A Region Decomposition Approach to Sum-Networks with Three Sources

Wentu Song, Kai Cai, Guangyue Han, Chau Yuen, Rongquan Feng and Kui Cai

Abstract

In this paper, we investigate the solvability of a sum-network with 3 sources and n terminals, or simply, a 3s/nt sum-network. More specifically, employing the region decomposition approach proposed in [9], we give necessary and sufficient conditions for the solvability of a so-called "terminal-separable" 3s/nt sum-network, which naturally yield sufficient conditions for that of a general 3s/nt sum-network. Based on this, we further give necessary and sufficient conditions for the solvability of a 3s/3t sum-network in terms of some forbidden structures, which we show can be determined using an $O(|E|)$ time algorithm, and thereby improving upon a previous algorithm [6] with time complexity $O(|E|^3)$.

Index Terms

Sum-network, linear network coding, region decomposition, function computation.

I. Introduction

A k-source n-terminal (ks/nt) sum-network is a finite, directed and acyclic graph G with k sources $\{s_1, \cdots, s_k\}$ and n terminals $\{t_1, \cdots, t_n\}$ where, at any time slot, each source s_i generates a message X_i from a finite field $\mathbb{F}$ and each terminal is tasked to compute the sum $\sum_{i=1}^{k} X_i$ using a linear network coding scheme [1]-[2]. As is typical in the literature, we assume that all sources generate independent messages and that each link in the network is free of errors and delays and can only carry one message for each use.

The aforementioned problem of communicating the sum over a network, which falls into the category of *distributed function computation* [18]-[20], has attracted increasing attention in recent years due to its potential applications in parallel processing, distributed data analysis and sensor networks [21]-[23]. More specifically, the sum-network problem has been investigated from several aspects including solvability [3]-[6], encoding fields [11]-[13], code constructions and capacity bounds [14]-[17].

The major concern of the present paper is to characterize the solvability of a ks/nt sum-network in terms of its topology, that is to say, to identify the network topology under which the desired sum can be successfully reconstructed by all terminals. In this direction, a first paper by A. Ramamoorthy [3] showed that a ks/2t or 2s/nt sum-network is solvable if and only if each terminal is reachable from all sources. Following this work, in [4]-[5], the authors considered a 3s/3t sum-network and showed that if each source-terminal pair is connected by two edge-disjoint paths, then the sum-network is solvable. To date, the best known result is a list of necessary and sufficient conditions for the solvability of a 3s/3t sum-network from Shenvi and Dey [6], which can be stated in terms of a collection of six "connection conditions" and have led to an $O(|E|^3)$ time algorithm.

The hardness of the above-mentioned problem can somehow be explained by a result in [7], where the authors established the equivalence between the solvability of a sum-network to that of the corresponding multiple-unicast network, and furthermore the problem of finding common

Wentu Song, Chau Yuen and Kui Cai are with Singapore University of Technology and Design, Singapore (e-mails: {wentu_song, yuenchau, cai_kui}@sutd.edu.sg).

Kai Cai and Guangyue Han are with Department of Mathematics, The University of Hong Kong. (e-mails: {kcai,ghan}@hku.hk).

Rongquan Feng is with the LMAM, School of Mathematical Sciences, Peking University, China (e-mail: fengrq@math.pku.edu.cn).

roots of a given set of polynomials. It is well-known that the latter two problems are extremely hard, which suggests the intricacy of the ks/nt sum-network solvability problem for generic k, n.

In this paper, employing the *region decomposition approach* proposed in [9], we give necessary and sufficient conditions (see Theorem 5.12) for the solvability of a class of so-called 3s/nt "terminal-separable" sum-networks (see Definition 4.4), which naturally lead to sufficient conditions for that of a general 3s/nt sum-network. Based on this, we further give necessary and sufficient conditions for the solvability of a 3s/3t sum-network in terms of some forbidden structures (see Theorem 6.1), which we show can be determined using an $O(|E|)$ time algorithm, and thereby improving upon a previous algorithm [6] with time complexity $O(|E|^3)$.

The region decomposition approach used in this paper is rooted in [8]. Roughly speaking, the idea of this approach is that instead of the original network, one can consider the corresponding simpler "region network", which can be generated in $O(|E|)$ time and has the same solvability as the original network. This method has been successfully applied to some difficult multiple source network coding problems [9]-[10].

It has been noted before that in this paper the topology of an unsolvable 3s/3t sum-network will be characterized in terms of forbidden structures. Network topology characterization via forbidden structures has a long history and can be arguably traced back to Kuratowski's Theorem [24], which asserts that a graph is planar if and only if it does not contain a subgraph homeomorphic to the complete graph K_5 or the complete bipartite graph $K_{3,3}$. A widespread adoption of this approach has been seen in graph theory, which notably includes the celebrated *Graph Structure Theorem* [25]. It is not surprising that the essence of this approach has been found applicable in the theory of network coding; for example, a similar characterization has been used to determine the solvability of a single rate 2-pair network [26].

The rest of the paper is organized as follows. We will introduce our notations and network model in Section II and recall the region decomposition approach in Section III. In Section IV, we will introduce the notion of terminal-separable sum-network, whose solvability will be characterized in Section V. Then, characterizations of solvable 3s/3t sum-networks using the forbidden structures will be presented in Section VI. Finally, the paper is concluded in Section VII.

II. Notations and Network Model

Throughout the paper, the ks/nt sum-network under consideration will be denoted by $G = (V, E)$, where V is the vertex set and E is the link set. For any link $e = (u, v) \in E$, u is called the *tail* of e, denoted by $u = \text{tail}(e)$; and v is called the *head* of e, denoted by $v = \text{head}(e)$; and e is said to be an *incoming link* of v and meanwhile an *outgoing link* of u. For two links $e, e' \in E$, we call e' an *incoming link* of e (or equivalently, e an *outgoing link* of e') if $\text{tail}(e) = \text{head}(e')$. For any $e \in E$, we will denote by $\text{In}(e)$ the set of all incoming links of e.

For ease of presentation, our analysis is actually done on an augmented version of the original network G, which, other than all the vertices and links in G, also includes a newly added link $(\hat{s}_i, s_i)$, henceforth termed as the i-th *source link*, to each s_i, and a newly added link $(t_j, \hat{t}_j)$, henceforth termed as the j-th *terminal link*, from each terminal j; see Fig. 2 (a) for an example. Evidently the original network communication problem on G naturally translates to one on its augmented version where the original message X_i is generated at the i-th source link, and the desirable message $\sum_{i=1}^{k} X_i$ is needed at each terminal link. For this reason, with a slight abuse of notation, we will still use G to denote its augmented version throughout the paper. Here we remark that a source link in G has no incoming links, i.e., $\text{In}(e) = \emptyset$ for any source link

e; moreover, w.l.o.g., we assume each non-source link e in G has at least one incoming link, namely, $\text{In}(e) \neq \emptyset$.

Let $\mathbb{F}^k$ denote the k-dimensional vector space over the finite field $\mathbb{F}$. For any subset $A \subseteq \mathbb{F}^k$, let $\langle A \rangle$ denote the subspace of $\mathbb{F}^k$ spanned by A. For any $i \in [k] \triangleq \{1, 2, \cdots, k\}$, let α_i denote the vector from $\mathbb{F}^k$ whose i-th component is 1 and other components are all 0. And let

$$\bar{\alpha} \triangleq \sum_{i=1}^{k} \alpha_i = (1, 1, \cdots, 1). \tag{1}$$

Under a linear network coding scheme, the message M_e transmitted along any link e is a linear combination of the messages X_i, taking the form of $M_e = \sum_{i=1}^{k} c_i X_i$ where $c_i \in \mathbb{F}$ and $(c_1, c_2, \ldots, c_k) \neq \mathbf{0}$, the zero vector in $\mathbb{F}^k$; in particular, $M_e = X_i$ if e is the i-th source link. We will use the coefficient vector $d_e = (c_1, c_2, \cdots, c_k)$, the *global encoding vector* on e, to represent the message M_e. Under this representation, a *linear network code* C on G is a collection of vectors $\{d_e \in \mathbb{F}^k; e \in V\}$ such that 1) $d_e = \alpha_i$ if e is the i-th source link; and 2) $d_e \in \langle d_{e'}; e' \in \text{In}(e) \rangle$ if e is a non-source link. We say that the linear network code C *solves* G, or equivalently, C is a *solution* to G, if $d_e = \bar{\alpha}$ for any terminal link e, and we say that G is *solvable* if there is a linear network code with respect to some finite field $\mathbb{F}$ solving G, and *unsolvable* otherwise.

Throughout the paper, we will assume the following (since otherwise G is obviously unsolvable):

Assumption 2.1: For any source s_i and any sink t_j, there exists a directed path from s_i to t_j.

III. The Region Decomposition Approach

This section recalls the region decomposition approach first proposed in [9]. Here, we remark that although we only examine ks/nt sum-networks, the approach actually applies to any general directed acyclic network.

A. Basic Region Decomposition and Basic Region Graph

Consider Algorithm 1 as in Fig. 1 applied to G, which yields a collection of non-empty subsets of E, termed as the *basic region decomposition* of G and denoted by $D = \{R_1, R_2, \cdots, R_K\}$. As elaborated in [9], D satisfies the following properties:

(1) $D = \{R_1, R_2, \cdots, R_K\}$ is a *partition* of E, that is, $R_1, R_2, \cdots, R_K$ are pairwise disjoint and $\bigcup_{j=1}^{K} R_j = E$ (each R_j will be referred to as a *region* of G);
(2) For each $R_j \in D$, there exists a link $e_j \in R_j$ such that $\text{In}(e) \subseteq R_j$ for any $e \in R_j \backslash \{e_j\}$ (e_j will be referred to as the *leader* of R_j, denoted by $e_j = \text{lead}(R_j)$);

Some remarks about the above decomposition are in order. It follows from the acyclicity of G that the links in G can be labelled as $e_1, e_2, e_3, \cdots, e_{|E|}$ such that 1) for each $i \in [k]$, e_i is the i-th source link; and 2) if e_j is an incoming link of e_ℓ, then $j < \ell$. The output of Algorithm 1 is determined by G and thereby the basic region decomposition uniquely exists. Moreover, it can be easily seen that each region has a unique leader and Algorithm 1 runs in time $O(|E|)$ since it only makes $|\text{In}(e_j)|$ comparisons for each e_j, $j \geq k + 1$.

Obviously a region R of G can contain at most one source link, which is necessarily the leader of R. A subtle point is that in general a region may contain more than one terminal links. Since, as elaborated below, all links in the same region will "share" a common message under a linear network coding scheme, we assume that in this paper any region in G can only contain at most one terminal link. A region is called a *source region* if it contains a source link, a *terminal*

Algorithm 1: Region Decomposing ($G = (V, E)$)
$j \leftarrow$ from 1 to k
  $R_j = \{e_j\}$;
$K = k$;
$j = k + 1$;
while $j \leq |E|$ **do**
  if there is a $k \in \{1, \cdots, K\}$ such that $\text{In}(e_j) \subseteq R_k$ **then**
    $R_k = R_k \cup \{e_j\}$;
  else
    $K = K + 1$;
    $R_K = \{e_j\}$;
  end if
  $j = j + 1$;
end while
Return $D = \{R_1, \cdots, R_K\}$;

Fig. 1. The algorithm for computing the basic region decomposition, where the links are labelled as $e_1, e_2, e_3, \cdots, e_{|E|}$ such that 1) for each $i \in [k]$, e_i is the i-th source link; and 2) if e_j is an incoming link of e_ℓ, then $j < \ell$.

region if a terminal link. Therefore, under the above-mentioned assumption, there are exactly k source regions and exactly n terminal regions of G.

We now define the basic region graph of G, a fundamental notion in this paper.

Definition 3.1 (Basic Region Graph [9]): Let $D = \{R_1, R_2, \cdots, R_K\}$ be the basic region decomposition of G. The *basic region graph* of G, denoted by RG(D), is a directed, simple graph with vertex set D and link set $\mathcal{E}_D$, where and $\mathcal{E}_D$ is the set of all ordered pairs (R_j, R_ℓ) such that $R_j \cap \text{In}(\text{lead}(R_\ell)) \neq \emptyset$. We say R_j is a *parent* of R_ℓ (or, R_ℓ is a *child* of R_j), if $(R_j, R_\ell) \in \mathcal{E}_D$. For each region $R_\ell \in D$, we use $\text{In}(R_\ell)$ to denote the set of all the parents of R_ℓ.

For the sake of convenience, from now on, we use S_i to denote the i-th source region, the one that contains the i-th source link, and T_j the j-th terminal region, the one that contains the j-th terminal link.

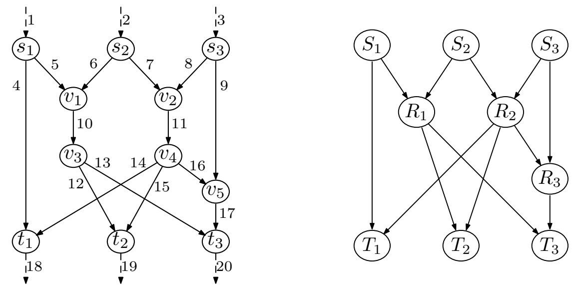
Example 3.2: Consider the network G in Fig. 2 (a). $D = \{S_1, S_2, S_3, R_1, R_2, R_3, T_1, T_2, T_3\}$ is the basic region decomposition of G, where $S_1 = \{1, 4, 5\}$, $S_2 = \{2, 6, 7\}$, $S_3 = \{3, 8, 9\}$, $R_1 = \{10, 12, 13\}$, $R_2 = \{11, 14, 15, 16\}$, $R_3 = \{17\}$, $T_1 = \{18\}$, $T_2 = \{19\}$ and $T_3 = \{20\}$. And Fig. 2(b) shows the basic region graph of G.

We note that the uniqueness of RG(D) trivially follows from that of D, and moreover, Algorithm 1 actually implicitly yields the RG(D) as a by-product. These observations immediately lead to the following theorem.

Theorem 3.3: RG(D) uniquely exists and it can be constructed in time $O(|E|)$.

We make the following two observations of RG(D): 1) It can be easily verified that RG(D) inherits acyclicity from G; 2) $|\text{In}(R_j)| \geq 2$ for any non-source region $R_j \in D$. To see this, note that if R_j is a non-source region, then $e_j = \text{lead}(R_j)$ is a non-source link, by Algorithm 1, $\text{In}(e_j)$ intersects with at least two regions in D. Hence, by Definition 3.1, R_j has at least two parents, that is, $|\text{In}(R_j)| \geq 2$.

