# On Mergings in Acyclic Directed Graphs *

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**Abstract**

Consider an acyclic directed graph $G$ with sources $s_1, s_2, \ldots, s_n$ and sinks $r_1, r_2, \ldots, r_n$. For $i = 1, 2, \ldots, n$, let $c_i$ denote the size of the minimum edge cut between $s_i$ and $r_i$, which, by Menger's theorem, implies that there exists a group of $c_i$ edge-disjoint paths from $s_i$ to $r_i$. Although they are edge-disjoint within the same group, the above-mentioned edge-disjoint paths from different groups may merge with each other (or, roughly speaking, share a common subpath). In this paper we show that by choosing these paths appropriately, the number of mergings among all these edge-disjoint paths is always bounded by a function $\mathcal{M}(c_1, c_2, \ldots, c_n)$, which is independent of the size of $G$. Moreover, we prove some elementary properties of $\mathcal{M}(c_1, c_2, \ldots, c_n)$, derive exact values of $\mathcal{M}(1, c)$ and $\mathcal{M}(2, c)$, and establish a scaling law of $\mathcal{M}(c_1, c_2)$ when one of the parameters is fixed.

## 1 Introduction

Let $G = (E, V)$ be an acyclic directed graph with the edge set $E$ and vertex set $V$. In this paper, an edge $e \in E$ linking a vertex $v_1 \in V$ to another vertex $v_2 \in V$ will be represented by $(v_1, v_2)$; and more generally, a directed path $\beta$ consisting of vertices $v_1, v_2, \ldots, v_{\ell+1}$, ordered according to the direction of the path, will be represented by $(v_1, v_2, \ldots, v_{\ell+1})$, where each $v_j$ will be referred to as the *direct predecessor* of $v_{j+1}$ on $\beta$ (see Figure 1 (a) for an quick example). We say $m \geq 2$ directed paths $\beta_1, \beta_2, \ldots, \beta_m$ *merge* at an edge $e = (u, v) \in E$ if 1) $e$ belongs to all $\beta_j$; and 2) there exist $j_1 \neq j_2$ such that the direct predecessors of $u$ on $\beta_{j_1}$ and $\beta_{j_2}$ are distinct; see Figure 1 for some illustrative examples.

We are primarily interested in the case that $G$ has $n$ distinct sources $s_1, s_2, \ldots, s_n$, $n$ distinct sinks $r_1, r_2, \ldots, r_n$, and for each $i$, the size of the minimum edge cut between $s_i$ and $r_i$ is $c_i$, which, by Menger's theorem [9], implies the existence of $\alpha_i = \{\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,c_i}\}$, a set of edge-disjoint paths from $s_i$ and $r_i$. Throughout the paper, for each feasible $i$, an element in $\alpha_i$ will be referred to as an *$\alpha_i$-path*, and an edge on some $\alpha_i$-path will be referred

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to as an $\alpha_i$-*edge*, and the set of all $\alpha_i$-edges will be denoted by $E(\alpha_i)$. For any edge $e$, let $\alpha_i(e)$ denote the set of all the $\alpha_i$-paths passing through $e$, which may be an empty set. Let $\mathcal{G}(c_1, c_2, \ldots, c_n)$ denote the set of all such $G$.

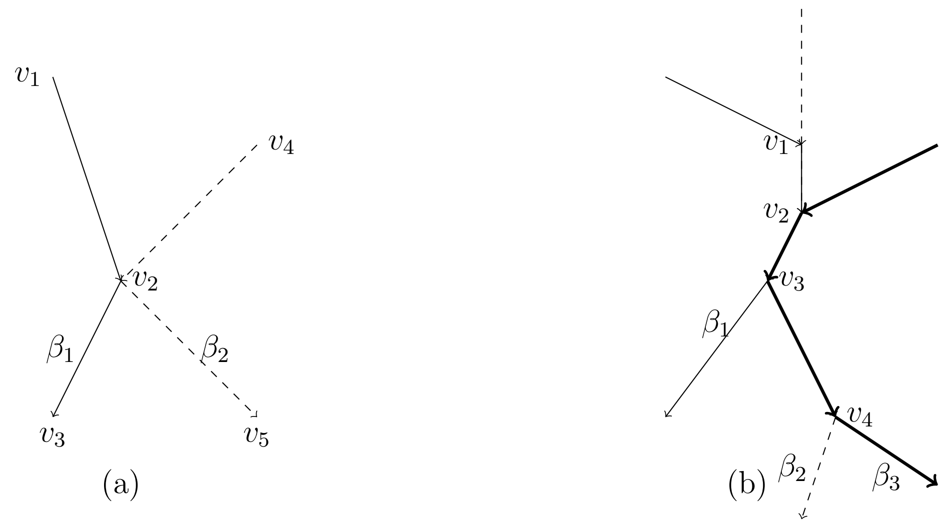


Figure 1: In (a), the path $\beta_1 = (v_1, v_2, v_3)$ consists of two concatenated edges: $e_1 = (v_1, v_2)$ and $e_2 = (v_2, v_3)$. Note that though paths $\beta_1$ and $\beta_2$ share the vertex $v_2$, they do not share any edges, so $\beta_1$ and $\beta_2$ do not merge. In (b), $\beta_1$ and $\beta_2$ merge at the edge $(v_1, v_2)$, however not at $(v_2, v_3)$; $\beta_1$ and $\beta_3$ merge at the edge $(v_2, v_3)$; $\beta_2$ and $\beta_3$ merge at the edge $(v_2, v_3)$; $\beta_1$, $\beta_2$ and $\beta_3$ merge at the edge $(v_2, v_3)$.

Apparently, an $\alpha_{i_1}$-path merges with an $\alpha_{i_2}$-path only if $i_1 \neq i_2$. An edge $e \in E$ is said to be a *merging* with respect to $\alpha_1, \alpha_2, \ldots, \alpha_n$ if there exist $i_1 \neq i_2$ such that some $\alpha_{i_1}$-path and some $\alpha_{i_2}$-path merge at $e$. Let $M(G; \alpha_1, \alpha_2, \ldots, \alpha_n)$ denote the number of mergings in $G$ with respect to $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Noting that the choices of each $\alpha_i$ may not be unique, we let $\Lambda_i(G)$ denote the set of all possible $\alpha_i$. The main result of this paper is that one can choose $\alpha_i \in \Lambda_i(G)$ for all feasible $i$ such that $M(G; \alpha_1, \alpha_2, \ldots, \alpha_n)$ is always finite, regardless of the size of the graph $G$. Here we remark that for any fixed $i$, the the Edmonds-Karp algorithm [5] (an efficient implementation of the classical Ford-Fulkerson method [6]) can find a minimum edge cut and a set of edge-disjoint paths between $s_i$ and $r_i$ in polynomial time. On other hand though, the fact that the *link disjoint problem*, which essentially asks if there are two edge-disjoint paths from $s_i$, $s_j$ to $r_i$, $r_j$ for any $i \neq j$, is NP-complete [7] suggests the intricacy of the scenarios where multiple pairs of sources and sinks are involved.

The following definition introduce some fundamental notions to be examined in this paper.

**Definition 1.1.** For any $G \in \mathcal{G}(c_1, c_2, \ldots, c_n)$, we define

$$M(G) \triangleq \min_{\alpha_i \in \Lambda_i(G):\ i=1,2,\ldots,n} M(G; \alpha_1, \alpha_2, \ldots, \alpha_n),$$

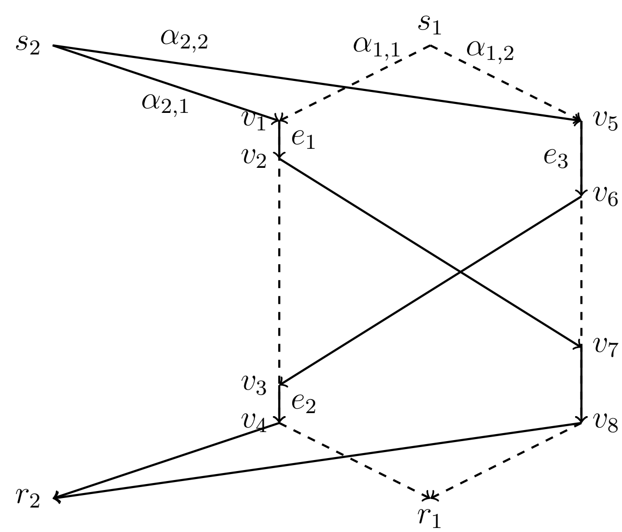


Figure 2: an illustrative example

and furthermore, for any $c_1, c_2, \ldots, c_n$, we define

$$\mathcal{M}(c_1, c_2, \ldots, c_n) = \sup_{G \in \mathcal{G}(c_1, c_2, \ldots, c_n)} M(G).$$

Colloquially, for the purpose of minimizing the number of mergings by appropriately choosing all $\alpha_i$-paths, $M(G)$ is the result of our best decision for a given graph $G$, and $\mathcal{M}(c_1, c_2, \ldots, c_n)$ is the worst performance of our best decisions among all possible graphs $G \in \mathcal{G}(c_1, c_2, \ldots, c_n)$.

