}$ as $h(D) = f({}^{\perp}D) + w \cdot \lambda_{\Delta}(D) - k$. Then, (N, h, Δ) is a w-demi-polymatroid with $h(N) = w \cdot \rho_{\Omega}(M) - k$. Moreover, for any $C \in \mathcal{P}(M)$, it holds that $f(C) = h(C^{\perp}) + w \cdot \rho_{\Omega}(C) - h(N)$.*
(2) *Let $\Phi \subseteq \mathcal{P}(M)$ such that $\{0\} \in \Phi$, $M \in \Phi$, $((\Phi, \subseteq), (\rho_{\Omega}) \mid_{\Phi})$ is an abundance, and set $\Theta = \{X^{\perp} \mid X \in \Phi\}$. Then, $\{0\} \in \Theta$, $N \in \Theta$, and $((\Theta, \subseteq), (\lambda_{\Delta}) \mid_{\Theta})$ is an abundance.*

Adopting the notations in Proposition 5.1. Then, (N, h, Δ) can be regarded as the dual w-demi-polymatroid of (M, f, Ω). We now establish a Wei-type duality theorem for generalized weights and profiles of $((M, f, \Omega), \Phi)$ and $((N, h, \Delta), \Theta)$.

Set $g = (\rho_{\Omega}) \mid_{\Phi}$. Then, $((\Phi, \subseteq), g)$ is an abundance with $g(M) = \rho_{\Omega}(M)$. Let $f_{\Phi} = f \mid_{\Phi}$. Then, $f_{\Phi}(M) = f(M) = k$, and by Definition 5.1, we have

(1) $f_{\Phi}(\{0\}) = 0$;

(2) For any $X, Y \in \Phi$ with $X \subseteq Y$, $0 \leqslant f_{\Phi}(Y) - f_{\Phi}(X) \leqslant w(g(Y) - g(X))$.

By Lemma 5.2, there uniquely exists a bijective map $\sigma : \Theta \longrightarrow \Phi$ such that $\sigma(D) = {}^{\perp}D$ for any $D \in \Theta$.

By now, we have the tuple $(\Phi, \rho_{\Omega}(M), g, w, k, f_{\Phi}, \Theta, \sigma)$ that is needed for applying Theorem 3.2. To be more specific, define $\mu : \Theta \longrightarrow [0, \rho_{\Omega}(M)]$ as $\mu(D) = \rho_{\Omega}(M) - g(\sigma(D))$. Then, by Lemma 5.2, we deduce that for any $D \in \Theta$, $\mu(D) = \lambda_{\Delta}(D)$, $f_{\Phi}(\sigma(D)) + w \cdot \mu(D) - k = h(D)$. Now define

φ, ψ, τ, η exactly in the way as in (3.5)–(3.8). Then, by Definition 5.2, for any $a \in [0,k]$, $b \in [0, \rho_\Omega(M)]$, we have $\varphi(a) = \mathbf{d}_a((M,f,\Omega),\Phi)$, $\psi(b) = \mathbf{K}_b((M,f,\Omega),\Phi)$. Similarly, for any $c \in [0, w\cdot\rho_\Omega(M)-k]$, $b \in [0,\rho_\Omega(M)]$, it holds that $\tau(c) = \mathbf{d}_c((N,h,\Delta),\Theta)$, $\eta(b) = \mathbf{K}_b((N,h,\Delta),\Theta)$.

The above discussions, together with Theorem 3.2, have implied the following Wei-type duality theorem.

Theorem 5.1. **(1)** *For any $l \in [0,\rho_\Omega(M)]$, it holds that*

$$\mathbf{K}_l((N,h,\Delta),\Theta) = \mathbf{K}_{\rho_\Omega(M)-l}((M,f,\Omega),\Phi) + wl - k.$$

(2) *Define $\mathcal{A}_{(\gamma)} = \{\mathbf{d}_a((M,f,\Omega),\Phi) \mid a \in [1,k], a \equiv \gamma + k \pmod{w}\}$, $\mathcal{B}_{(\gamma)} = \{\rho_\Omega(M) + 1 - \mathbf{d}_c((N,h,\Delta),\Theta) \mid c \in [1, w\cdot\rho_\Omega(M)-k], c \equiv \gamma \pmod{w}\}$ for any $\gamma \in \mathbb{Z}$. Then, we have $\mathcal{A}_{(\gamma)} \cap \mathcal{B}_{(\gamma)} = \emptyset$, $\mathcal{A}_{(\gamma)} \cup \mathcal{B}_{(\gamma)} = [1, \rho_\Omega(M)]$.*

Remark 5.2. *Let $\mathbb{F}$ be a field, E a nonempty finite set, and set $R = S = U = \mathbb{F}$, $M = N = \mathbb{F}^E$, ϖ the standard inner product on $\mathbb{F}^E$ and $\Phi = \mathcal{P}(M)$. Then, by Remark 5.1, $\rho_\Omega = \lambda_\Delta = \dim_{\mathbb{F}}$, and furthermore, Theorem 5.1 recovers [7, Theorem 4] and [17, Theorem 39].*

6 Generalized weights and profiles of codes with various metrics

In this section, we will consider codes defined over modules [35, 37] and present a unified approach to treat their generalized weights and profiles with respect to different metrics, including poset metric, Gabidulin-Roth rank metric and Delsarte rank metric (see [9], [16], [12]). It turns out that given a code, each of the aforementioned metrics gives rise to an associated w-demimatroid or w-demi-polymatroid, and the generalized weights/profiles of the code can be redefined as those of the associated w-demimatroid or w-demi-polymatroid. The aforementioned approach was first introduced in [5, 6] for linear codes over division rings, and was then extended to Gabidulin-Roth and Delsate rank metric codes in [7, 17]. Our presentation will be in terms of left modules, which can be readily translated to one in terms of right modules.