We can define network code $\tilde{C}$ on the region graph RG(D) as a collection of vectors $\{d_R \in \mathbb{F}^k; R \in D\}$ satisfying 1) $d_{S_i} = \alpha_i$ for any i; and 2) $d_R \in \langle d_{R'}; R' \in \text{In}(R)\rangle$ for any non-source



(a) (b)

Fig 2. (a) shows the augmented version of G, where the links $1, 2, 3$ are the source links, and $18, 19, 20$ are the terminal links. (b) shows the basic region graph $\text{RG}(D)$ of G, where $S_1 = \{1, 4, 5\}, S_2 = \{2, 6, 7\}, S_3 = \{3, 8, 9\}, R_1 = \{10, 12, 13\}, R_2 = \{11, 14, 15, 16\}, R_3 = \{17\}, T_1 = \{18\}, T_2 = \{19\}, T_3 = \{20\}$ and $D = \{S_1, S_2, S_3, R_1, R_2, R_3, T_1, T_2, T_3\}$.

region R. We say that $\tilde{C}$ *solves* $\text{RG}(D)$, or equivalently, $\tilde{C}$ is a *solution* to $\text{RG}(D)$, if $d_{T_j} = \bar{\alpha}$ for any j, and that $\text{RG}(D)$ is *solvable* if there is a linear network code solving $\text{RG}(D)$, and *unsolvable* otherwise.

Note that given any linear network code $C = \{d_e \in \mathbb{F}^k; e \in E\}$ of G, it is easy to verify that for any region $R \in D$ and any $e \in R$, $d_e \in \langle d_{\text{lead}(R)} \rangle$, which immediately means that, up to a scalar multiplication, all d_e with $e \in R$ are the same. Now, let $\tilde{C} = \{d_R \triangleq d_{\text{lead}(R)}; R \in D\}$. It is easy to see that $\tilde{C}$ is a linear network code on $\text{RG}(D)$, and if C solves G, then $\tilde{C}$ solves $\text{RG}(D)$. On the other hand, any linear network code solving $\text{RG}(D)$ can be extended to a linear solution of G by letting $d_e = d_R$ if $e \in R$, for any $e \in E$. Hence, we have the following theorem, which is parallel to Theorem 4.5 of [9] and of central importance in our treatment.

Theorem 3.4: G is solvable if and only if $\text{RG}(D)$ is solvable.

B. Super Region

Theorem 3.4 equates the solvability of G with that of $\text{RG}(D)$. To determine the latter, we need the following notion.

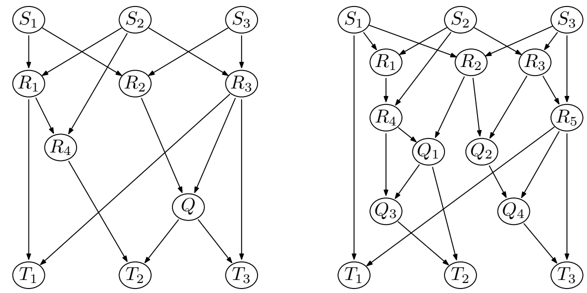
Definition 3.5 (Super Region [10]): For each non-empty $\Theta \subseteq D$, the *super region* of Θ, denoted by $\text{reg}(\Theta)$, is recursively defined as follows: (1) $\Theta \subseteq \text{reg}(\Theta)$; and (2) If $R \in D$ and $\text{In}(R) \subseteq \text{reg}(\Theta)$, then $R \in \text{reg}(\Theta)$. Furthermore, we define $\text{reg}^\circ(\Theta) = \text{reg}(\Theta) \setminus \Theta$.

For example, for the region graph in Fig. 3 (a), to obtain $\text{reg}(S_1, S_2)$, we first have $\{S_1, S_2\} \subseteq \text{reg}(S_1, S_2)$. Then, since $\text{In}(R_1) = \{S_1, S_2\} \subseteq \text{reg}(S_1, S_2)$, we have $R_1 \in \text{reg}(S_1, S_2)$. Further, since $\text{In}(R_4) = \{R_1, S_2\} \subseteq \text{reg}(S_1, S_2)$, we have $R_4 \in \text{reg}(S_1, S_2)$. Note that $S_3 \notin \text{reg}(S_1, S_2)$ because $\text{In}(S_3) = \emptyset \nsubseteq \text{reg}(S_1, S_2)$. Further, $R_2 \notin \text{reg}(S_1, S_2)$ because $\text{In}(R_2) = \{S_1, S_3\} \nsubseteq \text{reg}(S_1, S_2)$, and so on. Finally, we find $\text{reg}(S_1, S_2) = \{S_1, S_2, R_1, R_4\}$. By the same observation, for the region graph in Fig. 3 (b), we find $\text{reg}(R_2, R_4) = \{R_2, R_4, Q_1, Q_3, T_2\}$ and $\text{reg}(R_2, R_3, R_5) = \{R_2, R_3, R_5, Q_2, Q_4, T_3\}$. We add a remark that it is always true that $\text{reg}(\Theta) \neq \emptyset$ because $\Theta \subseteq \text{reg}(\Theta)$, but it is possible that $\text{reg}^\circ(\Theta) = \emptyset$.

Let $\text{RG}(D)$ be the basic region graph where D is labelled as $D = \{R_1, R_2, R_3, \cdots, R_N\}$ such that $R_i = S_i$ for $i \in \{1, 2, 3\}$ and $\ell < \ell'$ if R_ℓ is a parent of $R_{\ell'}$. Then, Algorithm 2 given in Fig. 4 computes $\text{reg}(\Theta)$ for any $\Theta \subseteq D$ in time $O(|D|)$.

Let $\tilde{C} = \{d_R; R \in D\}$ be a linear network code on $\text{RG}(D)$. The following simple but important lemma is the motivation for the definition of super region.

Lemma 3.6: For any $\Theta \subseteq D$ and $R \in \text{reg}(\Theta)$, $d_R \in \langle d_Q; Q \in \Theta \rangle$.



(a) (b)

Fig 3. Two examples of region graph.

Algorithm 2: Super-Region $(\mathrm{RG}(D), \Theta)$

$\mathrm{reg}(\Theta) = \Theta$;

$\ell \leftarrow$ from 1 to N

  If $\mathrm{In}(R_\ell) \subseteq \mathrm{reg}(\Theta)$ **then**

    $\mathrm{reg}(\Theta) = \mathrm{reg}(\Theta) \cup \{R_\ell\}$;

Return $\mathrm{reg}(\Theta)$;

Fig 4. An algorithm for computing a super region, where N is the number of regions in D.

IV. Terminal-separable Sum-network

Throughout this section, we focus on the case $k = 3$, that is, G is a 3s/nt sum-network. Let RG(D) be the basic region graph of G. As before, we use S_1, S_2, S_3 to denote the three source regions of G, and $T_1, T_2, \cdots, T_n$ to denote the n terminal regions of G. Note that it immediately follows from Assumption 2.1 that $T_\ell \notin \mathrm{reg}(S_i, S_j)$ for all $\ell \in [n] = \{1, 2, \cdots, n\}$ and all $\{i, j\} \subseteq \{1, 2, 3\}$.

For any two regions $R, R' \in D$, we say that R' is *reachable* from R, denoted by $R \rightarrow R'$, if there exists a directed path in RG(D) from R to R'; otherwise, R' is not reachable from R, which will be denoted by $R \nrightarrow R'$. In particular, we have $R \rightarrow R$ for all $R \in D$.

Definition 4.1: We define

$$\Pi \triangleq \mathrm{reg}(S_1, S_2) \cup \mathrm{reg}(S_1, S_3) \cup \mathrm{reg}(S_2, S_3).$$

And for each $I = \{i_1, \cdots, i_\ell\} \subseteq [n]$, we define $\Omega_I = \Omega_{i_1, \cdots, i_\ell}$ as the set of all $R \in D \backslash \Pi$ such that $R \rightarrow T_j$ for all $j \in I$ and $R \nrightarrow T_{j'}$ for all $j' \in [n] \setminus I$.

It is possible that $\Omega_I = \emptyset$ for some $I \subseteq [n]$; moreover, for two distinct subsets I and I' of $[n]$, it is easy to see that $\Omega_I \cap \Omega_{I'} = \emptyset$.

Example 4.2: Take the region graphs in Fig. 3 as an example.

1) For the region graph in Fig. 3 (a). By Definition 3.5, $\mathrm{reg}(S_1, S_2) = \{S_1, S_2, R_1, R_4\}$, $\mathrm{reg}(S_1, S_3) = \{S_1, S_3, R_2\}$ and $\mathrm{reg}(S_2, S_3) = \{S_2, S_3, R_3\}$. So $\Pi = \{S_1, S_2, S_3, R_1, R_2, R_3, R_4\}$. Note that $D \backslash \Pi = \{Q, T_1, T_2, T_3\}$. We can further find $\Omega_i = \{T_i\}$ for $i \in \{1, 2, 3\}$, $\Omega_{2,3} = \{Q\}$ and $\Omega_{1,2} = \Omega_{1,3} = \Omega_{1,2,3} = \emptyset$.
2) For the region graph in Fig. 3 (b), by Definition 3.5, $\mathrm{reg}(S_1, S_2) = \{S_1, S_2, R_1, R_4\}$, $\mathrm{reg}(S_1, S_3) = \{S_1, S_3, R_2\}$ and $\mathrm{reg}(S_2, S_3) = \{S_2, S_3, R_3, R_5\}$. So $\Pi = \{S_1, S_2, S_3, R_1, R_2, R_3, R_4, R_5\}$ and $D \backslash \Pi = \{Q_1, Q_2, Q_3, Q_4, T_1, T_2, T_3\}$. Furthermore, we find that $\Omega_1 = \{T_1\}$, $\Omega_2 = \{Q_1, Q_3, T_2\}$, $\Omega_3 = \{Q_2, Q_4, T_3\}$ and $\Omega_{1,2} = \Omega_{1,3} = \Omega_{2,3} = \Omega_{1,2,3} = \emptyset$.

Remark 4.3: From Definition 3.5, for any $\{i, j, \ell\} = \{1, 2, 3\}$, we have

$$\text{reg}(S_i, S_j) \cap \text{reg}(S_i, S_\ell) = \{S_i\}. \tag{2}$$

Hence, $\text{reg}^\circ(S_1, S_2)$, $\text{reg}^\circ(S_1, S_3)$ and $\text{reg}^\circ(S_2, S_3)$ are mutually disjoint.

Now, we define the notion of the so-called terminal-separable region graph, the main concern of this section.

Definition 4.4 (Terminal-Separable Region Graph): The region graph $\text{RG}(D)$ is said to be *terminal-separable* if $\Omega_I = \emptyset$ for all $I \subseteq [n]$ of size $|I| > 1$. Furthermore, G is said to be *terminal-separable* if $\text{RG}(D)$ is terminal-separable.

Theorem 4.5: There is an algorithm running in time $O(|D|)$ to determine whether $\text{RG}(D)$ is terminal-separable.

Proof: Firstly, we note that Π can be found in time $O(|D|)$ because for all $\{i, j\} \subseteq \{1, 2, 3\}$, $\text{reg}(S_i, S_j)$ can be found in time $O(|D|)$ by Algorithm 2.

Secondly, since $\text{RG}(D)$ is acyclic, then regions in $D\backslash\Pi$ can be labelled as $D\backslash\Pi = \{R_1, R_2, \cdots, R_N\}$ such that $\ell < \ell'$ if R_ℓ is a parent of $R_{\ell'}$. Then for each $\ell \in [n]$, by the following Algorithm 3, we can label all regions $R \in D\backslash\Pi$ in time $O(|D|)$ such that for each $m \in [n]$, $R \to T_m$ if and only if R is labelled with m. Hence, the time complexity for finding the subset Ω_I for all $I \subseteq [n]$ and determining whether $\text{RG}(D)$ is terminal-separable is $O(|D|)$. ■

Algorithm 3: Labelling ($\text{RG}(D)$)

$\ell \leftarrow$ from N to 1
  if $R_\ell = T_m$ for some $m \in [n]$ **then**
    Label R_ℓ with m;
  else if R_ℓ has a child $R_{\ell'}$ such that $R_{\ell'}$ is labelled with m **then**
    Label R_ℓ with m;

Fig 5. Labelling the regions in $D\backslash\Pi$, where $D\backslash\Pi = \{R_1, R_2, \cdots, R_N\}$ such that $\ell < \ell'$ if R_ℓ is a parent of $R_{\ell'}$.

The following is a motivating example for the notion of terminal-separability.

Example 4.6: The region graph $\text{RG}(D)$ in Fig. 3 (a) is not terminal-separable because $\Omega_{2,3} = \{Q\} \neq \emptyset$. However, we can consider the subgraph $\text{RG}(\overline{D})$ of $\text{RG}(D)$, where $\overline{D} = D\backslash\{T_2, T_3\}$ and view T_1 and Q as the terminal regions, then $\text{RG}(\overline{D})$ is terminal separable (To see this, note that in $\text{RG}(\overline{D})$, $\Omega_{1,2} = \emptyset$ with T_2 set to be Q). Note that $\text{RG}(\overline{D})$ can be solved by $\tilde{C}_{\overline{D}} = \{d_R; R \in \overline{D}\}$ with $d_{R_1} = d_{R_2} = \alpha_1$, $d_{R_3} = \alpha_2 + \alpha_3$ and $d_{T_1} = d_Q = \alpha_1 + \alpha_2 + \alpha_3$ (Recall that $\alpha_1 = (1, 0, 0), \alpha_2 = (0, 1, 0), \alpha_3 = (0, 0, 1)$ denote the encoding vectors of the source messages.). Clearly, we can extend $\tilde{C}_{\overline{D}}$ to a linear network code solving $\text{RG}(D)$ by letting $d_{T_2} = d_{T_3} = \alpha_1 + \alpha_2 + \alpha_3$.

In general, we have the following theorem.

Theorem 4.7: If $\text{RG}(D)$ is not terminal-separable, then there exists a subgraph $\text{RG}(\overline{D})$ of $\text{RG}(D)$ such that $\text{RG}(\overline{D})$ is terminal-separable with $n' < n$ terminal regions. Moreover, if $\text{RG}(\overline{D})$ is solvable then $\text{RG}(D)$ is solvable.

