

Reflexivity of Partitions Induced by Weighted Poset Metric and Combinatorial Metric*

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Abstract—Let \mathbf{H} be the Cartesian product of a family of finite abelian groups. Via a polynomial approach, we give sufficient conditions for a partition of \mathbf{H} induced by weighted poset metric to be reflexive, which also become necessary for some special scenarios. Moreover, by examining the roots of the Krawtchouk polynomials, we give sufficient conditions for a partition of \mathbf{H} induced by combinatorial metric to be non-reflexive, and then give several examples of non-reflexive partitions.

When \mathbf{H} is a vector space over a finite field \mathbb{F} , we consider the property of admitting MacWilliams identity (PAMI) and the MacWilliams extension property (MEP) for partitions of \mathbf{H} . More specifically, under some invariance assumptions, we show that two partitions of \mathbf{H} admit MacWilliams identity if and only if they are mutually dual and reflexive, and any partition of \mathbf{H} satisfying MEP is in fact an orbit partition induced by some subgroup of $\text{Aut}_{\mathbb{F}}(\mathbf{H})$, which is necessarily reflexive. Furthermore, we show that the aforementioned non-reflexive partitions induced by combinatorial metric do not satisfy MEP, which further enables us to disprove a conjecture

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proposed by Pinheiro, Machado and Firer in [39].

1 Introduction

MacWilliams identities based on partitions of finite abelian groups have been established by Zinoviev and Ericson in [45], by Gluesing-Luerssen in [18], and by Gluesing-Luerssen and Ravagnani in [21]. These identities provide a general framework for recovering known or deriving new MacWilliams identities in more explicit forms. There are also many closely related identities such as MacWilliams identities based on association schemes and numerical weights; see Delsarte [10] and Ravagnani [40], respectively.

The notion of *reflexive partition* has been introduced by Gluesing-Luerssen in [18]. Roughly speaking, reflexive partitions are ones which coincide with their bi-duals, and alternatively, they can be characterized in terms of association schemes (see [10, 11, 46]). Reflexive partitions arise naturally from various weights and metrics in coding theory such as poset metric (see [5, 22, 35]), rank metric (see [17, 21]) and homogeneous weight (see [19]). We refer the reader to [18, 19, 21, 40, 44, 45, 46] for more results and examples. A MacWilliams identity based on a reflexive partition is invertible, and the inverse is essentially the MacWilliams identity based on the dual partition. As stated in [18, Section 2], reflexive partitions provide a symmetric situation and form the most appealing case.

In this paper, we study partitions induced by weighted poset metric and combinatorial metric, and examine when such partitions are reflexive or non-reflexive. Moreover, we study the relations among reflexivity, the property of admitting MacWilliams identity (PAMI) and the MacWilliams extension property (MEP), three widely explored properties in coding theory.

The notion of *weighted poset metric* has been introduced by Hyun, Kim and Park in [23], where the authors have classified all the weighted posets and directed graphs that admit the extended Hamming code $\tilde{\mathcal{H}}_3$ to be a 2-perfect code, and relevant results for more general $\tilde{\mathcal{H}}_k$, $k \geq 3$, have also been established. A weighted poset metric is determined by a poset and a weight function, both defined on the coordinate set (see [23] or Section 2.2 for more details). Weighted poset metric boils down to poset metric (see [5, 22, 35]) if the weight function is identically 1, and to weighted Hamming metric (see [2]) if the poset is an anti-chain. As has been stated in [23, Section 1], weighted poset metric can be viewed as an algebraic version of directed graph metric introduced by Etzion, Firer and Machado in [13]. More recently in [29], Machado and Firer have proposed and studied labeled-

poset-block metric, which, in our terminology, is also a weighted poset metric (see Section 2.2). Weighted poset metric can be useful to model some specific kind of channels for which the error probability depends on a codeword position, i.e., the distribution of errors is nonuniform, and can also be useful to perform bitwise or messagewise unequal error protection (see the abstract of [2] and [13, Section I, Paragraph 6]).

The notion of *combinatorial metric* has been introduced by Gabidulin in [15, 16]. A combinatorial metric is determined by a covering of the coordinate set (see Section 2.3 for more details). If the covering consists of singletons, then combinatorial metric boils down to Hamming metric. Several subclasses of combinatorial metric have been studied in the literature, such as block metric (see [14]), b -burst metric (see [4]) and translational metric (see [33]). In [3], Bossert and Sidorenko have derived a Singleton-type bound for combinatorial metric. In [39], Pinheiro, Machado and Firer have studied PAMI, the group of isometries and MEP for combinatorial metric. They have also proposed a conjecture in [39, Section 5] on MEP, which we will disprove in this paper. Our approach towards the conjecture is based on non-reflexive partitions induced by combinatorial metric and the relation between reflexivity and MEP.

PAMI has been first introduced by Kim and Oh in [24], where the authors have proved that being hierarchical is a necessary and sufficient condition for a poset to admit MacWilliams identity. The original property has since been extended and generalized to poset-block metric by Pinheiro and Firer in [38], to combinatorial metric by Pinheiro, Machado and Firer in [39], to directed graph metric by Etzion, Machado and Firer in [13], and to labeled-poset-block metric by Machado and Firer in [29]. In [8], Choi, Hyun, Kim and Oh have proposed and studied MacWilliams-type equivalence relations, which, roughly speaking, are defined as equivalence relations which admit MacWilliams identities, where such relations are defined on the ideal lattice of a given poset on the coordinate set.

MEP has become a topic of interest in coding theory since MacWilliams proved in [30] that a Hamming weight preserving map between two linear codes can be extended to the whole ambient space (see [6] for a different proof). Such a property has been extended, generalized and discussed extensively in the literature: with respect to other weights and metrics; with respect to codes over ring and module alphabets; and with respect to partitions of finite modules; see, among many others, [1], [12], [13], [17], [20], [21], [28], [29], [39], [42] and [43].

The remainder of the paper is organized as follows. In Section 2, we present some definitions, notations and basic facts on partitions of finite

abelian groups, weighted poset metric and combinatorial metric. Here we note that the conjecture proposed in [39] is stated in Section 2.3 as Conjecture 2.1. In Section 3, for a partition induced by weighted poset metric, we give sufficient conditions for two codewords to belong to the same member of its dual partition, and give a sufficient condition for its reflexivity. By relating each codeword with a polynomial, we show that such sufficient conditions are also necessary if the poset is hierarchical and the weight function is integer-valued. In Section 4, we consider partitions induced by combinatorial metric. After giving a sufficient yet not necessary condition for reflexivity in Section 4.1, we turn to a particular subclass of partitions related to Conjecture 2.1 in Section 4.2. Adopting the polynomial approach given in Section 3, we characterize the dual partitions of such partitions in terms of the classical Krawtchouk polynomials. Then, using the properties of Krawtchouk polynomials, especially those of their roots, we give sufficient conditions for non-reflexivity of such partitions, and derive several examples of non-reflexive partitions. In Section 5, we consider \mathbb{F} -invariant partitions of a finite vector space over a finite field \mathbb{F} , and study the relations among reflexivity, PAMI and MEP. More precisely, we prove that MEP is stronger than reflexivity, and reflexivity is equivalent to PAMI. Finally, as an application of these results, we show that the non-reflexive partitions given in Section 4.2 do not satisfy MEP, which further provides counter-examples to Conjecture 2.1.

In Figure 1 “Relationships among all the results”, we present the relationships among all the propositions and theorems established in the paper.

2 Preliminaries

Throughout the remainder of the paper, we let \mathbb{Z} , \mathbb{Z}^+ , \mathbb{R} , \mathbb{R}^+ and \mathbb{C} denote the set of all the integers, positive integers, real numbers, positive real numbers and complex numbers, respectively. Furthermore, we let $\mathbb{N} = \mathbb{Z}^+ \cup \{0\}$, $\mathbb{C}^* = \mathbb{C} - \{0\}$. For any $a, b \in \mathbb{Z}$, we use $[a, b]$ to denote the set of all the integers between a and b , i.e., $[a, b] = \{i \in \mathbb{Z} \mid a \leq i \leq b\}$.

Let E be a finite set. A *covering* of E is a collection of its subsets whose union is E , and a *partition* of E is a covering of E whose members are nonempty and disjoint. Consider a partition Γ of E . For any $u, v \in E$, we write $u \sim_\Gamma v$ if u and v belong to the same member of Γ . For any $D \subseteq E$, the Γ -*distribution* of D is defined as the sequence $(|D \cap B| \mid B \in \Gamma)$, and for any $D, L \subseteq E$, we write $D \approx_\Gamma L$ if D and L have the same Γ -distribution. For two partitions Γ, Ψ of E , we say that Γ is *finer than* Ψ , if for any $u, v \in E$,

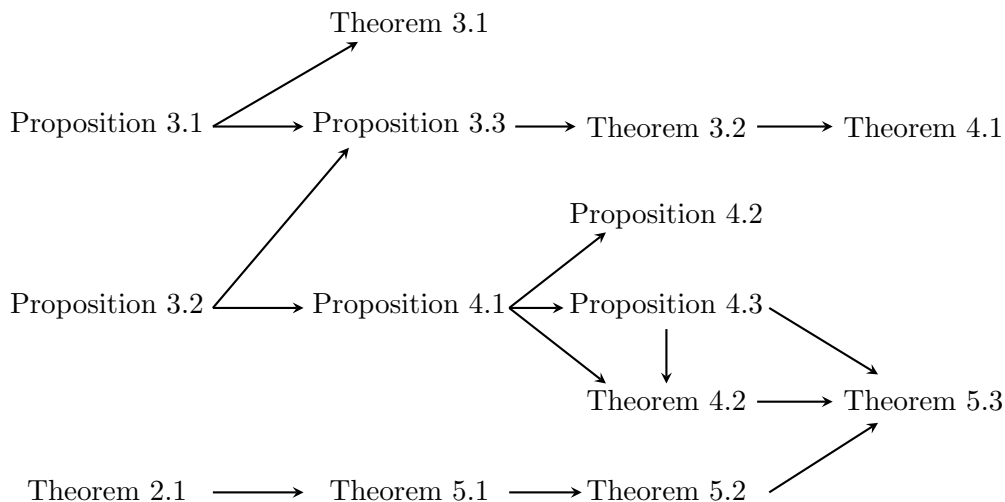


Figure 1: Relationships among all the results

$u \sim_{\Gamma} v$ implies $u \sim_{\Psi} v$. One can verify that Γ is finer than Ψ if and only if any member of Γ is contained in some member of Ψ .

2.1 Partitions of finite abelian groups

Let G and H be finite abelian groups, and let $f : G \times H \rightarrow \mathbb{C}^*$ be a *pairing*, i.e., for any $a, c \in G$ and $b, d \in H$, it holds that $f(ac, b) = f(a, b)f(c, b)$, $f(a, bd) = f(a, b)f(a, d)$ (see [34, Definition 11.7]). For any *additive codes* (i.e., subgroups) $C \leq G$ and $D \leq H$, define the codes $C^{\dagger} \leq H$ and ${}^{\dagger}D \leq G$ as

$$C^{\dagger} \triangleq \{b \in H \mid f(a, b) = 1 \text{ for all } a \in C\}, \quad (2.1)$$

$${}^{\dagger}D \triangleq \{a \in G \mid f(a, b) = 1 \text{ for all } b \in D\}. \quad (2.2)$$

In this paper, we always consider the case that f is *non-degenerate*, i.e., $G^{\dagger} = \{1_H\}$, ${}^{\dagger}H = \{1_G\}$ (see [34, Definition 11.7]). Note that the non-degenerate condition implies that $G \cong H$ as groups, and conversely, $G \cong H$ implies the existence of such a non-degenerate pairing (see [34, Lemma 11.8]).

For a partition Γ of H , the *left dual partition of Γ with respect to f* , denoted by $\mathbf{l}(\Gamma)$, is the partition of G such that for any $a, c \in G$, $a \sim_{\mathbf{l}(\Gamma)} c$ if and only if

$$\sum_{b \in B} f(a, b) = \sum_{b \in B} f(c, b) \text{ for all } B \in \Gamma. \quad (2.3)$$

For a partition Λ of G , the *right dual partition of Λ with respect to f* , denoted by $\mathbf{r}(\Lambda)$, is the partition of H such that for any $b, d \in H$, $b \sim_{\mathbf{r}(\Lambda)} d$ if and only if

$$\sum_{a \in A} f(a, b) = \sum_{a \in A} f(a, d) \text{ for all } A \in \Lambda. \quad (2.4)$$

Equations (2.3) and (2.4) are closely related to the notion of *dual partition* and *bi-dual partition* proposed in [18]. Indeed, by [34, Lemma 11.8], $\tau : G \rightarrow \text{Hom}(H, \mathbb{C}^*)$ defined as $\tau_{(a)}(b) = f(a, b)$ for all $b \in H$ is a group isomorphism, and consequently, for a partition Γ of H , we have

$$\{\tau[A] \mid A \in \mathbf{l}(\Gamma)\} = \widehat{\Gamma}, \quad \mathbf{r}(\mathbf{l}(\Gamma)) = \widehat{\widehat{\Gamma}},$$

where $\widehat{\Gamma}$ and $\widehat{\widehat{\Gamma}}$ are the *dual partition* and the *bi-dual partition* of Γ proposed in [18, Definition 2.1], respectively (cf. [1, Proposition 4.4], [7, Section 2.1], [19, Proposition 2.4]).

The following definition follows [18, Definition 2.1] and [46, Definition 2].

Definition 2.1. *Let Γ be a partition of H , and let Λ be a partition of G . If Λ is finer than $\mathbf{l}(\Gamma)$, then the left generalized Krawtchouk matrix of (Λ, Γ) with respect to f , denoted by $\rho : \Lambda \times \Gamma \rightarrow \mathbb{C}$, is defined as*

$$\rho(A, B) = \sum_{b \in B} f(a, b) \text{ for any chosen } a \in A. \quad (2.5)$$

If Γ is finer than $\mathbf{r}(\Lambda)$, then the right generalized Krawtchouk matrix of (Λ, Γ) with respect to f , denoted by $\varepsilon : \Lambda \times \Gamma \rightarrow \mathbb{C}$, is defined as

$$\varepsilon(A, B) = \sum_{c \in A} f(c, d) \text{ for any chosen } d \in B. \quad (2.6)$$

(Λ, Γ) is said to be mutually dual with respect to f if both Λ is finer than $\mathbf{l}(\Gamma)$ and Γ is finer than $\mathbf{r}(\Lambda)$. Finally, Γ is said to be reflexive if $\Gamma = \widehat{\widehat{\Gamma}}$.

The following lemma is a consequence of [18, Theorem 2.4].