Throughout this section, we let R be a ring, M a left R-module with a composition series, and E a nonempty finite set with $|E| = m$. Any left R-submodule of $C \subseteq M^E$ will be referred to as a *code*.

Rather than a single code, We will treat a family of codes. To be more specific, let I be a nonempty set and $(u(l) \mid l \in I)$ be a family of positive integers. Let $\mathbf{C} = (\mathbf{C}(l,j) \mid l \in I,\ j \in [1,u(l)])$ such that for any $l \in I$, $(\mathbf{C}(l,1),\ldots,\mathbf{C}(l,u(l)))$ is a *code flag*, i.e., $\mathbf{C}(l,u(l)) \subseteq \mathbf{C}(l,u(l)-1) \subseteq \cdots \subseteq \mathbf{C}(l,1)$ (see [5], [17]).

Also let Ω be a nonempty collection of simple left R-modules such that $w \triangleq \rho_\Omega(M) \geqslant 1$. Then, by Lemma 5.1, we have $\rho_\Omega(M^E) = wm$, and $k \triangleq \max\left\{\sum_{i=1}^{u(l)}(-1)^{i-1}\rho_\Omega(\mathbf{C}(l,i)) \mid l \in I\right\}$ is a nonnegative integer in $[0,wm]$.

First of all, we define $f_0 : \mathcal{P}(M^E) \longrightarrow \mathbb{Z}$ as

$$f_0(L) = \max\left\{\sum_{i=1}^{u(l)}(-1)^{i-1}\rho_\Omega(\mathbf{C}(l,i) \cap L) \mid l \in I\right\}. \tag{6.1}$$

It immediately follows from (4) of Lemma 5.1 that (M^E, f_0, Ω) is a 1-demi-polymatroid and $k = f_0(M^E)$, which will be referred to as *the* 1*-demi-polymatroid associated to* $(\mathbf{C},\Omega)$ *with respect to Gabidulin-Roth rank metric.* As detailed later, (M^E, f_0, Ω) will be the corresponding 1-demi-polymatroid with respect to Gabidulin-Roth rank metric, and can be used to derive the corresponding w-demimatroid with respect to poset metric, as well as the corresponding m-demi-polymatroid with respect to Delsarte rank metric.

6.1 Poset metric

Similarly as in Section 1, we define:

(1) For any $D \subseteq M^E$, let $\chi(D) = \{i \mid i \in E,\ \alpha_{(i)} \neq 0 \text{ for some } \alpha \in D\}$;

(2) Define $\delta : 2^E \longrightarrow \mathcal{P}(M^E)$ as $\delta(J) = \{\alpha \mid \alpha \in M^E,\ \chi(\{\alpha\}) \subseteq J\}$.

Moreover, we define $f_1 : 2^E \longrightarrow \mathbb{Z}$ as $f_1 = f_0 \circ \delta$. It then follows from Lemma 5.1 and a routine computation that (E, f_1) is a w-demimatroid with $f_1(E) = k$, which will be referred to as *the w-demimatroid associated to* $(\mathbf{C},\Omega)$ *with respect to poset metric.*

Definition 6.1. *Consider a poset* $\mathbf{P} = (E, \preccurlyeq_\mathbf{P})$*. For any* $a \in [0,k]$*, the a-th generalized weight of* $(\mathbf{C},\Omega,\mathbf{P})$*, denoted by* $\mathbf{d}_a(\mathbf{C},\Omega,\mathbf{P})$*, is defined as* $\mathbf{d}_a(f_1,\mathbf{P})$*, and for any* $b \in [0,m]$*, the b-th profile of* $(\mathbf{C},\Omega,\mathbf{P})$*, denoted by* $\mathbf{K}_b(\mathbf{C},\Omega,\mathbf{P})$*, is defined as* $\mathbf{K}_b(f_1,\mathbf{P})$*.*

Adopting the notations in Definition 6.1, we consider the case that $\mathbf{C}$ consists of one code, say, $C \subseteq M^E$. Then, by straightforward verification, for any $r \in [0,k]$, we have

$$\mathbf{d}_r(C,\Omega,\mathbf{P}) = \min\{|\langle\chi(D)\rangle_\mathbf{P}| \mid D \in \mathcal{P}(M^E),\ D \subseteq C,\ \rho_\Omega(D) = r\}.$$

Therefore, for the case that R is a field and $M = R$, one sees that by Remark 5.1, $\rho_\Omega = \dim_R$, and Definition 6.2 boils down to the original definition of generalized weights for a code with a poset metric ([9], [29]). Furthermore, with $\mathbf{P}$ set to be an anti-chain, for any $r \in [0, k]$, we have

$$\mathbf{d}_r(C, \Omega, \mathbf{P}) = \min\{|\chi(D)| \mid D \in \mathcal{P}(R^E),\ D \subseteq C,\ \dim_R(D) = r\}.$$

Hence, Definition 6.2 coincides with the GHWs of C, as presented in (1.1).