Proof: Since $\text{RG}(D)$ is not terminal-separable, there exists an $I \subseteq [n]$ such that $|I| > 1$ and $\Omega_I \neq \emptyset$. Let $D_1 = D \setminus \{T_i; i \in I\}$ and $\text{RG}(D_1)$ be the subgraph of $\text{RG}(D)$ induced by D_1 and pick an $R_{i_1} \in \Omega_I$ as a terminal region of $\text{RG}(D_1)$. Then $\text{RG}(D_1)$ has $n_1 = n - |I| + 1 < n$ terminal regions and clearly, any linear network code solving $\text{RG}(D_1)$ can be extended to one that solves $\text{RG}(D)$. So, if $\text{RG}(D_1)$ is solvable, then $\text{RG}(D)$ is solvable.

If $\mathrm{RG}(D_1)$ is terminal-separable, we are done. Otherwise, by a similar discussion, there exists a subgraph $\mathrm{RG}(D_2)$ such that $\mathrm{RG}(D_2)$ has $n_2 < n_1$ terminal regions and if $\mathrm{RG}(D_2)$ is solvable, then $\mathrm{RG}(D_1)$ is solvable, and so on. Note that any region graph with one terminal region is terminal-separable. We can finally find a sequence of subgraphs $\mathrm{RG}(D_j), j = 1, 2, \cdots, t$, where $1 \leq t < n$, satisfying: 1) $\mathrm{RG}(D_t)$ is terminal-separable; 2) For $j = 1, 2, \cdots, t$, $\mathrm{RG}(D_j)$ has $n_j < n_{j-1}$ terminal regions and if $\mathrm{RG}(D_j)$ is solvable, then $\mathrm{RG}(D_{j-1})$ is solvable, where $D_0 = D$ and $n_0 = n$. ■

In what follows, for a terminal-separable $\mathrm{RG}(D)$, we use Λ_i to denote the set of all $R \in \Pi$ such that R has a child $Q \in \Omega_i$. As an example, for the region graph in Fig. 3 (b), we have $\Lambda_1 = \{S_1, R_5\}$, $\Lambda_2 = \{R_2, R_4\}$ and $\Lambda_3 = \{R_2, R_3, R_5\}$. Since $T_\ell \notin \mathrm{reg}(S_i, S_j)$ for all $\ell \in [n]$ and all $\{i, j\} \subseteq \{1, 2, 3\}$, we have $\Lambda_\ell \nsubseteq \mathrm{reg}(S_i, S_j)$, and hence, $|\Lambda_\ell| \geq 2$.

Definition 4.8: For a terminal-separable network G, we say a linear network code $\tilde{C}_\Pi = \{d_R; R \in \Pi\}$ *solves* Π, or equivalently, $\tilde{C}_\Pi$ is a *solution* to Π, if

$$\bar{\alpha} \in \langle d_R; R \in \Lambda_j \rangle \text{ for all } j \in [n]. \tag{3}$$

And we say Π is *solvable* if there exists a linear network code that solves Π, and *unsolvable* otherwise.

The following theorem gives a necessary and sufficient condition for the solvability of $\mathrm{RG}(D)$, which, together with Theorem 3.4, further gives one for the solvability of a terminal-separable network G.

Theorem 4.9: For a terminal-separable network G, $\mathrm{RG}(D)$ is solvable if and only if Π is solvable.

Proof: Suppose that $\tilde{C} = \{d_R; R \in D\}$ solves $\mathrm{RG}(D)$. Consider $\tilde{C}_\Pi = \{d_R; R \in \Pi\}$, the restriction of $\tilde{C}$ to Π. Clearly, for each $j \in [n]$ and $Q \in \Omega_j$, $d_Q \in \langle d_R; R \in \Lambda_j \rangle$ and in particular, $\bar{\alpha} = d_{T_j} \in \langle d_R; R \in \Lambda_j \rangle$, which means that $\tilde{C}_\Pi$ solves Π.

Conversely, suppose a linear network code $\tilde{C}_\Pi = \{d_R; R \in \Pi\}$ solves Π, i.e., for each $j \in [n]$, $\bar{\alpha} \in \langle d_R; R \in \Lambda_j \rangle$. Noticing that all Ω_j's are mutually exclusive, it is easy to see that $\tilde{C}_\Pi$ can be extended to each Ω_j and obtain a code $\tilde{C}_{\Omega_j}$ such that $d_{T_j} = \bar{\alpha}$. Hence, the newly constructed code $\tilde{C} = \tilde{C}_\Pi \cup \left(\cup_{j=1}^n \tilde{C}_{\Omega_j} \right)$ solves $\mathrm{RG}(D)$. ■

V. Solvability of Terminal-separable Sum-networks

In this section, we will give polynomial-time checkable necessary and sufficient conditions for the solvability of a terminal-separable $3\mathrm{s}/n\mathrm{t}$ sum-network G. By Theorem 4.9, this can be done by considering the solvability of Π.

The following simple lemma will be frequently used in our proofs.

Lemma 5.1: Let a linear network code $\tilde{C}_\Pi = \{d_R \in \mathbb{F}^3; R \in \Pi\}$ be a solution to Π. If, for some $j \in [n]$, $\Lambda_j = \{R', R''\}$, then $\langle d_{R'}, d_{R''} \rangle = \langle \bar{\alpha}, d_{R'} \rangle = \langle \bar{\alpha}, d_{R''} \rangle$.

Proof: Since $\tilde{C}_\Pi$ solves Π, $\bar{\alpha} \in \langle d_{R'}, d_{R''} \rangle$. On the other hand though, by Definition 4.1 and Lemma 3.6, we have $d_{R'} \in \langle \alpha_i, \alpha_j \rangle$ and $d_{R''} \in \langle \alpha_i, \alpha_k \rangle$, for some $\{i, j, k\} = \{1, 2, 3\}$ and hence $\bar{\alpha} \notin \langle d_{R'} \rangle$ and $\bar{\alpha} \notin \langle d_{R''} \rangle$, which immediately implies the lemma. ■

A. Three Motivating Examples

Before stating our result and detailing the proof, we give three examples to illustrate the key ideas.

The following example typifies how one can construct a solution for a solvable G.

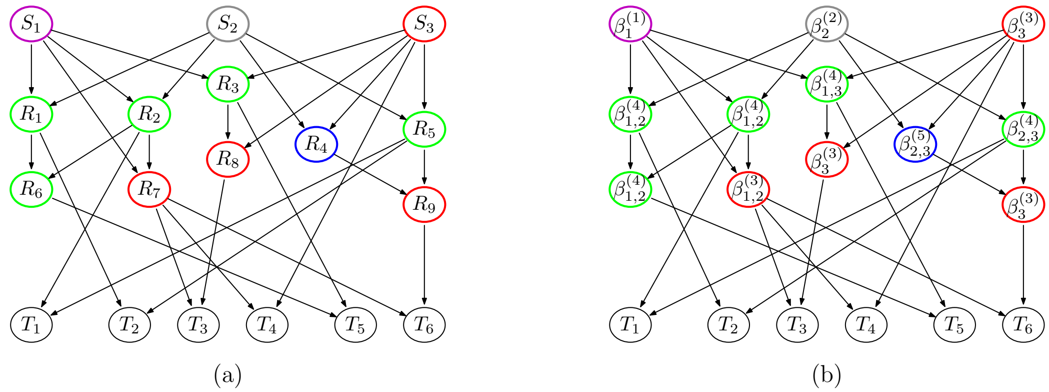


Fig 6. A solvable region graph and a solution on Π: (a) is the region graph, where the subset Π is partitioned into five classes (subsets) $\Delta_1 = \{S_1\}$, $\Delta_2 = \{S_2\}$, $\Delta_3 = \{S_3, R_7, R_8, R_9\}$, $\Delta_4 = \{R_1, R_2, R_3, R_5, R_6\}$ and $\Delta_5 = \{R_4\}$ (In this figure, regions in the same class have the same color); (b) shows the solution, where each region is labelled with its encoding vector.

Example 5.2: Consider the region graph in Fig. 6 (a). It can be easily checked that $\text{reg}^\circ(S_1, S_2) = \{R_1, R_2, R_6, R_7\}$, $\text{reg}^\circ(S_1, S_3) = \{R_3, R_8\}$, $\text{reg}^\circ(S_2, S_3) = \{R_4, R_5, R_9\}$ and hence $\Pi = \{S_1, S_2, S_3, R_1, \ldots, R_9\}$. Clearly, $\text{RG}(D)$ is terminal-separable and $\Omega_j = \{T_j\}$ for $j = 1, \cdots, 6$. Moreover, $\Lambda_1 = \{R_2, R_5\}$, $\Lambda_2 = \{R_1, R_5\}$, $\Lambda_3 = \{R_7, R_8\}$, $\Lambda_4 = \{S_3, R_7\}$, $\Lambda_5 = \{R_3, R_6\}$, $\Lambda_6 = \{R_7, R_9\}$.

We next determine the solvability of $\text{RG}(D)$, which, by Theorem 4.9, is equivalent to determine whether there exists a linear network code satisfying the condition (3). To this end, we first show that Π can be partitioned into five *classes* such that if $\tilde{C}_\Pi = \{d_R \in \mathbb{F}^3; R \in \Pi\}$ is a solution to Π, then the encoding vectors of all regions in each class is uniquely determined by the encoding vector of a fixed region in the same class (up to a scalar multiplication).

♦ Note that $\{S_3, R_7\} = \Lambda_4$, which, by Lemma 5.1, implies that $\langle d_{S_3}, \bar{\alpha}\rangle = \langle d_{R_7}, \bar{\alpha}\rangle$. Similarly, it follows from $\{R_7, R_8\} = \Lambda_3$ and $\{R_7, R_9\} = \Lambda_6$ that $\langle d_{R_7}, \bar{\alpha}\rangle = \langle d_{R_8}, \bar{\alpha}\rangle$ and $\langle d_{R_7}, \bar{\alpha}\rangle = \langle d_{R_9}, \bar{\alpha}\rangle$. So,

$$\langle d_{S_3}, \bar{\alpha}\rangle = \langle d_{R_7}, \bar{\alpha}\rangle = \langle d_{R_8}, \bar{\alpha}\rangle = \langle d_{R_9}, \bar{\alpha}\rangle,$$

and we obtain a class of regions $\{S_3, R_7, R_8, R_9\}$.

Now, we show that d_{R_7}, d_{R_8} and d_{R_9} are uniquely determined (up to a scalar multiplication). Note that for any linear network code solution, $d_{S_3} = \alpha_3$ and hence $\langle d_{S_3}, \bar{\alpha}\rangle = \langle \alpha_3, \bar{\alpha}\rangle$.

From Fig. 6 (a), we have $R_7 \in \text{reg}^\circ(S_1, S_2)$, and thereby $d_{R_7} \in \langle d_{S_1}, d_{S_2}\rangle = \langle \alpha_1, \alpha_2\rangle$, and hence,

$$d_{R_7} \in \langle d_{S_3}, \bar{\alpha}\rangle \cap \langle \alpha_1, \alpha_2\rangle = \langle \alpha_3, \bar{\alpha}\rangle \cap \langle \alpha_1, \alpha_2\rangle = \langle \alpha_1 + \alpha_2\rangle.$$

Similarly, it follows from $R_8 \in \text{reg}^\circ(S_1, S_3)$ and $R_9 \in \text{reg}^\circ(S_2, S_3)$ that

$$d_{R_8} \in \langle d_{S_3}, \bar{\alpha}\rangle \cap \langle \alpha_1, \alpha_3\rangle = \langle \alpha_3, \bar{\alpha}\rangle \cap \langle \alpha_1, \alpha_3\rangle = \langle \alpha_3\rangle$$

and

$$d_{R_9} \in \langle d_{S_3}, \bar{\alpha}\rangle \cap \langle \alpha_2, \alpha_3\rangle = \langle \alpha_3, \bar{\alpha}\rangle \cap \langle \alpha_2, \alpha_3\rangle = \langle \alpha_3\rangle.$$

♦ Similarly as above, from the fact that $\{R_1, R_5\} = \Lambda_2$, $\{R_2, R_5\} = \Lambda_1$ and $\{R_1, R_2\} \subseteq \text{reg}^\circ(S_1, S_2)$, we deduce that

$$\langle d_{R_1}, \bar{\alpha}\rangle = \langle d_{R_5}, \bar{\alpha}\rangle = \langle d_{R_2}, \bar{\alpha}\rangle$$

and

$$d_{R_1}, d_{R_2} \in \langle d_{R_5}, \bar{\alpha}\rangle \cap \langle \alpha_1, \alpha_2\rangle.$$

Note that $\langle d_{R_5}, \bar{\alpha}\rangle$ and $\langle \alpha_1, \alpha_2\rangle$ are two different 2-dimensional subspaces of $\mathbb{F}^3$, which immediately implies that

$$\langle d_{R_1}\rangle = \langle d_{R_2}\rangle = \langle d_{R_5}, \bar{\alpha}\rangle \cap \langle \alpha_1, \alpha_2\rangle.$$

From Fig. 6 (a), one verifies that $R_6 \in \mathrm{reg}(R_1, R_2)$, which implies that $d_{R_6} \in \langle d_{R_1}, d_{R_2}\rangle$. It then follows that

$$\langle d_{R_6}\rangle = \langle d_{R_1}\rangle = \langle d_{R_2}\rangle = \langle d_{R_5}, \bar{\alpha}\rangle \cap \langle \alpha_1, \alpha_2\rangle$$

and

$$\langle d_{R_6}, \bar{\alpha}\rangle = \langle d_{R_1}, \bar{\alpha}\rangle = \langle d_{R_2}, \bar{\alpha}\rangle = \langle d_{R_5}, \bar{\alpha}\rangle.$$

Moreover, from the fact that $\{R_3, R_6\} = \Lambda_5$ and $R_3 \in \mathrm{reg}^\circ(S_1, S_3)$, we deduce that

$$\langle d_{R_3}, \bar{\alpha}\rangle = \langle d_{R_6}, \bar{\alpha}\rangle = \langle d_{R_1}, \bar{\alpha}\rangle = \langle d_{R_2}, \bar{\alpha}\rangle = \langle d_{R_5}, \bar{\alpha}\rangle.$$

and

$$\langle d_{R_3}\rangle = \langle d_{R_5}, \bar{\alpha}\rangle \cap \langle \alpha_1, \alpha_3\rangle,$$

obtaining another class $\{R_1, R_2, R_3, R_5, R_6\}$ such that the encoding vectors of all regions can be determined by the encoding vector of a fixed region, say, e.g., R_5.