Lemma 2.1. *Let Γ be a partition of H , and let Λ be a partition of G . Then, we have $\{1_G\} \in \mathbf{l}(\Gamma)$, $|\Gamma| \leq |\mathbf{l}(\Gamma)|$, $\mathbf{r}(\mathbf{l}(\Gamma))$ is finer than Γ . Moreover, it holds true that Γ is reflexive $\iff \Gamma = \mathbf{r}(\mathbf{l}(\Gamma)) \iff |\Gamma| = |\mathbf{l}(\Gamma)|$. In addition, the following four statements are equivalent to each other:*

- (1) (Λ, Γ) is mutually dual with respect to f ;
- (2) Γ is Fourier-reflexive and $\Lambda = \mathbf{l}(\Gamma)$;
- (3) Λ is Fourier-reflexive and $\Gamma = \mathbf{r}(\Lambda)$;
- (4) $|\Lambda| \leq |\Gamma|$ and Λ is finer than $\mathbf{l}(\Gamma)$.

For partitions Γ of H and Λ of G such that Λ is finer than $\mathbf{l}(\Gamma)$, let $\rho : \Lambda \times \Gamma \rightarrow \mathbb{C}$ be the left generalized Krawtchouk matrix of (Λ, Γ) with respect to f . It has been proven in [18, Theorem 2.7] that for an additive code $C \leq G$, the Λ -distribution of C determines the Γ -distribution of C^\ddagger via the following MacWilliams identity

$$\forall B \in \Gamma : |C||C^\ddagger \cap B| = \sum_{A \in \Lambda} |C \cap A| \cdot \rho(A, B). \quad (2.7)$$

Consequently, we have

$$C_1 \approx_\Lambda C_2 \implies C_1^\ddagger \approx_\Gamma C_2^\ddagger \text{ for any } C_1, C_2 \leq G. \quad (2.8)$$

The following theorem can be viewed as a partial converse of the fact that “ Λ is finer than $\mathbf{l}(\Gamma)$ implies (2.8)”.

Theorem 2.1. *Let \mathbf{S} be a collection of non-identity subgroups of G with the same cardinality, and let Δ be a partition of G such that $\{1_G\} \in \Delta$, and for any $A \in \Delta$ with $A \neq \{1_G\}$, there exists $C \in \mathbf{S}$ such that $C - \{1_G\} \subseteq A$. Let Γ be a partition of H such that Δ is finer than $\mathbf{l}(\Gamma)$, and let Λ be a partition of G such that $\{1_G\} \in \Lambda$, Δ is finer than Λ . Further assume that for any $C, M \in \mathbf{S}$, we have $C \approx_\Lambda M \implies C^\ddagger \approx_\Gamma M^\ddagger$. Then, Λ is finer than $\mathbf{l}(\Gamma)$.*

Proof. Letting $W \in \Lambda$ and $a, c \in W$, we will show that $a \sim_{\mathbf{l}(\Gamma)} c$, which immediately yields the desired result. Let $U, V \in \Delta$ such that $a \in U$, $c \in V$. Since Δ is finer than Λ , we have $U \subseteq W$, $V \subseteq W$. If $W = \{1_G\}$, then $a = c = 1_G$, as desired. Therefore we assume in the following that $W \neq \{1_G\}$. By $\{1_G\} \in \Lambda$, we have $1_G \notin W$, which further implies that $U \neq \{1_G\}$, $V \neq \{1_G\}$. Hence we can choose $C, M \in \mathbf{S}$ such that $C - \{1_G\} \subseteq U$, $M - \{1_G\} \subseteq V$. Here we note that $|C| = |M| \geq 2$. It is straightforward to verify that $C \cap W = C - \{1_G\}$, $M \cap W = M - \{1_G\}$, $|C \cap \{1_G\}| = |M \cap \{1_G\}| = 1$, and for any $A \in \Lambda$ such that $A \neq W$, $A \neq \{1_G\}$, it holds that $C \cap A = M \cap A = \emptyset$. Therefore we have $C \approx_\Lambda M$, and hence $C^\ddagger \approx_\Gamma M^\ddagger$. Consider an arbitrary $B \in \Gamma$. Noticing that Δ is finer than $\mathbf{l}(\Gamma)$, we apply (2.7) to $C \leq G$ and $M \leq G$ to deduce that

$$|C||C^\ddagger \cap B| = |B| + (|C| - 1) \left(\sum_{b \in B} f(a, b) \right),$$

$$|M||M^\ddagger \cap B| = |B| + (|M| - 1) \left(\sum_{b \in B} f(c, b) \right).$$

Since $C^\ddagger \approx_\Gamma M^\ddagger$, $B \in \Gamma$, we have $|C^\ddagger \cap B| = |M^\ddagger \cap B|$, which, along with $|C| = |M| \geq 2$, immediately implies that $\sum_{b \in B} f(a, b) = \sum_{b \in B} f(c, b)$. It then follows from the arbitrariness of B that $a \sim_{\mathcal{I}(\Gamma)} c$, as desired. \square

Remark 2.1. *Theorem 2.1 is largely inspired by [8, Corollary 3.2 and Theorem 3.3], and will be used in Section 5 to establish the equivalence between reflexivity and PAMI for \mathbb{F} -invariant partitions.*

2.2 Weighted poset metric

Throughout this subsection, we let Ω be a nonempty finite set, and let $\mathbf{P} = (\Omega, \preceq_{\mathbf{P}})$ be a poset. A subset $B \subseteq \Omega$ is said to be an *ideal* of \mathbf{P} if for any $v \in B$ and $u \in \Omega$, $u \preceq_{\mathbf{P}} v$ implies that $u \in B$. The set of all the ideals of \mathbf{P} is denoted by $\mathcal{I}(\mathbf{P})$. For $B \subseteq \Omega$, we let $\max_{\mathbf{P}}(B)$ (*resp.*, $\min_{\mathbf{P}}(B)$) denote the set of all the maximal (*resp.*, minimal) elements of B , and let $\langle B \rangle_{\mathbf{P}}$ denote the ideal $\{u \in \Omega \mid \exists v \in B \text{ s.t. } u \preceq_{\mathbf{P}} v\}$. In addition, B is said to be a *chain* in \mathbf{P} if for any $u, v \in B$, either $u \preceq_{\mathbf{P}} v$ or $v \preceq_{\mathbf{P}} u$ holds, and B is said to be an *anti-chain* in \mathbf{P} if for any $u, v \in B$, $u \preceq_{\mathbf{P}} v$ implies $u = v$. For any $u \in \Omega$, we let $\text{len}_{\mathbf{P}}(u)$ denote the largest cardinality of a chain in \mathbf{P} containing u as its greatest element. The set of all the order automorphisms of \mathbf{P} will be denoted by $\text{Aut}(\mathbf{P})$. The *dual poset* of \mathbf{P} is defined as $\bar{\mathbf{P}} = (\Omega, \preceq_{\bar{\mathbf{P}}})$, where

$$u \preceq_{\bar{\mathbf{P}}} v \iff v \preceq_{\mathbf{P}} u \text{ for all } (u, v) \in \Omega \times \Omega.$$

The following definition will be used frequently in our discussion.

Definition 2.2. (1) \mathbf{P} is said to be *hierarchical* if for any $u, v \in \Omega$ such that $\text{len}_{\mathbf{P}}(u) + 1 \leq \text{len}_{\mathbf{P}}(v)$, it holds that $u \preceq_{\mathbf{P}} v$.

(2) For $\omega : \Omega \rightarrow \mathbb{R}^+$, we say that (\mathbf{P}, ω) satisfies the *unique decomposition property (UDP)* if for any $I, J \in \mathcal{I}(\mathbf{P})$ such that $\sum_{i \in I} \omega(i) = \sum_{j \in J} \omega(j)$, there exists $\lambda \in \text{Aut}(\mathbf{P})$ such that $J = \lambda[I]$ and $\omega(i) = \omega(\lambda(i))$ for all $i \in \Omega$.

The following lemma is an immediate consequence of Definition 2.2 and the fact that $\mathcal{I}(\bar{\mathbf{P}}) = \{\Omega - I \mid I \in \mathcal{I}(\mathbf{P})\}$ (see [22, Lemma 1.2]).

Lemma 2.2. *Let $\omega : \Omega \rightarrow \mathbb{R}^+$. Then, (\mathbf{P}, ω) satisfies UDP if and only if $(\bar{\mathbf{P}}, \omega)$ satisfies UDP.*

Now we let $(H_i \mid i \in \Omega)$ be a family of finite abelian groups, and let $\mathbf{H} \triangleq \prod_{i \in \Omega} H_i$. For any *codeword* $\beta \in \mathbf{H}$, we let

$$\text{supp}(\beta) \triangleq \{i \in \Omega \mid \beta_i \neq 1_{H_i}\}. \quad (2.9)$$

Consider $\omega : \Omega \longrightarrow \mathbb{R}^+$. For any $\beta \in \mathbf{H}$, the (\mathbf{P}, ω) -weight of β is defined as

$$\text{wt}_{(\mathbf{P}, \omega)}(\beta) \triangleq \sum_{i \in \langle \text{supp}(\beta) \rangle_{\mathbf{P}}} \omega(i). \quad (2.10)$$

It has been proven in [23, Lemma I.2] that $d_{(\mathbf{P}, \omega)} : \mathbf{H} \times \mathbf{H} \longrightarrow \mathbb{R}$ defined as

$$d_{(\mathbf{P}, \omega)}(\alpha, \beta) = \text{wt}_{(\mathbf{P}, \omega)}(\alpha^{-1}\beta)$$

induces a metric on \mathbf{H} , which is henceforth referred to as weighted poset metric. We note that if \mathbf{H} is a vector space over a finite field and the weight function is integer-valued, then weighted poset metric coincides with labeled-poset-block metric proposed in [29, Section III].

Finally, we introduce partitions induced by weighted poset metric.

Notation 2.1. For $\omega : \Omega \longrightarrow \mathbb{R}^+$, we let $\mathcal{Q}(\mathbf{H}, \mathbf{P}, \omega)$ denote the partition of \mathbf{H} such that for any $\beta, \theta \in \mathbf{H}$,

$$\beta \sim_{\mathcal{Q}(\mathbf{H}, \mathbf{P}, \omega)} \theta \iff d_{(\mathbf{P}, \omega)}(1_{\mathbf{H}}, \beta) = d_{(\mathbf{P}, \omega)}(1_{\mathbf{H}}, \theta) \iff \text{wt}_{(\mathbf{P}, \omega)}(\beta) = \text{wt}_{(\mathbf{P}, \omega)}(\theta).$$

2.3 Combinatorial metric

Throughout this subsection, we let Ω be a nonempty finite set. For a covering T of Ω , define $\omega_T : 2^\Omega \longrightarrow \mathbb{N}$ as

$$\forall A \subseteq \Omega : \omega_T(A) = \min\{|S| \mid S \subseteq T, A \subseteq \bigcup_{I \in S} I\}. \quad (2.11)$$

For any $r \in \mathbb{N}$, we let $\mathcal{P}(r, \Omega)$ denote the set of all the subsets of Ω with cardinality r , i.e.,

$$\mathcal{P}(r, \Omega) = \{A \subseteq \Omega \mid |A| = r\}. \quad (2.12)$$

We collect some basic facts in the following lemma.

Lemma 2.3. (1) Let T be a covering of Ω , and let R denote the set of all the maximal elements of (T, \subseteq) . Then, R is a covering of Ω , (R, \subseteq) is an anti-chain, and it holds that $\omega_T = \omega_R$.

(2) Let T and R be coverings of Ω such that both (T, \subseteq) and (R, \subseteq) are anti-chains. Further assume that for any $A \subseteq \Omega$, $\omega_T(A) = 1 \iff \omega_R(A) = 1$. Then, it holds that $T = R$.

(3) Let $k \in [1, |\Omega|]$. Then, $\mathcal{P}(k, \Omega)$ is a covering of Ω , $(\mathcal{P}(k, \Omega), \subseteq)$ is an anti-chain, and for any $A \subseteq \Omega$, it holds that $\omega_{\mathcal{P}(k, \Omega)}(A) = \left\lceil \frac{|A|}{k} \right\rceil$.

Proof. We note that (1) and (2) have been stated in [39, Propositions 1 and 2]. Since all the proofs are straightforward, the details are omitted. \square

Now we let $(H_i \mid i \in \Omega)$ be a family of finite abelian groups, and let $\mathbf{H} \triangleq \prod_{i \in \Omega} H_i$. Consider a covering T of Ω . For any codeword $\beta \in \mathbf{H}$, the T -weight of β is defined as

$$\text{wt}_T(\beta) = \omega_T(\text{supp}(\beta)) = \min\{|S| \mid S \subseteq T, \text{supp}(\beta) \subseteq \bigcup_{I \in S} I\}. \quad (2.13)$$

It has been proven in [15] that $d_T : \mathbf{H} \times \mathbf{H} \rightarrow \mathbb{N}$ defined as

$$d_T(\alpha, \beta) = \text{wt}_T(\alpha^{-1}\beta)$$

induces a metric on \mathbf{H} , which is henceforth referred to as the T -combinatorial metric. Based on (1) of Lemma 2.3, one can assume that (T, \subseteq) is an anti-chain without loss of generality. As a special case, for any $k \in [1, |\Omega|]$, by (3) of Lemma 2.3 and (2.13), we have

$$\forall \beta \in \mathbf{H} : \text{wt}_{\mathcal{P}(k, \Omega)}(\beta) = \left\lceil \frac{|\text{supp}(\beta)|}{k} \right\rceil. \quad (2.14)$$

We remark that $\mathcal{P}(1, \Omega)$ -combinatorial metric is exactly Hamming metric.

Now we introduce partitions induced by combinatorial metric.

Notation 2.2. For a covering T of Ω , we let $\mathcal{CO}(\mathbf{H}, T)$ denote the partition of \mathbf{H} such that for any $\beta, \theta \in \mathbf{H}$,

$$\beta \sim_{\mathcal{CO}(\mathbf{H}, T)} \theta \iff d_T(\mathbf{1}_{\mathbf{H}}, \beta) = d_T(\mathbf{1}_{\mathbf{H}}, \theta) \iff \text{wt}_T(\beta) = \text{wt}_T(\theta).$$

Now we recall some relevant results by Pinheiro, Machado and Firer in [39] on MEP for combinatorial metric. Throughout the rest of this subsection, we set $H_i = \mathbb{F}_2$ for all $i \in \Omega$, and so

$$\mathbf{H} = \mathbb{F}_2^\Omega,$$

where \mathbb{F}_2 denotes the binary field. For a covering T of Ω where (T, \subseteq) is an anti-chain, we say that the T -combinatorial metric satisfies MEP, if for any additive code $C \leq \mathbf{H}$ and $f \in \text{Hom}(C, \mathbf{H})$ such that $\text{wt}_T(\alpha) = \text{wt}_T(f(\alpha))$ for all $\alpha \in C$, there exists $\varphi \in \text{Aut}(\mathbf{H})$ such that $\varphi|_C = f$ and

$$\text{wt}_T(\alpha) = \text{wt}_T(\varphi(\alpha)) \text{ for all } \alpha \in \mathbf{H}.$$

We note that MEP will be defined more generally from a partition perspective in Section 5 (see Definition 5.1).