**Example 1.2.** It can be easily verified that the graph $G$ in Figure 2 belongs to $\mathcal{G}(2, 2)$. Let

$$\alpha_1 = \{\alpha_{1,1}, \alpha_{1,2}\} = \{(s_1, v_1, v_2, v_3, v_4, r_1), (s_1, v_5, v_6, v_7, v_8, r_1)\},$$

$$\alpha_2 = \{\alpha_{2,1}, \alpha_{2,2}\} = \{(s_2, v_1, v_2, v_7, v_8, r_2), (s_2, v_5, v_6, v_3, v_4, r_2)\}.$$

Apparently, $e_1, e_2, e_3, e_4$ are the only 4 mergings with respect to $\alpha_1, \alpha_2$ and so $M(G; \alpha_1, \alpha_2) = 4$. Now, with the following alternative set of two edge-disjoint paths from $s_1$ to $r_1$:

$$\alpha_1' = \{\alpha_{1,1}', \alpha_{1,2}'\} = \{(s_1, v_1, v_2, v_7, v_8, r_1), (s_1, v_5, v_6, v_3, v_4, r_1)\},$$

one verifies that $e_1, e_3$ are the only 2 mergings with respect to $\alpha_1', \alpha_2$, and thereby $M(G; \alpha_1', \alpha_2) = 2$, and furthermore, the choice of $\alpha_1'$ and $\alpha_2$ achieves $M(G) = 2$. Note that it will be established in Theorem 4.4 that $\mathcal{M}(2, 2) = 5$, that is to say, for any graph in $\mathcal{G}(2, 2)$, there is a way to choose $\alpha_1, \alpha_2$ such that the number of mergings in the graph with respect to $\alpha_1, \alpha_2$ is at most 5, which can be achieved by some graph in $\mathcal{G}(2, 2)$.

At first glance, $\mathcal{M}(c_1, c_2, \ldots, c_n)$ may be infinite. However, the following theorem, which is the main result in this paper, asserts its finiteness.

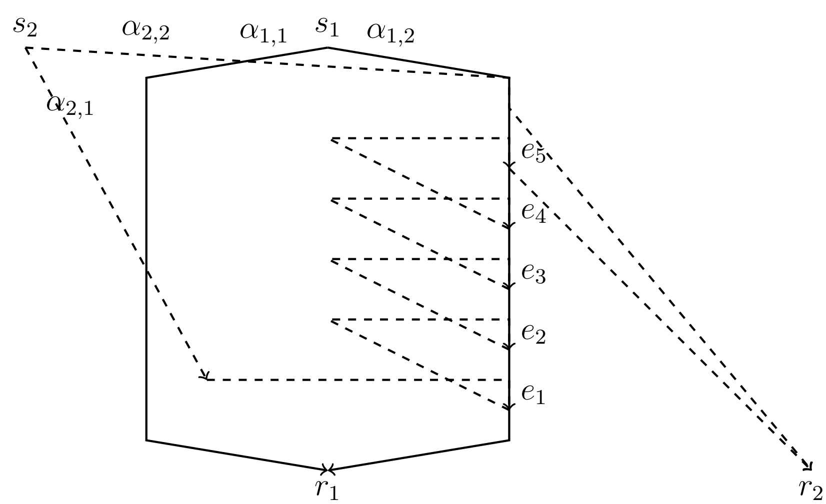


Figure 3: an illustrative example

**Theorem 1.3.** *For any* $c_1, c_2, \ldots, c_n$*, we have*

$$\mathcal{M}(c_1, c_2, \ldots, c_n) \; < \; \infty.$$

Theorem 1.3 follows from Proposition 3.5, which establishes the finiteness of $\mathcal{M}$ with two parameters via an explicit upper bound, and Proposition 3.6, which upper bounds $\mathcal{M}$ with multiple parameters using a sum of $\mathcal{M}$ with two parameters.

**Remark 1.4.** Theorem 1.3 does not hold for cyclic directed graphs: For the cyclic directed graph $G$ in Figure 3, $\alpha_{2,1}$ merges with $\alpha_{1,2}$ at $e_1, e_2, e_3, e_4, e_5$ and it can be verified that $M(G) = 5$. Moreover, in a similar fashion, one can construct a graph $G'$ such that $\alpha_{2,1}$ merges with $\alpha_{1,2}$ at $e_1, e_2, \ldots, e_\ell$ with $M(G') = \ell$ for an arbitrary $\ell$, which means that $\mathcal{M}(2,2)$, if defined on cyclic directed graphs, is in fact infinity.

**Remark 1.5.** Theorem 1.3 has a number of variants. For one example, the definition of $\mathcal{M}$ carries over almost verbatim to the cases that some of the sources and/or some of the sinks are identical, for which Theorem 1.3 still holds true. For another example, we note that (I) each $c_i$ can be redefined as the size of mininum vertex cut between $s_i$ and $r_i$; and (II) each $\alpha_i$ can be redefined as a set of $c_i$ vertex-disjoint paths; and (III) the notion of merging can be redefined as follows: we say $m \geq 2$ directed paths $\beta_1, \beta_2, \ldots, \beta_m$ *merge* at $v \in V$ if 1) $v$ belong to all $\beta_j$; and 2) there exist $j_1 \neq j_2$ such that the direct predecessors of $v$ on $\beta_{j_1}$ and $\beta_{j_2}$ are distinct. It can be easily verified that with the above-mentioned new definitions in place, the definition of $\mathcal{M}$ carries over in a straightforward fashion and Theorem 1.3 still holds true.

As elaborated below, Theorem 1.3 and its variants can be of use to certain practical situations where a network feature multiple sources and/or multiple sinks.

In particular, the case that all $r_i$ are identical is of relevance to transportation networks. More specifically, consider the traffics at a mono-centric city during the morning rush hour (see, e.g., [11, 1, 2]), where a very large number of commuters travel from home (presumably in the suburban areas) to workplace (presumably in the downtown area). Apparently, the bulk of the suburban traffics (i.e., traffics **outside** the downtown area) in such a situation is "largely" loopless: before reaching the downtown area, it makes no sense for any traveler to first go substantially further away from the workplace in order to eventually get to work, or take a circuitous detour in any part of his/her journey. As a result, we can assume, through contracting the downtown area to a single destination and orientating the occupied roads, that the suburban morning traffics are acyclic with multiple origins $s_1, s_2, \ldots, s_n$ and one single destination $r$. Assume the travel demand from $s_i$ to $r$ is $c_i$, which in turn demands traffic assignments on a set of $c_i$ edge-disjoint paths of unit capacity from $s_i$ to $r$. Then, Theorem 1.3 implies that under the optimal routing strategy, the minimum number of traffic mergings is always upper bounded by $\mathcal{M}(c_1, c_2, \ldots, c_n)$, which is independent of the size of the underlying transportation network. Generally speaking, noting that traffic mergings naturally give rise to congestions in transportation networks [3, 10], we expect that Theorem 1.3 can lead to an enhanced understanding on the level of traffic congestion in transportation networks under system-optimum route choices.

In constrast, the case that all $s_i$ are identical is of relevance to communication networks. Below, we briefly mention the connection between our work and the theory of network coding [12]: Let $G$ be an acyclic directed communication network with one sender $s$ and $n$ receivers $r_1, r_2, \ldots, r_n$. Again, let $c_i$ denote the size of the minimum edge cut between $s$ and $r_i$. It has been shown that if network coding is employed at some intermediate nodes, then information can be simultaneously transmitted from $s$ to $r_i$ at full rate $c_i$ for all $i$. The so-called *network encoding complexity* refers to the least number of encoding nodes required for a feasible network coding scheme (see [8] and references therein). It turns out for the above-mentioned network, its network encoding complexity is always bounded by $\mathcal{M}(c_1, c_2, \ldots, c_n)$.

Here we remark that an independent work [4] has conducted an in-depth analysis on routings on vertex-disjoint paths for graphs of a given tree-width, which has implications to relevant aspects of graph theory and computational complexity. It has been observed that a slightly modified proof of the main results in [4] can be used to establish $\mathcal{M}(c, c) = O(c^4)$, whereas, by comparison, our Proposition 3.5 implies that $\mathcal{M}(c, c) = O(c^3)$. It however remains to be seen that if our arguments can be modified to yield a sharper upper bound for the setting in [4] since the graphs considered therein are more general and in particular may not necessarily be acyclic.

The remainder of the paper is organized as follows. First of all, notations and terminology that will be used in our proofs will be introduced in Section 2. And in Section 3, we will prove Theorem 1.3, the main result in this paper. Some properties, exact values and a scaling law of $\mathcal{M}$ will be established in Sections 4; more specifically, we will show that for any positive integer $c$, $\mathcal{M}(1, c) = c$ (Theorem 4.3) and $\mathcal{M}(2, c) = 3c - 1$ (Theorem 4.4) and a scaling law of $\mathcal{M}(c_1, c_2)$ with one of the parameters fixed (Theorem 4.6).

## 2 Notations and Terminologies

Let $G \in \mathcal{G}(c_1, c_2, \ldots, c_n)$. For an edge $e$ in $G$, we will use $h(e)$ and $t(e)$ to denote its *head* and *tail*, respectively. And we say a vertex $v$ is *reachable* from another vertex $u$, if there is a directed path from $u$ to $v$. For a directed path $\beta$ containing two vertices $u, v$ with $v$ reachable from $u$, let $\beta[u, v]$ denote the *segment* of $\beta$ starting from $u$ and ending at $v$. For two distinct vertices $u, v$, we say $u$ is *smaller* than $v$ (or equivalently, $v$ is *larger* than $u$) if $v$ is reachable from $u$. Similarly, for two distinct edges $e, f$ in $G$, we say $e$ is *smaller* than $f$ (or equivalently, $f$ is *larger* than $e$) if $t(f)$ is reachable from $h(e)$; if, in addition, $e$, $f$ and the connecting path from $h(e)$ to $t(f)$ all belong to a path $\beta$, we say $e$ is *smaller* than $f$ on $\beta$. For a set of vertices $v_1, v_2, \ldots, v_k$ in $G$, define $G|v_1, \ldots, v_k)$ to be the subgraph of $G$ induced on the set of vertices, each of which is smaller or equal to some $v_i$, $i = 1, 2, \ldots, k$.

