Shifts of finite type defined by forbidding one block

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  - Mutual replaceability and bijection between sets of periodic points

Preliminaries on symbolic dynamics

Let $\mathcal{A}$ be a finite alphabet and $\mathcal{A}^{\mathbb{Z}}$ (the full-shift) be the set of all bi-infinite sequences over $\mathcal{A}$.

- A *shift apsce* is a shift invariant subset of $\mathcal{A}^{\mathbb{Z}}$.
- A shift space X can be define by a list of forbidden words $\mathcal{F}$:

$$X = X_{\mathcal{F}} = \{x \in \mathcal{A}^{\mathbb{Z}} : x \text{ contains no words from } \mathcal{F}\}$$

- If $\mathcal{F}$ can be chosen to be finite, then $X = X_{\mathcal{F}}$ is a *shift of finite type* (SFT).
- Example: $\mathcal{A} = \{0, 1\}$, $\mathcal{F} = \{11\}$, $X_{\mathcal{F}}$ is the so-called *golden mean shift*.
- Any SFT can be represented by a finite directed graph, where any point in the SFT is represented by a unique bi-infinite path on the graph.

Preliminaries on symbolic dynamics

- A shift space X is call *irreducible* if any two allowed words in X can be connected by another allowed word.

  $\rightarrow$ If X is an SFT, X is irreducible if the adjacency matrix of some representing graph is irreducible.

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- Sliding block code ($\phi : X \rightarrow Y$): continuous, shift commuting map between shift spaces. Usually defined by a block code.

$$\cdots x_{i-m-1} \boxed{x_{i-m} x_{i-m+1} \cdots x_{i+n-1} x_{i+n}} \, x_{i+n+1} \cdots$$

$$\downarrow \Phi$$

$$\cdots y_{i-1} \boxed{y_i} \, y_{i+1} \cdots$$

  - ϕ is an *embedding* if it is one-to-one.
  - ϕ is a *factor code* if it is onto.
  - ϕ is a *conjugacy* if it is one-to-one and onto.

SFTs obtained by forbidding one word in one-dimension

- Notation: for $w \in [q]^k$, let $X_{\{w\}}$ be the SFT obtained by forbidding w from the (ambient) full shift.
- Motivating question: let $u = 10110, v = 10100$ and consider $X_{\{u\}}$, $X_{\{v\}}$. Which one has larger entropy? Can we determine this immediately with only little computation?
- It turns out that the answer to the question is related to the "self-overlap" (auto-correlation) of words u and v.

Quantities of interest

Let $w = w_1 w_2 \cdots w_k$.

- Self-overlap set:
  $\mathrm{overlap}(w, w) := \{i \in [1, k] : w_{[1,i]} = w_{[k-i+1,k]}\}$.
  (w has trivial self-overlap if $\mathrm{overlap}(w, w) = \{k\}$.)

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- Correlation polynomial: $\phi_w(t) = \sum_{i \in \mathrm{overlap}(w,w)} t^{i-1}$.
- Characteristic polynomial: $\chi_w^*(t) = t^{-d} p_G(t)$ where G is a conjugacy presentation of $X_{\{w\}}$ and d is the unique constant such that $\chi_w^*(t)$ has a non-zero constant term.
- Zeta function: $\zeta_w(t) = \exp\left(\sum_{n=1}^{\infty} \frac{p_n(X_{\{w\}})}{n} t^n\right) = \frac{1}{t^r P_G(t^{-1})}$
- Number of allowed words of length n: $|\mathcal{B}_n(X_{\{w\}})|$.
- Topological entropy:
  $h(X_{\{w\}}) = \lim_{n \to \infty} \frac{1}{n} \log |\mathcal{B}_n(X_{\{w\}})| = \log(\lambda_{A_G})$.

Earlier results (equality version)

Proposition (from Guibas&Odlyzko 81, Lind 89 and Erikkson 97)

Let u, v be two strings of length k over the q-ary alphabet ($q \geq 2$). Then the following are equivalent:

1. $\mathrm{overlap}(u, u) = \mathrm{overlap}(v, v)$;
2. $\phi_u(q) = \phi_v(q)$;
3. $\zeta_u(t) = \zeta_v(t)$ *for all* t;
4. $|\mathcal{B}_n(X_{\{u\}})| = |\mathcal{B}_n(X_{\{v\}})|$ *for all* n;
5. $h(X_{\{u\}}) = h(X_{\{v\}})$.

- (2) $\Rightarrow$ (3) is due to Lind's formula:

$$\chi_w^*(t) = (t - q)\phi_w(t) + 1 \qquad \text{for any } w.$$

Earlier results (inequality version)

Proposition

Let $u, v \in [q]^k$. Then, the following are equivalent:

(a) $\phi_u(q) > \phi_v(q)$;

(b) $|\mathcal{B}_n(X_{\{u\}})| \geq |\mathcal{B}_n(X_{\{v\}})|$ *for all* n *and* $|\mathcal{B}_n(X_{\{u\}})| > |\mathcal{B}_n(X_{\{v\}})|$ *for all sufficiently large* n;

(c) $h(X_{\{u\}}) > h(X_{\{v\}})$.