Now let T be a covering of Ω such that (T, \subseteq) is an anti-chain, and let \sim be an equivalence relation on T defined as follows: for any $A, B \in T$, $A \sim B$ if and only if there exists $s \in \mathbb{Z}^+$ and $C_1, \dots, C_s \in T$ such that $C_1 = A$, $C_s = B$, and $C_i \cap C_{i+1} \neq \emptyset$ for all $i \in [1, s-1]$. Following [39, Definition 7], every equivalence class of (T, \sim) is referred to as a connected component of T . Moreover, T is said to be connected if T has exactly one connected component (which is necessarily T itself), and T is said to be disconnected if T is not connected.

When T is disconnected, a necessary and sufficient condition for the T -combinatorial metric to satisfy MEP has been established in the following theorem.

Theorem 2.2. ([39, Theorem 3]) *If T is disconnected, then the T -combinatorial metric satisfies MEP if and only if either $T = \mathcal{P}(1, \Omega)$ or $T = \{A, \Omega - A\}$ for some $A \subseteq \Omega$ with $|A| = |\Omega|/2$.*

When T is connected, a necessary condition for the T -combinatorial metric to satisfy MEP has been established in the following theorem.

Theorem 2.3. ([39, Theorem 4]) *If T is connected and the T -combinatorial metric satisfies MEP, then $T = \mathcal{P}(k, \Omega)$ for some $k \in [2, |\Omega|]$.*

By Theorems 2.2 and 2.3, to complete the characterization of the combinatorial metrics over \mathbb{F}_2^Ω that satisfy MEP, it suffices to examine the case that the covering T is equal to $\mathcal{P}(k, \Omega)$ for some $k \in [2, |\Omega|]$. Indeed, it has been conjectured in [39] that the converse of Theorem 2.3 also holds. More precisely, we state the following conjecture proposed in [39] using our notation.

Conjecture 2.1. *If T is connected, then the T -combinatorial metric satisfies MEP if and only if $T = \mathcal{P}(k, \Omega)$ for some $k \in [2, |\Omega|]$.*

Note that the “only if” part of Conjecture 2.1 follows from Theorem 2.3. We will prove in Section 5 that the “if” part of Conjecture 2.1 does not hold in general, i.e., the converse of Theorem 2.3 is not always true. Indeed, we will show that for a fixed $k \geq 3$, if $|\Omega|$ is sufficiently large, then the $\mathcal{P}(k, \Omega)$ -combinatorial metric does not satisfy MEP (see Theorem 5.3). Our approach is based on non-reflexivity of $\mathcal{CO}(\mathbf{H}, \mathcal{P}(k, \Omega))$ and the relation between reflexivity and MEP, as detailed in Section 4.2 and Section 5, respectively.

3 Partitions induced by weighted poset metric

Throughout this and the next section, we let Ω be a nonempty finite set, and let $(G_i \mid i \in \Omega)$ and $(H_i \mid i \in \Omega)$ be two families of finite abelian groups such that $G_i \cong H_i$ and $|H_i| \triangleq h_i$ for all $i \in \Omega$. Write

$$\mathbf{G} \triangleq \prod_{i \in \Omega} G_i, \quad \mathbf{H} \triangleq \prod_{i \in \Omega} H_i.$$

For any $i \in \Omega$, let $\pi_i : G_i \times H_i \rightarrow \mathbb{C}^*$ be a non-degenerate pairing. We define the non-degenerate pairing $f : \mathbf{G} \times \mathbf{H} \rightarrow \mathbb{C}^*$ as

$$\forall (\alpha, \beta) \in \mathbf{G} \times \mathbf{H} : f(\alpha, \beta) = \prod_{i \in \Omega} \pi_i(\alpha_i, \beta_i). \quad (3.1)$$

For any partition Γ of \mathbf{H} , we let $\mathbf{l}(\Gamma)$ denote the left dual partition of Γ with respect to f , as defined in (2.3). For any partition Δ of \mathbf{G} , we let $\mathbf{r}(\Delta)$ denote the right dual partition of Δ with respect to f , as defined in (2.4).

Throughout this section, we fix a poset $\mathbf{P} = (\Omega, \preccurlyeq_{\mathbf{P}})$. For any $D, I \subseteq \Omega$, define $\varphi(D, I) \in \mathbb{Z}$ and $\psi(D, I) \in \mathbb{Z}$ as follows:

$$\varphi(D, I) = \begin{cases} (-1)^{|I \cap D|} \left(\prod_{i \in I - \max_{\mathbf{P}}(I)} h_i \right) \left(\prod_{i \in \max_{\mathbf{P}}(I) - D} (h_i - 1) \right), & I \cap D \subseteq \max_{\mathbf{P}}(I); \\ 0, & I \cap D \not\subseteq \max_{\mathbf{P}}(I), \end{cases}$$

$$\psi(D, I) = \begin{cases} (-1)^{|I \cap D|} \left(\prod_{i \in D - \min_{\mathbf{P}}(D)} h_i \right) \left(\prod_{i \in \min_{\mathbf{P}}(D) - I} (h_i - 1) \right), & I \cap D \subseteq \min_{\mathbf{P}}(D); \\ 0, & I \cap D \not\subseteq \min_{\mathbf{P}}(D). \end{cases}$$

We also fix $\omega : \Omega \rightarrow \mathbb{R}^+$, and define $\varpi : 2^\Omega \rightarrow \mathbb{R}$ as $\varpi(I) = \sum_{i \in I} \omega(i)$. Moreover, we write $\Lambda = \mathbf{l}(\mathcal{Q}(\mathbf{H}, \mathbf{P}, \omega))$, $\Theta = \mathbf{r}(\mathcal{Q}(\mathbf{G}, \overline{\mathbf{P}}, \omega))$.

3.1 A sufficient condition for $\mathcal{Q}(\mathbf{H}, \mathbf{P}, \omega)$ to be reflexive

We begin by computing the left generalized Krawtchouk matrix of $(\Lambda, \mathcal{Q}(\mathbf{H}, \mathbf{P}, \omega))$ with respect to f . By [44, Proposition II.1], for any $\alpha \in \mathbf{G}$ and $I \in \mathcal{I}(\mathbf{P})$, we have

$$\sum_{(\beta \in \mathbf{H}, \langle \text{supp}(\beta) \rangle_{\mathbf{P}} = I)} f(\alpha, \beta) = \varphi(\langle \text{supp}(\alpha) \rangle_{\overline{\mathbf{P}}}, I). \quad (3.2)$$

Hence by (2.10), for any $\alpha \in \mathbf{G}$ and $b \in \mathbb{R}$, it holds that

$$\sum_{(\beta \in \mathbf{H}, \text{wt}_{(\mathbf{P}, \omega)}(\beta) = b)} f(\alpha, \beta) = \sum_{(I \in \mathcal{I}(\mathbf{P}), \varpi(I) = b)} \varphi(\langle \text{supp}(\alpha) \rangle_{\overline{\mathbf{P}}}, I). \quad (3.3)$$

The right generalized Krawtchouk matrix of $(\mathcal{Q}(\mathbf{G}, \overline{\mathbf{P}}, \omega), \Theta)$ with respect to f can be computed in a parallel fashion. More precisely, for any $\theta \in \mathbf{H}$ and $D \in \mathcal{I}(\overline{\mathbf{P}})$, we have

$$\sum_{(\gamma \in \mathbf{G}, \langle \text{supp}(\gamma) \rangle_{\overline{\mathbf{P}}} = D)} f(\gamma, \theta) = \psi(D, \langle \text{supp}(\theta) \rangle_{\mathbf{P}}). \quad (3.4)$$

Hence for any $\theta \in \mathbf{H}$ and $b \in \mathbb{R}$, it holds that

$$\sum_{(\gamma \in \mathbf{G}, \text{wt}_{(\overline{\mathbf{P}}, \omega)}(\gamma) = b)} f(\gamma, \theta) = \sum_{(D \in \mathcal{I}(\overline{\mathbf{P}}), \varpi(D) = b)} \psi(D, \langle \text{supp}(\theta) \rangle_{\mathbf{P}}). \quad (3.5)$$

Using (3.3) and (3.5), we give the following sufficient conditions for two codewords to belong to the same member of Λ or Θ .

Proposition 3.1. *Let $\lambda \in \text{Aut}(\mathbf{P})$ such that $h_i = h_{\lambda(i)}$, $\omega(i) = \omega(\lambda(i))$ for all $i \in \Omega$. Then, we have:*

- (1) *For $\alpha, \gamma \in \mathbf{G}$ with $\langle \text{supp}(\gamma) \rangle_{\overline{\mathbf{P}}} = \lambda[\langle \text{supp}(\alpha) \rangle_{\overline{\mathbf{P}}}]$, it holds that $\alpha \sim_{\Lambda} \gamma$;*
- (2) *For $\beta, \theta \in \mathbf{H}$ with $\langle \text{supp}(\theta) \rangle_{\mathbf{P}} = \lambda[\langle \text{supp}(\beta) \rangle_{\mathbf{P}}]$, it holds that $\beta \sim_{\Theta} \theta$.*

Proof. We only prove (1) since the proof of (2) is similar. Let $D = \langle \text{supp}(\alpha) \rangle_{\overline{\mathbf{P}}}$. Then, we have $\langle \text{supp}(\gamma) \rangle_{\overline{\mathbf{P}}} = \lambda[D]$. With the assumptions that $\lambda \in \text{Aut}(\mathbf{P})$ and $h_i = h_{\lambda(i)}$ for all $i \in \Omega$, one verifies that for any $I \subseteq \Omega$, it holds that $\varphi(D, I) = \varphi(\lambda[D], \lambda[I])$. Consider an arbitrary $b \in \mathbb{R}$. With the assumptions that $\lambda \in \text{Aut}(\mathbf{P})$ and $\omega(i) = \omega(\lambda(i))$ for all $i \in \Omega$, one verifies that for any $I \in \mathcal{I}(\mathbf{P})$ with $\varpi(I) = b$, it holds that $\lambda[I] \in \mathcal{I}(\mathbf{P})$, $\varpi(\lambda[I]) = b$. Now by (3.3), we have

$$\begin{aligned} \sum_{(\beta \in \mathbf{H}, \text{wt}_{(\mathbf{P}, \omega)}(\beta) = b)} f(\gamma, \beta) &= \sum_{(I \in \mathcal{I}(\mathbf{P}), \varpi(I) = b)} \varphi(\lambda[D], I) \\ &= \sum_{(I \in \mathcal{I}(\mathbf{P}), \varpi(I) = b)} \varphi(\lambda[D], \lambda[I]) \\ &= \sum_{(I \in \mathcal{I}(\mathbf{P}), \varpi(I) = b)} \varphi(D, I) \\ &= \sum_{(\beta \in \mathbf{H}, \text{wt}_{(\mathbf{P}, \omega)}(\beta) = b)} f(\alpha, \beta), \end{aligned}$$

which immediately implies the desired result. \square

Now we prove the main result of this subsection.

Theorem 3.1. *Assume that (\mathbf{P}, ω) satisfies UDP, and for any $u, v \in \Omega$ such that $\text{len}_{\mathbf{P}}(u) = \text{len}_{\mathbf{P}}(v)$ and $\omega(u) = \omega(v)$, it holds that $h_u = h_v$. Then, we have $\Lambda = \mathcal{Q}(\mathbf{G}, \bar{\mathbf{P}}, \omega)$, $\Theta = \mathcal{Q}(\mathbf{H}, \mathbf{P}, \omega)$, and both $\mathcal{Q}(\mathbf{G}, \bar{\mathbf{P}}, \omega)$ and $\mathcal{Q}(\mathbf{H}, \mathbf{P}, \omega)$ are reflexive.*

Proof. First, consider $\alpha, \gamma \in \mathbf{G}$ with $\text{wt}_{(\bar{\mathbf{P}}, \omega)}(\alpha) = \text{wt}_{(\bar{\mathbf{P}}, \omega)}(\gamma)$. By (2.10), we have $\varpi(\langle \text{supp}(\alpha) \rangle_{\bar{\mathbf{P}}}) = \varpi(\langle \text{supp}(\gamma) \rangle_{\bar{\mathbf{P}}})$. Since (\mathbf{P}, ω) satisfies UDP, by Lemma 2.2, we can choose $\lambda \in \text{Aut}(\mathbf{P})$ such that $\langle \text{supp}(\gamma) \rangle_{\bar{\mathbf{P}}} = \lambda[\langle \text{supp}(\alpha) \rangle_{\bar{\mathbf{P}}}]$ and $\omega(i) = \omega(\lambda(i))$ for all $i \in \Omega$. For any $i \in \Omega$, it follows from the fact $\lambda \in \text{Aut}(\mathbf{P})$ that $\text{len}_{\mathbf{P}}(i) = \text{len}_{\mathbf{P}}(\lambda(i))$, which, along with $\omega(i) = \omega(\lambda(i))$, implies that $h_i = h_{\lambda(i)}$. By Proposition 3.1, we have $\alpha \sim_{\Lambda} \gamma$. It follows that $\mathcal{Q}(\mathbf{G}, \bar{\mathbf{P}}, \omega)$ is finer than Λ . A similar discussion leads to the fact that $\mathcal{Q}(\mathbf{H}, \mathbf{P}, \omega)$ is finer than Θ . Therefore $(\mathcal{Q}(\mathbf{G}, \bar{\mathbf{P}}, \omega), \mathcal{Q}(\mathbf{H}, \mathbf{P}, \omega))$ is mutually dual with respect to f , which, along with Lemma 2.1, immediately implies the desired result. \square

3.2 The case that \mathbf{P} is hierarchical

Throughout this subsection, we assume that ω is integer-valued, i.e.,

$$\omega(i) \in \mathbb{Z}^+ \text{ for all } i \in \Omega. \quad (3.6)$$

We also let m be the largest cardinality of a chain in \mathbf{P} , and for any $j \in [1, m]$, let $W_j \triangleq \{u \in \Omega \mid \text{len}_{\mathbf{P}}(u) = j\}$. Moreover, for any $D \subseteq \Omega$, we let $\sigma(D)$ denote the largest integer $r \in [1, m]$ such that $D \subseteq \bigcup_{j=r}^m W_j$.