Now, choose $\alpha_i$ from $\Lambda_i(G)$ for each $i = 1, 2, \ldots, n$. We say $\alpha_i$ is *reroutable* if there exists a different set $\alpha'_i$ of $c_i$ edge-disjoint paths from $s_i$ to $r_i$, and more specifically, we say $\alpha_i$ can be *rerouted* to $\alpha'_i$ through removing $E(\alpha_i) - E(\alpha'_i)$ from the $\alpha_i$-paths and then adding $E(\alpha'_i) - E(\alpha_i)$ to form the $\alpha'_i$-paths. And we say $G$ is *reroutable* with respect to $\alpha_1, \alpha_2, \ldots, \alpha_n$, if some $\alpha_i$ is reroutable. Note that for a non-reroutable $G$, the choice of $\alpha_i$ is unique, so $M(G)$ is simply $M(G; \alpha_1, \alpha_2, \ldots, \alpha_n)$. For any fixed $i$, reverse the directions of those edges which do not belong to any $\alpha_i$-path to obtain a new (possibly cyclic) graph $\tilde{G}$. For any two vertices $v, v'$ in $G$, if there is a directed path $(v, u_1, u_2, \ldots, u_\ell, v')$ in $\tilde{G}$, we say $v'$ is *semi-reachable* from $v$ along $\alpha_i$, and more specifically, we may also say $v$ *semi-reaches* $v'$ via $u_1, u_2, \ldots, u_\ell, v'$.

**Example 2.1.** Again, consider the graph in Figure 2. Note that $e_1$ is smaller than both $e_2$ and $e_4$, and so is $e_3$. $G|s_1, s_2)$ only consists of two isolated vertices $s_1, s_2$; $G|v_1, v_5)$ is the subgraph of $G$ induced on the set of vertices $\{s_1, s_2, v_1, v_5\}$; $G|v_4, v_8)$ is the subgraph of $G$ induced on the set of vertices $\{s_1, s_2, v_1, v_5, v_2, v_6, v_3, v_7, v_4, v_8\}$; and $G|r_1, r_2)$ is just $G$ itself.

Let $\alpha_1, \alpha_2, \alpha'_1$ be chosen as in Example 1.2. Note that $\alpha_1$ is reroutable, since there exists another group $\alpha'_1$ of two edge-disjoint paths from $s_1$ to $r_1$, and hence $G$ is reroutable with respect to $\alpha_1, \alpha_2$; similarly, one can verify that $\alpha_2$ is also reroutable. It is also easy to check, by definition, that $v_3$ semi-reaches $v_7$ along $\alpha_2$ via $v_2, v_7$; $v_2$ semi-reaches $v_7$ along $\alpha_2$ via $v_7$; and $v_7$ semi-reaches itself along $\alpha_1$ via $v_2, v_3, v_6, v_7$; and $v_7$ also semi-reaches itself along $\alpha_2$ via $v_6, v_3, v_2, v_7$.

Consider the following 4 types of operations on $G$ with $\alpha_1, \alpha_2, \ldots, \alpha_n$ chosen as above:

(A) If an vertex $v$ is isolated, then remove $v$;

(B) If an edge $e$ does not belong to any $\alpha_j$-path, $j = 1, 2, \ldots, n$, then remove $e$;

(C) If an $\alpha_{j_1}$-path and $\alpha_{j_2}$-path ($j_1$ may be equal to $j_2$) share a non-terminal vertex $v$ (not a source nor a sink), however do not share any edge incident with $v$ (for an example, see Figure 1(a)), we "detach" the two paths at $v$, or more precisely, we split $v$ into two vertices $v^{(1)}, v^{(2)}$ and let the corresponding new $\alpha_{j_1}$-path pass through $v^{(1)}$ and let the corresponding new $\alpha_{j_2}$-path pass through $v^{(2)}$;

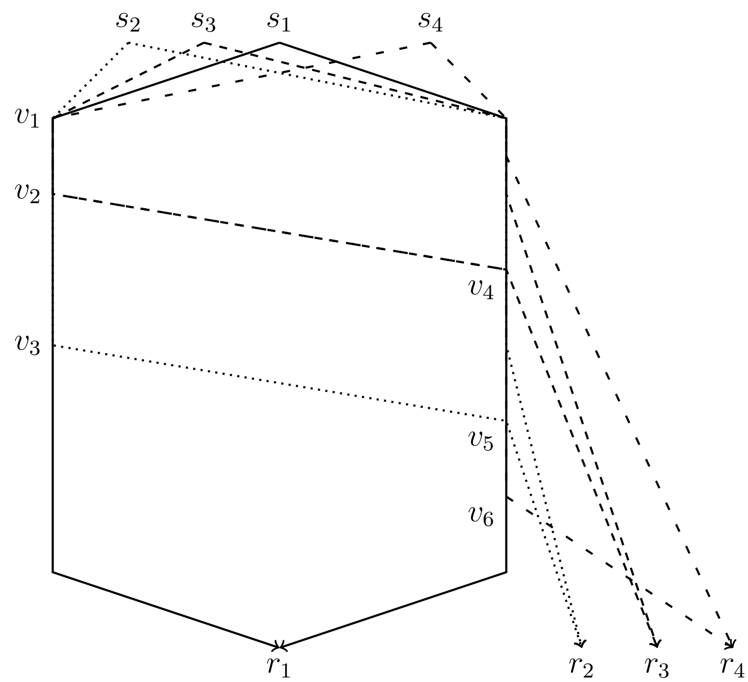


Figure 4: $\alpha_{4,1} = (s_4, v_1, v_2, v_4, v_5, v_6, r_4)$ can be rerouted to $\alpha'_{4,1} = (s_4, v_1, v_3, v_5, v_6, r_4)$, however the number of mergings does not decrease after the rerouting.

(D) If, for a vertex $v$, there is only one incoming edge $(v', v)$ and only one out-going edge $(v, v'')$, we then contract the edge $(v, v'')$ into one single vertex, which, for any $\alpha_j$-path passing through $(v, v'')$, naturally yields a corresponding new $\alpha_j$-path.

A repeated application of the above 4 types of operations to $G$ until there are no feasible operations left will yield $\hat{G} \in \mathcal{G}(c_1, c_2, \ldots, c_n)$, a *reduced* version of $G$, with $\hat{\alpha}_j \in \Lambda_j(\hat{G})$ corresponding to $\alpha_j$, $j = 1, 2, \ldots, n$. It can be easily verified that

$$M(G; \alpha_1, \alpha_2, \ldots, \alpha_n) = M(\hat{G}; \hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n). \tag{1}$$

And we say $G$ is *reduced* with respect to $\alpha_1, \alpha_2, \ldots, \alpha_n$ if all the 4 types of operations are unfeasible for $G$. If, in addition, $G$ is non-reroutable, we simply say $G$ is *reduced* and rewrite $M(G; \alpha_1, \alpha_2, \ldots, \alpha_n)$ as $|G|_{\mathcal{M}}$ without referencing $\alpha_1, \alpha_2, \ldots, \alpha_n$ since each $\alpha_j$ is the only element in $\Lambda_i(G)$.

# 3 Proof of Theorem 1.3

We first need the following lemma.

**Lemma 3.1.** *Let $G \in \mathcal{G}(c_1, c_2, \ldots, c_n)$ be reduced with respect to $\alpha_1, \alpha_2, \ldots, \alpha_n$, where $\alpha_i \in \Lambda_i(G)$, $i = 1, 2, \ldots, n$.*

*a) For any $i = 1, 2, \ldots, n$, a rerouting of $\alpha_i$ to $\alpha_i'$ will decrease the number of mergings in $G$, that is,*

$$M(G; \alpha_1, \alpha_2, \ldots, \alpha_i', \ldots, \alpha_n) \leq M(G; \alpha_1, \alpha_2, \ldots, \alpha_i, \ldots, \alpha_n). \tag{2}$$

*b) If, in addition, $n = 2$, the above inequality strictly holds.*

*Proof.* Let $V_0$ denote the set of vertices in $G$ whose in-degrees are at least 2. Since $G$ is reduced with respect to $\alpha_1, \alpha_2, \ldots, \alpha_n$, $M(G; \alpha_1, \alpha_2, \ldots, \alpha_n)$ is equal to $|V_0|$. Let $G'$ denote the subgraph of $G$ induced on the $\alpha_i'$-path and all the $\alpha_j$-paths, $j \neq i$, and let $V_0'$ denote the set of vertices in $G'$ whose in-degrees are at least 2. Similarly as above, it holds that $M(G; \alpha_1, \alpha_2, \ldots, \alpha_i', \ldots, \alpha_n) = |V_0'|$. The desired inequality (2) asserted in a) then immediately follows from the fact $V_0' \subseteq V_0$. Here we note that there exists at least one vertex $v \in V_0$ whose in-degree strictly decreases.

If, in addition, $n = 2$, one verifies that each non-terminal vertex in $G$ either (has exactly two incoming edges and one outgoing edge) or (has exactly one incoming edge and two outgoing edges), which then implies that there exists at least one vertex $v \in V_0$ such that $v \notin V_0'$, and thereby $V_0' \subset V_0$, which further implies the strict inequality asserted in b). □

**Remark 3.2.** The inequality as in (2) may not strictly hold for $n \geq 3$; see, e.g., Figure 4.

**Remark 3.3.** It follows from Lemma 3.1 b) that to compute $\mathcal{M}(c_1, c_2)$, it is enough to take the supremum in Definition 1.1 over all graph $G$ such that $G$ is non-reroutable and reduced.