⧫ Following from the above discussions, we obtain a partition $\mathcal{I} = \{\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5\}$ of Π, where $\Delta_1 = \{S_1\}$, $\Delta_2 = \{S_2\}$, $\Delta_3 = \{S_3, R_7, R_8, R_9\}$, $\Delta_4 = \{R_1, R_2, R_3, R_5, R_6\}$ and $\Delta_5 = \{R_4\}$ are the resulting classes such that the encoding vector of all regions in each class is uniquely determined by the encoding vector of a fixed region in the same class (up to a scalar multiplication). Furthermore, each class of $\mathcal{I}$ can be partitioned into *subclasses* such that all regions in the same subclass share a same encoding vector (again, up to a scalar multiplication). That is to say,

$$\Delta_3 = [\Delta_3]_3 \cup [\Delta_3]_{1,2},$$

where,

$$[\Delta_3]_{1,2} = \Delta_3 \cap \mathrm{reg}(S_1, S_2) = \{R_7\},$$
$$[\Delta_3]_3 = \Delta_3 \cap (\mathrm{reg}(S_1, S_3) \cup \mathrm{reg}(S_2, S_3)) = \{S_3, R_8, R_9\};$$

and

$$\Delta_4 = [\Delta_4]_{1,2} \cup [\Delta_4]_{1,3} \cup [\Delta_4]_{2,3},$$

where

$$[\Delta_4]_{1,2} = \Delta_4 \cap \mathrm{reg}(S_1, S_2) = \{R_1, R_2, R_6\},$$
$$[\Delta_4]_{1,3} = \Delta_4 \cap \mathrm{reg}(S_1, S_3) = \{R_3\},$$
$$[\Delta_4]_{2,3} = \Delta_4 \cap \mathrm{reg}(S_2, S_3) = \{R_5\}.$$

Note that, adopting the above notions, the singleton class Δ_1, Δ_2 and Δ_5 can also be written into the union of subclasses. That is,

$$\Delta_1 = [\Delta_1]_1 \cup [\Delta_1]_{2,3},$$

where,

$$[\Delta_1]_{2,3} = \Delta_1 \cap \mathrm{reg}(S_2, S_3) = \emptyset,$$
$$[\Delta_1]_1 = \Delta_1 \cap (\mathrm{reg}(S_1, S_2) \cup \mathrm{reg}(S_1, S_3)) = \{S_1\};$$

$$\Delta_2 = [\Delta_2]_2 \cup [\Delta_2]_{1,3},$$

where,

$$[\Delta_2]_{1,3} = \Delta_2 \cap \text{reg}(S_2, S_3) = \emptyset,$$
$$[\Delta_2]_2 = \Delta_2 \cap (\text{reg}(S_1, S_2) \cup \text{reg}(S_2, S_3)) = \{S_2\};$$

and

$$\Delta_5 = [\Delta_5]_{1,2} \cup [\Delta_5]_{1,3} \cup [\Delta_5]_{2,3},$$

where

$$[\Delta_5]_{1,2} = \Delta_5 \cap \text{reg}(S_1, S_2) = \emptyset,$$
$$[\Delta_5]_{1,3} = \Delta_5 \cap \text{reg}(S_1, S_3) = \emptyset,$$
$$[\Delta_5]_{2,3} = \Delta_5 \cap \text{reg}(S_2, S_3) = \{R_4\}.$$

♦ Based on the subclasses of the partition $\mathcal{I} = \{\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5\}$, we construct a code $\tilde{C}_\Pi = \{d_R \in \mathbb{F}^3; R \in \Pi\}$ as illustrated in Fig. 6 (b), where $\beta_1^{(1)} = \alpha_1$, $\beta_2^{(2)} = \alpha_2$, $\beta_3^{(3)} = \alpha_3$, $\beta_{1,2}^{(3)} = \alpha_1 + \alpha_2$, $\beta_{1,2}^{(4)} = \alpha_1 + 3\alpha_2$, $\beta_{1,3}^{(4)} = 2\alpha_1 + 3\alpha_3$, $\beta_{2,3}^{(4)} = 2\alpha_2 - \alpha_3$, $\beta_{2,3}^{(5)} = \alpha_2 - 2\alpha_3$, and $\mathbb{F} = GF(5)$. It is easy to check that $\tilde{C}_\Pi$ solves Π and hence RG(D) is solvable.

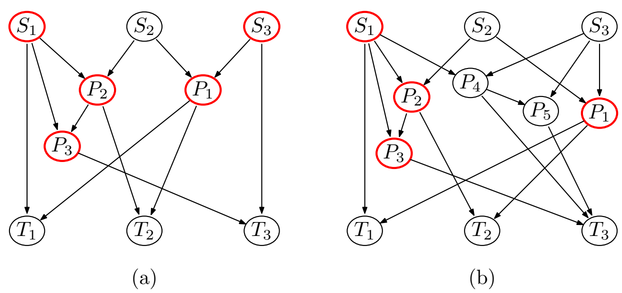


Fig 7. Two unsolvable region graphs

Each of the following two examples typifies a reason that G fails to be solvable.

Example 5.3: Let RG(D) be the region graph as in Fig. 7 (a). It can be verified that RG(D) is terminal-separable with reg$(S_1, S_2) = \{S_1, S_2, P_2, P_3\}$, reg$(S_1, S_3) = \{S_1, S_3\}$, reg$(S_2, S_3) = \{S_2, S_3, P_1\}$, $\Omega_j = \{T_j\}$ for $j \in \{1, 2, 3\}$, $\Lambda_1 = \{S_1, P_1\}$, $\Lambda_2 = \{P_1, P_2\}$, $\Lambda_3 = \{S_3, P_3\}$.

We will show that RG(D) is unsolvable. Suppose, by way of contradiction, that there is a solution $\tilde{C}_\Pi = \{d_R \in \mathbb{F}^3; R \in \Pi\}$ to Π. Noting that $\{S_1, P_1\} = \Lambda_1$ and $\{P_1, P_2\} = \Lambda_2$ and applying Lemma 5.1, we infer that

$$\langle d_{P_2}, \bar{\alpha} \rangle = \langle d_{P_1}, \bar{\alpha} \rangle = \langle d_{S_1}, \bar{\alpha} \rangle.$$

And from the fact that $P_2 \in \text{reg}(S_1, S_2)$, we further deduce

$$\langle d_{P_2} \rangle = \langle d_{S_1}, \bar{\alpha} \rangle \cap \langle \alpha_1, \alpha_2 \rangle = \langle d_{S_1} \rangle. \tag{4}$$

Moreover, we note that $P_3 \in \text{reg}(S_1, P_2)$ and thereby $\langle d_{P_3} \rangle \in \langle d_{P_2}, d_{S_1} \rangle$, which, together with (4), immediately leads to

$$\langle d_{P_3} \rangle = \langle d_{P_2} \rangle = \langle d_{S_1} \rangle$$

and

$$\langle d_{P_3}, \bar{\alpha} \rangle = \langle d_{P_2}, \bar{\alpha} \rangle = \langle d_{P_1}, \bar{\alpha} \rangle = \langle d_{S_1}, \bar{\alpha} \rangle.$$

Furthermore, noting $\{S_3, P_3\} = \Lambda_3$ and applying Lemma 5.1, we have

$$\langle d_{S_3}, \bar{\alpha} \rangle = \langle d_{P_3}, \bar{\alpha} \rangle = \langle d_{S_1}, \bar{\alpha} \rangle. \tag{5}$$

So, similarly as in Example 5.2, we can partition Π into classes $\Delta_1 = \{S_1, P_1, P_2, P_3, S_3\}$ and $\Delta_2 = \{S_2\}$.

On the other hand, from the fact that $\langle d_{S_1}, d_{S_3}, \bar{\alpha} \rangle = \langle \alpha_1, \alpha_3, \bar{\alpha} \rangle$ has dimension 3, we infer that

$$\langle d_{S_3}, \bar{\alpha} \rangle \neq \langle d_{S_1}, \bar{\alpha} \rangle,$$

which however contradicts (5).

Example 5.4: Let $\mathrm{RG}(D)$ be the region graph as in Fig. 7 (b). It can be verified that $\mathrm{RG}(D)$ is terminal-separable with $\mathrm{reg}(S_1, S_2) = \{S_1, S_2, P_2, P_3\}$, $\mathrm{reg}(S_1, S_3) = \{S_1, S_3, P_4, P_5\}$, $\mathrm{reg}(S_2, S_3) = \{S_2, S_3, P_1\}$, $\Omega_j = \{T_j\}$ for $j \in \{1, 2, 3\}$, $\Lambda_1 = \{S_1, P_1\}$, $\Lambda_2 = \{P_1, P_2\}$, $\Lambda_3 = \{P_3, P_4, P_5\}$.

Now, we show that $\mathrm{RG}(D)$ is unsolvable. Suppose, by way of contradiction, there is a solution $\tilde{C}_\Pi = \{d_R \in \mathbb{F}^3; R \in \Pi\}$ to Π. Starting from the fact $\Lambda_1 = \{S_1, P_1\}$, $\Lambda_2 = \{P_1, P_2\}$ and $P_3 \in \mathrm{reg}(S_1, P_2)$ and going through similar arguments as in Example 5.2, we obtain a class $\{S_1, P_1, P_2, P_3\}$ such that

$$\langle d_{P_3}, \bar{\alpha} \rangle = \langle d_{P_2}, \bar{\alpha} \rangle = \langle d_{P_1}, \bar{\alpha} \rangle = \langle d_{S_1}, \bar{\alpha} \rangle.$$

So, parallel to Example 5.2, we can partition Π into classes $\Delta_1 = \{S_1, P_1, P_2, P_3\}$, $\Delta_2 = \{S_2\}$, $\Delta_3 = \{S_3\}$, $\Delta_4 = \{P_4\}$ and $\Delta_5 = \{P_5\}$. Note that from Fig. 7 (b), we know that

$$P_3 \in [\Delta_1]_1 = \Delta_1 \cap (\mathrm{reg}(S_1, S_2) \cup \mathrm{reg}(S_1, S_3))$$

and we have

$$\langle d_{P_3} \rangle \in \langle d_{S_1}, \bar{\alpha} \rangle \cap (\langle \alpha_1, \alpha_2 \rangle \cup \langle \alpha_1, \alpha_3 \rangle) = \langle \alpha_1 \rangle.$$

On the other hand, from $\{P_4, P_5\} \subseteq \mathrm{reg}(S_1, S_3)$, we deduce that $d_{P_4}, d_{P_5} \in \langle d_{S_1}, d_{S_3} \rangle = \langle \alpha_1, \alpha_3 \rangle$, and hence $\langle d_{P_3}, d_{P_4}, d_{P_5} \rangle \subseteq \langle \alpha_1, \alpha_3 \rangle$. This, together with the fact that $\Lambda_3 = \{P_3, P_4, P_5\}$, implies that

$$\langle d_R; R \in \Lambda_3 \rangle = \langle d_{P_3}, d_{P_4}, d_{P_5} \rangle \subseteq \langle \alpha_1, \alpha_3 \rangle,$$

which however contradicts the condition (3).

B. Characteristic Partition

As hinted by the three examples above, the basic idea for determining the solvability of a terminal-separable region graph is to check whether certain partition of Π has some desired properties. In this section, we formalize this idea in the general setting, which will lead to necessary and sufficient conditions for the solvability in the next subsection.

First of all, we introduce more new notations and definitions about partitions of Π as follows.

Let $\mathcal{I} = \{\Delta_1, \Delta_2, \ldots, \Delta_K\}$ be a *partition* of Π, where each Δ_j will be referred to as a *class* of $\mathcal{I}$ in the sequel. For any $R \in \Pi$, we use $[R]$ to denote the class that contains R. A partition is said to be *singular* if $[S_i] = [S_j]$ for some distinct $i, j \in \{1, 2, 3\}$, and *non-singular* otherwise. For ease of presentation, throughout the paper, for any aforementioned non-singular partition, we assume that $\Delta_1 = [S_1]$, $\Delta_2 = [S_2]$ and $\Delta_3 = [S_3]$, as in the three previous examples.

Given a class $\Delta_\ell \in \mathcal{I}$ and $\{j, k\} \subseteq \{1, 2, 3\}$, letting

$$[\Delta_\ell]_{j,k} := \Delta_\ell \cap \mathrm{reg}(S_j, S_k),$$

we further partition Δ_ℓ into *subclasses* as follows:

- If $\Delta_\ell \cap \{S_1, S_2, S_3\} = \emptyset$, then Δ_ℓ can be partitioned into three disjoint subclasses $[\Delta_\ell]_{1,2}, [\Delta_\ell]_{1,3}$ and $[\Delta_\ell]_{2,3}$.
- If $\Delta_\ell = [S_i]$ and $\Delta_\ell \neq [S_j]$, $\Delta_\ell \neq [S_k]$, where $\{i, j, k\} = \{1, 2, 3\}$, then Δ_ℓ can be partitioned into two subclasses $[S_i]_i := [S_i]_{i,j} \cup [S_i]_{i,k}$ and $[S_i]_{j,k}$.
- If $\Delta_\ell = [S_i] = [S_j]$ for some $\{i, j\} \subseteq \{1, 2, 3\}$, then Δ_ℓ has only one subclass, that is, itself.

Clearly, all the subclasses still form a partition, which is a refinement of the partition $\mathcal{I}$. We use $[[R]]$ to denote the subclass which contains region $[R]$.

Now, we introduce an equivalent relation between two classes of a non-singular partition $\mathcal{I}$, which is a key notion of our treatment. Two classes Δ_r and Δ_s of $\mathcal{I}$ are said to be *equivalent* if one of the following conditions holds: 1) There is a subclass $[[R']]$ of Δ_r and a subclass $[[R'']]$ of Δ_s such that $\Lambda_j \subseteq [[R']] \cup [[R'']]$ for some $j \in [n]$; or 2) There exists $\{i, j\} \subseteq \{1, 2, 3\}$ such that $\text{reg}([\Delta_r]_{i,j}) \cap \text{reg}([\Delta_s]_{i,j}) \neq \emptyset$.

Definition 5.5 (Characteristic Partition): Starting from the partition $\mathcal{I}_0 \triangleq \{[R] = \{R\}; R \in \Pi\}$, Algorithm 4 in Fig. 8 repeatedly merges two equivalent classes until $[S_i] = [S_j]$ for some distinct $i, j \in \{1, 2, 3\}$, or there are no equivalent classes left. The resulting partition, denoted by $\mathcal{I}_c$, is called a *characteristic partition* of Π.