As a generalization of [44, Notation II.1], we can relate each $\alpha \in \mathbf{G}$ with a polynomial $F(\omega, \alpha)$ defined as

$$F(\omega, \alpha) \triangleq \sum_{l=0}^{\varpi(\Omega)} \sum_{(\beta \in \mathbf{H}, \text{wt}_{(\mathbf{P}, \omega)}(\beta)=l)} f(\alpha, \beta) x^l. \quad (3.7)$$

By the definition of Λ , we have

$$\forall \alpha, \gamma \in \mathbf{G} : \alpha \sim_{\Lambda} \gamma \iff F(\omega, \alpha) = F(\omega, \gamma). \quad (3.8)$$

In addition, we can derive a more explicit form of $F(\omega, \alpha)$, as detailed in the following proposition.

Proposition 3.2. (1) *Let $\alpha \in \mathbf{G}$, and write $D = \langle \text{supp}(\alpha) \rangle_{\bar{\mathbf{P}}}$, $X = (\Omega - D) \cup \min_{\mathbf{P}}(D)$. Then, we have*

$$F(\omega, \alpha) = \sum_{(I \in \mathcal{I}(\mathbf{P}), I \subseteq X)} (-1)^{|I \cap D|} \left(\prod_{i \in I - \max_{\mathbf{P}}(I)} h_i \right) \left(\prod_{i \in \max_{\mathbf{P}}(I) - D} (h_i - 1) \right) x^{\varpi(I)}.$$

In addition, if $h_i \geq 2$ for all $i \in \Omega$, then $\deg(F(\omega, \alpha)) = \varpi(X)$.

(2) Suppose that \mathbf{P} is hierarchical. Let $\alpha \in \mathbf{G}$, and write $D = \langle \text{supp}(\alpha) \rangle_{\overline{\mathbf{P}}}$, $r = \sigma(D)$. Then, we have

$$\begin{aligned} F(\omega, \alpha) &= \left(\prod_{i \in (\bigcup_{j=1}^{r-1} W_j)} h_i \cdot x^{\omega(i)} \right) \left(\prod_{i \in W_r \cap D} (1 - x^{\omega(i)}) \right) \left(\prod_{i \in W_r - D} ((h_i - 1)x^{\omega(i)} + 1) \right) \\ &\quad + \sum_{t=1}^{r-1} \left(\prod_{i \in (\bigcup_{j=1}^{t-1} W_j)} h_i \cdot x^{\omega(i)} \right) \left(\prod_{i \in W_t} ((h_i - 1)x^{\omega(i)} + 1) \right) \\ &\quad - \sum_{t=2}^r \left(\prod_{i \in (\bigcup_{j=1}^{t-1} W_j)} h_i \cdot x^{\omega(i)} \right). \end{aligned}$$

In addition, if $h_i \geq 2$ for all $i \in \Omega$, then $\deg(F(\omega, \alpha)) = \varpi(\bigcup_{j=1}^r W_j)$.

Proof. (1) For any $I \in \mathcal{I}(\mathbf{P})$, it is observed that

$$I \cap D \subseteq \max_{\mathbf{P}}(I) \iff I \subseteq X.$$

With this observation, the first part is a direct consequence of (3.3) and (3.7). Then, the second part follows from the first part and the fact that $X \in \mathcal{I}(\mathbf{P})$, as desired.

(2) Define $g : \{(t, V) \mid t \in [1, m], V \subseteq W_t, V \neq \emptyset\} \rightarrow 2^\Omega$ as

$$g(t, V) = \left(\bigcup_{j=1}^{t-1} W_j \right) \cup V.$$

From \mathbf{P} is hierarchical, we infer that g is injective and the range of g is equal to $\mathcal{I}(\mathbf{P}) - \{\emptyset\}$. Moreover, for any $t \in [1, m]$, $V \subseteq W_t$, $V \neq \emptyset$, we have $\max_{\mathbf{P}}(g(t, V)) = V$. The fact that \mathbf{P} is hierarchical, together with $\sigma(D) = r$, implies that $\min_{\mathbf{P}}(D) = W_r \cap D$, $(\Omega - D) \cup \min_{\mathbf{P}}(D) = \bigcup_{j=1}^r W_j$. By (1), we have

$$\begin{aligned} F(\omega, \alpha) - 1 &= \sum_{t=1}^r \sum_{(V \subseteq W_t, V \neq \emptyset)} (-1)^{|V \cap D|} \left(\prod_{i \in (\bigcup_{j=1}^{t-1} W_j)} h_i \right) \left(\prod_{i \in V - D} (h_i - 1) \right) x^{\varpi((\bigcup_{j=1}^{t-1} W_j) \cup V)} \\ &= \sum_{t=1}^r \left(\prod_{i \in (\bigcup_{j=1}^{t-1} W_j)} h_i \cdot x^{\omega(i)} \right) \left(\sum_{(V \subseteq W_t, V \neq \emptyset)} (-1)^{|V \cap D|} \left(\prod_{i \in V - D} (h_i - 1) \right) x^{\varpi(V)} \right) \\ &= \sum_{t=1}^r \left(\prod_{i \in (\bigcup_{j=1}^{t-1} W_j)} h_i \cdot x^{\omega(i)} \right) \left(\left(\prod_{i \in W_t \cap D} (1 - x^{\omega(i)}) \right) \left(\prod_{i \in W_t - D} ((h_i - 1)x^{\omega(i)} + 1) \right) - 1 \right). \end{aligned}$$

Since for any $t \in [1, r-1]$, it holds true that $W_t \cap D = \emptyset$, $W_t - D = W_t$, the first part immediately follows from the above computation. Moreover, the second part immediately follows from (1), as desired. \square

We are in a position to derive a necessary and sufficient condition for two codewords of \mathbf{G} to belong to the same member of Λ when \mathbf{P} is hierarchical.

Proposition 3.3. *Assume that $h_i \geq 2$ for all $i \in \Omega$ and \mathbf{P} is hierarchical. Let $\alpha, \gamma \in \mathbf{G}$, and write $D = \langle \text{supp}(\alpha) \rangle_{\overline{\mathbf{P}}}$, $B = \langle \text{supp}(\gamma) \rangle_{\overline{\mathbf{P}}}$. Then, $\alpha \sim_{\Lambda} \gamma$ if and only if there exists $\lambda \in \text{Aut}(\mathbf{P})$ such that $D = \lambda[B]$ and $h_i = h_{\lambda(i)}$, $\omega(i) = \omega(\lambda(i))$ for all $i \in \Omega$.*

Proof. Since the “if” part follows from Proposition 3.1, it remains to establish the “only if” part. Suppose that $\alpha \sim_{\Lambda} \gamma$, and write $r = \sigma(D)$, $s = \sigma(B)$. By (3.8), we have $F(\omega, \alpha) = F(\omega, \gamma)$. From Proposition 3.2, we deduce that $\varpi(\bigcup_{j=1}^r W_j) = \deg(F(\omega, \alpha)) = \deg(F(\omega, \gamma)) = \varpi(\bigcup_{j=1}^s W_j)$, which implies that $r = s$. This, together with Proposition 3.2, implies that

$$\left(\prod_{i \in W_r \cap D} (x^{\omega(i)} - 1) \right) \left(\prod_{i \in W_r - D} (x^{\omega(i)} + (h_i - 1)^{-1}) \right) = \left(\prod_{i \in W_r \cap B} (x^{\omega(i)} - 1) \right) \left(\prod_{i \in W_r - B} (x^{\omega(i)} + (h_i - 1)^{-1}) \right).$$

By Proposition A.1, which we state and prove in Appendix A, we can choose a bijection $\varepsilon : W_r \cap B \rightarrow W_r \cap D$ such that $h_i = h_{\varepsilon(i)}$, $\omega(i) = \omega(\varepsilon(i))$ for all $i \in W_r \cap B$. Now we can further choose a permutation ε_1 of W_r such that $\varepsilon_1|_{W_r \cap B} = \varepsilon$ and $h_i = h_{\varepsilon_1(i)}$, $\omega(i) = \omega(\varepsilon_1(i))$ for all $i \in W_r$. Define $\lambda : \Omega \rightarrow \Omega$ as $\lambda|_{W_r} = \varepsilon_1$ and $\lambda|_{\Omega - W_r} = \text{id}_{\Omega - W_r}$. Since \mathbf{P} is hierarchical, it is straightforward to verify that $\lambda \in \text{Aut}(\mathbf{P})$, $\lambda[B] = D$, and $h_i = h_{\lambda(i)}$, $\omega(i) = \omega(\lambda(i))$ for all $i \in \Omega$, as desired. \square

Now we give necessary and sufficient conditions for $\mathcal{Q}(\mathbf{H}, \mathbf{P}, \omega)$ to be reflexive when \mathbf{P} is hierarchical. The following is the main result of this subsection.

Theorem 3.2. *Assume that $h_i \geq 2$ for all $i \in \Omega$ and \mathbf{P} is hierarchical. Then, Λ is finer than $\mathcal{Q}(\mathbf{G}, \overline{\mathbf{P}}, \omega)$. Moreover, the following four statements are equivalent to each other:*

- (1) (\mathbf{P}, ω) satisfies UDP, and for any $u, v \in \Omega$ such that $\text{len}_{\mathbf{P}}(u) = \text{len}_{\mathbf{P}}(v)$ and $\omega(u) = \omega(v)$, it holds that $h_u = h_v$;
- (2) $(\mathcal{Q}(\mathbf{G}, \overline{\mathbf{P}}, \omega), \mathcal{Q}(\mathbf{H}, \mathbf{P}, \omega))$ is mutually dual with respect to f ;
- (3) $\mathcal{Q}(\mathbf{H}, \mathbf{P}, \omega)$ is reflexive;
- (4) $\Lambda = \mathcal{Q}(\mathbf{G}, \overline{\mathbf{P}}, \omega)$.

Proof. First of all, it follows from Proposition 3.3 that Λ is finer than $\mathcal{Q}(\mathbf{G}, \bar{\mathbf{P}}, \omega)$. Since $\mathcal{I}(\bar{\mathbf{P}}) = \{\Omega - I \mid I \in \mathcal{I}(\mathbf{P})\}$ and $h_i \geq 2$ for all $i \in \Omega$, we have $|\mathcal{Q}(\mathbf{G}, \bar{\mathbf{P}}, \omega)| = |\mathcal{Q}(\mathbf{H}, \mathbf{P}, \omega)|$. Now (1) \implies (2) follows from Theorem 3.1, and (2) \implies (3) follows from Lemma 2.1. Suppose that $\mathcal{Q}(\mathbf{H}, \mathbf{P}, \omega)$ is reflexive. Then, by Lemma 2.1, we have $|\Lambda| = |\mathcal{Q}(\mathbf{H}, \mathbf{P}, \omega)| = |\mathcal{Q}(\mathbf{G}, \bar{\mathbf{P}}, \omega)|$, which, along with the fact that Λ is finer than $\mathcal{Q}(\mathbf{G}, \bar{\mathbf{P}}, \omega)$, implies that $\Lambda = \mathcal{Q}(\mathbf{G}, \bar{\mathbf{P}}, \omega)$, which further establishes (3) \implies (4). Therefore it remains to prove (4) \implies (1).

(4) \implies (1) First, we let $D, B \in \mathcal{I}(\bar{\mathbf{P}})$ with $\varpi(D) = \varpi(B)$. Since $h_i \geq 2$ for all $i \in \Omega$, we can choose $\alpha, \gamma \in \mathbf{G}$ such that $\langle \text{supp}(\alpha) \rangle_{\bar{\mathbf{P}}} = D$, $\langle \text{supp}(\gamma) \rangle_{\bar{\mathbf{P}}} = B$. From $\varpi(D) = \varpi(B)$, we infer that $\text{wt}_{(\bar{\mathbf{P}}, \omega)}(\alpha) = \text{wt}_{(\bar{\mathbf{P}}, \omega)}(\gamma)$, which further implies that $\alpha \sim_{\Lambda} \gamma$. By Proposition 3.3, we can choose $\lambda \in \text{Aut}(\mathbf{P})$ such that $D = \lambda[B]$ and $h_i = h_{\lambda(i)}$, $\omega(i) = \omega(\lambda(i))$ for all $i \in \Omega$. It follows from Lemma 2.2 that (\mathbf{P}, ω) satisfies UDP. Next, we let $u, v \in \Omega$ such that $\text{len}_{\mathbf{P}}(u) = \text{len}_{\mathbf{P}}(v)$ and $\omega(u) = \omega(v)$. Consider $B_1 = \langle \{u\} \rangle_{\bar{\mathbf{P}}}$, $D_1 = \langle \{v\} \rangle_{\bar{\mathbf{P}}}$. Since \mathbf{P} is hierarchical, it is straightforward to verify that $B_1, D_1 \in \mathcal{I}(\bar{\mathbf{P}})$, $\varpi(B_1) = \varpi(D_1)$. Hence we can choose $\mu \in \text{Aut}(\mathbf{P})$ such that $D_1 = \mu[B_1]$ and $h_i = h_{\mu(i)}$ for all $i \in \Omega$. Since $\mu \in \text{Aut}(\mathbf{P})$, we have $v = \mu(u)$, which further implies that $h_u = h_v$, as desired. \square

Remark 3.1. *If ω is the constant 1 map, then Theorem 3.2 recovers [18, Theorem 5.5] and part of [18, Theorem 5.4].*

4 Partitions induced by combinatorial metric

Throughout this section, for a covering T of Ω , we define $\omega_T : 2^{\Omega} \rightarrow \mathbb{N}$ as in (2.11), and for any $\alpha \in \mathbf{G}$, $\beta \in \mathbf{H}$, we let $\text{wt}_T(\alpha) \triangleq \omega_T(\text{supp}(\alpha))$, $\text{wt}_T(\beta) \triangleq \omega_T(\text{supp}(\beta))$, as in (2.13). For any polynomial $g \in \mathbb{C}[x]$, we let $g_{[i]}$ denote the coefficient of x^i in g .

4.1 Sufficient yet not necessary conditions for reflexivity

In this subsection, we prove the following theorem.

Theorem 4.1. *Assume that $h_i \geq 2$ for all $i \in \Omega$. Let T be a covering of Ω such that (T, \subseteq) is an anti-chain. Then, the following three statements are equivalent to each other:*

- (1) $\mathcal{CO}(\mathbf{G}, T)$ is finer than $\mathbf{l}(\mathcal{CO}(\mathbf{H}, T))$;
- (2) $\mathcal{CO}(\mathbf{G}, T) = \mathbf{l}(\mathcal{CO}(\mathbf{H}, T))$;
- (3) T is a partition of Ω , and $\prod_{i \in U} h_i = \prod_{j \in V} h_j$ for all $U, V \in T$.