We also need the following key lemma, which gives the necessary and sufficient conditions for a reduced graph in $\mathcal{G}(c_1, c_2)$ to be reroutable.

**Lemma 3.4.** *Let $G \in \mathcal{G}(c_1, c_2)$ be reduced with respect to $\alpha_1, \alpha_2$, where $\alpha_i \in \Lambda_i(G)$. The following statements are equivalent:*

*a) $\alpha_1$ (resp. $\alpha_2$) is reroutable.*

*b) there exists a merging $e$ such that $h(e)$ semi-reaches itself along $\alpha_1$ (resp. $\alpha_2$).*

*c) there exists a merging $e$ such that $t(e)$ semi-reaches itself along $\alpha_1$ (resp. $\alpha_2$).*

*Proof.* We first establish the equivalence between b) and c). If $h(e)$ semi-reaches itself via $u_1, u_2, \ldots, u_\ell, h(e)$, then $u_1$ must be a tail of certain merging $\hat{e}$, which semi-reaches itself via $u_2, u_3, \ldots, h(e), u_1$, establishing b) $\Rightarrow$ c). A similar argument can be used to establish c) $\Rightarrow$ b). So, in the following, we only prove the equivalence between a) and b).

**a) $\Rightarrow$ b).** Suppose that $\alpha_1$ can be rerouted to $\alpha_1'$ and that $E(\alpha_1) - E(\alpha_1')$ consists of the following $\alpha_1$-edges: $e_1, e_2, \ldots, e_\ell$. Without loss of generality, assume that $\{\alpha_{1,1}, \alpha_{1,2}, \ldots, \alpha_{1,k}\}$ is the set of $\alpha_1$-paths which contain at least one $e_j$, that is,

$$\bigcup_{j=1}^{\ell} \alpha_1(e_j) = \{\alpha_{1,1}, \alpha_{1,2}, \ldots, \alpha_{1,k}\};$$

and assume that $e_1, e_2, \ldots, e_k$ are the smallest edges from $E(\alpha_1) - E(\alpha_1')$ on $\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,k}$, respectively. Since $G$ is reduced with respect to $\alpha_1, \alpha_2$, $E(\alpha_1') - E(\alpha_1)$ only consists of $\alpha_2$-edges, and furthermore, there are some $\alpha_2$-edges $f_1, f_2, \ldots, f_k$ such that for each $i = 1, 2, \ldots k$,

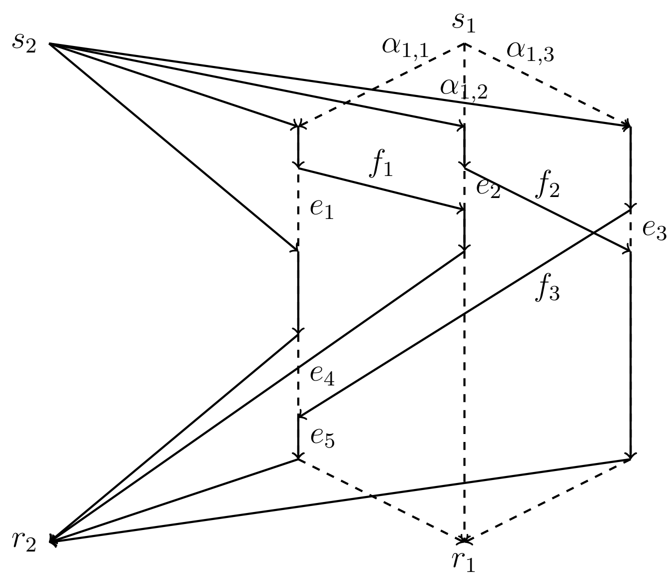


Figure 5: For the merging $e_5$, $t(e_5)$ semi-reaches itself along $\alpha_1$ via $t(f_3), h(e_3), t(f_2), h(e_2), t(f_1), h(e_1), t(e_4), t(e_5)$. One verifies that $\alpha_1$ can be rerouted to skip $e_1, e_2, e_3, e_4$ and pass through $f_1, f_2, f_3$ instead.

there exists $1 \leq k_i \leq \ell$ such that $t(f_i) = t(e_i)$, $h(f_i) = h(e_{k_i})$. It then follows that one can find a subset $\{\hat{k}_1, \hat{k}_2, \ldots, \hat{k}_s\}$ of $\{1, 2, \ldots, k\}$ such that $h(f_{\hat{k}_1}) \in \alpha_{1,\hat{k}_2}$, $h(f_{\hat{k}_2}) \in \alpha_{1,\hat{k}_3}$, $\ldots$, $h(f_{\hat{k}_s}) \in \alpha_{1,\hat{k}_1}$, which implies that there is a merging whose head semi-reaches itself along $\alpha_1$ (see Figure 5 for an illustrative example).

**b) $\Rightarrow$ a).** Assume that $h(e)$ semi-reaches itself along $\alpha_1$ via $v_1, v_2, \ldots, v_\ell$. Let

$$C = \{(h(e), v_1), (v_1, v_2), (v_2, v_3), \ldots, (v_{\ell-1}, v_\ell), (v_\ell, h(e))\}.$$

Let $A = \{(u, w) \in C : (w, u) \text{ is a reversed } \alpha_2\text{-edge}\}$, and let $B = C - A$. Then, it can be readily verified that the subgraph of $G$ induced on $(E(\alpha_i) \cup A) - B$ consists of $c_i$ connected components, each of which is a path from $s_i$ to $r_i$ (see Figure 5 for an illustrative example). □

Before the proof of Theorem 1.3, we will first prove the following proposition, which will establish the finiteness of $\mathcal{M}(c_1, c_2)$ .

**Proposition 3.5.** *For any $c_1, c_2$,*

$$\mathcal{M}(c_1, c_2) \leq c_1 c_2 (c_1 + c_2)/2.$$

*Proof.* By Remark 3.3, it suffices to prove that for any non-reroutable and reduced $G \in \mathcal{G}(c_1, c_2)$,

$$M(G; \alpha_1, \alpha_2) \leq c_1 c_2 (c_1 + c_2)/2.$$

The proof is by way of contradiction. We will prove that if

$$M(G; \alpha_1, \alpha_2) \geq c_1c_2(c_1+c_2)/2 + 1,$$

then either $\alpha_1$ or $\alpha_2$ is reroutable, a contradiction to the assumption that $G$ is non-reroutable.

Consider the following operations on $G$: we first delete all the edges that are both $\alpha_1$-edges and $\alpha_2$-edges, which are necessarily mergings due to the assumption that $G$ is reduced, then we reverse the directions of the remaining $\alpha_2$-edges. Note that after the above operations, for any directed path in $\hat{G}$, each edge is either an $\alpha_1$-edge or a reversed $\alpha_2$-edge. Suppose that there is a directed cycle $(v_1, v_2, \ldots, v_\ell)$ in $\hat{G}$, where $v_\ell = v_1$ and $e_i \triangleq (v_i, v_{i+1})$ is a reversed $\alpha_2$-edge for any odd $i$ and an $\alpha_1$-edge for any even $i$. It can be verified that all $v_j$ belong to $V_{\mathcal{M}}$, where $V_{\mathcal{M}}$ denotes the set of all the tails and heads of all the mergings. It then follows that $v_1$ semi-reaches itself along $\alpha_1$ via $v_2, v_3, \ldots, v_{\ell-1}, v_\ell, v_1$, which implies $\alpha_1$ is reroutable, a contradiction.

So, in the following, we assume that $\hat{G}$ is acyclic. Note that in $\hat{G}$, $s_1, r_2$ have out-degree $c_1, c_2$, respectively, $s_2, r_1$ has in-degree $c_1, c_2$, respectively, and any vertex in $V_{\mathcal{M}}$ has in-degree 1 and out-degree 1. It then immediately follows that $\hat{G}$ consists of $c_1 + c_2$ pairwise vertex-disjoint paths, each of which starts from either $s_1$ or $r_2$, ends at either $s_2$ or $r_1$ and consists of a sequence of concatenated edges that alternates between an $\alpha_1$-edge and a reversed $\alpha_2$-edge. It then follows from

$$|V_{\mathcal{M}}| = 2M(G; \alpha_1, \alpha_2) \geq c_1c_2(c_1+c_2) + 1,$$

that out of the $c_1 + c_2$ edge-disjoint paths, there must be at least one path, say, $\gamma$, that contains more than $c_1c_2$ vertices in $V_{\mathcal{M}}$. It then follows that there are two distinct vertices $u, v \in V_{\mathcal{M}}$ on $\gamma$ and $i_0, j_0$ such that $u$ corresponds to a merging by $\alpha_{1,i_0}$ and $\alpha_{2,j_0}$, and so does $v$. Note that if $u$ is smaller (*resp.* larger) than $v$ on $\alpha_{1,i_0}$, then $u$ will be also smaller (*resp.* larger) than $v$ on $\alpha_{2,j_0}$, otherwise we would have a cycle formed by concatenating $\alpha_{1,i_0}[u, v]$ and $\alpha_{2,j_0}[v, u]$ in $G$, which contradicts the assumption that $G$ is acyclic.

In what follows, we assume that $\gamma[u, v] = (u, w_1, w_2, \ldots, w_\ell, v)$ and we consider the following conditions:

- $u$ is smaller (larger) than $v$ on $\alpha_{1,i_0}$;
- $u$ is the tail (head) of the corresponding merging in $G$, $v$ is the tail (head) of the corresponding merging in $G$.