Now consider the case that $\mathbf{C}$ is a code flag (i.e., $|I| = 1$), R is a division ring and $M = R$. Then, Definition 6.2 recovers the generalized weights and profiles for a code flag ([5, Section III]); if we further assume that $u(1) = 2$, i.e, $\mathbf{C}$ is a code flag consists of two codes C_1 and C_2 with $C_2 \subseteq C_1$, Definition 6.2 then boils down to the relative generalized Hamming weight and relative dimension/length profile defined in [25].

6.2 Gabidulin-Roth rank metric

We begin with the following definition.

Definition 6.2. *Let $\Phi \subseteq \mathcal{P}(M^E)$ such that $\{0\} \in \Phi$, $M^E \in \Phi$ and $((\Phi, \subseteq), (\rho_\Omega) \mid_\Phi)$ is an abundance. For any $a \in [0, k]$, the a-th Gabidulin-Roth generalized rank weight of $(\mathbf{C}, \Omega, \Phi)$, denoted by ${\mathbf{d}^{\mathbf{G}}}_a(\mathbf{C}, \Omega, \Phi)$, is defined as $\mathbf{d}_a((M^E, f_0, \Omega), \Phi)$. For any $b \in [0, wm]$, the b-th Gabidulin-Roth profile of $(\mathbf{C}, \Omega, \Phi)$, denoted by ${\mathbf{K}^{\mathbf{G}}}_b(\mathbf{C}, \Omega, \Phi)$, is defined as $\mathbf{K}_b((M^E, f_0, \Omega), \Phi)$.*

Definition 6.2 includes the generalized weights and profiles of Gabidulin-Roth rank metric codes as special cases, as detailed in the following remark.

Remark 6.1. *Adopting the notations in Definition 6.2, we first mention that it is straightforward to verify that for any $a \in [0, k]$,*

$$\begin{aligned} {\mathbf{d}^{\mathbf{G}}}_a(\mathbf{C}, \Omega, \Phi) &= \min\{\rho_\Omega(L) \mid L \in \Phi,\ a \leqslant f_0(L)\} \\ &= \min\{\rho_\Omega(L) \mid L \in \Phi,\ f_0(L) = a\}. \end{aligned}$$

Now let $\mathbb{F}$ be a finite field with $|\mathbb{F}| = q$. Suppose that R is a finite field extension of $\mathbb{F}$ and $M = R$. Then, $\rho_\Omega = \dim_R$, $w = \rho_\Omega(M) = 1$, and any code $V \subseteq R^E$ is a de facto Gabidulin-Roth rank metric code (see [7], [14]). We now let Φ denote the following collection of codes,

$$\Phi = \{V \mid V \text{ is spanned by } V \cap \mathbb{F}^E \text{ as an } R\text{-vector space}\}.$$

It can be readily verified that $\{0\} \in \Phi$, $M^E \in \Phi$ and $((\Phi, \subseteq), (\dim_R) \mid_\Phi)$ is an abundance. Assume that $\mathbf{C}$ is a code flag consists of two codes C_1 and C_2 with $C_2 \subseteq C_1$. Then, for any $a \in [0, k]$, we have

$$\mathbf{d^G}_a(\mathbf{C}, \Omega, \Phi) = \min\left\{\dim_R(L) \mid L \in \Phi,\ \dim_R(C_1 \cap L) - \dim_R(C_2 \cap L) = a\right\},$$

and for any $b \in [0, m]$, we have

$$\mathbf{K^G}_b(\mathbf{C}, \Omega, \Phi) = \max\{\dim_R(C_1 \cap L) - \dim_R(C_2 \cap L) \mid L \in \Phi,\ \dim_R(L) = b\}.$$

These two expressions recover the relative generalized rank weights and relative dimension/intersection profiles of Gabidulin-Roth rank metric codes (see [24, Definitions 4 and 5] and [14]).

6.3 Delsarte rank metric

First of all, we define $f_2 : \mathcal{P}(M) \longrightarrow \mathbb{Z}$ as $f_2(W) = f_0(W^E)$. It then follows from Lemma 5.1 that (M, f_2, Ω) is an m-demi-polymatroid with $f_2(M) = k$, which will be referred to as *the m-demi-polymatroid associated to $(\mathbf{C}, \Omega)$ with respect to Delsarte rank metric.*