Proof. First of all, since $h_i \geq 2$ for all $i \in \Omega$, we have

$$|\mathcal{CO}(\mathbf{G}, T)| = |\mathcal{CO}(\mathbf{H}, T)| = |\{\omega_T(A) \mid A \subseteq \Omega\}|,$$

which, along with Lemma 2.1, implies that (1) \iff (2). Next, suppose that (2) holds true, and we will show that T is a partition of Ω . By way of contradiction, we suppose that T is not a partition of Ω . Since (T, \subseteq) is an anti-chain, we have $\emptyset \notin T$. Hence we can choose $A, B \in T$ such that $A \cap B \neq \emptyset$, $B \not\subseteq A$. Therefore we can further choose $u \in B - A$, $v \in A \cap B$. Apparently, we have $\omega_T(\{u\}) = \omega_T(\{u, v\}) = 1$. Since $h_i \geq 2$ for all $i \in \Omega$, we can choose $\alpha, \gamma \in \mathbf{G}$ such that $\text{supp}(\alpha) = \{u\}$, $\text{supp}(\gamma) = \{u, v\}$. Applying (3.2) to the anti-chain $(\Omega, =)$, we have

$$a \triangleq \sum_{(\beta \in \mathbf{H}, \text{wt}_T(\beta) \leq 1)} f(\alpha, \beta) = \sum_{(I \subseteq \Omega, \omega_T(I) \leq 1)} (-1)^{|I \cap \{u\}|} \left(\prod_{i \in I - \{u\}} (h_i - 1) \right),$$

$$b \triangleq \sum_{(\beta \in \mathbf{H}, \text{wt}_T(\beta) \leq 1)} f(\gamma, \beta) = \sum_{(I \subseteq \Omega, \omega_T(I) \leq 1)} (-1)^{|I \cap \{u, v\}|} \left(\prod_{i \in I - \{u, v\}} (h_i - 1) \right).$$

By $\text{wt}_T(\alpha) = \text{wt}_T(\gamma) = 1$ and (2), we have $\alpha \sim_{\mathcal{I}(\mathcal{CO}(\mathbf{H}, T))} \gamma$, which further implies that $a = b$. On the other hand, some straightforward computation yields that

$$a - b = \left(\sum_{(J \subseteq \Omega - \{u, v\}, \omega_T(J \cup \{v\}) \leq 1, \omega_T(J \cup \{u, v\}) \geq 2)} \prod_{i \in J} (h_i - 1) \right) h_v. \quad (4.1)$$

Noticing that $u \notin A$, we have $A - \{v\} \subseteq \Omega - \{u, v\}$. Since $v \in A$, $A \in T$, we have $(A - \{v\}) \cup \{v\} = A$, $\omega_T(A) = 1$. Again by $v \in A$, we have $(A - \{v\}) \cup \{u, v\} = A \cup \{u\}$. If $\omega_T(A \cup \{u\}) \leq 1$, then we can choose $C \in T$ such that $A \cup \{u\} \subseteq C$, which, along with $u \notin A$, further implies that $A \subsetneq C$, which is impossible since $A, C \in T$, (T, \subseteq) is an anti-chain. It then follows that $\omega_T(A \cup \{u\}) \geq 2$. By the above discussion and $h_i \geq 2$ for all $i \in \Omega$, (4.1) immediately implies that $a - b \geq 1$, a contradiction to $a = b$, as desired. Therefore we have shown that T is a partition of Ω .

Now we prove (2) \iff (3). By the discussion in the previous paragraph, we assume that T is a partition of Ω . Now let

$$\mathbf{G}_1 \triangleq \prod_{A \in T} \left(\prod_{i \in A} G_i \right), \quad \mathbf{H}_1 \triangleq \prod_{A \in T} \left(\prod_{i \in A} H_i \right).$$

For any $A \in T$, let $\varsigma_A : (\prod_{i \in A} G_i) \times (\prod_{i \in A} H_i) \longrightarrow \mathbb{C}^*$ denote the non-degenerate pairing defined as

$$\varsigma_A(\gamma, \theta) = \prod_{i \in A} \pi_i(\gamma_i, \theta_i).$$

Now let $f_1 : \mathbf{G}_1 \times \mathbf{H}_1 \longrightarrow \mathbb{C}^*$ denote the non-degenerate pairing defined as

$$f_1(\lambda, \mu) = \prod_{A \in T} \varsigma_A(\lambda_A, \mu_A),$$

and let $\tilde{\omega}$ denote the constant 1 map defined on T . Since T is a partition of Ω , some straightforward computation implies that $\mathcal{CO}(\mathbf{G}, T) = \mathcal{I}(\mathcal{CO}(\mathbf{H}, T))$ holds true if and only if $\mathcal{Q}(\mathbf{G}_1, (T, =), \tilde{\omega})$ is the left dual partition of $\mathcal{Q}(\mathbf{H}_1, (T, =), \tilde{\omega})$ with respect to f_1 . For any $A \in T$, by $A \neq \emptyset$ and $h_i \geq 2$ for all $i \in \Omega$, we have $|\prod_{i \in A} H_i| = \prod_{i \in A} h_i \geq 2$. It follows from an application of Theorem 3.2 to \mathbf{G}_1 , \mathbf{H}_1 , f_1 and $((T, =), \tilde{\omega})$ that $\mathcal{Q}(\mathbf{G}_1, (T, =), \tilde{\omega})$ is the left dual partition of $\mathcal{Q}(\mathbf{H}_1, (T, =), \tilde{\omega})$ with respect to f_1 if and only if $\prod_{i \in U} h_i = \prod_{j \in V} h_j$ for all $U, V \in T$, which completes the proof of (2) \iff (3). \square

We remark that by Lemma 2.1, each of (1)–(3) of Theorem 4.1 is a sufficient condition for $\mathcal{CO}(\mathbf{H}, T)$ to be reflexive. We will show in Section 4.2 that such sufficient conditions are not necessary.

4.2 Non-reflexive partitions of the form $\mathcal{CO}(\mathbf{H}, \mathcal{P}(k, \Omega))$

Throughout this subsection, we fix $q \in \mathbb{Z}^+$ with $q \geq 2$.

From now on, we will focus on $\mathcal{P}(k, \Omega)$ -combinatorial metric, where $k \in [1, |\Omega|]$. In addition, we will always consider the case that all the H_i 's have order q . Such an additional assumption will enable us to relate the partitions with the well known Krawtchouk polynomials, which we first recall in the following definition (see, e.g., [9, 25, 32]).

Definition 4.1. *For any $(n, k) \in \mathbb{N} \times \mathbb{N}$, define the Krawtchouk polynomial $\mathbf{KU}_{(n, k)}$ as*

$$\mathbf{KU}_{(n, k)} = \frac{(-1)^k}{k!} \sum_{t=0}^k \binom{k}{t} (q-1)^{k-t} \left(\prod_{i=0}^{t-1} (x-i) \right) \left(\prod_{i=0}^{k-t-1} (x-n+i) \right).$$

We collect all the properties of the Krawtchouk polynomials that we need in the following lemma.

Lemma 4.1. (1) Let $(n, k) \in \mathbb{N} \times \mathbb{N}$. Then, we have $\deg(\mathbf{KU}_{(n,k)}) = k$. Moreover, for any $s \in [0, n]$, it holds that

$$\mathbf{KU}_{(n,k)}(s) = \sum_{t=0}^k (-1)^t (q-1)^{k-t} \binom{s}{t} \binom{n-s}{k-t} = ((1-x)^s (1+(q-1)x)^{n-s})_{[k]}.$$

(2) Let $n \in \mathbb{Z}^+$, $k \in \mathbb{N}$. Then, for any $s \in [1, n]$, it holds that

$$\sum_{l=0}^k \mathbf{KU}_{(n,l)}(s) = \mathbf{KU}_{(n-1,k)}(s-1).$$

(3) Suppose that $q = 2$. Let $(n, k) \in \mathbb{N} \times \mathbb{N}$. Then, for any $s \in [0, n]$, it holds that

$$\mathbf{KU}_{(n,k)}(n-s) = (-1)^k \mathbf{KU}_{(n,k)}(s).$$

(4) Let $n \in \mathbb{Z}^+$, $k \in [1, n]$. Then, $\mathbf{KU}_{(n,k)}$ has k distinct roots in \mathbb{R} , all of which lie between 0 and n . Moreover, $\mathbf{KU}'_{(n,k)}$ has $k-1$ distinct roots in \mathbb{R} , all of which lie between the smallest root and the largest root of $\mathbf{KU}_{(n,k)}$.

(5) Fix $k \in \mathbb{Z}^+$. For any $n \in \mathbb{N}$ with $n \geq k$, let $u_{(n)}$ denote the smallest root of $\mathbf{KU}_{(n,k)}$. Then, the sequence $(u_{(n)}/n \mid n \in \mathbb{N}, n \geq k)$ converges to $(q-1)/q$.

Proof. We note that (1)–(3) and the first part of (4) are well known and can be found in [9, 25], and the second part of (4) follows from the first part of (4) and the fact that $\deg(\mathbf{KU}_{(n,k)}) = k$. Hence it remains to establish (5). Fix $k \in \mathbb{Z}^+$, and let $T = \{\lambda = (\lambda_0, \dots, \lambda_{k-1}) \in \mathbb{R}^k \mid \sum_{i=0}^{k-1} \lambda_i^2 = 1\}$. For any $n \in \mathbb{N}$ with $n \geq k$, define

$$c_{(n)} \triangleq \max \left\{ (q-2) \left(\sum_{i=0}^{k-1} i \lambda_i^2 \right) + 2\sqrt{q-1} \left(\sum_{i=0}^{k-2} \lambda_i \lambda_{i+1} \sqrt{(i+1)(n-i)} \right) \mid \lambda \in T \right\}.$$

By [25, Theorem 6.1], for any $n \in \mathbb{N}$ with $n \geq k$, we have

$$\frac{u_{(n)}}{n} = \frac{q-1}{q} - \frac{c_{(n)}}{qn},$$

and moreover, some straightforward computation yields that

$$-2(k-1)\sqrt{(q-1)(k-1)n} \leq c_{(n)} \leq (q-2)(k-1) + 2(k-1)\sqrt{(q-1)(k-1)n}.$$

It follows that the sequence $(c_{(n)}/qn \mid n \in \mathbb{N}, n \geq k)$ converges to 0, which immediately implies the desired result. \square

Next, we characterize $\mathcal{I}(\mathcal{CO}(\mathbf{H}, \mathcal{P}(k, \Omega)))$ in terms of the Krawtchouk polynomials, and give some sufficient conditions for $\mathcal{CO}(\mathbf{H}, \mathcal{P}(k, \Omega))$ to be non-reflexive in terms of the Krawtchouk polynomials, especially in terms of their roots.

Proposition 4.1. *Suppose that $h_i = q$ for all $i \in \Omega$. Fix $k \in [1, |\Omega|]$, and let $\Lambda = \mathcal{I}(\mathcal{CO}(\mathbf{H}, \mathcal{P}(k, \Omega)))$. Then, the following five statements hold true:*

- (1) $|\mathcal{CO}(\mathbf{H}, \mathcal{P}(k, \Omega))| = \lceil \frac{|\Omega|}{k} \rceil + 1$, $|\Lambda| \geq \lceil \frac{|\Omega|}{k} \rceil + 1$, $\{1_{\mathbf{G}}\} \in \Lambda$. Moreover, $\mathcal{CO}(\mathbf{H}, \mathcal{P}(k, \Omega))$ is non-reflexive if and only if $|\Lambda| \geq \frac{|\Omega|}{k} + 2$;
- (2) Let $\alpha, \gamma \in \mathbf{G} - \{1_{\mathbf{G}}\}$, and let $t = |\text{supp}(\alpha)|$, $r = |\text{supp}(\gamma)|$. Then, $\alpha \sim_{\Lambda} \gamma$ if and only if for any $s \in [1, |\Omega| - 1]$ with $k \mid s$, it holds that

$$\mathbf{KU}_{(|\Omega|-1, s)}(t-1) = \mathbf{KU}_{(|\Omega|-1, s)}(r-1);$$

- (3) Let $s \in [1, |\Omega| - 1]$ such that $k \mid s$. Then, it holds that

$$|\Lambda| \geq |\{\mathbf{KU}_{(|\Omega|-1, s)}(j) \mid j \in [0, |\Omega| - 1]\}| + 1.$$

Assume in addition that

$$|\{\mathbf{KU}_{(|\Omega|-1, s)}(j) \mid j \in [0, |\Omega| - 1]\}| - 1 \geq \frac{|\Omega|}{k}.$$

Then, $\mathcal{CO}(\mathbf{H}, \mathcal{P}(k, \Omega))$ is non-reflexive;

- (4) Let $s \in [1, |\Omega| - 1]$ such that $k \mid s$, and let u denote the smallest root of $\mathbf{KU}_{(|\Omega|-1, s)}$. Assume in addition that

$$\lfloor u \rfloor \geq \frac{|\Omega|}{k}.$$

Then, $\mathcal{CO}(\mathbf{H}, \mathcal{P}(k, \Omega))$ is non-reflexive;

- (5) Suppose that $|\Omega| \geq 3$. Let $s \in [2, |\Omega| - 1]$ such that $k \mid s$, and let w denote the smallest root of $\mathbf{KU}'_{(|\Omega|-1, s)}$. Assume in addition that

$$\lfloor w \rfloor \geq \frac{|\Omega|}{k}.$$

Then, $\mathcal{CO}(\mathbf{H}, \mathcal{P}(k, \Omega))$ is non-reflexive.