Ignoring the parenthesized words for the moment, one verifies that $v$ semi-reaches itself along $\alpha_2$ via $w_\ell, w_{\ell-1}, \ldots, w_1$ and vertices on $\alpha_{1,i_0}[u, v]$ (ordered by the direction of the path $\alpha_{1,i_0}$), implying $\alpha_2$ is reroutable, a contradiction. A similar argument can be applied to other cases when any parenthesized words replace the words before them, which completes the proof. □

We are now ready for the proof of Theorem 1.3. With Proposition 3.5 established, it suffices to prove the following proposition.

**Proposition 3.6.** *For any $c_1, c_2, \ldots, c_n$, we have*

$$\mathcal{M}(c_1, c_2, \ldots, c_n) \leq \sum_{i<j} \mathcal{M}(c_i, c_j).$$

*Proof.* It suffices to prove that for $n \geq 3$

$$\mathcal{M}(c_1, c_2, \ldots, c_n) \leq \mathcal{M}(c_1, c_2, \ldots, c_{n-1}) + \sum_{i<n} \mathcal{M}(c_i, c_n). \tag{3}$$

Consider any graph $G \in \mathcal{G}(c_1, c_2, \ldots, c_n)$ that is reduced with respect to $\alpha_1, \alpha_2, \ldots, \alpha_n$, where $\alpha_i \in \Lambda_i(G)$. Let $\bar{G}$ denote the subgraph of $G$ induced on all $\alpha_j$-paths, $j = 1, 2, \ldots, n-1$. Note that, through rerouting if necessary, we can assume that $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are chosen such that

$$M(\bar{G}; \alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \leq \mathcal{M}(c_1, c_2, \ldots, c_{n-1}).$$

A merging is said to be *new* if $\alpha_n(e) \neq \emptyset$ merges with each non-empty $\alpha_j(e)$, $j = 1, 2, \ldots, n-1$, at $e$. We will prove that if the number of new mergings between $\alpha_n$ and $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ is larger than or equal to

$$l \triangleq \mathcal{M}(c_1, c_n) + \mathcal{M}(c_2, c_n) + \cdots + \mathcal{M}(c_{n-1}, c_n) + 1,$$

certain reroutings can be done to strictly reduce the number of new mergings. Apparently, this is sufficient to imply (3) and then the proposition. For ease of presentation, in the remainder of this proof, we assume that all mergings in $G$ are new.

Now, label all the new mergings as $e_1, e_2, \ldots, e_l$. Then, by the Pigeonhole principle, there exists some $i = 1, 2, \ldots, n-1$ such that $\alpha_i$ merges with $\alpha_n$ for more than $\mathcal{M}(c_i, c_n)$ times. As a consequence, the subgraph of $G$ induced on $\{\alpha_i, \alpha_n\}$ is reroutable; in other words, either $\alpha_i$ or $\alpha_n$ can be rerouted in this induced subgraph of $G$. If such a rerouting is in fact a rerouting of $\alpha_n$, then the number of new mergings between $\alpha_n$ and $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ will be strictly decreased after such a rerouting. So in the following we assume that the rerouting between every $\alpha_i$ and $\alpha_n$, if exists, is a rerouting of $\alpha_i$. Then, after the rerouting of $\alpha_i$, there are at least

$$\mathcal{M}(c_1, c_{k+1}) + \cdots + \mathcal{M}(c_{i-1}, c_{k+1}) + \mathcal{M}(c_{i+1}, c_{k+1}) + \cdots + \mathcal{M}(c_k, c_{k+1}) + 1$$

of the $e_j$'s, at which the new $\alpha_i$ does not merge. This implies that there exists at least one $e_j$ such that none of $\alpha_i$'s, $i = 1, 2, \ldots, n-1$, merge with $\alpha_n$ at $e_j$. So the number of new mergings between $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ and $\alpha_n$ strictly decreases after the reroutings of all $\alpha_i$'s. □

# 4 Properties, Exact Values and a Scaling Law

With the finiteness of $\mathcal{M}$ established in Theorem 1.3, one natural question is to compute the values of $\mathcal{M}$, which seems to be fairly difficult, even for small parameters. In this section, among other results, we will derive the values of $\mathcal{M}$ for certain special parameters. Some propositions on properties of $\mathcal{M}$ as a function of its parameters will be established as well; these propositions, besides helping to derive the values of $\mathcal{M}$, are of interest in their own right.

The following proposition shows that $\mathcal{M}$ is "sup-linear" in all its parameters.

**Proposition 4.1.** *For any* $c_{1,0}, c_{1,1}, c_2, \ldots, c_n$*, we have*

$$\mathcal{M}(c_{1,0} + c_{1,1}, c_2, \ldots, c_n) \geq \mathcal{M}(c_{1,0}, c_2, \ldots, c_n) + \mathcal{M}(c_{1,1}, c_2, \ldots, c_n).$$

*Proof.* We only prove the proposition for the case $n = 2$, the case with a generic $n$ being similar.

For any $c_{1,0}, c_{1,1}$ and $c_2$, consider the following acyclic directed graph $G$ (see Figure 6 for an illustrative example) with 2 sources $s_1, s_2$ and 2 sinks $r_1, r_2$ such that

1. there is a set $\alpha_1$ of $c_{1,0} + c_{1,1}$ edge-disjoint paths from $s_1$ to $r_1$, here $\alpha_1 = \alpha_1^{(0)} \cup \alpha_1^{(1)}$, where $\alpha_1^{(0)}$ and $\alpha_1^{(1)}$ are mutually exclusive, consisting of $c_{1,0}$, $c_{1,1}$ edge-disjoint paths, respectively, and there is a set $\alpha_2$ of $c_2$ edge-disjoint paths from $s_2$ to $r_2$;

2. mergings by $\alpha_1^{(0)}, \alpha_2$ and mergings by $\alpha_1^{(1)}, \alpha_2$ are "sequentially isolated" on $\alpha_2$ in the sense that on each $\alpha_2$-path, the smallest $\alpha_1^{(1)}$-merging is larger than the largest $\alpha_1^{(0)}$-merging;

3. the number of mergings in the subgraph of $G$ induced on $\alpha_1^{(0)}$ and $\alpha_2$ achieves $\mathcal{M}(c_{1,0}, c_2)$, and the number of mergings in the subgraph of $G$ induced on $\alpha_1^{(1)}$ and $\alpha_2$ achieves $\mathcal{M}(c_{1,1}, c_2)$.

It can be verified that for such $G$, the size of the minimum edge cut between $s_1$ and $r_1$ is $c_{1,0} + c_{1,1}$, and the size of the minimum edge cut between $s_2$ and $r_2$ is $c_2$, and

$$M(G; \alpha_1, \alpha_2) = \mathcal{M}(c_{1,0}, c_2) + \mathcal{M}(c_{1,1}, c_2),$$

which implies that

$$\mathcal{M}(c_{1,0} + c_{1,1}, c_2) \geq \mathcal{M}(c_{1,0}, c_2) + \mathcal{M}(c_{1,1}, c_2).$$

□

The following proposition gives a lower bound on $\mathcal{M}$ with multiple parameters using $\mathcal{M}$ with two parameters.

**Proposition 4.2.** *For any* $c_1, c_2, \ldots, c_n$ *and any fixed* $k$ *with* $1 \leq k \leq n$*, we have*

$$\mathcal{M}(c_1, c_2, \ldots, c_n) \geq \sum_{i \leq k, j \geq k+1} \mathcal{M}(c_i, c_j).$$

*Proof.* For any $c_1, c_2, \ldots, c_n$, consider the following directed graph $G$ (see Figure 7 for an illustrative example) with $n$ sources $s_1, s_2, \ldots, s_n$ and $n$ sinks $r_1, r_2, \ldots, r_n$ such that for any fixed $k$ with $1 \leq k \leq n$,

1. there is a set $\alpha_i$ of $c_i$ edge-disjoint paths from $s_i$ to $r_i$ for each $i$;

2. all $\alpha_i$'s, $i \leq k$, do not merge with each other, and all $\alpha_j$'s, $j \geq k+1$, do not merge with each other;

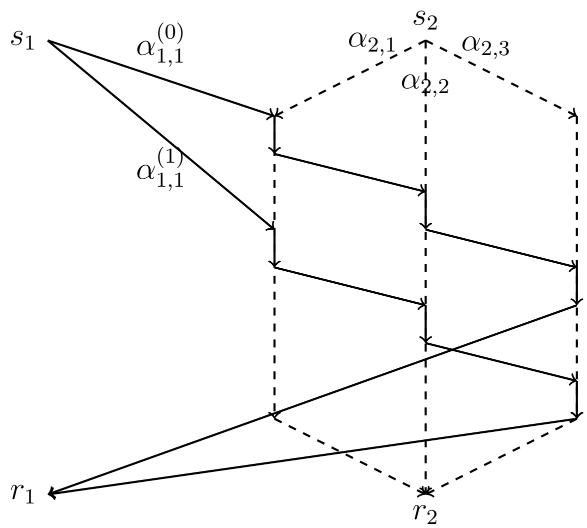


Figure 6: Each merging by $\alpha_{1,1}^{(0)}$ and a $\alpha_2$-path is smaller than than that by $\alpha_{1,2}^{(0)}$ and the same $\alpha_2$-path.