Definition 6.3. *Let $\Phi \subseteq \mathcal{P}(M)$ such that $\{0\} \in \Phi$, $M \in \Phi$ and $((\Phi, \subseteq), (\rho_\Omega) \mid_\Phi)$ is an abundance. For any $a \in [0, k]$, the a-th Delsarte generalized rank weight of $(\mathbf{C}, \Omega, \Phi)$, denoted by $\mathbf{d^R}_a(\mathbf{C}, \Omega, \Phi)$, is defined as $\mathbf{d}_a((M, f_2, \Omega), \Phi)$, and for any $b \in [0, w]$, the b-th Delsarte profile of $(\mathbf{C}, \Omega, \Phi)$, denoted by $\mathbf{K^R}_b(\mathbf{C}, \Omega, \Phi)$, is defined as $\mathbf{K}_b((M, f_2, \Omega), \Phi)$.*

For any $D \subseteq M^E$, we let $\mathrm{CSupp}\,(D)$ denote the left R-submodule of M generated by $\{\alpha_{(e)} \mid \alpha \in D,\ e \in E\}$. Then, the following lemma follows from a routine computation.

Lemma 6.1. *Assume $\mathbf{C}$ consists of one code, say, $C \subseteq M^E$. Then, for any $a \in [0, k]$, we have*

$$\mathbf{d^R}_a(C, \Omega, \mathcal{P}(M)) = \min\{\rho_\Omega(\mathrm{CSupp}\,(D)) \mid D \in \mathcal{P}(M^E),\ D \subseteq C,\ \rho_\Omega(D) = a\}.$$

In the following remark, we show how Definition 6.3 recovers some known generalized rank weights of Delsarte rank metric codes.

Remark 6.2. *Assume $E = [1, m]$, R is a field and $M = R^w$. With each vector in M in its column form, we identify M^E with the set of all matrices over R, with w rows and m columns. Then, a code $C \subseteq M^E$ becomes a Delsarte rank metric code (see [18, Definition 1.1]). Moreover, we set Φ in Definition 6.3 as $\mathcal{P}(M)$.*

First, we let $w = m$, $I = \{1, 2\}$, $u(1) = u(2) = 1$, and for a fixed code C, define $\mathbf{C}(1,1) = C$, $\mathbf{C}(2,1) = \{\theta^T \mid \theta \in C\}$. Then, Definition 6.3 boils down to Ravagnani's definition for generalized rank weights of C, which is originally proposed by an optimal anticodes approach (see [33, Definition 23], [18, Remark 5.8]).

Second, we consider the case $|I| = 1$, i.e., $\mathbf{C}$ is a code flag. Then, Definition 6.3 becomes generalized rank weights of Delsarte rank metric code flags [17, Definition 46]. If the code flag consists of two codes, then Definition 6.3 becomes the relative generalized matrix weights and relative dimension/rank support profiles proposed in [28, Definitions 10 and 11]. If $\mathbf{C}$ consists of one code, say, $C \subseteq M^E$, then by Lemma 6.1, Definition 6.3 becomes the generalized matrix weight of C [28, Definitions 10], and moreover, as long as $k \geqslant 1$, ${\mathbf{d}^{\mathbf{R}}}_1(C, \Omega, \mathcal{P}(M)) = \min\{\mathrm{rank}\,(\theta) \mid \theta \in C,\ \theta \neq 0\}$ is exactly the minimal rank distance of C (see [18, Definition 3.1]), where rank *denotes the rank of a matrix.*

7 Wei-type duality theorems for codes with various metrics

In this section, we prove Wei-type duality theorems for codes with Gabidulin-roth metric, poset metric, Delsarte rank metric and Generalized Hamming weight with respect to rank (see [20]).

Throughout this section, we let R and S be rings, and let U be an R-S bimodule satisfying (5.3) and (5.4). Let M be a left R-module with a composition series, and let N be a right S-module. E will denote a nonempty finite set with $|E| = m$, and any left R-submodule $L \subseteq M^E$ or right S-submodule $V \subseteq N^E$ will be referred to as a *left linear code* or a *right linear code*, respectively.

Two closely related bilinear maps will be needed for our discussion. To be more specific, we let $\varpi : M \times N \longrightarrow U$ be a non-degenerated bilinear map, and define $\langle\ ,\ \rangle : M^E \times N^E \longrightarrow U$ as $\langle \alpha, \beta \rangle = \sum_{e \in E} \varpi(\alpha_{(e)}, \beta_{(e)})$. It can be readily verified that $\langle\ ,\ \rangle$ is also non-degenerated. For any $X \subseteq M$ and $Y \subseteq N$, we define $X^\top$ and ${}^\top Y$ as follows:

(1) $X^\top = \{y \mid y \in N,\ \varpi(x, y) = 0 \text{ for all } x \in X\}$;

(2) ${}^\top Y = \{x \mid x \in M,\ \varpi(x, y) = 0 \text{ for all } y \in Y\}$.

Similarly, for any $C \subseteq M^E$ and $D \subseteq N^E$, we define $C^\perp$ and ${}^\perp D$ as follows:

(3) $C^\perp = \big\{\beta \mid \beta \in N^E,\ \langle \alpha, \beta \rangle = 0 \text{ for all } \alpha \in C\big\}$;

(4) ${}^\perp D = \big\{\alpha \mid \alpha \in M^E,\ \langle \alpha, \beta \rangle = 0 \text{ for all } \beta \in D\big\}$.