Proof. (1) By (2.14) and the fact that $h_i \geq 2$ for all $i \in \Omega$, we have

$$\begin{aligned} |\mathcal{CO}(\mathbf{H}, \mathcal{P}(k, \Omega))| &= \left| \left\{ \left\lceil \frac{|\text{supp}(\beta)|}{k} \right\rceil \mid \beta \in \mathbf{H} \right\} \right| = \left| \left\{ \left\lceil \frac{s}{k} \right\rceil \mid s \in [0, |\Omega|] \right\} \right| \\ &= \left| \left[0, \left\lceil \frac{|\Omega|}{k} \right\rceil \right] \right| = \left\lceil \frac{|\Omega|}{k} \right\rceil + 1, \end{aligned}$$

which, together with Lemma 2.1, can be used to complete the proof of (1).
(2) Applying Proposition 3.2 to the anti-chain $(\Omega, =)$ and the constant 1 map, we have

$$\sum_{l=0}^{|\Omega|} \sum_{(\beta \in \mathbf{H}, |\text{supp}(\beta)|=l)} f(\alpha, \beta) x^l = (1-x)^t (1+(q-1)x)^{|\Omega|-t},$$

which, along with (1) of Lemma 4.1, implies that for any $l \in \mathbb{N}$,

$$\sum_{(\beta \in \mathbf{H}, |\text{supp}(\beta)|=l)} f(\alpha, \beta) = ((1-x)^t (1+(q-1)x)^{|\Omega|-t})_{[l]} = \mathbf{KU}_{(|\Omega|, l)}(t). \quad (4.2)$$

For an arbitrary $b \in \mathbb{N}$, (2.14) implies that

$$\forall \beta \in \mathbf{H} : \text{wt}_{\mathcal{P}(k, \Omega)}(\beta) \leq b \iff |\text{supp}(\beta)| \leq bk,$$

which, in combination with (4.2) and (2) of Lemma 4.1, further implies that

$$\begin{aligned} \sum_{(\beta \in \mathbf{H}, \text{wt}_{\mathcal{P}(k, \Omega)}(\beta) \leq b)} f(\alpha, \beta) &= \sum_{(\beta \in \mathbf{H}, |\text{supp}(\beta)| \leq bk)} f(\alpha, \beta) \\ &= \sum_{l=0}^{bk} \sum_{(\beta \in \mathbf{H}, |\text{supp}(\beta)|=l)} f(\alpha, \beta) \\ &= \sum_{l=0}^{bk} \mathbf{KU}_{(|\Omega|, l)}(t) = \mathbf{KU}_{(|\Omega|-1, bk)}(t-1). \end{aligned}$$

A parallel argument for γ leads to the fact that

$$\forall b \in \mathbb{N} : \sum_{(\beta \in \mathbf{H}, \text{wt}_{\mathcal{P}(k, \Omega)}(\beta) \leq b)} f(\gamma, \beta) = \mathbf{KU}_{(|\Omega|-1, bk)}(r-1).$$

From the above discussion and the definition of Λ , we deduce that

$$\alpha \sim_{\Lambda} \gamma \iff (\forall b \in \mathbb{N} : \mathbf{KU}_{(|\Omega|-1, bk)}(t-1) = \mathbf{KU}_{(|\Omega|-1, bk)}(r-1)).$$

By (1) of Lemma 4.1, we have $\mathbf{KU}_{(|\Omega|-1, 0)}(t-1) = \mathbf{KU}_{(|\Omega|-1, 0)}(r-1) = 1$ and $\mathbf{KU}_{(|\Omega|-1, s)}(t-1) = \mathbf{KU}_{(|\Omega|-1, s)}(r-1) = 0$ for all $s \in \mathbb{N}$ with $s \geq |\Omega|$, which immediately implies the desired result.

(3) The first part follows from (2) and the fact that $\{1_{\mathbf{G}}\} \in \Lambda$, and the second part follows from (1) and the first part, as desired.

(4) By (4) of Lemma 4.1, we have

$$|\{\mathbf{KU}_{(|\Omega|-1,s)}(j) \mid j \in [0, |\Omega| - 1]\}| \geq \lfloor u \rfloor + 1,$$

which, together with (3), immediately implies the desired result.

(5) By (4) of Lemma 4.1, we have

$$|\{\mathbf{KU}_{(|\Omega|-1,s)}(j) \mid j \in [0, |\Omega| - 1]\}| \geq \lfloor w \rfloor + 1,$$

and hence the desired result again follows from (3). \square

As a first application of Proposition 4.1, we show that the sufficient conditions for reflexivity given in Theorem 4.1 are not necessary.

Proposition 4.2. *Suppose that $h_i = q = 2$ for all $i \in \Omega$. Then, the following two statements hold:*

(1) *Assume that $|\Omega| \geq 2$, and let $\Lambda = \mathcal{U}(\mathcal{CO}(\mathbf{H}, \mathcal{P}(|\Omega| - 1, \Omega)))$. Then, for any $\alpha, \gamma \in \mathbf{G} - \{1_{\mathbf{G}}\}$, it holds that*

$$\alpha \sim_{\Lambda} \gamma \iff |\text{supp}(\alpha)| \equiv |\text{supp}(\gamma)| \pmod{2}.$$

Moreover, $\mathcal{CO}(\mathbf{H}, \mathcal{P}(|\Omega| - 1, \Omega))$ is reflexive;

(2) *Assume that $|\Omega| \geq 2$, and let $\Lambda = \mathcal{U}(\mathcal{CO}(\mathbf{H}, \mathcal{P}(2, \Omega)))$. Then, for any $\alpha, \gamma \in \mathbf{G}$, it holds that*

$$\alpha \sim_{\Lambda} \gamma \iff (|\text{supp}(\alpha)| = |\text{supp}(\gamma)| \text{ or } |\text{supp}(\alpha)| + |\text{supp}(\gamma)| = |\Omega| + 1).$$

Moreover, $\mathcal{CO}(\mathbf{H}, \mathcal{P}(2, \Omega))$ is reflexive.

Proof. (1) Let $\alpha, \gamma \in \mathbf{G} - \{1_{\mathbf{G}}\}$, and let $t = |\text{supp}(\alpha)|$, $r = |\text{supp}(\gamma)|$. By (2) of Proposition 4.1 and (1) of Lemma 4.1, we have

$$\begin{aligned} \alpha \sim_{\Lambda} \gamma &\iff \mathbf{KU}_{(|\Omega|-1,|\Omega|-1)}(t-1) = \mathbf{KU}_{(|\Omega|-1,|\Omega|-1)}(r-1) \\ &\iff (-1)^{t-1} = (-1)^{r-1} \\ &\iff t \equiv r \pmod{2}, \end{aligned}$$

as desired. It then follows from (1) of Proposition 4.1 that

$$|\mathcal{CO}(\mathbf{H}, \mathcal{P}(|\Omega| - 1, \Omega))| = |\Lambda| = 3,$$

which implies that $\mathcal{CO}(\mathbf{H}, \mathcal{P}(|\Omega| - 1, \Omega))$ is reflexive, as desired.

(2) Let Δ denote the partition of \mathbf{G} such that for any $\alpha, \gamma \in \mathbf{G}$, $\alpha \sim_{\Delta} \gamma$ if and only if either $|\text{supp}(\alpha)| = |\text{supp}(\gamma)|$ or $|\text{supp}(\alpha)| + |\text{supp}(\gamma)| = |\Omega| + 1$

holds true. Let $\alpha, \gamma \in \mathbf{G}$ such that $\alpha \sim_{\Delta} \gamma$, and let $t = |\text{supp}(\alpha)|$, $r = |\text{supp}(\gamma)|$. We will show that $\alpha \sim_{\Lambda} \gamma$. If $t = r$, then $\alpha \sim_{\Lambda} \gamma$ follows from (2) of Proposition 4.1. Hence we assume in the following that $t + r = |\Omega| + 1$. By (3) of Lemma 4.1, we have

$$\mathbf{KU}_{(|\Omega|-1,s)}(t-1) = \mathbf{KU}_{(|\Omega|-1,s)}(r-1) \text{ for all } s \in [1, |\Omega| - 1] \text{ with } 2 \mid s,$$

which, along with (2) of Proposition 4.1, implies that $\alpha \sim_{\Lambda} \gamma$, as desired. It then follows that Δ is finer than Λ . Also noticing that

$$|\Delta| = |\mathcal{CO}(\mathbf{H}, \mathcal{P}(2, \Omega))| = \left\lceil \frac{|\Omega|}{2} \right\rceil + 1,$$

from Lemma 2.1, we deduce that $\Lambda = \Delta$ and $\mathcal{CO}(\mathbf{H}, \mathcal{P}(2, \Omega))$ is reflexive, as desired. \square

Remark 4.1. *If $|\Omega| \geq 3$, then neither $\mathcal{P}(|\Omega|-1, \Omega)$ nor $\mathcal{P}(2, \Omega)$ is a partition of Ω . Hence Proposition 4.2 gives sufficient conditions for reflexivity which are not covered by those presented in Theorem 4.1.*

Now we give some criterions for non-reflexivity of $\mathcal{CO}(\mathbf{H}, \mathcal{P}(k, \Omega))$.

Proposition 4.3. *Suppose that $h_i = q$ for all $i \in \Omega$. Then, the following four statements hold:*

(1) *Assume that $q \geq 3$, $|\Omega| \geq 3$. Fix $k \in [2, |\Omega| - 1]$ such that*

$$|\Omega| \equiv 1 \pmod{k},$$

and let $\Lambda = \mathcal{I}(\mathcal{CO}(\mathbf{H}, \mathcal{P}(k, \Omega)))$. Then, for any $\alpha, \gamma \in \mathbf{G}$, it holds that

$$\alpha \sim_{\Lambda} \gamma \iff |\text{supp}(\alpha)| = |\text{supp}(\gamma)|.$$

Consequently, $\mathcal{CO}(\mathbf{H}, \mathcal{P}(k, \Omega))$ is non-reflexive;

(2) *If $q \geq 3$, $|\Omega| \geq 4$, then $\mathcal{CO}(\mathbf{H}, \mathcal{P}(|\Omega| - 2, \Omega))$ is non-reflexive;*

(3) *Assume that $q = 2$, $|\Omega| \geq 5$. Then, for any $k \in [\lceil \frac{|\Omega|}{2} \rceil, |\Omega| - 2]$, $\mathcal{CO}(\mathbf{H}, \mathcal{P}(k, \Omega))$ is non-reflexive;*

(4) *Assume that $q = 2$, $|\Omega| \geq 7$. Then, for any $k \in [\lceil \frac{|\Omega|}{5} \rceil, |\Omega| - 2]$ such that $2 \nmid k$, $\mathcal{CO}(\mathbf{H}, \mathcal{P}(k, \Omega))$ is non-reflexive.*

Proof. (1) Let $\alpha, \gamma \in \mathbf{G}$, and let $t = |\text{supp}(\alpha)|$, $r = |\text{supp}(\gamma)|$. By (2) of Proposition 4.1, $t = r$ implies $\alpha \sim_{\Lambda} \gamma$. Now we suppose that $\alpha \sim_{\Lambda} \gamma$. If $1_{\mathbf{G}} \in \{\alpha, \gamma\}$, then by $\{1_{\mathbf{G}}\} \in \Lambda$, we have $\alpha = \gamma = 1_{\mathbf{G}}$, and hence

$t = r = 0$. Therefore we assume in the following that $\alpha \neq 1_{\mathbf{G}}$, $\gamma \neq 1_{\mathbf{G}}$. By $|\Omega| \equiv 1 \pmod{k}$ and (2) of Proposition 4.1, we have

$$\mathbf{KU}_{(|\Omega|-1, |\Omega|-1)}(t-1) = \mathbf{KU}_{(|\Omega|-1, |\Omega|-1)}(r-1),$$

which, along with (1) of Lemma 4.1, implies that

$$(-1)^{t-1}(q-1)^{|\Omega|-t} = (-1)^{r-1}(q-1)^{|\Omega|-r}.$$

Since $q \geq 3$, we have $|\Omega| - t = |\Omega| - r$, and hence $t = r$, as desired. It follows that Λ is the partition induced by Hamming weight, which, together with $|\Omega| \geq 3$, $k \geq 2$, implies that

$$|\Lambda| = |\Omega| + 1 > \frac{|\Omega|}{k} + 2.$$

Now the non-reflexivity of $\mathcal{CO}(\mathbf{H}, \mathcal{P}(k, \Omega))$ immediately follows from (1) of Proposition 4.1.

(2) Let $n = |\Omega|$. Since $q \geq 3$, $n \geq 4$, one can check that $\mathbf{KU}_{(n-1, n-2)}$ takes different values on $0, n-2, n-1$, which further implies that

$$|\{\mathbf{KU}_{(n-1, n-2)}(j) \mid j \in [0, n-1]\}| - 1 \geq 2 \geq \frac{n}{n-2}.$$

It then follows from (3) of Proposition 4.1 that $\mathcal{CO}(\mathbf{H}, \mathcal{P}(n-2, \Omega))$ is non-reflexive, as desired.

(3) and (4) Let $n = |\Omega|$. Suppose that $n \geq 5$, and fix $k \in [3, n-2]$. It follows from (1) of Lemma 4.1 and some straightforward computation that

$$\mathbf{KU}_{(n-1, k)}(0) = \binom{n-1}{k} \triangleq a, \quad \mathbf{KU}_{(n-1, k)}(1) = \binom{n-2}{k} - \binom{n-2}{k-1} \triangleq b,$$

$$\mathbf{KU}_{(n-1, k)}(2) = \binom{n-3}{k} - 2\binom{n-3}{k-1} + \binom{n-3}{k-2} \triangleq c,$$

$$\mathbf{KU}_{(n-1, k)}(3) = \binom{n-4}{k} - 3\binom{n-4}{k-1} + 3\binom{n-4}{k-2} - \binom{n-4}{k-3} \triangleq d.$$

It is straightforward to verify the following facts:

$$a > |b|, \quad a > |c|, \quad a > |d|, \quad (b = c \iff n = 2k), \quad (b = d \iff n = 2k + 1). \quad (4.3)$$

From (4.3), we infer that $|\{a, b, c, d\}| \geq 3$. If $k \in [\lceil \frac{n}{2} \rceil, n-2]$, then we have

$$|\{\mathbf{KU}_{(n-1, k)}(j) \mid j \in [0, n-1]\}| - 1 \geq 2 \geq \frac{n}{k},$$

which, along with (3) of Proposition 4.1, implies that $\mathcal{CO}(\mathbf{H}, \mathcal{P}(k, \Omega))$ is non-reflexive, which further establishes (3). Hence it remains to prove (4). From now on, we assume that $n \geq 7$, $k \in [\lceil \frac{n}{5} \rceil, n-2]$, $2 \nmid k$. It follows from (3) of Lemma 4.1 that $\mathbf{KU}_{(n-1,k)}(n-1) = -a$, $\mathbf{KU}_{(n-1,k)}(n-2) = -b$, $\mathbf{KU}_{(n-1,k)}(n-3) = -c$, $\mathbf{KU}_{(n-1,k)}(n-4) = -d$. From (4.3), we infer that $|\{\pm a, b, c, d\}| \geq 4$. Hence if $k \geq n/3$, then we have

$$|\{\mathbf{KU}_{(n-1,k)}(j) \mid j \in [0, n-1]\}| - 1 \geq 3 \geq \frac{n}{k},$$

and (4) follows from (3) of Proposition 4.1. Therefore we assume in the following that $k \leq (n-1)/3$. By straightforward computation, we have

$$b+c = 2 \binom{n-3}{k} - 2 \binom{n-3}{k-1}, \quad b-c = 2 \binom{n-3}{k-1} - 2 \binom{n-3}{k-2}.$$

Since $n \geq 7$, we have $k \leq (n-1)/3 \leq (n-3)/2$. It then follows that $b+c > 0$, $b-c > 0$, and hence $b > \pm c$. The above discussion yields that

$$a > b > \pm c > -b > -a. \quad (4.4)$$

From (4.4), we infer that $|\{\pm a, \pm b, \pm c\}| \geq 5$. Hence if $k \geq n/4$, then we have

$$|\{\mathbf{KU}_{(n-1,k)}(j) \mid j \in [0, n-1]\}| - 1 \geq 4 \geq \frac{n}{k},$$

and (4) follows from (3) of Proposition 4.1. Therefore we further assume in the following that $k \leq (n-1)/4$. Then, some straightforward computation yields that $c \neq 0$, which, along with (4.4), implies that $|\{\pm a, \pm b, \pm c\}| = 6$. It then follows from $k \geq n/5$ that

$$|\{\mathbf{KU}_{(n-1,k)}(j) \mid j \in [0, n-1]\}| - 1 \geq 5 \geq \frac{n}{k},$$

and hence an application of (3) of Proposition 4.1 completes the proof. \square

The following theorem is the main result of this subsection.