Algorithm 4: Partitioning (Π)
$L = 0$;
while there are $\Delta_r, \Delta_s \in \mathcal{I}_L$ which are equivalent **do**
  Let $\mathcal{I}_{L+1}$ be a contraction of $\mathcal{I}_L$ by combining Δ_r and Δ_s;
  $L = L + 1$;
  If $[S_i] = [S_j]$ for some $\{i, j\} \subseteq \{1, 2, 3\}$ **then**
    break;
$\mathcal{I}_c = \mathcal{I}_L$;
Return $\mathcal{I}_c$;

Fig 8. An algorithm for computing the characteristic partition

Remark 5.6: Clearly, the while-loop of Algorithm 4 has at most $|\mathcal{I}_0| = |\Pi|$ rounds, which means that, in each round, it needs a polynomial in n time to determine whether there are two equivalent classes. And overall, Algorithm 4 can output $\mathcal{I}_c$ in a polynomial in $\{|\Pi|, n\}$ time.

We also remark that if $\text{RG}(D)$ is solvable, then Π has a unique characteristic partition, the proof of which is omitted since it is tedious and not needed for the main result of this paper. On the other hand though, if $\text{RG}(D)$ is unsolvable, the characteristic partition of Π may not be unique. For example, consider the region graph in Fig. 9. Let $\mathcal{I}_1 = \{[S_1], [S_3], [P_1]\}$ such that $[S_1] = \{S_1, P_3, P_2, S_2\}$, $[S_3] = \{S_3\}, [P_1] = \{P_1\}$ and $\mathcal{I}_2 = \{[S_1], [S_2], [P_2]\}$ such that $[S_1] = \{S_1, P_3, P_1, S_3\}, [S_2] = \{S_2\}, [P_2] = \{P_2\}$. It can be checked that $\mathcal{I}_1$ and $\mathcal{I}_2$ are both characteristic partitions of Π.

We have the following Lemma for the characteristic partition.

Lemma 5.7: Let $\tilde{C}_\Pi = \{d_R; R \in \Pi\}$ be a solution to Π and $\mathcal{I}_c$ be a characteristic partition of Π. Then, for any $R \in \Pi$, the following statements hold:

1) If $Q, Q' \in [R]$, then $\langle d_{Q'}, \bar{\alpha} \rangle = \langle d_Q, \bar{\alpha} \rangle$;
2) If $Q, Q' \in [[R]]$, then $\langle d_{Q'} \rangle = \langle d_Q \rangle$.

Proof: First of all, we prove that for any partition $\mathcal{I}$ of Π and any class $[R] \in \mathcal{I}$, if 1) holds then 2) holds. To this end, we consider the following two cases:

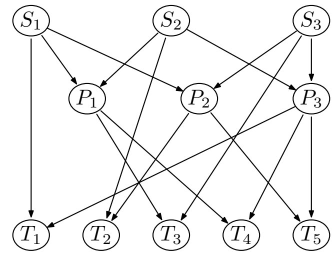


Fig 9. An example of unsolvable region graph

(1) If $[[R]] = [S_i]_i$, then, for any $Q \in [S_i]_i$, by 1), $\langle d_Q, \bar{\alpha}\rangle = \langle d_{S_i}, \bar{\alpha}\rangle = \langle \alpha_i, \bar{\alpha}\rangle$, so $d_Q \in \langle \alpha_i, \bar{\alpha}\rangle$. Moreover, by the definition of subclass $[S_i]_i$, we have $d_Q \in \langle \alpha_i, \alpha_j\rangle \cup \langle \alpha_i, \alpha_k\rangle$, where $\{i, j, k\} = \{1, 2, 3\}$. Then, $d_Q \in \langle \alpha_i, \bar{\alpha}\rangle \cap (\langle \alpha_i, \alpha_j\rangle \cup \langle \alpha_i, \alpha_k\rangle) = \langle \alpha_i\rangle$.

(2) If $[[R]] = [R]_{j,k}$ and $R \neq S_j, S_k$, then, for any $Q, Q' \in [R]_{j,k}$, by 1), $\langle d_Q, \bar{\alpha}\rangle = \langle d_{Q'}, \bar{\alpha}\rangle$. Moreover, by definition, $d_Q, d_{Q'} \in \langle \alpha_j, \alpha_k\rangle$. Then, $\langle d_Q\rangle = \langle d_Q, \bar{\alpha}\rangle \cap \langle \alpha_j, \alpha_k\rangle = \langle d_{Q'}, \bar{\alpha}\rangle \cap \langle \alpha_j, \alpha_k\rangle = \langle d_{Q'}\rangle$.

So we only need to prove 1). Let $\mathcal{I}_0, \mathcal{I}_1, \dots, \mathcal{I}_L = \mathcal{I}_c$ be a sequence of partitions yielded by Algorithm 4 where each $\mathcal{I}_\ell$ is obtained by merging two equivalent classes of $\mathcal{I}_{\ell-1}$. Below, we will prove, by way of induction, that 1) holds for each $\mathcal{I}_\ell$.

Obviously, 1) holds for $\mathcal{I}_0$ since for each class $[R] \in \mathcal{I}_0$, either $[[R]] = \{R\}$ or $[[R]] = \emptyset$. Now, suppose 1) holds (and thereby so does 2) by previous discussions) for $\mathcal{I}_{\ell-1}$, we will prove that it also holds for $\mathcal{I}_\ell$. If $[R] \in \mathcal{I}_\ell$ is also a class of $I_{\ell-1}$, then there is nothing to prove, so suppose $[R] = [R'] \cup [R'']$ for some distinct $[R'], [R''] \in \mathcal{I}_{\ell-1}$. We need deal with the following two cases:

Case 1: There is a $\Lambda_j \subseteq [[R']] \cup [[R'']]$. For this case, note that by the induction assumption, all the regions of $[[R']]$ (also $[[R'']]$) share a same encoding vector, so we can assume $\bar{\alpha} \in \langle d_R; R \in \Lambda_j\rangle = \langle d_{P'}, d_{P''}\rangle$ for some $P' \in \Lambda_j \cap [[R']]$ and $P'' \in \Lambda_j \cap [[R'']]$, and so $\langle d_{P'}, \bar{\alpha}\rangle = \langle d_{P''}, \bar{\alpha}\rangle$. Again by the induction assumption, for each $W' \in [R']$ and $W'' \in [R'']$, we have $\langle d_{W'}, \bar{\alpha}\rangle = \langle d_{P'}, \bar{\alpha}\rangle$ and $\langle d_{W''}, \bar{\alpha}\rangle = \langle d_{P''}, \bar{\alpha}\rangle$. Hence, for any $Q, Q' \in [R] = [R'] \cup [R'']$, we have $\langle d_{Q'}, \bar{\alpha}\rangle = \langle d_Q, \bar{\alpha}\rangle$.

Case 2: There exists $\{i, j\} \subseteq \{1, 2, 3\}$ such that $\text{reg}([R']_{i,j}) \cap \text{reg}([R'']_{i,j}) \neq \emptyset$. Without loss of generality, we can assume that there exists a $Q_0 \in \text{reg}([R']_{i,j}) \cap \text{reg}([R'']_{i,j})$. By Lemma 3.6 and the induction assumption, we have $d_{Q_0} \in \langle d_{P'}; P' \in \text{reg}([R']_{i,j})\rangle = \langle d_{P'}\rangle$ for any $P' \in \text{reg}([R']_{i,j})$. Similarly, $d_{Q_0} \in \langle d_{P''}; P'' \in \text{reg}([R'']_{i,j})\rangle = \langle d_{P''}\rangle$ for any $P'' \in \text{reg}([R'']_{i,j})$. Hence, by the induction assumption, for any $Q, Q' \in [R] = [R'] \cup [R'']$, $\langle d_{Q'}, \bar{\alpha}\rangle = \langle d_{Q_0}, \bar{\alpha}\rangle = \langle d_Q, \bar{\alpha}\rangle$.

Having dealt with the two cases above, we conclude that 1) holds for $\mathcal{I}_\ell$, and thereby holds for $\mathcal{I}_c$, as desired. ■

C. Necessary and Sufficient Conditions for the Solvability

First of all, we introduce the notion of regular partition as follows.

Definition 5.8 (Regular Partition): A non-singular partition $\mathcal{I} = \{\Delta_1, \Delta_2, \dots, \Delta_K\}$ is said to be *regular* if there are no equivalent classes in $\mathcal{I}$ and for all $\ell \in [n]$ and all $\{j, k\} \subseteq \{1, 2, 3\}$,

$$\Lambda_\ell \nsubseteq [S_j]_j \cup [S_k]_k \cup \text{reg}(S_j, S_k).$$

For example, it can be verified that the characteristic partition obtained in Example 5.2 is regular, and the one obtained in Example 5.3 is not regular since it is singular and the one obtained in Example 5.4 is not regular since $\Lambda_3 \subseteq [S_1]_1 \cup \text{reg}(S_1, S_3)$.

For $i = 1, 2, 3$, let $\mathcal{B}_i = \{\beta_i^{(i)}, \beta_{j,k}^{(i)}\}$ with $\beta_i^{(i)} = \alpha_i$ and $\beta_{j,k}^{(i)} = \alpha_j + \alpha_k$, where $\{j, k\} = \{1, 2, 3\}\backslash\{i\}$. The following lemma lays the foundation for our code construction.

Lemma 5.9: Let $K \geq 4$. For a sufficiently large finite field $\mathbb{F}$, there exist $K - 3$ sets, $\mathcal{B}_\ell = \{\beta_{1,2}^{(\ell)}, \beta_{1,3}^{(\ell)}, \beta_{2,3}^{(\ell)}\} \subseteq \mathbb{F}^3$, $\ell = 4, \cdots, K$, such that the following properties are satisfied:

1) For any $\ell \in [K]$ and $\{i, j\} \subseteq \{1, 2, 3\}$, $\beta_{i,j}^{(\ell)} \in \langle \alpha_i, \alpha_j \rangle$;
2) For any $\ell \in [K]$ and $\{\gamma, \gamma'\} \subseteq \mathcal{B}_\ell$, $\bar{\alpha} \in \langle \gamma, \gamma' \rangle$;
3) For any pair $\{\gamma, \gamma'\} \subseteq \bigcup_{\ell=1}^{K} \mathcal{B}_\ell$, γ and γ' are linearly independent;
4) For any triple $\{\gamma, \gamma', \gamma''\} \subseteq \bigcup_{\ell=1}^{K} \mathcal{B}_\ell$ satisfying $\{\gamma, \gamma', \gamma''\} \nsubseteq \langle \alpha_i, \alpha_j \rangle$ $(\forall \{i, j\} \subseteq \{1, 2, 3\})$ and $\{\gamma, \gamma', \gamma''\} \neq \{\beta_{1,2}^{(\ell)}, \beta_{1,3}^{(\ell)}, \beta_{2,3}^{(\ell)}\}$ $(\forall \ell \in \{4, \cdots, K\})$, γ, γ' and γ'' are linearly independent.

Proof: We prove the lemma by way of induction. First of all, it is easy to check that $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ satisfy the properties 1)−4). Now, for $K \geq 4$ suppose there are $K-1$ sets $\mathcal{B}_1, \cdots, \mathcal{B}_{K-1}$ satisfy the properties 1)−4). In the following, we construct a subset $\mathcal{B}_K = \{\beta_{1,2}^{(K)}, \beta_{1,3}^{(K)}, \beta_{2,3}^{(K)}\}$ such that $\mathcal{B}_1, \cdots, \mathcal{B}_K$ satisfy the properties 1)−4). To do this, let Φ_{K-1} be the set of all pairs $\{\gamma', \gamma''\} \subseteq \bigcup_{\ell=1}^{K-1} \mathcal{B}_\ell$ such that $\{\gamma', \gamma''\} \nsubseteq \langle \alpha_i, \alpha_j \rangle$ for all $\{i, j\} \subseteq \{1, 2, 3\}$ and let

$$\Psi_{K-1} = \bigcup_{\{\gamma', \gamma''\} \in \Phi_{K-1}} \{\langle \gamma', \gamma'' \rangle_{1,2}, \langle \gamma', \gamma'' \rangle_{1,3}, \langle \gamma', \gamma'' \rangle_{2,3}\},$$

where for each $\{i, j\} \subseteq \{1, 2, 3\}$,

$$\langle \gamma', \gamma'' \rangle_{i,j} = \langle \gamma', \gamma'' \rangle \cap \langle \alpha_i, \alpha_j \rangle.$$

Since $\mathbb{F}$ is sufficiently large, there exists a $\beta^{(K)} \in \mathbb{F}^3$ such that $\beta^{(K)} \notin \langle \bar{\alpha}, \gamma \rangle$ for all $\gamma \in \Psi_{K-1}$. Then for each $\{i, j\} \subseteq \{1, 2, 3\}$, choose $\beta_{i,j}^{(K)}$ such that

$$\mathbf{0} \neq \beta_{i,j}^{(K)} \in \langle \beta^{(K)}, \bar{\alpha} \rangle \cap \langle \alpha_i, \alpha_j \rangle,$$

and let $\mathcal{B}_K = \{\beta_{1,2}^{(K)}, \beta_{1,3}^{(K)}, \beta_{2,3}^{(K)}\}$. Then, it is easy to see that $\mathcal{B}_1, \cdots, \mathcal{B}_K$ satisfy the properties 1)−4), which completes the proof. ■

Example 5.10: We give several examples to illustrate the construction in Lemma 5.9.