We let Ω be a nonempty collection of simple left R-modules such that $w = \rho_{\Omega}(M) \geqslant 1$, and let $\Delta = \{\mathrm{Hom}_R(X, U) \mid X \in \Omega\}$.

Let I be a nonempty set, and let $(u(l) \mid l \in I)$ be a family of **odd** positive integers. Choose $\mathbf{C} = (\mathbf{C}(l,j) \mid l \in I,\ j \in [1, u(\lambda)])$ such that for any $l \in I$, $(\mathbf{C}(l,1), \ldots, \mathbf{C}(l, u(l)))$ is a flag of left linear codes. We further assume there exists $k \in [0, wm]$ such that for any $l \in I$, $\sum_{i=1}^{u(l)} (-1)^{i-1} \rho_{\Omega}(\mathbf{C}(l,i)) = k$. Define $\mathbf{D} = (\mathbf{D}(l,j) \mid l \in I,\ j \in [1, u(l)])$ as $\mathbf{D}(l,j) = \mathbf{C}(l, u(l)+1-j)^{\perp}$. Naturally, $\mathbf{D}$ can be regarded as the dual of $\mathbf{C}$.

We have the following corollary of Lemmas 5.1 and 5.2.

Lemma 7.1. *For any $l \in I$, $(\mathbf{D}(l,1), \ldots, \mathbf{D}(l, u(l)))$ is a right linear code flag with $\sum_{i=1}^{u(l)} (-1)^{i-1} \lambda_{\Delta}(\mathbf{D}(l,j)) = wm - k$, and moreover, for any $V \in \mathcal{P}(N^E)$, it holds that*

$$\left(\sum_{i=1}^{u(l)} (-1)^{i-1} \lambda_{\Delta}(\mathbf{D}(l,i) \cap V)\right) - \left(\sum_{i=1}^{u(l)} (-1)^{i-1} \rho_{\Omega}(\mathbf{C}(l,i) \cap ({}^{\perp}V))\right) = \lambda_{\Delta}(V) - k.$$

Remark 7.1. *The assumption "for any $l \in I$, $u(l)$ is odd" is essential for our discussion and can not be removed, since otherwise Lemma 7.1 (which we heavily rely on) would fail to hold true (see [5, Theorem 10], [17, Proposition 47]).*

On the other hand, Ghorpade and Johnsen proposed an alternative approach to establish modified Wei-type duality theorems for Delsarte rank metric code flags of even length in [17] (in particular, for relative generalized matrix weights, see [28]). Their approach, which is based on Wei-type duality theorems for code flags of odd length, can also be adopted to poset metric and Gabidulin-Roth rank metric (in particular, to relative generalized Hamming weights, see [25]).

7.1 Gabidulin-Roth rank metric

We let (M^E, f_0, Ω) be the associated 1-demi-polymatroid of $(\mathbf{C}, \Omega)$ with respect to Gabidulin-Roth rank metric, and let (N^E, h_0, Δ) be the associated 1-demi-polymatroid of $(\mathbf{D}, \Delta)$ with respect to Gabidulin-Roth rank metric.

Proposition 7.1. *For any $V \in \mathcal{P}(N^E)$, $h_0(V) = f_0({}^{\perp}V) + \lambda_{\Delta}(V) - k$.*

Proof. By (6.1), for any $L \in \mathcal{P}(M^E)$, we have

$$f_0(L) = \max\left\{\sum_{i=1}^{u(l)} (-1)^{i-1} \rho_{\Omega}(\mathbf{C}(l,i) \cap L) \mid l \in I\right\},$$

and for any $V \in \mathcal{P}(N^E)$, we have

$$h_0(V) = \max\left\{\sum_{i=1}^{u(l)}(-1)^{i-1}\lambda_\Delta(\mathbf{D}(l,i) \cap V) \mid l \in I\right\}.$$

Now the proposition follows immediately from Lemma 7.1. $\square$

Combining Theorem 5.1 and Proposition 7.1, we have the following Wei-type duality theorem for Gabidulin-Roth rank metric codes.

Theorem 7.1. *Let $\Phi \subseteq \mathcal{P}(M^E)$ such that $\{0\} \in \Phi$, $M^E \in \Phi$ and $((\Phi, \subseteq), (\rho_\Omega) \mid_\Phi)$ is an abundance, and let $\Theta = \{X^\perp \mid X \in \Phi\}$. Then, we have:*
(1) *For any $b \in [0, wm]$, $\mathbf{K^G}_b(\mathbf{D}, \Delta, \Theta) = \mathbf{K^G}_{wm-b}(\mathbf{C}, \Omega, \Phi) + b - k$;*
(2) $\{\mathbf{d^G}_a(\mathbf{C}, \Omega, \Phi) \mid a \in [1, k]\}$ *and* $\{wm+1-\mathbf{d^G}_c(\mathbf{D}, \Delta, \Theta) \mid c \in [1, wm-k]\}$ *form a partition of* $[1, wm]$.