- The equivalence between item (a) and item (c) answers the question we raised before:
  $u = 10110 \to \phi_u(t) = t^4 + t$, $\phi_u(2) = 18$
  $v = 10100 \to \phi_v(t) = t^4$, $\phi_v(2) = 16$
  Since $\phi_u(2) > \phi_v(2)$, we must have $h(X_{\{u\}}) > h(X_{\{v\}})$.
  (Indeed, $h(X_{\{u\}}) \approx 0.954$, $h(X_{\{v\}}) = 0.947$)

When will $X_{\{u\}}$ conjugate to $X_{\{v\}}$?

- Lind's formula implies that $X_{\{u\}}, X_{\{v\}}$ have the same zeta function when $\mathrm{overlap}(u, u) = \mathrm{overlap}(v, v)$.

Q: Will $X_{\{u\}}$ be conjugate to $X_{\{v\}}$ if u, v have the same self-overlap set?

- Observation: Let G_u and G_v be the unlabelled follower set graph of $X_{\{u\}}$ and $X_{\{v\}}$, respectively. If G_u is graph isomorphic to G_v, then $X_{\{u\}}$ is conjugate to $X_{\{v\}}$.

Proposition

G_u and G_v are isomorphic if and only if there is a permutation on $[q]$ that takes u to v.

- Thus, if u and v are essentially different, there is no "graph isomorphic conjugacy" between $X_{\{u\}}$ and $X_{\{v\}}$.

Swap conjugacies

Q: Are there other conjugacies?

- Idea of conjugacy given by replacement: take a point $x \in X_{\{u\}}$. We want to replace appearances of v in x with u to obtain a point $y \in X_{\{v\}}$. If this replacement gives a one-to-one correspondence, then $X_{\{u\}}$ is conjugate to $X_{\{v\}}$.

  $\rightarrow$ When is this possible?

Fact

If u and v both have trivial self-overlap and they have no cross-correlation, then the replacement gives a conjugacy between $X_{\{u\}}$ and $X_{\{v\}}$.

- We define *swap conjugacy* to be either a graph isomorphic conjugacy or a conjugacy given by replacement.

Main results on swap conjugacies

Theorem (Chandgotia, Marcus, Richey, Wu 24')

Let $u, v \in [q]^k$ both have only trivial self-overlap. Then, there exist (other than four exceptional cases given below) a positive integer $N \leq 5$ and a set of swap conjugacies $\phi_1, \phi_2, \cdots, \phi_N$ such that

$$\phi_N \circ \phi_{N-1} \circ \cdots \circ \phi_1(X_{\{u\}}) = X_{\{v\}}.$$

- Exceptional cases: $q = 2$, and $w \in \{10^{k-1}, 1^{k-1}0, 01^{k-1}, 0^{k-1}1\}$. Indeed, $X_{\{w\}}$ is reducible iff $q = 2$ and $w \in \{10^{k-1}, 1^{k-1}0, 01^{k-1}, 0^{k-1}1\}$.
- The theorem requires no assumption on the cross-correlation between u and v.
- Trivial self-overlap is important in this result: if $u = 100100$, $v = 110110$, then there is no swap conjugacy chain between $X_{\{u\}}$ and $X_{\{v\}}$.

Idea of the proof

- Idea of the proof for the binary alphabet case (and $k > 4$): find interpolating words $w^{(1)}, w^{(2)}, w^{(3)}, w^{(4)}$ to form a swap conjugacy chain between u and v.
  - Test words: $p(j) = 1^j 0^{k-j}$ $(2 \leq j \leq k-2)$.
  - The "connector" for test words: $q = 1010^{k-3}$
  - $u \to p(j) \to q \to p(j') \to \overline{v} \to v$
- Example: $u = 110010$, $v = 000101$.
  - They both have trivial self-overlap; and they do have non-trivial cross-correlation.
  - But there is a swap conjugacy chain:

$$u = 110010 \to 110000 \to 101000 \to 111000 \to 111010 \to 000101 = v$$

and therefore $X_{\{u\}}$ is (chain-swap) conjugate to $X_{\{v\}}$.

Extension of swap conjugacy result to the GM ambient shift

What if the ambient SFT is not a full shift?

- Lind's formula for 1-step ambient SFT (presented by G):

$$\chi_w^*(t) = P_G(t)\phi_w(t) + \mathrm{cof}_{ij}(tId - A_G).$$

- So, for example, let the ambient shift be the GM shift and let $u, v \in \mathcal{B}_k(X_{\{11\}})$ be such that
  - $\mathrm{overlap}(u, u) = \mathrm{overlap}(v, v)$ and
  - $u_1 = v_1, u_k = v_k$,

  then $\chi_u^*(t) = \chi_v^*(t)$ and therefore $(X_{\{11\}})_{\{u\}}$ and $(X_{\{11\}})_{\{v\}}$ have the same zeta function.