For $K = 4$, we have $\Phi_3 = \{\{\alpha_1, \alpha_2+\alpha_3\}, \{\alpha_2, \alpha_1+\alpha_3\}, \{\alpha_3, \alpha_1+\alpha_2\}, \{\alpha_1+\alpha_2, \alpha_1+\alpha_3\}, \{\alpha_1+\alpha_2, \alpha_2+\alpha_3\}, \{\alpha_1+\alpha_3, \alpha_2+\alpha_3\}\}$. Correspondingly, $\Psi_3 = \{\langle \alpha_1 \rangle, \langle \alpha_2+\alpha_3 \rangle\} \cup \{\langle \alpha_2 \rangle, \langle \alpha_1+\alpha_3 \rangle\} \cup \{\langle \alpha_3 \rangle, \langle \alpha_1+\alpha_2 \rangle\} \cup \{\langle \alpha_1+\alpha_2 \rangle, \langle \alpha_1+\alpha_3 \rangle, \langle \alpha_2-\alpha_3 \rangle\} \cup \{\langle \alpha_1+\alpha_2 \rangle, \langle \alpha_2+\alpha_3 \rangle, \langle \alpha_1-\alpha_3 \rangle\} \cup \{\langle \alpha_1+\alpha_3 \rangle, \langle \alpha_2+\alpha_3 \rangle, \langle \alpha_1-\alpha_2 \rangle\} = \{\langle \alpha_1 \rangle, \langle \alpha_2 \rangle, \langle \alpha_3 \rangle, \langle \alpha_1+\alpha_2 \rangle, \langle \alpha_1+\alpha_3 \rangle, \langle \alpha_2+\alpha_3 \rangle, \langle \alpha_1-\alpha_2 \rangle, \langle \alpha_1-\alpha_3 \rangle, \langle \alpha_2-\alpha_3 \rangle\}$. Then we can choose $\beta^{(4)} = \alpha_1 + 3\alpha_2$ and $\beta_{1,2}^{(4)} = \alpha_1 + 3\alpha_2$, $\beta_{1,3}^{(4)} = 2\alpha_1 + 3\alpha_3$, $\beta_{2,3}^{(4)} = 2\alpha_2 - \alpha_3$, and furthermore, we obtain $\mathcal{B}_4 = \{\alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_3, 2\alpha_2 - \alpha_3\}$.

For $K = 5$, we have $\Phi_4 = \Phi_3 \cup \{\{\alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_3\}, \{\alpha_1 + 3\alpha_2, 2\alpha_2 - \alpha_3\}, \{2\alpha_1 + 3\alpha_3, 2\alpha_2-\alpha_3\}, \{\alpha_1+3\alpha_2, \alpha_3\}, \{\alpha_1+3\alpha_2, \alpha_1+\alpha_3\}, \{\alpha_1+3\alpha_2, \alpha_2+\alpha_3\}, \{2\alpha_1+3\alpha_3, \alpha_2\}, \{2\alpha_1+3\alpha_3, \alpha_1+\alpha_2\}, \{2\alpha_1+3\alpha_3, \alpha_2+\alpha_3\}, \{2\alpha_2-\alpha_3, \alpha_1\}, \{2\alpha_2-\alpha_3, \alpha_1+\alpha_2\}, \{2\alpha_2-\alpha_3, \alpha_1+\alpha_3\}\}$. Correspondingly, $\Psi_4 = \Psi_3 \cup \{\langle \alpha_1+3\alpha_2 \rangle, \langle 2\alpha_1+3\alpha_3 \rangle, \langle 2\alpha_2-\alpha_3 \rangle\} \cup \{\langle \alpha_1+3\alpha_2 \rangle, \langle \alpha_3 \rangle\} \cup \{\langle \alpha_1+3\alpha_2 \rangle, \langle \alpha_1+\alpha_3 \rangle, \langle 3\alpha_2-\alpha_3 \rangle\} \cup \{\langle \alpha_1+3\alpha_2 \rangle, \langle \alpha_2+\alpha_3 \rangle, \langle \alpha_1-3\alpha_3 \rangle\} \cup \{\langle 2\alpha_1+3\alpha_3 \rangle, \langle \alpha_2 \rangle\} \cup \{\langle 2\alpha_1+3\alpha_3 \rangle, \langle \alpha_1+\alpha_2 \rangle, \langle 3\alpha_3-2\alpha_2 \rangle\} \cup \{\langle 2\alpha_1+3\alpha_3 \rangle, \langle \alpha_2+\alpha_3 \rangle, \langle 2\alpha_1-3\alpha_2 \rangle\} \cup \{\langle 2\alpha_2-\alpha_3 \rangle, \langle \alpha_1 \rangle\} \cup \{\langle 2\alpha_2-\alpha_3 \rangle, \langle \alpha_1+\alpha_2 \rangle, \langle 2\alpha_1+\alpha_3 \rangle\} \cup \{\langle 2\alpha_2-\alpha_3 \rangle, \langle \alpha_1+\alpha_3 \rangle, \langle \alpha_1+2\alpha_2 \rangle\} = \Psi_3 \cup \{\langle \alpha_1+3\alpha_2 \rangle, \langle 2\alpha_1+3\alpha_3 \rangle, \langle 2\alpha_2-\alpha_3 \rangle, \langle 3\alpha_2-\alpha_3 \rangle, \langle \alpha_1-3\alpha_3 \rangle, \langle 3\alpha_3-2\alpha_2 \rangle, \langle 2\alpha_1-3\alpha_2 \rangle, \langle 2\alpha_1+\alpha_3 \rangle, \langle \alpha_1+2\alpha_2 \rangle\}$. Then we can choose $\beta^{(5)} = 2\alpha_1 + 3\alpha_2$ and $\beta_{1,2}^{(5)} = 2\alpha_1 + 3\alpha_2$, $\beta_{1,3}^{(5)} = \alpha_1 + 3\alpha_3$, $\beta_{2,3}^{(5)} = \alpha_2 - 2\alpha_3$, and furthermore, we obtain $\mathcal{B}_5 = \{2\alpha_1 + 3\alpha_2, \alpha_1 + 3\alpha_3, \alpha_2 - 2\alpha_3\}$.

In a similar fashion, we can inductively construct $\mathcal{B}_6, \mathcal{B}_7$, and so on.

We next state and prove a key lemma in this section, which will also be used as a basic tool in the sequel.

Lemma 5.11: If there exists a regular partition of Π, then we can construct a solution to Π and hence $\text{RG}(D)$ is solvable.

Proof: Let $\mathcal{I} = \{\Delta_1, \Delta_2, \ldots, \Delta_K\}$ be a regular partition of Π, where we have chosen Δ_i to be $[S_i]$ for any $i = 1, 2, 3$. For $i \in \{1, 2, 3\}$, let

$$d_R = \begin{cases} \beta_i^{(i)} = \alpha_i, & \text{if } R \in [S_i]_i, \\ \beta_{j,k}^{(i)} = \alpha_j + \alpha_k, & \text{if } R \in [S_i]_{j,k}, \end{cases} \tag{6}$$

where $\{j, k\} = \{1, 2, 3\} \backslash \{i\}$. If $4 \le \ell \le K$, then for any $\{j, k\} \subseteq \{1, 2, 3\}$ and any $R \in [\Delta_\ell]_{j,k}$, let

$$d_R = \beta_{j,k}^{(\ell)},$$

where $\{\beta_{1,2}^{(\ell)}, \beta_{1,3}^{(\ell)}, \beta_{2,3}^{(\ell)}\}$, $\ell = 4, \cdots, K$, are constructed by Lemma 5.9. We will prove that $\tilde{C}_\Pi := \{d_R; R \in \Pi\}$ is a solution to Π in the following three steps:

1) Since $\mathcal{I}$ is a regular partition, $\mathcal{I}$ is non-singular, and so $[S_i] \neq [S_j]$ if $i \neq j$. It then follows that $d_{S_i} = \alpha_i, i = 1, 2, 3$.

2) In the following, for any $R \in \Pi \setminus \{S_1, S_2, S_3\}$, we prove $d_R \in \langle d_Q; Q \in \text{In}(R) \rangle$. Firstly, it follows from $\mathcal{I}$ has no equivalent classes that we have $[\Delta_\ell]_{i,j} = \text{reg}([\Delta_\ell]_{i,j})$ for each $\ell \in [K]$ and $\{i, j\} \subseteq \{1, 2, 3\}$ (since otherwise there would exist an $R \in \text{reg}([\Delta_\ell]_{i,j}) \backslash [\Delta_\ell]_{i,j}$ and therefore $[R]$ and $[\Delta_\ell]$ are equivalent, a contradiction). Hence, we need consider the following two cases:

- $\text{In}(R) \subseteq [[R]]$. In this case, R and $\text{In}(R)$ are in the same subclass. Clearly, $d_R \in \langle d_Q; Q \in \text{In}(R) \rangle$.
- There exist $R', R'' \in \text{In}(R)$ that belong to two different subclasses. In this case, $\{R, R, R''\} \subseteq \text{reg}(S_j, S_k)$ for some $\{j, k\} \subseteq \{1, 2, 3\}$. Then, by 1) of Lemma 5.9, $d_R, d_{R'}, d_{R''} \in \langle \alpha_j, \alpha_k \rangle$. By 3) of Lemma 5.9, we infer that $d_{R'}$, $d_{R''}$ are linearly independent, and thereby $d_R \in \langle d_{R'}, d_{R''} \rangle = \langle \alpha_j, \alpha_k \rangle = \langle d_Q; Q \in \text{In}(R) \rangle$.

Based on 1) and 2), we conclude now that $\tilde{C}_\Pi$ is a linear network code on Π. We next show that $\tilde{C}_\Pi$ is a solution to Π, for which we need to deal with the following two cases:

- Λ_j intersects with at least two different subclasses of some $\Delta_\ell \in \mathcal{I}$. Assume $\{Q', Q''\} \subseteq \Lambda_j$ and Q', Q'' belong to different subclasses of Δ_ℓ. By the construction of $\tilde{C}_\Pi$, we have $\{d_{Q'}, d_{Q''}\} \subseteq \mathcal{B}_\ell$, so by 2) of Lemma 5.9, $\bar{\alpha} \in \langle d_{Q'}, d_{Q''} \rangle \subseteq \langle d_R; R \in \Lambda_j \rangle$.
- For each class Δ_ℓ, Λ_j intersects with at most one subclass of Δ_ℓ. Then Λ_j intersects with at least three distinct classes because if only two classes intersect with Λ_j, they would be equivalent and hence $\mathcal{I}$ would not be regular. Moreover, by Definition 5.8, $\Lambda_j \nsubseteq [S_{i_1}]_{i_1} \cup [S_{i_2}]_{i_2} \cup \text{reg}(S_{i_1}, S_{i_2})$ for all $\{i_1, i_2\} \subseteq \{1, 2, 3\}$. So, suppose there exist three distinct subclasses $[[Q]], [[Q']], [[Q'']]$ intersect with Λ_j such that $[[Q]] \cup [[Q']] \cup [[Q'']] \nsubseteq [S_{i_1}]_{i_1} \cup [S_{i_2}]_{i_2} \cup \text{reg}(S_{i_1}, S_{i_2})$, Then, by construction of the code and 4) of Lemma 5.9,$d_Q, d_{Q'}$ and $d_{Q''}$ are linearly independent, and hence $\bar{\alpha} \in \mathbb{F}^3 = \langle d_Q, d_{Q'}, d_{Q''} \rangle \subseteq \langle d_R; R \in \Lambda_j \rangle$.

Finally, we note that the above discussions immediately lead to the conclusion that $\tilde{C}_\Pi = \{d_R; R \in \Pi\}$ is a solution to Π, as desired. ■

Now, we are ready to state and prove the main result of this section.

Theorem 5.12: The following statements are equivalent for a terminal-separable $3\text{s}/n\text{t}$ sum-network G:

1) $\text{RG}(D)$ is solvable.
2) Any characteristic partition of Π is regular.
3) There exists a regular partition of Π.

Proof: Note that $2) \Rightarrow (3)$ is trivial and $3) \Rightarrow (1)$ is nothing but Lemma 5.11. So in the following we only need to prove $1) \Rightarrow 2)$.

Suppose that RG(D) is solvable. Let $\tilde{C}_\Pi = \{d_R \in \mathbb{F}^3; R \in \Pi\}$ is a solution to Π and let $\mathcal{I}_c = \{\Delta_1, \Delta_2, \ldots, \Delta_K\}$ be a characteristic partition of Π.

First of all, we note that $\mathcal{I}_c$ is non-singular, that is, $[S_i] \neq [S_j]$ for any distinct $i, j \subseteq \{1,2,3\}$, since otherwise we would have, by 1) of Lemma 5.7, $\langle \alpha_i, \bar{\alpha} \rangle = \langle \alpha_j, \bar{\alpha} \rangle$, which is impossible since $\langle \alpha_i, \alpha_j, \bar{\alpha} \rangle = \mathbb{F}^3$. Then, by Definition 5.5, there is no equivalent classes in $\mathcal{I}_c$. Moreover, we can have that $\Lambda_\ell \nsubseteq [S_j]_j \cup [S_k]_k \cup \text{reg}(S_j, S_k)$ for any $\ell \in [n]$ and any $\{j,k\} \subseteq \{1,2,3\}$ since otherwise, noticing that $d_R = \alpha_i$ for all $R \in [S_i]_i$ $(i \in \{1,2,3\})$, we will have $\bar{\alpha} \in \langle \alpha_j, \alpha_k \rangle$, a contradiction. ■

Remark 5.13: By Theorem 5.12, the regularity of characteristic partition $\mathcal{I}_c$ determines the solvability of a terminal-separable 3s/nt network. By Remark 5.6, this can be done in a polynomial in $\{|\Pi|, n\}$ time.

As a corollary of Theorem 5.12, some sufficient conditions for the solvability of a 3s/nt sum-network G, which will be used in the next section, can be derived as follows.

Corollary 5.14: G is solvable if one of the following conditions hold.

1) $\Lambda_{j_1} \cap \Lambda_{j_2} = \emptyset$ for all $\{j_1, j_2\} \subseteq [n]$ with $|\Lambda_{j_1}| = |\Lambda_{j_2}| = 2$;
2) $\Lambda_j \subseteq \Pi \backslash \{S_1, S_2, S_3\}$ for all $j \in [n]$;
3) There exists a subset $\{i_1, i_2\} \subseteq \{1,2,3\}$ such that for all $j \in [n]$, $\Lambda_j \cap \text{reg}^\circ(S_{i_1}, S_{i_2}) \neq \emptyset$.

Proof: (1) Let A be the set of all $j \in [n]$ such that $|\Lambda_j| = 2$ and let

$$\mathcal{I} = \{\Lambda_j; j \in A\} \cup \{[R]; R \in \Pi \backslash (\cup_{j \in A} \Lambda_j)\},$$

where $[R] = \{R\}$ for all $R \in \Pi \backslash (\cup_{j \in A} \Lambda_j)$. It is easy to check by definition that $\mathcal{I}$ is regular, hence by Lemma 5.11, RG(D) and thereby G is solvable (Fig. 10 gives such an example and a solution).

(2) Let $\mathcal{I} = \{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$, where $\Delta_i = \{S_i\}$, $i = 1,2,3$, and $\Delta_4 = \Pi \backslash \{S_1, S_2, S_3\}$. Then $\mathcal{I}$ is regular and hence G is solvable (Fig. 11 (a) gives such an example).