Theorem 4.2. *Let X be a finite abelian group with $|X| = q$. Then, the following four statements hold:*

- (1) *Fix $k \in \mathbb{Z}^+$ such that $k \geq 2$, $(k, q) \neq (2, 2)$. Then, there exists $m \in \mathbb{Z}^+$ such that for any $n \in \mathbb{Z}^+$ with $n \geq m$, $n \geq k+1$, the partition $\mathcal{CO}(X^n, \mathcal{P}(k, [1, n]))$ is non-reflexive;*
- (2) *If $q \geq 3$, then for any $n \in \mathbb{Z}^+$ with $n \geq 3$, $\mathcal{CO}(X^n, \mathcal{P}(2, [1, n]))$ is non-reflexive;*

(3) Let $n \in \mathbb{Z}^+$ such that one of the following three conditions holds:

3.1) $3 \mid n$ and

$$n \geq \frac{9(q-1) + \sqrt{48q^4 - 144q^3 + 189q^2 - 162q + 81}}{2(2q-3)^2} + 3;$$

3.2) $n \equiv 1 \pmod{3}$, $n \geq 4$, $q \geq 3$;

3.3) $n \equiv 2 \pmod{3}$ and

$$n \geq \frac{4q^2 + 3q - 9 + \sqrt{48q^4 - 72q^3 + 9q^2 - 54q + 81}}{2(2q-3)^2} + 3.$$

Then, it holds that $n \geq 4$ and $\mathcal{CO}(X^n, \mathcal{P}(3, [1, n]))$ is non-reflexive;

(4) If $q = 2$, then for any $n \in \mathbb{Z}^+$ with $n \geq 5$, $\mathcal{CO}(X^n, \mathcal{P}(3, [1, n]))$ is non-reflexive.

Proof. (1) For any $n \in \mathbb{N}$ with $n \geq k$, let $u_{(n)}$ denote the smallest root of $\mathbf{KU}_{(n,k)}$. By (5) of Lemma 4.1, the sequence $(\lfloor u_{(n-1)} \rfloor / n \mid n \in \mathbb{N}, n \geq k+1)$ converges to $(q-1)/q$. Since $k, q \geq 2$, $(k, q) \neq (2, 2)$, we have

$$\frac{q-1}{q} > \frac{1}{k}.$$

Hence we can choose $m \in \mathbb{Z}^+$ such that for any $n \in \mathbb{Z}^+$ with $n \geq m$, $n \geq k+1$, it holds that $\lfloor u_{(n-1)} \rfloor > n/k$. Now for any $n \in \mathbb{Z}^+$ such that $n \geq m$, $n \geq k+1$, an application of (4) of Proposition 4.1 to $[1, n]$ and X^n leads to the non-reflexivity of $\mathcal{CO}(X^n, \mathcal{P}(k, [1, n]))$, which further establishes (1).

(2) Suppose that $q \geq 3$, and fix $n \in \mathbb{Z}^+$ with $n \geq 3$. It follows from some straightforward computation that the only root of $\mathbf{KU}'_{(n-1,2)}$ is equal to

$$\frac{q-1}{q}n - \frac{3}{2} + \frac{2}{q} \triangleq v.$$

If $2 \nmid n$, then (2) follows from (1) of Proposition 4.3; if $2 \mid n$, $n \geq 5$, then along with $q \geq 3$, one can readily verify that $\lfloor v \rfloor \geq n/2$, and hence (2) follows from (5) of Proposition 4.1; and if $n = 4$, then by $q \geq 3$ and (2) of Proposition 4.3, the partition $\mathcal{CO}(X^4, \mathcal{P}(2, [1, 4]))$ is non-reflexive, which completes the proof of (2).

(3) Apparently, we have $n \geq 4$. Let w denote the smallest root of $\mathbf{KU}'_{(n-1,3)}$. Some straightforward computation yields that

$$w = \frac{q-1}{q}n - 2 + \frac{3}{q} - \frac{\sqrt{(q-1)(n-3) + \frac{1}{3}q^2}}{q}. \quad (4.5)$$

If either 3.1) or 3.3) holds true, then from (4.5) and some straightforward computation, we have $\lfloor w \rfloor \geq n/3$, and (3) follows from (5) of Proposition 4.1; and if 3.2) holds true, then (3) follows from (1) of Proposition 4.3, as desired.

(4) Suppose that $q = 2$. Let $n \in \mathbb{Z}^+$ with $n \geq 5$, and let w denote the smallest root of $\mathbf{KU}'_{(n-1,3)}$. If $n \in \{5, 6\}$, then (4) follows from (3) of Proposition 4.3; if $n \in [7, 15]$, then (4) follows from (4) of Proposition 4.3; if $n \geq 16$, $n \not\equiv 1 \pmod{3}$, then we note that n satisfies either 3.1) or 3.3), and hence (4) follows from (3); if $n \geq 20$, $n \equiv 1 \pmod{3}$, then by (4.5), we have $\lfloor w \rfloor \geq n/3$, and hence (4) follows from (5) of Proposition 4.1; and if $n \in \{16, 19\}$, then some straightforward computation yields that

$$|\{\mathbf{KU}_{(15,3)}(j) \mid j \in [0, 6]\}| = 7, \quad |\{\mathbf{KU}_{(18,3)}(j) \mid j \in [0, 7]\}| = 8,$$

and hence an application of (3) of Proposition 4.1 completes the proof. \square

Remark 4.2. *In Section 5, we will use Proposition 4.3 and Theorem 4.2 to provide counter-examples to Conjecture 2.1 (see Theorem 5.3).*

5 Reflexivity, PAMI and MEP

Throughout this section, we let \mathbb{F} be a finite field, Ω be a nonempty finite set, and $(k_i \mid i \in \Omega)$ be a family of positive integers. We consider the \mathbb{F} -vector space

$$\mathbf{H} \triangleq \prod_{i \in \Omega} \mathbb{F}^{k_i}.$$

Define the inner product $\langle \cdot, \cdot \rangle : \mathbf{H} \times \mathbf{H} \rightarrow \mathbb{F}$ as

$$\langle \alpha, \beta \rangle = \sum_{i \in \Omega} \sum_{t=1}^{k_i} \alpha_{i,t} \cdot \beta_{i,t},$$

where for $\alpha \in \mathbf{H}$ and $i \in \Omega$, $\alpha_{i,t}$ denote the t -th entry of $\alpha_i \in \mathbb{F}^{k_i}$. For any linear code (i.e., \mathbb{F} -subspace) $C \subseteq \mathbf{H}$, we let

$$C^\perp \triangleq \{\beta \in \mathbf{H} \mid \langle \alpha, \beta \rangle = 0 \text{ for all } \alpha \in C\}$$

denote the dual code of C .

We also fix a non-trivial additive character χ of \mathbb{F} , and define the non-degenerate pairing $f : \mathbf{H} \times \mathbf{H} \rightarrow \mathbb{C}^*$ as

$$f(\alpha, \beta) = \chi(\langle \alpha, \beta \rangle).$$

It is well known that for any linear code $C \subseteq \mathbf{H}$, we have

$$C^\perp = \{\beta \in \mathbf{H} \mid f(\alpha, \beta) = 1 \text{ for all } \alpha \in C\}. \quad (5.1)$$

For a partition Δ of \mathbf{H} , we let $\mathbf{l}(\Delta)$ denote the left dual partition of Δ with respect to f , and let $\text{inv}(\Delta)$ denote the following subgroup of $\text{Aut}_{\mathbb{F}}(\mathbf{H})$:

$$\text{inv}(\Delta) = \{\sigma \in \text{Aut}_{\mathbb{F}}(\mathbf{H}) \mid \beta \sim_{\Delta} \sigma(\beta) \text{ for all } \beta \in \mathbf{H}\}. \quad (5.2)$$

For a subgroup $K \leq \text{Aut}_{\mathbb{F}}(\mathbf{H})$, we let $\text{orb}(K)$ denote the *orbit partition* of K acting on \mathbf{H} , i.e., for any $\alpha, \beta \in \mathbf{H}$, $\alpha \sim_{\text{orb}(K)} \beta$ if and only if there exists $\sigma \in K$ with $\beta = \sigma(\alpha)$.

Definition 5.1. (1) Let Γ and Λ be partitions of \mathbf{H} . We say that (Λ, Γ) admits MacWilliams identity if for any linear codes $C_1, C_2 \subseteq \mathbf{H}$ such that $C_1 \approx_{\Lambda} C_2$, it holds that $C_1^\perp \approx_{\Gamma} C_2^\perp$.

(2) Let Δ be a partition of \mathbf{H} . We say that Δ satisfies the MacWilliams extension property (MEP) if for any linear code $C \subseteq \mathbf{H}$ and $g \in \text{Hom}_{\mathbb{F}}(C, \mathbf{H})$ such that g is injective and $\alpha \sim_{\Delta} g(\alpha)$ for all $\alpha \in C$, there exists $\varphi \in \text{inv}(\Delta)$ with $\varphi|_C = g$.

(3) A partition Δ of \mathbf{H} is said to be \mathbb{F} -invariant if for any $B \in \Delta$ and $c \in \mathbb{F} - \{0\}$, it holds that $B = \{c \cdot \beta \mid \beta \in B\}$.

We first examine the relations between reflexivity and PAMI. The following lemma is an immediate consequence of Lemma 2.1, (2.8) and (5.1).

Lemma 5.1. Let Γ and Λ be partitions of \mathbf{H} such that Λ is finer than $\mathbf{l}(\Gamma)$. Then, we have $\{0\} \in \Lambda$ and (Λ, Γ) admits MacWilliams identity. Furthermore, for a reflexive partition Δ of \mathbf{H} , we have $\{0\} \in \Delta$, and both $(\mathbf{l}(\Delta), \Delta)$ and $(\Delta, \mathbf{l}(\Delta))$ admit MacWilliams identity.

Now we improve Lemma 5.1 for \mathbb{F} -invariant partitions. We begin with some basic properties. By [21, Remark 1.4], for an \mathbb{F} -invariant partition Θ of \mathbf{H} , $\mathbf{l}(\Theta)$ is again \mathbb{F} -invariant and is independent of the choice of the non-trivial additive character χ , and the left generalized Krawtchouk matrix of $(\mathbf{l}(\Theta), \Theta)$ is independent of the choice of χ as well.

The following is our first main result of this section.

Theorem 5.1. Let Γ and Λ be \mathbb{F} -invariant partitions of \mathbf{H} . Then, the following three statements are equivalent to each other:

- (1) Λ is finer than $\mathbf{l}(\Gamma)$;
- (2) $\{0\} \in \Lambda$, and (Λ, Γ) admits MacWilliams identity;

(3) $\{0\} \in \Lambda$, and for any 1-dimensional linear codes $C_1, C_2 \subseteq \mathbf{H}$ such that $C_1 \approx_\Lambda C_2$, it holds that $C_1^\perp \approx_\Gamma C_2^\perp$.

Assume in addition that $|\Lambda| \leq |\Gamma|$. Then, (2) holds true if and only if Γ is reflexive and $\Lambda = \mathbf{l}(\Gamma)$, if and only if (Λ, Γ) is mutually dual with respect to f .

Proof. We note that (1) \implies (2) follows from Lemma 5.1 and (2) \implies (3) is trivial. Now we prove (3) \implies (1) with the help of Theorem 2.1. Let \mathbf{S} denote the set of all the 1-dimensional linear codes, and let

$$\Delta = \{C - \{0\} \mid C \in \mathbf{S}\} \cup \{\{0\}\}.$$

Apparently, \mathbf{S} is a collection of non-identity subgroups of \mathbf{H} with the same cardinality, Δ is a partition of \mathbf{H} containing $\{0\}$, and for any $A \in \Delta$ with $A \neq \{0\}$, there exists $C \in \mathbf{S}$ such that $C - \{0\} = A$. Since Γ is \mathbb{F} -invariant, $\mathbf{l}(\Gamma)$ is again \mathbb{F} -invariant and hence Δ is finer than $\mathbf{l}(\Gamma)$. Moreover, it follows from Λ is \mathbb{F} -invariant that Δ is finer than Λ . Now we apply Theorem 2.1 and (5.1) and reach the fact that Λ is finer than $\mathbf{l}(\Gamma)$, which further establishes (3) \implies (1). Finally, if $|\Lambda| \leq |\Gamma|$, then the rest is a direct consequence of Lemma 2.1 and the proven part (1) \iff (2). \square

Now we apply Theorem 5.1 to MacWilliams-type equivalence relations proposed in [8] as well as partitions induced by weighted poset metric and combinatorial metric, as detailed in the following three examples.

Example 5.1. (*Characterizing MacWilliams-type equivalence relations*) Fix a poset $\mathbf{P} = (\Omega, \preceq_{\mathbf{P}})$, and consider an equivalence relation E on $\mathcal{I}(\mathbf{P})$. Let Γ denote the partition of \mathbf{H} such that for any $\beta, \theta \in \mathbf{H}$,

$$\beta \sim_\Gamma \theta \iff (\langle \text{supp}(\beta) \rangle_{\mathbf{P}}, \langle \text{supp}(\theta) \rangle_{\mathbf{P}}) \in E,$$

and let Λ denote the partition of \mathbf{H} such that for any $\alpha, \gamma \in \mathbf{H}$,

$$\alpha \sim_\Lambda \gamma \iff (\Omega - \langle \text{supp}(\alpha) \rangle_{\overline{\mathbf{P}}}, \Omega - \langle \text{supp}(\gamma) \rangle_{\overline{\mathbf{P}}}) \in E.$$

Since Γ and Λ are \mathbb{F} -invariant partitions with $|\Lambda| = |\Gamma|$, Theorem 5.1 implies that (Λ, Γ) is mutually dual with respect to f if and only if (Λ, Γ) admits MacWilliams identity and $\{\Omega\}$ is an equivalence class of E , which recovers (i) \iff (ii) of [8, Theorem 3.3]. In addition, the general MacWilliams identity (2.7) along with (5.1) recovers (ii) \iff (iii) of [8, Theorem 3.3]. In [8], E is referred to as a MacWilliams-type equivalence relation if (Λ, Γ) admits MacWilliams identity.