3. for any $i$ with $i \leq k$, mergings by $\alpha_i$ and all $\alpha_j$'s, $j \geq k+1$, are "sequentially isolated" on $\alpha_i$ in the sense that on each $\alpha_i$-path, for any $j_1 < j_2$ with $j_1, j_2 \geq k+1$, the smallest $\alpha_{j_2}$-merging is larger than the largest $\alpha_{j_1}$-merging. Similarly for any $j$ with $j \geq k+1$, mergings by $\alpha_j$ and all $\alpha_i$'s, $i \leq k$, are sequentially isolated on $\alpha_j$;

4. the number of mergings in the subgraph of $G$ induced on any $\alpha_i$, $i \leq k$, and any $\alpha_j$, $j \geq k+1$, achieves $\mathcal{M}(c_i, c_j)$.

One checks that for such a graph $G$, the size of the minimum edge cut between $s_i$ and $r_i$ is $c_i$, and

$$M(G) = \sum_{i \leq k, j \geq k+1} \mathcal{M}(c_i, c_j),$$

which implies that

$$\mathcal{M}(c_1, c_2, \ldots, c_n) \geq \sum_{i \leq k, j \geq k+1} \mathcal{M}(c_i, c_j).$$

□

The following theorem is straightforward. We give a proof for completeness.

**Theorem 4.3.** *For any $c$,*

$$\mathcal{M}(1, c) = c.$$

*Proof.* Consider a non-reroutable and reduced graph $G \in \mathcal{G}(1, c)$. If $\alpha_{1,1}$ merges with some $\alpha_2$-path, say, $\alpha_{2,j}$, at mergings $e$ and $\hat{e}$. Then we can reroute $\alpha_{1,1}$ by replacing $\alpha_{1,1}[t(e), t(\hat{e})]$, the subpath of $\alpha_{1,1}$ starting from $t(e)$ to $t(\hat{e})$, by $\alpha_{2,j}[t(e), t(\hat{e})]$, the subpath of $\alpha_{2,j}$ starting

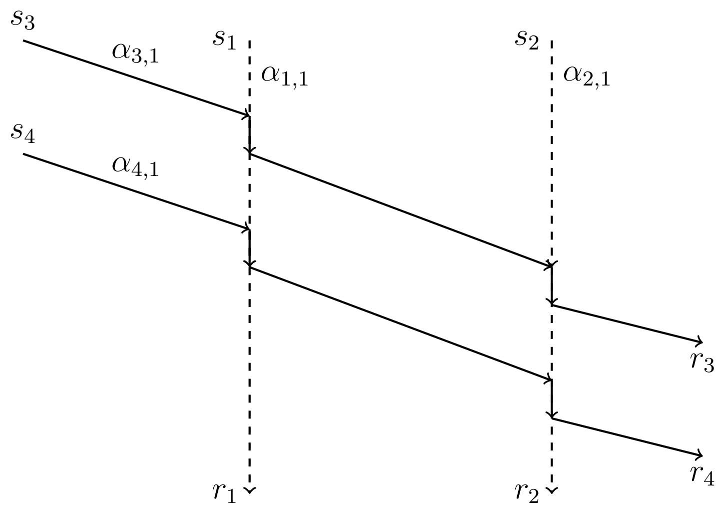


Figure 7: On the path $\alpha_{i,1}$, $i = 1, 2$, the merging with $\alpha_{3,1}$ is smaller than that with $\alpha_{4,1}$, and on the path $\alpha_{i,1}$, $i = 3, 4$, the merging with $\alpha_{1,1}$ is smaller than that with $\alpha_{2,1}$.

from $t(e)$ to $t(\hat{e})$. This contradicts the assumption that $G$ is non-reroutable, which implies that $\alpha_{1,1}$ can be chosen to merge with each $\alpha_2$-path for at most once, which further implies that

$$\mathcal{M}(1, c) \leq c,$$

For the other direction, by Proposition 4.1, we have

$$\mathcal{M}(1, c) \geq \sum_{i=1}^{c} \mathcal{M}(1, 1) = c,$$

the last equality follows from the simple fact that $\mathcal{M}(1, 1) = 1$. □

**Theorem 4.4.** *For any $c$,*

$$\mathcal{M}(2, c) = 3c - 1.$$

*Proof.* **The upper bound direction.** We first show that

$$\mathcal{M}(2, c) \leq 3c - 1.$$

Consider a non-reroutable and reduced $G \in \mathcal{G}(2, c)$. In this proof, for notational convenience, we rewrite $\alpha_1, \alpha_2$ as $\psi, \phi$, respectively. Assume that out of $c$ $\phi$-paths, there are $k$ $\phi$-paths, say $\phi_1, \phi_2, \ldots, \phi_k$, each of which merges for at least 3 times. Notice that when $k = 0$, the total number of mergings in $G$ is upper bounded by $2c$, which trivially implies the upper bound direction. So, in this proof, we only consider the case when $k \geq 1$. For $i = 1, 2, \ldots, k$, assume that $\phi_i$ sequentially merges at edges $e_{i,1}, e_{i,2}, \ldots, e_{i,m_i}$. Let $\ell(i, j)$

denote the index of the $\psi$-path which $e_{i,j}$ belongs to. Since each $\phi$-path has to merge with $\psi_1, \psi_2$ alternately, we have $\ell(i,j) = \ell(i,k)$ if $j = k \mod 2$.

Note that for each pair of mergings $e_{i,j}, e_{i,j+2}$, there must exist one merging, say $f_{i,j}$, which is in between $e_{i,j}$ and $e_{i,j+2}$ on $\psi_{\ell(i,j)}$. It can readily verified that

- for any $i, j$, $\phi(f_{i,j})$ merges with $\psi$-paths at most twice; and
- if $\phi(f_{i,j}) = \phi(f_{i,k})$ for any $j < k$, then necessarily $k = j + 1$.

Now, for the associated $\phi$-paths of all $f_{i,j}$, we claim that:

**Claim 4.5.** For any fixed $i$, one can choose all $f_{i,j}$ such that for $j \neq k$, $\phi(f_{i,j}) \neq \phi(f_{i,k})$.

The claim can be shown via an inductive argument on the length of path $\phi_i$. The case when $m_i = 3$ is trivial. Now suppose the claim is established for $m_i = l$ and assume that $f_{i,2}, f_{i,3}, \ldots, f_{i,l+1}$ all belong to different $\phi$-paths. We next show that the claim is also true for $m_i = l + 1$. We will consider each of the following cases:

Case 1: $\phi(f_{i,1}) \neq \phi(f_{i,2})$. For this case, by induction assumptions, the claim is trivially true.

Case 2: $\phi(f_{i,1}) = \phi(f_{i,2})$. For this case, either (in between $e_{i,1}$ and $f_{i,1}$ on $\psi_{\ell(i,1)}$) or (in between $f_{i,2}$ and $e_{i,4}$ on $\psi_{\ell(i,2)}$), there must be a merging, whose associated $\phi$-path is different from that of $f_{i,1}$ and $f_{i,2}$. Otherwise, $t(e_{i,4})$ would semi-reach itself along $\phi$ via $h(f_{i,2}), t(f_{i,1}), h(e_{i,4}), t(e_{i,4})$, which implies $\phi$ is reroutable, a contradiction.

Case 2.1: in between $e_{i,1}$ and $f_{i,1}$ on $\psi_{\ell(i,1)}$, there is a merging $f'_{i,1}$ such that $\phi(f'_{i,1}) \neq \phi(f_{i,1})$. For this case, one can simply reset $f_{i,1}$ to be $f'_{i,1}$, then the claim immediately follows.

Case 2.2: in between $f_{i,2}$ and $e_{i,4}$ on $\psi_{\ell(i,2)}$, there is a merging $f'_{i,2}$ such that $\phi(f'_{i,2}) \neq \phi(f_{i,1})$. For this case, we have the following subcases.

Case 2.2.1: $\phi(f'_{i,2}) \neq \phi(f_{i,3})$. For this case, we can simply reset $f_{i,2}$ to be $f'_{i,2}$ to establish the claim.

Case 2.2.2: $\phi(f'_{i,2}) = \phi(f_{i,3})$. For $j = 2, 3, \ldots, m_i - 3$, we say $f_{i,j}$ is of *type I* if (there exists exactly one merging $f'_{i,j}$ in between $f_{i,j}$ and $e_{i,j+2}$) and ($f'_{i,j}$, $f_{i,j+1}$ belong to the same $\phi$-path).

Case 2.2.2.1: all $f_{i,j}$, $j = 2, 3, \ldots, m_i - 3$, are of type I. For this case, consider $f_{i,m_i-2}$. One checks that there must exist a merging, say, $f'_{i,m_i-2}$, in between $f_{i,m_i-2}$ and $e_{i,m_i}$ on $\psi_{\ell(i,m_i-2)}$, since otherwise $t(e_{i,m_i})$ would semi-reach itself along $\phi$ via

$$h(f_{i,m_i-2}), t(f'_{i,m_i-3}), h(f_{i,m_i-3}), \ldots, t(f_{i,1}) \text{ and vertices on } \phi_i[h(e_{i,1}), t(e_{i,m_i})],$$

which implies $\phi$ is reroutable, a contradiction. Then we can reset $f_{i,m_i}$ to be $f'_{i,m_i}$, $f_{i,m_i-1}$ to be $f'_{i,m_i-1}$, $\ldots$, $f_{i,2}$ to be $f'_{i,2}$. One checks that each of newly defined $f_{i,j}$ belongs to different $\phi$-paths.