Remark 7.2. *Similarly as in Remark 6.1, let $U = M = N = R = S$, and let ϖ be multiplication within R. Then, $\langle\ ,\ \rangle$ becomes the standard inner product of R^E. By [14, Lemma III.1], we have $\{V^\perp \mid V \in \Phi\} = \Phi$. Hence, Theorem 7.1 recover [14, Theorem I.3] for Gabidulin-Roth rank metric codes.*

7.2 Poset metric

We let (E, f_1) be the associated w-demimatroid of $(\mathbf{C}, \Omega)$ with respect to poset metric, and let (E, h_1) be the associated w-demimatroid of $(\mathbf{D}, \Delta)$ with respect to poset metric.

Proposition 7.2. *For any $J \subseteq E$, $h_1(J) = f_1(E - J) + w|J| - k$.*

Proof. First of all, choose f_0 and h_0 as defined in Section 7.1. For any $B \subseteq E$, define $\delta(B) \subseteq M^E$ as in Section 6.2, and let $\varepsilon(B) \subseteq N^E$ denote the set of all the codewords in N^E whose positions outside of B are zeros. For any $J \subseteq E$, by Proposition 7.1, we have

$$h_1(J) = h_0(\varepsilon(J)) = f_0(^\perp\varepsilon(J)) + \lambda_\Delta(\varepsilon(J)) - k.$$

Noticing that $^\perp\varepsilon(J) = \delta(E - J)$, $\lambda_\Delta(N) = \rho_\Omega(M) = w$ (Lemma 5.2) and $\lambda_\Delta(\varepsilon(J)) = \lambda_\Delta(N) \cdot |J| = w|J|$ (Lemma 5.1), we conclude that

$$h_1(J) = f_0(\delta(E - J)) + w|J| - k = f_1(E - J) + w|J| - k,$$

completing the proof. $\square$

Note that $f_1(E) = k$, we combine Theorem 4.2 and Proposition 7.2 and reach the following Wei-type duality theorem for codes with a poset metric.

Theorem 7.2. *Let* $\mathbf{P} = (E, \preccurlyeq_{\mathbf{P}})$ *be a poset. Then, we have:*
(1) *For any* $b \in [0, m]$, $\mathbf{K}_b(\mathbf{D}, \Delta, \overline{\mathbf{P}}) = \mathbf{K}_{m-b}(\mathbf{C}, \Omega, \mathbf{P}) + wb - k$;
(2) *For any* $\gamma \in \mathbb{Z}$, *let* $\mathcal{A}_{(\gamma)} = \{\mathbf{d}_a(\mathbf{C}, \Omega, \mathbf{P}) \mid a \in [1, k], a \equiv \gamma + k \pmod{w}\}$, $\mathcal{B}_{(\gamma)} = \{m + 1 - \mathbf{d}_c(\mathbf{D}, \Delta, \overline{\mathbf{P}}) \mid c \in [1, wm - k],\ c \equiv \gamma \pmod{w}\}$. *Then, we have* $\mathcal{A}_{(\gamma)} \cap \mathcal{B}_{(\gamma)} = \emptyset$, $\mathcal{A}_{(\gamma)} \cup \mathcal{B}_{(\gamma)} = [1, m]$.

Remark 7.3. *As in Example 5.1, let* R *be a quasi-Frobenius ring,* $U = M = N = S = R$, *and* ϖ *be multiplication within* R. *With these assumptions, Theorem 7.2 includes [3, Lemma 2.2], [6, Theorem 11] and [29, Theorem 2] for code over fields or division rings with a poset metric. Since a Galois ring is quasi-Frobenius, Theorem 7.2 can also be regarded as a poset metric generalization of [2, Theorem 2] for Galois ring linear codes.*

7.3 Delsarte rank metric

We let (M, f_2, Ω) be the associated m-demi-polymatroid of $(\mathbf{C}, \Omega)$ with respect to Delsarte rank metric, and let (N, h_2, Δ) be the associated m-demi-polymatroid of $(\mathbf{D}, \Delta)$ with respect to Delsarte rank metric.

Proposition 7.3. *For any* $L \in \mathcal{P}(N)$, $h_2(L) = f_2(^{\top}L) + m \cdot \lambda_{\Delta}(L) - k$.

Proof. Let f_0 and h_0 as defined in Section 7.1. For any $L \in \mathcal{P}(N)$, by Proposition 7.1, we have

$$h_2(L) = h_0(L^E) = f_0(^{\perp}(L^E)) + \lambda_{\Delta}(L^E) - k.$$

Since $^{\perp}(L^E) = (^{\top}L)^E$, we have $f_0(^{\perp}(L^E)) = f_0((^{\top}L)^E) = f_2(^{\top}L)$, which, together with $\lambda_{\Delta}(L^E) = m \cdot \lambda_{\Delta}(L)$ by Lemma 5.1, implies the proposition. □

A combination of Theorem 5.1 and Proposition 7.3 yield the following Wei-type duality theorem for Delsarte rank metric codes.