Example 5.2. Let $\mathbf{P} = (\Omega, \preceq_{\mathbf{P}})$ be a poset, and fix $\omega : \Omega \rightarrow \mathbb{R}^+$. Since $\mathcal{Q}(\mathbf{H}, \overline{\mathbf{P}}, \omega)$ and $\mathcal{Q}(\mathbf{H}, \mathbf{P}, \omega)$ are \mathbb{F} -invariant partitions which have the same cardinality and contain $\{0\}$, Theorem 5.1 implies that $(\mathcal{Q}(\mathbf{H}, \overline{\mathbf{P}}, \omega), \mathcal{Q}(\mathbf{H}, \mathbf{P}, \omega))$ admits MacWilliams identity if and only if $\mathcal{Q}(\mathbf{H}, \overline{\mathbf{P}}, \omega) = \mathcal{I}(\mathcal{Q}(\mathbf{H}, \mathbf{P}, \omega))$. Further assume that \mathbf{P} is hierarchical and ω is integer-valued. Then, by Theorem 3.2, $(\mathcal{Q}(\mathbf{H}, \overline{\mathbf{P}}, \omega), \mathcal{Q}(\mathbf{H}, \mathbf{P}, \omega))$ admits MacWilliams identity if and only if $\mathcal{Q}(\mathbf{H}, \mathbf{P}, \omega)$ is reflexive, if and only if (\mathbf{P}, ω) satisfies UDP, and for any $u, v \in \Omega$ such that $\text{len}_{\mathbf{P}}(u) = \text{len}_{\mathbf{P}}(v)$ and $\omega(u) = \omega(v)$, it holds that $k_u = k_v$. The latter equivalence has also been established in [29, Theorem 7] for labeled-poset-block metric by using different methods.

Example 5.3. Let T be a covering of Ω such that (T, \subseteq) is an anti-chain. Since $\mathcal{CO}(\mathbf{H}, T)$ is an \mathbb{F} -invariant partition containing $\{0\}$, Theorems 4.1 and 5.1 imply that the following three statements are equivalent to each other:

- (1) $(\mathcal{CO}(\mathbf{H}, T), \mathcal{CO}(\mathbf{H}, T))$ admits MacWilliams identity;
- (2) $\mathcal{CO}(\mathbf{H}, T) = \mathcal{I}(\mathcal{CO}(\mathbf{H}, T))$;
- (3) T is a partition of Ω , and $\sum_{i \in U} k_i = \sum_{j \in V} k_j$ for all $U, V \in T$.

In addition, (1) \iff (3) recovers [39, Theorem 1] if $k_i = 1$ for all $i \in \Omega$.

Now we summarize the relations among reflexivity, PAMI and MEP in the following theorem.

Theorem 5.2. Let Γ be an \mathbb{F} -invariant partition of \mathbf{H} such that $\{0\} \in \Gamma$, and consider the following five statements:

- (1) Γ satisfies MEP;
- (2) $\Gamma = \text{orb}(\text{inv}(\Gamma))$;
- (3) Γ is reflexive;
- (4) $(\Gamma, \mathcal{I}(\Gamma))$ admits MacWilliams identity;
- (5) There exists an \mathbb{F} -invariant partition Λ of \mathbf{H} such that $\{0\} \in \Lambda$ and both (Λ, Γ) and (Γ, Λ) admit MacWilliams identity.

Then, it holds true that (1) \implies (2), (2) \implies (3) and (3) \iff (4) \iff (5).

Proof. We begin by noting that $\{0\} \in \mathcal{I}(\Gamma)$ and $\mathcal{I}(\Gamma)$ is \mathbb{F} -invariant. Now (2) \implies (3) follows from [18, Theorem 2.6] and (3) \implies ((4) \wedge (5)) follows from Lemma 5.1. Next, we prove (4) \implies (3) and (5) \implies (3). If (4) holds true, then Theorem 5.1 implies that Γ is finer than $\mathcal{I}(\mathcal{I}(\Gamma))$, which, along with Lemma 2.1, further implies that Γ is reflexive, as desired. If (5) holds true, then Theorem 5.1 implies that (Λ, Γ) is mutually dual with respect to f , which, along with Lemma 2.1, further implies that Γ is reflexive, as desired. Hence it remains to establish (1) \implies (2). From now on, we assume that Γ satisfies MEP. By definition, it can be readily verified that $\text{orb}(\text{inv}(\Gamma))$ is

finer than Γ . Next, we will show that Γ is finer than $\text{orb}(\text{inv}(\Gamma))$, which immediately implies that $\Gamma = \text{orb}(\text{inv}(\Gamma))$. Indeed, consider $\alpha, \beta \in \mathbf{H}$ with $\alpha \sim_{\Gamma} \beta$. If $\alpha = 0$, then by $\{0\} \in \Gamma$, we have $\beta = \alpha = 0$, and hence $\alpha \sim_{\text{orb}(\text{inv}(\Gamma))} \beta$. Therefore in what follows, we assume that $\alpha \neq 0$. Then, by $\{0\} \in \Gamma$, we have $\beta \neq 0$. Define $g \in \text{Hom}_{\mathbb{F}}(\mathbb{F} \cdot \alpha, \mathbf{H})$ as

$$g(\alpha) = \beta.$$

Apparently, g is injective. Moreover, it follows from Γ is \mathbb{F} -invariant that $\gamma \sim_{\Gamma} g(\gamma)$ for all $\gamma \in \mathbb{F} \cdot \alpha$. Since Γ satisfies MEP, we can choose $\varphi \in \text{inv}(\Gamma)$ such that $\varphi|_{\mathbb{F} \cdot \alpha} = g$. It then follows that $\varphi(\alpha) = g(\alpha) = \beta$, which immediately implies that $\alpha \sim_{\text{orb}(\text{inv}(\Gamma))} \beta$, as desired. By now, we have shown that $\Gamma = \text{orb}(\text{inv}(\Gamma))$, which further establishes (2). \square

Remark 5.1. *In Theorem 5.2, (2) \implies (1) does not hold in general. We refer the reader to [1, Example 2.9], [17, Example 1.12] and [21, Section 8] for counter-examples arising from partitions of matrix spaces and rank metric codes.*

Now we are in a position to disprove Conjecture 2.1. The following theorem immediately follows from Proposition 4.3, Theorem 4.2 and Theorem 5.2, where we collect all the counter-examples to Conjecture 2.1 that we have obtained.

Theorem 5.3. *Fix $k \in \mathbb{Z}^+$, $k \geq 2$, and consider the set*

$$Q \triangleq \{n \in \mathbb{Z}^+ \mid n \geq k + 1, \mathcal{CO}(\mathbb{F}^n, \mathcal{P}(k, [1, n])) \text{ satisfies MEP}\}.$$

Then, it holds that:

- (1) *If $(k, |\mathbb{F}|) \neq (2, 2)$, then Q is finite;*
- (2) *If $|\mathbb{F}| = 2$, $k = 3$, then $Q \subseteq \{4\}$;*
- (3) *If $|\mathbb{F}| = 2$, $k \geq 3$, then $Q \cap [k + 2, 2k] = \emptyset$;*
- (4) *If $|\mathbb{F}| = 2$, $2 \nmid k$, then $Q \cap [\max\{k + 2, 7\}, 5k] = \emptyset$;*
- (5) *If $|\mathbb{F}| \geq 3$, $k = 2$, then $Q = \emptyset$;*
- (6) *If $|\mathbb{F}| \geq 3$, then $k + 2 \notin Q$, $Q \cap \{ak + 1 \mid a \in \mathbb{Z}^+\} = \emptyset$.*

We remark that (1)–(4) of Theorem 5.3 disprove Conjecture 2.1. More specifically, it follows from (1) of Theorem 5.3 that for any given $k \geq 3$, there are only finitely many $n \geq k + 1$ such that the $\mathcal{P}(k, [1, n])$ -combinatorial metric over \mathbb{F}^n satisfies MEP. However, we have not found an explicit way to determine all such n 's; moreover, Theorem 5.3 does not consider MEP for the $\mathcal{P}(2, [1, n])$ -combinatorial metric over \mathbb{F}_2^n . **So, Theorem 5.3 leaves open the question of finding all the combinatorial metrics that satisfy MEP.**

6 Conclusion

We have given sufficient conditions for a partition induced by weighted poset metric to be reflexive, which also becomes necessary when the poset is hierarchical and the weight function is integer-valued. With the help of the Krawtchouk polynomials, we have given various sufficient conditions for a partition induced by combinatorial metric to be reflexive or non-reflexive, and have given several classes of reflexive or non-reflexive partitions induced by combinatorial metric. In particular, with some additional assumptions, we have shown that for a fixed $k \geq 2$ and any sufficiently large n , the partition induced by the $\mathcal{P}(k, [1, n])$ -combinatorial metric is non-reflexive. We have also studied the relations among reflexivity, PAMI and MEP, and have shown that for \mathbb{F} -invariant partitions of a finite vector space over a finite field \mathbb{F} , MEP implies reflexivity, and reflexivity is equivalent to PAMI. Finally, we have disproved a conjecture proposed by Pinheiro, Machado and Firer in [39] (Conjecture 2.1) by showing that for a fixed $k \geq 3$ and any sufficiently large n , the $\mathcal{P}(k, [1, n])$ -combinatorial metric does not satisfy MEP. **On the other hand, it remains open to classify all the combinatorial metrics that satisfy MEP.**

Appendix

A Proof of Proposition A.1

In this appendix, we prove the following Proposition A.1, which has been used in the proof of Proposition 3.3.

Proposition A.1. (1) *Let I, J be finite sets, $(n_i \mid i \in I)$, $(m_j \mid j \in J)$ be two families of positive integers, and $(a_i \mid i \in I)$, $(b_j \mid j \in J)$ be two families of positive real numbers. Assume that one of the following two equations holds:*

$$\prod_{i \in I} (x^{n_i} + a_i) = \prod_{j \in J} (x^{m_j} + b_j); \quad (\text{A.1})$$

$$\left(\prod_{i \in I} (x^{n_i} - a_i) \right) \left(\prod_{j \in J} (x^{m_j} + b_j) \right) = \left(\prod_{i \in I} (x^{n_i} + a_i) \right) \left(\prod_{j \in J} (x^{m_j} - b_j) \right). \quad (\text{A.2})$$

Then, there exists a bijection $\sigma : I \rightarrow J$ such that for any $i \in I$, $n_i = m_{\sigma(i)}$, $a_i = b_{\sigma(i)}$.

(2) Let Y be a finite set, $(n_i \mid i \in Y)$ be a family of positive integers, and $(a_i \mid i \in Y)$ be a family of positive real numbers. Fix $C, D \subseteq Y$ such that

$$\left(\prod_{i \in D} (x^{n_i} - 1) \right) \left(\prod_{i \in Y-D} (x^{n_i} + a_i) \right) = \left(\prod_{i \in C} (x^{n_i} - 1) \right) \left(\prod_{i \in Y-C} (x^{n_i} + a_i) \right).$$

Then, there exists a bijection $\sigma : C \rightarrow D$ such that for any $i \in C$, $n_i = n_{\sigma(i)}$, $a_i = a_{\sigma(i)}$.

Proof. (1) Without loss of generality, we assume that there exists $\gamma \in I$ such that for any $i \in I$ and $j \in J$, $n_\gamma \geq n_i$, $n_\gamma \geq m_j$. Let $\lambda \in \mathbb{C}$ be a $2n_\gamma$ -th primitive root, and set $c = (a_\gamma)^{1/n_\gamma}$. Then, $c\lambda$ is a root of $x^{n_\gamma} + a_\gamma$. For any $i \in I$, the facts that $n_\gamma \geq n_i$ and $a_i > 0$ imply that $c\lambda$ is not a root of $x^{n_i} - a_i$. Therefore either (A.1) or (A.2) implies that there exists $\theta \in J$ such that $c\lambda$ is a root of $x^{m_\theta} + b_\theta$. By $n_\gamma \geq m_\theta$ and $b_\theta > 0$, we have $m_\theta = n_\gamma$, $b_\theta = a_\gamma$. We then deduce that either (A.1) or (A.2) remains valid if we replace I and J by $I - \{\gamma\}$ and $J - \{\theta\}$, respectively. Applying an induction argument to $I - \{\gamma\}$ and $J - \{\theta\}$, and noticing that $n_\gamma = m_\theta$, $a_\gamma = b_\theta$, we conclude that there exists a bijection $\sigma : I \rightarrow J$ such that $n_i = m_{\sigma(i)}$, $a_i = b_{\sigma(i)}$ for all $i \in I$, as desired.

(2) Since for any $i, j \in Y$ with $a_i \neq 1$, $\gcd(x^{n_i} + a_i, x^{n_j} \pm 1) = 1$, we have the following two equations:

$$\left(\prod_{i \in D} (x^{n_i} - 1) \right) \left(\prod_{(i \in C, a_i=1)} (x^{n_i} + 1) \right) = \left(\prod_{i \in C} (x^{n_i} - 1) \right) \left(\prod_{(i \in D, a_i=1)} (x^{n_i} + 1) \right), \quad (\text{A.3})$$

$$\prod_{(i \in C, a_i \neq 1)} (x^{n_i} + a_i) = \prod_{(i \in D, a_i \neq 1)} (x^{n_i} + a_i). \quad (\text{A.4})$$

By (1) and (A.4), we can choose a bijection τ from $\{i \in C \mid a_i \neq 1\}$ to $\{i \in D \mid a_i \neq 1\}$ such that $n_i = n_{\tau(i)}$, $a_i = a_{\tau(i)}$ for all $i \in C$ with $a_i \neq 1$. It follows that

$$\prod_{(i \in C, a_i \neq 1)} (x^{n_i} - 1) = \prod_{(i \in D, a_i \neq 1)} (x^{n_i} - 1),$$

which, together with (A.3), further implies that

$$\left(\prod_{(i \in D, a_i=1)} (x^{n_i} - 1) \right) \left(\prod_{(i \in C, a_i=1)} (x^{n_i} + 1) \right) = \left(\prod_{(i \in C, a_i=1)} (x^{n_i} - 1) \right) \left(\prod_{(i \in D, a_i=1)} (x^{n_i} + 1) \right).$$

By (1), we can choose a bijection η from $\{i \in C \mid a_i = 1\}$ to $\{i \in D \mid a_i = 1\}$ such that $n_i = n_{\eta(i)}$ for all $i \in C$ with $a_i = 1$. Define $\varepsilon : C \rightarrow D$ as $\varepsilon \upharpoonright_{\{i \in C \mid a_i \neq 1\}} = \tau$ and $\varepsilon \upharpoonright_{\{i \in C \mid a_i = 1\}} = \eta$. It follows that $\varepsilon : C \rightarrow D$ is a bijection such that $n_i = n_{\varepsilon(i)}$, $a_i = a_{\varepsilon(i)}$ for all $i \in C$, as desired. \square

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