Case 2.2.2.2: for some $2 \leq k \leq m_i - 3$, $f_{i,k}$ is not of type I. Let $2 \leq k \leq m_i - 3$ be the smallest index such that $f_{i,k}$ is not of type I, meaning either

- there is no merging in between $f_{i,k}$ and $e_{i,k+2}$ on $\psi_{\ell(i,k+2)}$; or
- there is a merging, say $f'_{i,k}$, in between $f_{i,k}$ and $e_{i,k+2}$ on $\psi_{\ell(i,k+2)}$, however $\phi(f'_{i,k}) \neq \phi(f_{i,k+1})$.

The first case implies that $t(e_{i,k+2})$ semi-reaches itself along $\phi$ by itself via

$$h(f_{i,k}), t(f'_{i,k-1}), h(f_{i,k-1}), \ldots, t(f_{i,1}) \text{ and vertices on } \phi_i[h(e_{i,1}), t(e_{i,k+2})],$$

a contradiction to the fact that $G$ is non-reroutable; while for the second case, one can reset $f_{i,k}$ to be $f'_{i,k}$, $f_{i,k-1}$ to be $f'_{i,k-1}$, $\ldots$, and $f_{i,2}$ to be $f'_{i,2}$ to establish the claim.

One also verifies that for any $i, j = 1, 2, \ldots, k$, $\phi_i$ and $\phi_j$ are "well-separated"; more precisely, one of the pair, say $\phi_i$, must be "smaller" than the other one, $\phi_j$, in the sense that the mergings by $\phi_i$ on $\psi_1, \psi_2$ must be smaller than the mergings by $\phi_j$ on $\psi_1, \psi_2$, respectively. Through renumbering, if necessary, we assume that for any $1 \leq i < j \leq k$, $\phi_i$ is always smaller than $\phi_j$. Then with this, one checks that for any $1 \leq i_1 < i_2 \leq k$, $f_{i_1,j_1}$ and $f_{i_2,j_2}$ share the same $\phi$-path if and only if $i_2 = i_1 + 1$ and $j_1 = m_{i_1} - 2, j_2 = 1$. Thus, by Claim 4.5, there must exist at least $(m_1 - 2 + m_2 - 2 + \cdots + m_k - 2) - (k-1)$ $\phi$-paths, each of which contains some $f_{i,j}$, and again each of these $\phi$-paths can merge at most twice.

Now, we conclude that

$$|G|_{\mathcal{M}} \leq m_1 + m_2 + \cdots + m_k + 2(c-k),$$

where

$$(m_1 - 2) + (m_2 - 2) + \cdots + (m_k - 2) - (k-1) \leq |\{\phi(f_{i,j})\}| \leq c - k,$$

which further implies that

$$|G|_{\mathcal{M}} \leq 3c - 1.$$

So, we have established the upper bound direction.

**The lower bound direction.** To show

$$\mathcal{M}(2, c) \geq 3c - 1,$$

it suffices to construct a non-reroutable graph $G$ with $M(G) = 3c - 1$. For instance, we can first choose $\phi_1$ to alternately merge with $\psi_1, \psi_2$ for $c+1$ times at $e_1, e_2, \ldots, e_{c+1}$. Next we choose each $\phi_i$, $i = 2, 3, \ldots, c$ to merge exactly twice, while ensuring that, for all $i < j$, $\phi$ is smaller than $\phi_j$ in the sense that the merged subpaths by $\phi_i$ on $\psi_1, \psi_2$ are smaller than the merged subpaths by $\phi_j$ on $\psi_1, \psi_2$, respectively. Moreover we also require that $\phi_{2i}$ first merges with $\psi_1$ in between $e_{2i-1}$ and $e_{2i+1}$, and then merge with $\psi_2$ in between $e_{2i-2}$ and $e_{2i}$, and that $\phi_{2i+1}$ first merges with $\psi_2$ in between $e_{2i}$ and $e_{2i+2}$, and then merges with $\psi_1$ in between $e_{2i-1}$ and $e_{2i+1}$ (see an example graph in Figure 8 for the case $c = 3$). It can be checked that such a graph is non-reroutable and the number of mergings is $3c - 1$.

□

We next prove that when fixing one parameter, $\mathcal{M}(c_1, c_2)$ grows at most linearly with respect to the other parameter.

**Theorem 4.6.** *For any fixed $c_1$, there exists a positive constant $C_{c_1}$ such that for all $c_2$,*

$$\mathcal{M}(c_1, c_2) \leq C_{c_1} c_2.$$

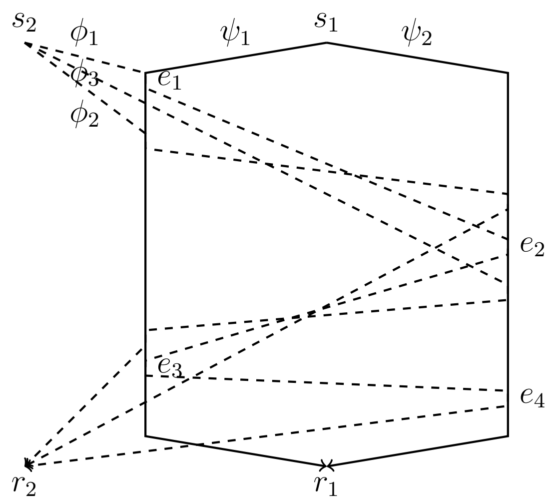


Figure 8: an example graph in $\mathcal{G}(2,3)$ achieving $\mathcal{M}(2,3)$

*Proof.* For notational simplicity, in this proof, we rewrite $c_1, c_2$ as $k, l$, respectively, that is, we will prove that for any fixed $k$, there exists a positive constant $C_k$ such that for all $l$,

$$\mathcal{M}(k,l) \leq C_k l.$$

We proceed by induction on $k$. It follows from $\mathcal{M}(1,l) = l$ (see Theorem 4.3) that for the case when $k=1$, the theorem is true with $C_1 = 1$. Now for any $k \geq 2$, assume that for any $i = 1, 2, \ldots, k-1$, there exists a positive constant $C_i$ such that for all $l$,

$$\mathcal{M}(i,l) \leq C_i l;$$

we next show that there exists a positive constant $C_k$ such that for all $l$,

$$\mathcal{M}(k,l) \leq C_k l.$$

Consider $G \in \mathcal{G}(k,l)$ and assume $G$ is non-reroutable and reduced. Rewriting

$$\alpha_1 = \{\alpha_{1,1}, \alpha_{1,2}, \ldots, \alpha_{1,k}\}, \quad \alpha_2 = \{\alpha_{2,1}, \alpha_{2,2}, \ldots, \alpha_{2,l}\}$$

as

$$\psi = \{\psi_1, \psi_2, \ldots, \psi_k\}, \quad \phi = \{\phi_1, \phi_2, \ldots, \phi_l\},$$

respectively. We only need to prove that there exists $C_k$ such that

$$|G|_{\mathcal{M}} \leq C_k l.$$

Consider the following iterative procedure on $G$, where, for notational simplicity, we treat a graph as the union of its vertex set and edge set.

**Initialization.** Set

$$\mathbb{S}^{(0)} = \emptyset, \qquad \mathbb{R}^{(0)} = G.$$

**Finding a normal block.** Now for an arbitrary yet fixed $K > 0$ (we will choose $K$ large enough later) and each $j = 1, 2, \ldots, k$, pick merging $e_{0,j}$ such that $e_{0,j}$ belongs to path $\psi_j$ and

$$|\mathbb{R}^{(0)}|h(e_{0,1}), h(e_{0,2}), \ldots, h(e_{0,k})|_{\mathcal{M}} = K.$$

Note that, without loss of generality, we can assume that, within $\mathbb{R}^{(0)}|h(e_{0,1}), h(e_{0,2}), \ldots, h(e_{0,k}))$, $e_{0,j}$ is the largest merging on $\psi_j$ (one can set $h(e_{0,j})$ to be $s_1$ if such merging does not exist on $\psi_j$). Now, set

$$\mathbb{L}^{(1)} = \mathbb{R}^{(0)}|h(e_{0,1}), h(e_{0,2}), \ldots, h(e_{0,k})),$$

and subsequently

$$\mathbb{S}^{(1)} = \mathbb{S}^{(0)} \cup \mathbb{L}^{(1)}, \qquad \mathbb{R}^{(1)} = \mathbb{R}^{(0)} - \mathbb{L}^{(1)}.$$

If a merging is the smallest (*resp.* the largest) on a $\phi$-path, we say it is an *x-terminal* (*resp. y-terminal*) merging on the $\phi$-path, or simply a *$\phi$-terminal* merging. Suppose that we already obtain

$$\mathbb{L}^{(i)} = \mathbb{R}^{(i-1)}|h(e_{i-1,1}), h(e_{i-1,2}), \ldots, h(e_{i-1,k})),$$

and

$$\mathbb{S}^{(i)} = \mathbb{S}^{(i-1)} \cup \mathbb{L}^{(i)}, \qquad \mathbb{R}^{(i)} = \mathbb{R}^{(i-1)} - \mathbb{L}^{(i)},$$

where $\mathbb{L}^{(i)}$ contains exactly $K$ mergings and at least one $\phi$-terminal merging. We then try to pick within $\mathbb{R}^{(i)}$ a merging $e_{i,j}$ on each $\psi_j$ such that

$$|\mathbb{R}^{(i)}|h(e_{i,1}), h(e_{i,2}), \ldots, h(e_{i,k})|_{\mathcal{M}} = K,$$

where each $e_{i-1,j}$, $j = 1, 2, \ldots, k$, is chosen to be largest merging on $\psi_j$, and there is at least one $\phi$-terminal merging in $\mathbb{R}^{(i)}|h(e_{i,1}), h(e_{i,2}), \ldots, h(e_{i,k}))$. If such $e_{i,j}$ exist or $|\mathbb{R}^{(i)}|_{\mathcal{M}} < K$, we then set

$$\mathbb{L}^{(i+1)} = \mathbb{R}^{(i)}|h(e_{i,1}), h(e_{i,2}), \ldots, h(e_{i,k})),$$

and subsequently

$$\mathbb{S}^{(i+1)} = \mathbb{S}^{(i)} \cup \mathbb{L}^{(i+1)}, \qquad \mathbb{R}^{(i+1)} = \mathbb{R}^{(i)} - \mathbb{L}^{(i+1)};$$

furthermore, for the case $|\mathbb{R}^{(i)}|_{\mathcal{M}} < K$, we will terminate the procedure. So far, for any obtained "block" $\mathbb{L}^{(i+1)}$, either we have $(|\mathbb{L}^{(i+1)}|_{\mathcal{M}} < K)$ or $(|\mathbb{L}^{(i+1)}|_{\mathcal{M}} = K$ and there are at least one $\psi$-terminal mergings in $\mathbb{L}^{(i+1)})$; such block $\mathbb{L}^{(i+1)}$ is said to be *normal*. If $|\mathbb{R}^{(i)}| \geq K$, however, we cannot find a normal block, we continue the procedure and define a *singular* $\mathbb{L}^{(i+1)}$ in the following.