$\rightarrow$ Q: Is $(X_{\{11\}})_{\{u\}}$ also conjugate to $(X_{\{11\}})_{\{v\}}$?

Extension of swap conjugacy result to the GM ambient shift

Theorem (Chandgotia, Marcus, Richey, Wu 24')

Let $u, v \in \mathcal{B}_k(X_{\{11\}})$. If $u_1 = v_1, u_k = v_k$ and both u and v have only trivial self-overlap (and neither of them is in the set of exceptional words $\{10^{k-1}, 0^{k-1}1, (10)^{\frac{k-1}{2}}0, 0(01)^{\frac{k-1}{2}}\}$), then there is a chain of swap conjugacies between $(X_{\{11\}})_{\{u\}}$ and $(X_{\{11\}})_{\{v\}}$. Furthermore, the length of the chain is upper bounded by 4.

- Idea of the proof is similar to the full-shift case, the main difference is the choice of "test words": instead of choosing 1^j0^{k-j}, we choose $(10)^j0^{k-2j}$ to be the test words.

SFTs obtained by forbidding one pattern in higher dimension

We want to generalize the definition of self-overlap set to a pattern whose support is a finite subset (possibly non-contiguous) in $\mathbb{Z}^d$.

Definition

Given a finite set $S \subset \mathbb{Z}^d$ and a pattern $u \in [q]^S$, we define $\mathrm{overlap}(u, u)$ to be the set of sites $i \in S - S$ for which u and its translate $\sigma^i(u)$ agree on $S \cap (S + i)$.

Mutual replaceability

Let u, v be allowed patterns in the ambient SFT X with the same support. Then u is *replaceable* by v in X if for all $x \in X$ and for any given sites where u occur, we can replace all the u by v simultaneously to obtain an allowed configuration in X. u, v are *mutually replaceable* if u is replaceable by v and v is replaceable by u.

- When X is the full shift, then u is replaceable by v iff $\mathrm{overlap}(u, u) \subset \mathrm{overlap}(v, v)$;
- If X is not the full shift, things become subtle: for example when the ambient is given by $\mathcal{F} = \{0101, 1010\}$ and $u = 101, v = 122$. Then, $\mathrm{overlap}(u, u) \not\subset \mathrm{overlap}(v, v)$, but u is still replaceable by v.

Mutual replaceability and bijection between set of allowed words

Given a finite set $S \subset \mathbb{Z}^d$, define

$$B_S(X_{\mathcal{F}}, u) = \{x|_S : x \in X_{\mathcal{F}} \text{ and } u \text{ does not appear in } x|_S\}.$$

Theorem (Chandgotia, Marcus, Richey, Wu 24')

Let $X_{\mathcal{F}}$ be the ambient SFT and u, v be mutually replaceable in $X_{\mathcal{F}}$. Then for any finite sets $S \subset \mathbb{Z}^d$ there is a bijection between $\mathcal{B}_S(X_{\mathcal{F}}, u)$ and $\mathcal{B}_S(X_{\mathcal{F}}, v)$.

- Idea of proof: mutually replaceable + inclusion-exclusion principle
- We don't know how to prove the converse (while in the 1d full-shift case we can), because some properties of the self-overlap set hold in 1-dimension but does not hold in higher dimension.

Replaceability and periodic points

Let $\Lambda \subset \mathbb{Z}^d$ be a subgroup and $X_{\mathcal{F}}$ be the ambient SFT in $\mathbb{Z}^d$. Define

$$per^{\Lambda}(X_{\mathcal{F}}) = \{x \in X_{\mathcal{F}} \ : \ \sigma^i(x) = x \text{ for all } i \in \Lambda\}$$
$$per^{\Lambda}_S(X_{\mathcal{F}}) = \{x|_S \ : \ x \in X_{\mathcal{F}}, \sigma^i(x) = x \text{ for all } i \in \Lambda\}.$$

Proposition

Let $D \subset \mathbb{Z}^d$ be a fundamental domain for Λ. Let u and v be patterns in the ambient SFT $X_{\mathcal{F}}$ which are mutually replaceable. Then for any finite sets $S \subset D$ there is a bijection between $per^{\Lambda}_S(X_{\mathcal{F}}, u)$ and $per^{\Lambda}_S(X_{\mathcal{F}}, v)$.

- If $(X_{\mathcal{F}})_{\{u\}}$ and $(X_{\mathcal{F}})_{\{v\}}$ both have finitely many periodic points, then $|per^{\Lambda}((X_{\mathcal{F}})_{\{u\}})| = |per^{\Lambda}((X_{\mathcal{F}})_{\{v\}})|$.
- If they have infinitely many periodic points, then the corresponding topological entropies are the same.