Theorem 7.3. *Let* $\Phi \subseteq \mathcal{P}(M)$ *such that* $\{0\} \in \Phi$, $M \in \Phi$ *and* $((\Phi, \subseteq), (\rho_{\Omega}) \mid_{\Phi})$ *is an abundance, and let* $\Theta = \{X^{\top} \mid X \in \Phi\}$. *Then, we have:*
(1) *For any* $b \in [0, w]$, $\mathbf{K^R}_b(\mathbf{D}, \Delta, \Theta) = \mathbf{K^R}_{w-b}(\mathbf{C}, \Omega, \Phi) + mb - k$;
(2) *For any* $\gamma \in \mathbb{Z}$, *define* $\mathcal{A}_{(\gamma)} = \{\mathbf{d^R}_a(\mathbf{C}, \Omega, \Phi) \mid a \in [1, k], a \equiv \gamma + k \pmod{m}\}$, $\mathcal{B}_{(\gamma)} = \{w + 1 - \mathbf{d^R}_c(\mathbf{D}, \Delta, \Theta) \mid c \in [1, wm - k],\ c \equiv \gamma \pmod{m}\}$. *Then, we have* $\mathcal{A}_{(\gamma)} \cap \mathcal{B}_{(\gamma)} = \emptyset$, $\mathcal{A}_{(\gamma)} \cup \mathcal{B}_{(\gamma)} = [1, w]$.

Remark 7.4. *Similarly as in the first paragraph of Remark 6.3, let $U = R = S$, $M = N = R^w$, and ϖ be the standard inner product of R^w. Then, for any two matrices $\gamma, \theta \in (R^w)^m$, we have $\langle \gamma, \theta \rangle = \operatorname{tr}(\gamma \cdot \theta^T)$ ([18, Definition 4.1]).*

Set $\Phi = \mathcal{P}(M)$. Then, for the case that $\mathbf{C}$ consists of one code, Theorem 7.3 becomes [28, Proposition 65]. If $|I| = 1$, then Theorem 7.3 becomes the Wei-type duality theorem for a code flag, which alternatively follows from [17, Theorem 39]. If $m = w$, $I = \{1, 2\}$, $u(1) = u(2) = 1$, and $\mathbf{C}(2,1) = \{\theta^T \mid \theta \in \mathbf{C}(1,1)\}$, then Theorem 7.3 recovers [33, Corollary 38].

7.4 Generalized weights with respect to rank of modules over finite chain rings

In [19], Horimoto and Shiromoto established a Wei-type duality theorem for Generalized Hamming weights with respect to rank (GHWR) (see [20, Theorem 3.12]). In this subsection, we study GHWR via a demimatroid approach, and give a poset metric generalization of their result. This subsection is different from the previous subsections in that the corresponding demimatroid will be defined by rank of modules, instead of ρ_Ω or λ_Δ, which are essentially defined by composition series of modules.

Assume that A is a finite chain ring, i.e., A is a finite local ring, and the Jacobson radical of A is a principle left ideal. For any finitely generated left (right) A-module X, we let $\operatorname{rank}(X)$ denote the minimum number of generators of X (see [20]).

Throughout, we set $R = S = U = M = N = A$ and let ϖ be multiplication within A. Then, $\langle\ ,\ \rangle$ becomes the standard inner product of A^E. For any A-A-submodule $L \subseteq A^E$, thanks to [20, Proposition 2.1], the minimum number of generators of L as a left A-module is equal to the minimum number of generators of L as a right A-module. In particular, for any $J \subseteq E$, we let $\delta(J)$ denote the set consisting of all codewords in A^E whose positions outside of J are zeros. Then, $\delta(J)$ is an A-A-submodule with $\operatorname{rank}(\delta(J)) = |J|$.

The following proposition provides the demimatroid we need.

Proposition 7.4. *Let $C \subseteq A^E$ be a left (or right) linear code with $t = \operatorname{rank}(C)$, and define $f : 2^E \longrightarrow \mathbb{Z}$ as $f(J) = \operatorname{rank}(C \cap \delta(J))$. Then, (E, f) is a demimatroid with $f(E) = t$.*

Proof. Since A is a finite chain ring, it holds true that for any finitely generated left (right) A-module W and A-submodule $V \subseteq W$, we have

$$0 \leqslant \operatorname{rank}(W) - \operatorname{rank}(V) \leqslant \operatorname{rank}(W/V). \tag{7.1}$$

Now without loss of generality, we assume C is a right linear code. Since $\delta(\emptyset) = \{0\}$, $\delta(E) = A^E$, we have $f(\emptyset) = 0$, $f(E) = \operatorname{rank}(C) = t$. For any B, D with $B \subseteq D \subseteq E$, we have $\delta(B) \subseteq \delta(D)$, which, together with (7.2), yields that

$$0 \leqslant f(D) - f(B) \leqslant \operatorname{rank}((C \cap \delta(D))/(C \cap \delta(B))) \leqslant \operatorname{rank}(\delta(D)/\delta(B)) = |D| - |B|.$$

Therefore (E, f) is a demimatroid, completing the proof. $\square$

Using the notations in Proposition 7.4, we give the following definition.