(3) Let $\{i_3\} = \{1,2,3\} \backslash \{i_1, i_2\}$ and let $\mathcal{I} = \{[S_{i_1}], [S_{i_2}], [S_{i_3}]\}$, where $[S_{i_1}] = \{S_{i_1}\}$, $[S_{i_2}] = \{S_{i_2}\}$ and $[S_{i_3}] = \Pi \backslash \{S_{i_1}, S_{i_2}\}$. The claimed result then follows from the fact that $\mathcal{I}$ is regular (see Fig. 11 (b) for an example). ■

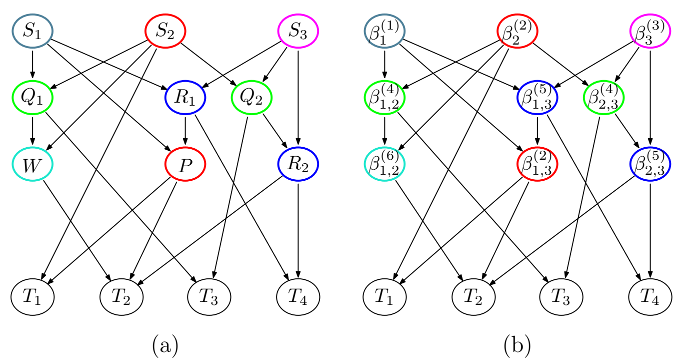


Fig 10. A network and a solution exemplifies (1) of Corollary 5.14: (a) is a region graph with $\Lambda_1 = \{S_2, P\}, \Lambda_2 = \{W, P, R_2\}$, $\Lambda_3 = \{Q_1, Q_2\}, \Lambda_4 = \{R_1, R_2\}$. Let $\mathcal{I} = \{[S_1], [S_2], [S_3], [Q_1], [R_1], [W]\}$, where $[S_1] = \{S_1\}$, $[S_2] = \{S_2, P\}$, $[S_3] = \{S_3\}$, $[Q_1] = \{Q_1, Q_2\}$, $[R_1] = \{R_1, R_2\}$ and $[W] = \{W\}$. (b) illustrates a solution to Π.

VI. Forbidden Structures of Solvable 3-source 3-terminal Sum-Networks

In this section, we focus on the case $k = n = 3$, that is, G is a 3s/3t sum-network (not necessarily terminal-separable). We will show that G is solvable if and only if its basic region graph RG(D) does not contain certain forbidden structures.

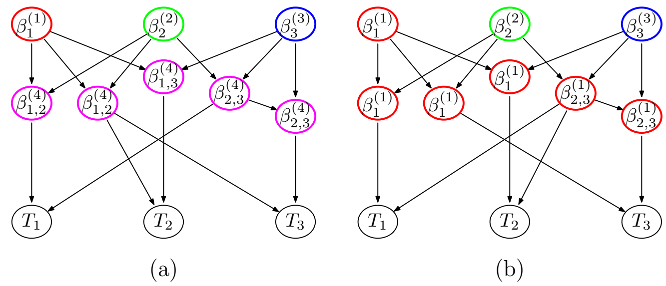


Fig 11. A network and its solutions exemplify (2) and (3) of Corollary 5.14: in (a), $\Lambda_\ell \subseteq \Pi \backslash \{S_1, S_2, S_3\}$ for $\ell = \{1, 2, 3\}$; in (b), $\Lambda_\ell \cap \text{reg}^\circ(S_2, S_3) \neq \emptyset$ for $\ell = 1, 2, 3$.

More specifically, we have the following theorem.

Theorem 6.1: G is unsolvable if and only if $\text{RG}(D)$ is terminal-separable and, by properly labelling the regions, the following condition (FS stands for forbidden structures) holds:

(**FS**) There exist $P_1 \in \text{reg}^\circ(S_2, S_3)$ and $P_2 \in \text{reg}^\circ(S_1, S_2)$ such that $\Lambda_i = \{S_1, P_1\}$, $\Lambda_j = \{P_1, P_2\}$ and $\Lambda_k \subseteq \text{reg}(S_1, P_2) \cup \text{reg}(S_1, S_3)$, where $\{i, j, k\} = \{1, 2, 3\}$. (see Fig. 7 for two typical examples satisfying the condition (FS) with $i = 1, j = 2, k = 3$).

By Theorem 3.3, $\text{RG}(D)$ can be computed from G in time $O(|E|)$. Moreover, by Theorem 4.5, determining whether $\text{RG}(D)$ is terminal-separable can be done in time $O(|D|)$. Hence, Theorem 6.1 suggests an $O(|E|)$ time algorithm to determine the solvability of a 3s/3t sum-network, which improves upon the previous $O(|E|^3)$ time complexity result in [6].

Before proving Theorem 6.1, we first prove the following lemma.

Lemma 6.2: If $\text{RG}(D)$ is unsolvable, then $\text{RG}(D)$ has three terminal regions and is terminal-separable.

Proof: Note that the sum-network with one terminal region is always solvable. We first prove that $\text{RG}(D)$ is solvable if it has two terminal regions. If $\text{RG}(D)$ is not terminal-separable, then by Theorem 4.7, there exists a terminal-separable subgraph $\text{RG}(\overline{D})$ with one terminal region, which is clearly solvable, hence, $\text{RG}(D)$ is solvable. If $\text{RG}(D)$ is terminal-separable, we consider the following cases.

- There exists $i \in \{1, 2\}$ such that $|\Lambda_i| \geq 3$ or $|\Lambda_1| = |\Lambda_2| = 2$, $\Lambda_1 \cap \Lambda_2 = \emptyset$. Then, $\text{RG}(D)$ is solvable by 1) of Corollary 5.14.
- $|\Lambda_1| = |\Lambda_2| = 2$ and $\Lambda_1 \cap \Lambda_2 \neq \emptyset$. In this case, we assume $\Lambda_1 = \{Q_1, Q_2\}$ and $\Lambda_2 = \{Q_1, Q_3\}$ and have the following cases.
  Case 1: $Q_1 = S_i$ for some $i \in \{1, 2, 3\}$. Without loss of generality, assume $Q_1 = S_1$. Then, since $\Lambda_1 \nsubseteq \text{reg}(S_1, S_2)$ and $\Lambda_1 \nsubseteq \text{reg}(S_1, S_3)$, we have $Q_2 \in \text{reg}^\circ(S_2, S_3)$. Similarly, we have $Q_3 \in \text{reg}^\circ(S_2, S_3)$. Hence, $\text{RG}(D)$ is solvable by 3) of Corollary 5.14.
  Case 2: $Q_1 \in \text{reg}^\circ(S_j, S_k)$ for distinct $\{j, k\} \subseteq \{1, 2, 3\}$. Again, by 3) of Corollary 5.14, $\text{RG}(D)$ is solvable.

Combine the above discussions, we conclude that $\text{RG}(D)$ is solvable if it has two terminal regions.

Now, suppose $\text{RG}(D)$ has three terminal regions and we prove that $\text{RG}(D)$ is solvable if it is not terminal-separable. By Theorem 4.7, there exists a terminal-separable subgraph $\text{RG}(\overline{D})$ with $n' < 3$ terminal regions, which is solvable by above discussions, and hence, $\text{RG}(D)$ is solvable, which completes the proof. ■

We also need the following four lemmas, for which we will assume that $\text{RG}(D)$ is terminal-separable.

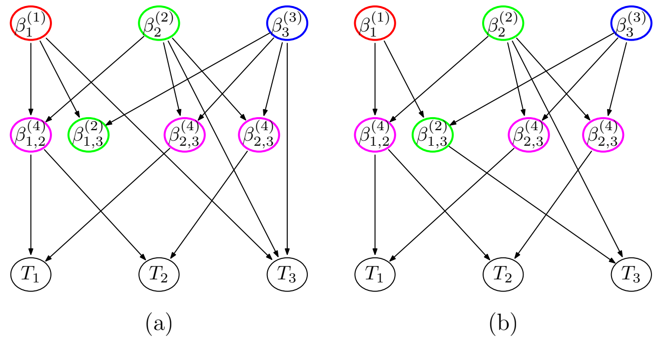


Fig 12. Example of the code construction for Lemma 6.3: (a) is for $\{S_1, S_2, S_3\} \subseteq \Lambda_3$; (b) is for $\{S_2, P\} \subseteq \Lambda_3$.

Lemma 6.3: Suppose $P_1 \in \text{reg}^\circ(S_1, S_2)$ and $P_2, P_3 \in \text{reg}^\circ(S_2, S_3)$ such that $\Lambda_1 = \{P_1, P_2\}$ and $\Lambda_2 = \{P_1, P_3\}$. Then, $\text{RG}(D)$ is solvable.

Proof: If $\Lambda_3 \cap (\text{reg}^\circ(S_1, S_2) \cup \text{reg}^\circ(S_2, S_3)) \neq \emptyset$, then $\text{RG}(D)$ satisfies the condition 3) of Corollary 5.14, and hence is solvable. So we assume in the following that

$$\Lambda_3 \subseteq \Pi \setminus (\text{reg}^\circ(S_1, S_2) \cup \text{reg}^\circ(S_2, S_3)) = \{S_2\} \cup \text{reg}(S_1, S_3).$$

Note that $\Lambda_3 \not\subseteq \text{reg}(S_1, S_3)$. Then we have $S_2 \in \Lambda_3$. Similarly, since $\Lambda_3 \not\subseteq \text{reg}(S_1, S_2)$ and $\Lambda_3 \not\subseteq \text{reg}(S_2, S_3)$, we deduce that either $\{S_1, S_2, S_3\} \subseteq \Lambda_3$ or $\{S_2, P\} \subseteq \Lambda_3$ for some $P \in \text{reg}^\circ(S_1, S_3)$. For both cases, letting $\mathcal{I} = \{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$, where $\Delta_1 = \{S_1\}$, $\Delta_2 = \{S_2\} \cup \text{reg}^\circ(S_1, S_3)$, $\Delta_3 = \{S_3\}$ and $\Delta_4 = \text{reg}^\circ(S_1, S_2) \cup \text{reg}^\circ(S_2, S_3)$, we verify that $\mathcal{I}$ is a regular partition of Π, and thereby $\text{RG}(D)$ is solvable (See Fig. 12 for an example). ■

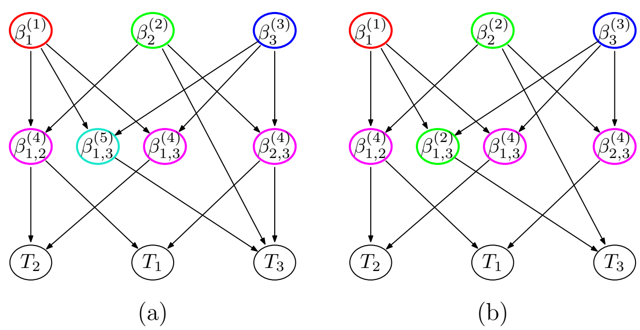


Fig 13. Examples of the code construction for Lemma 6.4: (a) illustrates a code for $|\Lambda_3| \geq 3$; (b) illustrates a solution for $\Lambda_3 = \{S_2, P\}$.

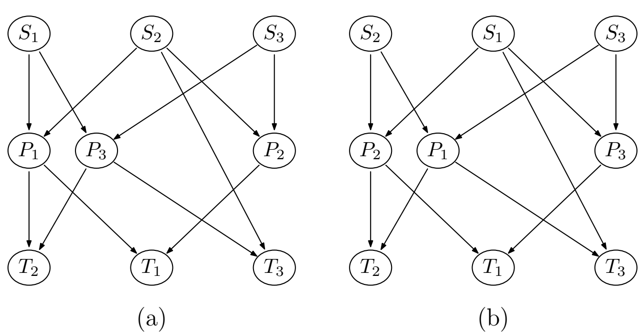


Fig 14. Illustration of region relabelling in the proof of Lemma 6.4: (b) is obtained from (a) by interchanging the label of S_1 and S_2, and relabel P_1, P_2 and P_3 as P_2, P_3 and P_1, respectively.

Lemma 6.4: If $P_1 \in \text{reg}^\circ(S_1, S_2)$, $P_2 \in \text{reg}^\circ(S_2, S_3)$ and $P_3 \in \text{reg}^\circ(S_1, S_3)$ such that $\Lambda_1 = \{P_1, P_2\}$ and $\Lambda_2 = \{P_1, P_3\}$, then $\text{RG}(D)$ is unsolvable only if the condition (FS) holds.

Proof: First of all, it holds that $\Lambda_3 \cap \{S_1, S_2, S_3\} \neq \emptyset$ (since otherwise, by (2) of Corollary 5.14, $\text{RG}(D)$ would be solvable, which contradicts the assumption) and $|\Lambda_3| = 2$ (since otherwise, the partition $\mathcal{I} = \{[P_1]\} \cup \{[R]; R \in \Pi \backslash [P_1]\}$, where $[P_1] = \{P_1, P_2, P_3\}$ and $[R] = \{R\}$ for all $R \in \Pi \backslash [P_1]$ would be regular and hence $\text{RG}(D)$ would be solvable, as exemplified in Fig. 13 (a), which contradicts the assumption). It then follows that $\Lambda_3 = \{S_i, P\}$ for some $i \in \{1, 2, 3\}$. Moreover, note that $\Lambda_3 \nsubseteq \text{reg}(S_i, S_j)$ and $\Lambda_3 \nsubseteq \text{reg}(S_i, S_k)$, where $\{j, k\} = \{1, 2, 3\} \backslash \{i\}$, therefore $P \in \text{reg}^\circ(S_j, S_k)$. Consequently, we only need to consider the following three cases:

Case 1: $\Lambda_3 = \{S_3, P\}$ and $P \in \text{reg}^\circ(S_1, S_2)$. Then $\text{RG}(D)$ satisfies the condition (3) of Corollary 5.14, and hence is solvable.

Case 2: $\Lambda_3 = \{S_2, P\}$ and $P \in \text{reg}^\circ(S_1, S_3)$. If $P \neq P_3$, then it can be verified that $\mathcal{I}$ is regular, where $\mathcal{I} = \{[S_2], [P_1]\} \cup \{[R]; R \in \Pi \backslash ([S_2] \cup [P_1])\}$ with $[S_2] = \{S_2, P\}$, $[P_1] = \{P_1, P_2, P_3\}$ and $[R] = \{R\}$ for all $R \in \Pi \backslash ([S_2] \cup [P_1])$. It then follows that $\text{RG}(D)$ is solvable, as exemplified in Fig. 13 (b). Hence, in this case, $\text{RG}(D)$ is unsolvable only if $P = P_3$.