**Finding a singular block.** A merging within $\mathbb{S}^{(i)}$ is said to be *critical* with respect to $\mathbb{S}^{(i)}$ if its associated $\phi$-path, after the said merging, does not merge anymore within $\mathbb{S}^{(i)}$. Now, let $\{\beta_j^{(i)}\}$ denote the set of all critical mergings with respect to $\mathbb{S}^{(i)}$, and let $\bar{\mathbb{T}}^{(i)}$ denote the set of all the mergings whose heads or tails are semi-reachable by the head of some $\beta_j^{(i)}$ along $\phi$. One verifies at least one $\psi$-path in $\{\psi(\beta_j^{(i)})\}$ does not contain any mergings within $\bar{\mathbb{T}}^{(i)}$ (since otherwise $\psi$ can be proven to be reroutable, a contradiction).

Assume that $f_{i,1}, f_{i,2}, \ldots, f_{i,m_i}$, $1 \leq m_i \leq k-1$, are the largest mergings within $\bar{T}^{(i)}$, and they belong to paths $\psi_{j_{i,1}}, \psi_{j_{i,2}}, \ldots, \psi_{j_{i,m_i}}$, respectively. Now, we set

$$\mathbb{L}^{(i+1)} = \mathbb{R}^{(i)}|h(f_{i,1}), h(f_{i,2}), \ldots, h(f_{i,m_i}))$$

and define

$$\mathbb{T}^{(i)} = \bigcup_{j=1}^{m_i} \psi_{j_{i,j}}[h(e_{i-1,j_{i,j}}), h(f_{i,j})].$$

Here, let us note that $\psi_{j_{i,j}}[h(e_{i-1,j_{i,j}}), h(f_{i,j})]$ is the segment of $\psi_{j_{i,j}}$ that is within $\mathbb{R}^{(i)}$ and before $h(f_{i,j})$; or more formally,

$$\psi_{j_{i,j}}[h(e_{i-1,j_{i,j}}), h(f_{i,j})] = \psi_{j_{i,j}}[s_1, h(f_{i,j})] \cap \mathbb{R}^{(i)}.$$

Let $x_i$ and $y_i$ denote the numbers of $x$-terminal and $y$-terminal mergings in the $\phi$-paths in $\mathbb{L}^{(i)}$, respectively. Note that for any $f_{i,j}$, $j = 1, 2, \ldots, m_i$, the associated $\phi$-path, from $f_{i,j}$, may merge outside $\mathbb{T}^{(i)}$ next time; if this $\phi$-path merge within $\mathbb{T}^{(i)}$ again after a number of mergings outside $\mathbb{T}^{(i)}$, we call it an *excursive* $\phi$-path (with respect to $f_{i,j}$; see Figure 9 for an illustrative example). One checks that there are at most $m_i - 1$ excursive $\phi$-paths (since, otherwise, we can find a cycle in $G$, which is a contradiction). Note that for any merging within $\mathbb{T}^{(i)}$ other than $f_{i,j}$, $j = 1, 2, \ldots, m_i$, say $g$, the associated $\phi$-path, from $g$, can only merge within $\mathbb{T}^{(i)}$. Now, consider all $\phi$-paths that contains at least one merging within $\mathbb{L}^{(i+1)}$, the number of connected components of such $\phi$-paths is upper bounded by $y_{i+1} + m_i$ (restricted to $\mathbb{T}^{(i)}$, an excursive path can be split into multiple connected components). Then, by the induction assumptions,

$$|\mathbb{L}^{(i+1)} \cap \mathbb{T}^{(i)}|_{\mathcal{M}} \leq C_{m_i}(y_{i+1} + m_i) \leq C_{k-1} y_{i+1} + C_{k-1}(k-1).$$

One also checks that there exists at least one $\psi_{j_{i,j}}$, $j = 1, 2, \ldots, m_i$, which does not merge with any $\phi$-paths within $\mathbb{L}^{(i+1)} - \mathbb{T}^{(i)}$ (since otherwise we can find a cycle in $G$, which is a contradiction). Also, it is clear that all non-excursive $\phi$-paths that contain at least one merging within $\mathbb{L}^{(i+1)} - \mathbb{T}^{(i)}$ must have $x$-terminal mergings in $\mathbb{L}^{(i+1)}$, and the number of involved connected components of $\phi$-paths is at most $x_{i+1} + m_i - 1$. Thus, by the induction assumptions,

$$|\mathbb{L}^{(i+1)} - \mathbb{T}^{(i)}|_{\mathcal{M}} \leq C_{k-1}(x_{i+1} + m_i - 1) \leq C_{k-1} x_{i+1} + C_{k-1}(k-2).$$

It then immediately follows that

$$|\mathbb{L}^{(i+1)}|_{\mathcal{M}} \leq C_{k-1}(x_{i+1} + y_{i+1}) + C_{k-1}(2k-3).$$

Now, we claim that if $K$ is chosen such that $K \geq \mathcal{M}(k-1, k) + 1$, then necessarily $x_{i+1} + y_{i+1} \geq 1$. Indeed, let $z_i = \sum_{j=1}^{i}(x_i - y_i)$, that is, $z_i$ is the number of $\phi$-paths that will continue to merge within $\mathbb{R}^{(i)}$. Then if $z_i \geq k$, one verifies that at least one $\phi$-path merges within $\mathbb{L}^{(i+1)}$, however not within $\mathbb{R}^{(i+1)}$, which means $y_{i+1} \geq 1$; if $z_i \leq k-1$, then an $x$-terminal merging must exist within $\mathbb{L}^{(i+1)}$, which implies that $x_{i+1} \geq 1$.

**Recursive application.** We continue these operations in an iterative fashion to further obtain normal blocks and singular blocks until there are no mergings left in $G$. ∎

Suppose that upon the termination of the above procedure, $l_1$ singular blocks $\mathbb{L}_{j_1}, \mathbb{L}_{j_2}, \ldots, \mathbb{L}_{j_{l_1}}$ and $l_2$ normal blocks are found. Note that except the last normal one, each block has at least one $\phi$-terminal merging, which implies that

$$l_1 + l_2 \leq 2l + 1.$$

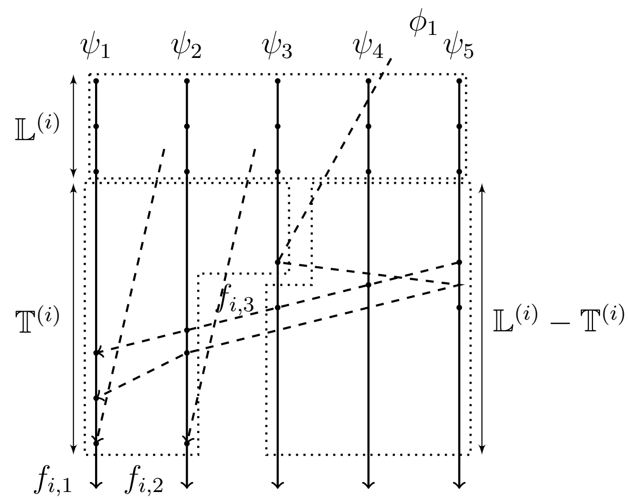


Figure 9: This diagram illustrates how to find a singular block, where mergings are represented by solid dots. Note that $\phi_1$ is an excursive path with respect to $f_{i,3}$.

In any case, we will have for some $C_k > 0$,

$$\begin{aligned}
|G|_{\mathcal{M}} &\leq Kl_2 + \sum_{i=1}^{l_1} [C_{k-1}(x_{j_i} + y_{j_i}) + C_{k-1}(2k-3)] \\
&\leq 2C_{k-1}l + (K - C_{k-1})l_2 + C_{k-1}(2k-3)l_1 \\
&\leq C_k l.
\end{aligned}$$

□

**Remark 4.7.** Theorem 4.6 partially confirms the following conjecture:

**Conjecture 4.8.** *There exists a positive constant $C$ such that for any $c_1, c_2$, we have*

$$\mathcal{M}(c_1, c_2) \leq Cc_1c_2.$$

Note that this conjecture, together with the easily verifiable fact that $\mathcal{M}(c, c) \geq c^2$, implies that $\mathcal{M}(c, c)$ is exactly of order $c^2$.

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