Definition 7.1. *Let $C \subseteq A^E$ be a left (or right) linear code with $t = \operatorname{rank}(C)$, and define $f : 2^E \longrightarrow \mathbb{Z}$ as $f(J) = \operatorname{rank}(C \cap \delta(J))$. Then, (E, f) is called the associated demimatroid of C with respect to rank. Moreover, given a poset $\mathbf{P} = (E, \preccurlyeq_{\mathbf{P}})$, for any $a \in [0, t]$, the a-th generalized weight of $(C, \mathbf{P})$ with respect to rank, denoted by $\widetilde{\mathbf{d}}_a(C, \mathbf{P})$, is defined as $\mathbf{d}_a(f, \mathbf{P})$, and for any $b \in [0, m]$, the b-th profile of $(C, \mathbf{P})$ with respect to rank, denoted by $\widetilde{\mathbf{K}}_b(C, \mathbf{P})$, is defined as $\mathbf{K}_b(f, \mathbf{P})$.*

The following Lemma shows that Definition 7.1 naturally extends the notion of GHWR.

Lemma 7.2. *Adopting the notations in Definition 7.1, for any $a \in [0, t]$, we have*

$$\widetilde{\mathbf{d}}_a(C, \mathbf{P}) = \min\{|\langle \chi(D) \rangle_{\mathbf{P}}| \mid D \text{ is a left (right) subcode of } C, \operatorname{rank}(D) = a\},$$

where $\chi(\cdot)$ is defined as in Section 6.1. In particular, Definition 7.1 becomes the GHWR of C ([20, Definition 3.1]) if $\mathbf{P}$ is set to be the anti-chain.

Now we consider the corresponding Wei-type duality theorem. The free code, i.e., the free A-submodule of A^E, will play a particularly important role in the discussion.

Throughout the rest of this subsection, we let $C \subseteq A^E$ be a right linear code with $t = \operatorname{rank}(C)$, and let $M \subseteq A^E$ be a free right linear code such that $C \subseteq M$ and $t = \operatorname{rank}(M)$. We note that such M always exists, moreover, ${}^{\perp}M$ is a free left linear code with $\operatorname{rank}({}^{\perp}M) = m - t$.

Let (E, f) and (E, h) be the associated demimatroids of M and ${}^{\perp}M$ with respect to rank, respectively. It turns out that (E, f) and (E, h) are dual dematroids in the sense of Proposition 4.1, which, albeit without mentioning demimatroids or profiles, has already been observed in the proof of [20, Theorem 3.12].

Proposition 7.5. *For any $J \subseteq E$, it holds that $h(J) = f(E-J) + |J| - t$.*

Proof. Since M is free, by [20, Lemma 3.11], for any $J \subseteq E$, we have

$$\mathrm{rank}\,(M \cap \delta(E-J)) = \mathrm{rank}\,(M) + \mathrm{rank}\,((^{\perp}M) \cap \delta(J)) - \mathrm{rank}\,(\delta(J)).$$

Note that $\mathrm{rank}\,(\delta(J)) = |J|$, $t = \mathrm{rank}\,(M)$, $\mathrm{rank}\,(M \cap \delta(E-J)) = f(E-J)$ and $\mathrm{rank}\,((^{\perp}M) \cap \delta(J)) = h(J)$, the proposition follows immediately. $\square$

We are now ready to state and prove the aforementioned Wei-type duality theorem.

Theorem 7.4. *For any given poset $\mathbf{P} = (E, \preccurlyeq_{\mathbf{P}})$, it holds true that:*
(1) $\widetilde{\mathbf{d}}_a(C,\mathbf{P}) = \widetilde{\mathbf{d}}_a(M,\mathbf{P})$ *for any* $a \in [0,t]$*;*
(2) $\widetilde{\mathbf{K}}_b(C,\mathbf{P}) = \widetilde{\mathbf{K}}_b(M,\mathbf{P})$ *for any* $b \in [0,m]$*;*
(3) $\widetilde{\mathbf{K}}_l(^{\perp}M,\overline{\mathbf{P}}) = \widetilde{\mathbf{K}}_{m-l}(C,\mathbf{P}) + l - t$ *for any* $l \in [0,m]$*;*
(4) $\{\widetilde{\mathbf{d}}_u(C,\mathbf{P}) \mid u \in [1,t]\}$ *and* $\{m+1-\widetilde{\mathbf{d}}_v(^{\perp}M,\overline{\mathbf{P}}) \mid v \in [1,m-t]\}$ *form a partition of* $[1,m]$*.*

Proof. **(1)** This follows from Lemma 3.3 and Theorem 3.4 of [20], we note that although these results were originally stated for GHWR, with the help of Lemma 7.2, one only needs to slightly modify the proofs in [20] for poset metric.
(2) This follows from (1) and Lemma 2.1, since generalized weights and profiles with respect to rank for any linear code form a Galois connection.
(3) and **(4)** These two statements follow from (1), (2), Proposition 7.5 and Theorem 4.2. $\square$

Remark 7.5. *Theorem 7.4 can be alternatively proved by using Proposition 7.5 and the Wei-type duality theorem for demimatroids [6, Theorem 6], and can be regarded as a poset metric generalization of [20, Theorem 3.12].*

We also note that Theorems 7.2 and 7.4 enable us to answer a question raised in [6, Section 3.4], namely, whether it is possible to find a poset metric generalization of Wei-type duality theorems established for codes over Galois rings and chain rings.

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