Case 3: $\Lambda_3 = \{S_1, P\}$ and $P \in \text{reg}^\circ(S_2, S_3)$. Via a parallel argument as in Case 2, we infer that $\text{RG}(D)$ is unsolvable only if $P = P_2$.

Combining the above three cases, we conclude that if $\text{RG}(D)$ is unsolvable, then either $\Lambda_3 = \{S_1, P_2\}$ or $\Lambda_3 = \{S_2, P_3\}$. If $\Lambda_3 = \{S_2, P_3\}$, then, upon an appropriate relabelling (see Fig. 14), the condition (FS) holds. Similarly, if $\Lambda_3 = \{S_1, P_2\}$, then by interchanging the label of P_1 and P_2, the condition (FS) also holds. ■

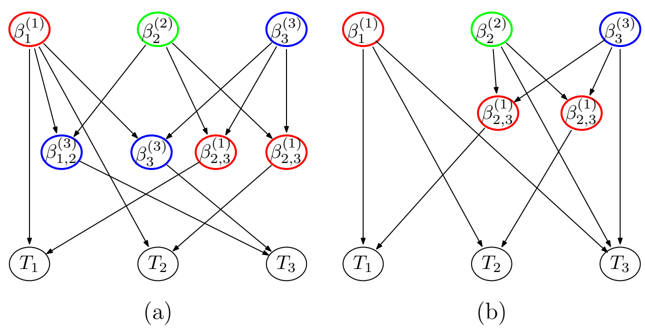


Fig 15. Examples of code construction for Lemma 6.5: (a) illustrates a solution for Case 1 and (b) illustrates a solution for Case 2.

Lemma 6.5: Suppose that $\Lambda_1 = \{S_1, P\}$ and $\Lambda_2 = \{S_1, Q\}$ where $P, Q \in \text{reg}^\circ(S_2, S_3)$. Then, $\text{RG}(D)$ is solvable.

Proof: If $P = Q$, then $\Lambda_1 = \Lambda_2$ and the proof is the same as the case that $\text{RG}(D)$ has two terminal regions. So we assume that $P \neq Q$.

If $\Lambda_3 \cap \text{reg}^\circ(S_2, S_3) \neq \emptyset$, then $\text{RG}(D)$ satisfies the condition (3) of Corollary 5.14, and hence is solvable. So in the following, we assume

$$\Lambda_3 \subseteq \Pi \setminus \text{reg}^\circ(S_2, S_3) = \text{reg}(S_1, S_2) \cup \text{reg}(S_1, S_3).$$

We need to consider the following two cases:

Case 1: $\Lambda_3 \cap (\text{reg}^\circ(S_1, S_2) \cup \text{reg}^\circ(S_1, S_3)) \neq \emptyset$. Without loss of generality, assume $Q_1 \in \Lambda_3 \cap \text{reg}^\circ(S_1, S_2)$. Since $\Lambda_3 \nsubseteq \text{reg}(S_1, S_2)$, there exists a $Q_2 \in \Lambda_3 \cap \text{reg}(S_1, S_3) \backslash \{S_1\}$. Let $\mathcal{I} = \{\Delta_1, \Delta_2, \Delta_3\}$, where $\Delta_1 = \{S_1\} \cup \text{reg}^\circ(S_2, S_3)$, $\Delta_2 = \{S_2\}$ and $\Delta_3 = \{S_3\} \cup \text{reg}^\circ(S_1, S_2) \cup \text{reg}^\circ(S_1, S_3)$. Then, $\mathcal{I}$ is regular and hence $\text{RG}(D)$ is solvable, as illustrated in Fig. 15 (a).

Case 2: $\Lambda_3 \cap (\text{reg}^\circ(S_1, S_2) \cup \text{reg}^\circ(S_1, S_3)) = \emptyset$. Then $\Lambda_3 \subseteq \{S_1, S_2, S_3\}$. Note that $\Lambda_3 \nsubseteq \text{reg}(S_j, S_k)$ for all $\{j, k\} \subseteq \{1, 2, 3\}$, we have that $\Lambda_3 = \{S_1, S_2, S_3\}$. Let $\mathcal{I} = \{[S_1]\} \cup \{[R]; R \in$

$\Pi\backslash[S_1]\}$, where $[S_1] = \{S_1\} \cup \text{reg}^\circ(S_2, S_3)$ and $[R] = \{R\}$ for all $R \in \Pi\backslash[S_1]$. Then $\mathcal{I}$ is regular and hence RG(D) is solvable, as illustrated in Fig. 15 (b). ■

Lemma 6.6: Suppose $P_1 \in \text{reg}^\circ(S_2, S_3)$ and $P_2 \in \text{reg}^\circ(S_1, S_2)$ such that $\Lambda_i = \{S_1, P_1\}$ and $\Lambda_j = \{P_1, P_2\}$, where $\{i, j\} \subseteq \{1, 2, 3\}$. If RG$(D)$ is unsolvable then the condition (FS) holds.

Proof: W.l.o.g., assume $i = 1$ and $j = 2$. If $\Lambda_3 \cap \text{reg}^\circ(S_2, S_3) \neq \emptyset$, then RG$(D)$ satisfies the condition (3) of Corollary 5.14, and hence is solvable, which contradicts the assumption. So we have

$$\Lambda_3 \subseteq \Pi \setminus \text{reg}^\circ(S_2, S_3) = \text{reg}(S_1, S_2) \cup \text{reg}(S_1, S_3). \tag{7}$$

Henceforth, assuming

$$\Lambda_3 \nsubseteq \text{reg}(S_1, P_2) \cup \text{reg}(S_1, S_3), \tag{8}$$

we will prove that RG(D) is solvable.

Firstly, since $P_2 \in \text{reg}^\circ(S_1, S_2)$, then $\text{reg}(S_1, P_2) \subseteq \text{reg}(S_1, S_2)$, and by (7) and (8), there exists a $P_3 \in \Lambda_3 \cap \text{reg}(S_1, S_2)\backslash\text{reg}(S_1, P_2)$. Moreover, since $\Lambda_3 \nsubseteq \text{reg}(S_1, S_2)$, there exists a $P_4 \in \Lambda_3 \cap \text{reg}(S_1, S_3)\backslash\{S_1\}$. We consider the following two cases.

Case 1: $|\Lambda_3| \geq 3$. In this case, let $\mathcal{I} = \{[S_1]\} \cup \{[R]; R \in \Pi\backslash[S_1]\}$, where $[S_1] = \text{reg}(S_1, P_2) \cup \{P_1\}$ and $[R] = \{R\}$ for all $R \in \Pi\backslash[S_1]\}$. It can be verified that $\mathcal{I}$ is regular, so, by Lemma 5.11, RG(D) is solvable (An illustrative example is given in Fig. 16).

Case 2: $|\Lambda_3| = 2$. In this case $\Lambda_3 = \{P_3, P_4\}$ and we let $\mathcal{I} = \{[S_1], [P_3]\} \cup \{[R]; R \in \Pi\backslash[S_1] \cup [P_3]\}$, where $[S_1] = \text{reg}(S_1, P_2) \cup \{P_1\}$, $[P_3] = \{P_3, P_4\}$ and $[R] = \{R\}$ for all $R \in \Pi\backslash[S_1] \cup [P_3]\}$. It can be verified that $\mathcal{I}$ is regular, so, by Lemma 5.11, RG(D) is solvable (An illustrative example is given in Fig. 17).

Finally, we conclude that if RG(D) is unsolvable, then (8) is violated, and so $\Lambda_3 \subseteq \text{reg}(S_1, P_2) \cup \text{reg}(S_1, S_3)$, which, in combination with the lemma assumptions, implies that the condition (FS) holds. ■

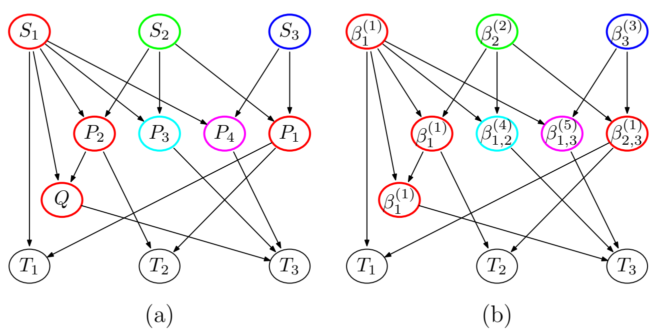


Fig 16. An example of the code construction for Case 1 of Lemma 6.6: (a) is a region graph and (b) is a solution.

We are now ready to prove Theorem 6.1. By Theorem 3.4, it suffices to prove that RG(D) is unsolvable if and only if it is terminal-separable and satisfies the condition (FS).

Proof of Necessity: Suppose RG(D) is unsolvable. First of all, by Lemma 6.2, RG(D) has three terminal regions and is terminal-separable. If for any $\{j_1, j_2\} \subseteq \{1, 2, 3\}$ with $|\Lambda_{j_1}| = |\Lambda_{j_2}| = 2$, $\Lambda_{j_1} \cap \Lambda_{j_2} = \emptyset$, then by (1) of Corollary 5.14, RG(D) is solvable. In what follows, w.l.o.g, we suppose there exist three regions $P_1, P_2, P_3 \in \Pi$ such that

$$\Lambda_1 = \{P_1, P_2\} \text{ and } \Lambda_2 = \{P_1, P_3\}.$$

Consider the following two cases:

Case 1: $\{P_1, P_2, P_3\} \subseteq \Pi\backslash\{S_1, S_2, S_3\}$.

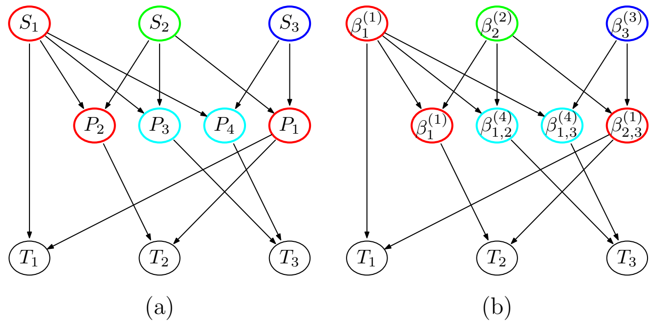


Fig 17. An example of the code construction for Case 2 of Lemma 6.6: (a) is the region graph and (b) is a solution.

Since $\Lambda_1 \nsubseteq \text{reg}(S_{i_1}, S_{i_2})$ for all $\{i_1, i_2\} \subseteq \{1, 2, 3\}$, so, w.l.o.g, we assume

$$P_1 \in \text{reg}^\circ(S_1, S_2) \text{ and } P_2 \in \text{reg}^\circ(S_2, S_3).$$

Similarly, from $\Lambda_2 \nsubseteq \text{reg}(S_{i_1}, S_{i_2})$ for any $\{i_1, i_2\} \subseteq \{1, 2, 3\}$ it follows that

$$P_3 \in \text{reg}^\circ(S_2, S_3) \text{ or } P_3 \in \text{reg}^\circ(S_1, S_3).$$

If $P_3 \in \text{reg}^\circ(S_2, S_3)$, then by Lemma 6.3, RG$(D)$ is solvable, which contradicts our assumption; if $P_3 \in \text{reg}^\circ(S_1, S_3)$, then, by Lemma 6.4, the condition (FS) holds.

Case 2: $\{P_1, P_2, P_3\} \cap \{S_1, S_2, S_3\} \neq \emptyset$.

W.l.o.g, we assume $S_1 \in \{P_1, P_2, P_3\}$. Then, we declare that $S_1 \neq P_1$, since otherwise, noting that $\Lambda_1 = \{P_1, P_2\}$ and $\Lambda_2 = \{P_1, P_3\}$, we would have that $P_2, P_3 \in \text{reg}^\circ(S_2, S_3)$, which, by Lemma 6.5, further implies that RG(D) is solvable, which contradicts our assumption. So $S_1 = P_2$ or $S_1 = P_3$. W.l.o.g, we assume $S_1 = P_3$. Since $\{P_1, S_3\} = \Lambda_2 \nsubseteq \text{reg}(S_1, S_2)$ and $\Lambda_2 \nsubseteq \text{reg}(S_1, S_3)$, we have $P_1 \in \text{reg}^\circ(S_2, S_3)$, which further indicates $P_2 \in \text{reg}^\circ(S_1, S_2) \cup \text{reg}^\circ(S_1, S_3) \cup \{S_1\}$. Note that $P_2 \neq S_1$, since otherwise, by Lemma 6.5, RG(D) is solvable, which contradicts our assumption, so $P_2 \in \text{reg}^\circ(S_1, S_2)$ or $P_2 \in \text{reg}^\circ(S_1, S_3)$. W.l.o.g, we assume $P_2 \in \text{reg}^\circ(S_1, S_2)$. Then by Lemma 6.6, the condition (FS) holds.

Having dealt with all the cases, we now conclude that if RG(D) is unsolvable, then the condition (FS) holds. ■

Proof of Sufficiency: Suppose RG(D) is terminal-separable and satisfies the condition (FS). We consider a characteristic partition $\mathcal{I}_c$ of RG(D). If $\mathcal{I}_c$ is singular, then RG(D) is unsolvable, so we assume that $\mathcal{I}_c$ is non-singular. By definition, $\mathcal{I}_c$ has no equivalent classes. Note that by condition (FS), $\Lambda_i = \{S_1, P_1\}, \Lambda_j = \{P_1, P_2\}$ and $P_2 \in \text{reg}^\circ(S_1, S_2)$. So $\{P_1, P_2\} \subseteq [S_1]$ and $\text{reg}(S_1, P_2) \subseteq [S_1]_1$. Again by condition (FS), $\Lambda_k \subseteq [S_1]_1 \cup \text{reg}(S_1, S_3)$, which indicates that $\mathcal{I}_c$ is not regular and consequentially we have that RG(D) is unsolvable by Theorem 5.12. ■

VII. Conclusions Remarks

Employing the region decomposition approach, we have in this paper obtained necessary and sufficient conditions for the solvability of the so-called "terminal-separable" 3s/nt sum-networks, based on which, we have characterized the solvability of a 3s/3t sum-network in terms of certain forbidden structures, which can be decided by an $O(|E|)$ time algorithm. A natural problem, which seems to be worth exploring given the results obtained in this paper, is to examine whether the same approach can be applied to deal with the solvability problem for more general sum-network families.

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