Polynomial time approximation of entropy of shifts of finite type

Stefan Adams, Raimundo Briceno, Brian Marcus, Ronnie Pavlov

Conference on Applied Mathematics
University of Hong Kong
August, 2016
Let \( \mathcal{A} \) be a finite alphabet. 
\( \mathcal{A}^{\mathbb{Z}^d} := \{ \text{all } d\text{-dimensional arrays of symbols from } \mathcal{A} \} \).

Shift of finite type (SFT):
Let \( \mathcal{F} \) is a finite list of “forbidden” patterns on finite sets, 
\[
\mathcal{X} = \mathcal{X}_{\mathcal{F}} = \{ x \in \mathcal{A}^{\mathbb{Z}^d} : x \text{ contains no translate of an element of } \mathcal{F} \}
\]

SFT’s also known as “finite memory constraints.”

Nearest neighbor (n.n.) SFT: an SFT where all forbidden patterns are patterns on edges of \( \mathbb{Z}^d \).

Main Example \((d = 2)\): hard square SFT
\[
\mathcal{A} = \{0, 1\}, \mathcal{F} = \{ 11, 1 \}
\]
Let $\mathcal{A}$ be a finite alphabet. 
$\mathcal{A}^{\mathbb{Z}^d} := \{ \text{all } d\text{-dimensional arrays of symbols from } \mathcal{A} \}$.  

**Shift of finite type (SFT):** 
Let $\mathcal{F}$ is a finite list of “forbidden” patterns on finite sets, 

$X = X_\mathcal{F} = \{ x \in \mathcal{A}^{\mathbb{Z}^d} : x \text{ contains no translate of an element of } \mathcal{F} \}$

SFT’s also known as “finite memory constraints.”

**Nearest neighbor (n.n.) SFT:** an SFT where all forbidden patterns are patterns on edges of $\mathbb{Z}^d$.

**Main Example** $(d = 2)$: *hard square SFT* 
$\mathcal{A} = \{0, 1\}, \mathcal{F} = \{11, 1\}$.  


Let $\mathcal{A}$ be a finite alphabet.
$\mathcal{A}^{\mathbb{Z}^d} := \{ \text{all } d\text{-dimensional arrays of symbols from } \mathcal{A} \}$.

**Shift of finite type (SFT):**
Let $\mathcal{F}$ is a finite list of “forbidden” patterns on finite sets,

$$X = X_\mathcal{F} = \{ x \in \mathcal{A}^{\mathbb{Z}^d} : x \text{ contains no translate of an element of } \mathcal{F} \}$$

SFT’s also known as “finite memory constraints.”

**Nearest neighbor (n.n.) SFT:** an SFT where all forbidden patterns are patterns on edges of $\mathbb{Z}^d$.

Main Example ($d = 2$): hard square SFT

$\mathcal{A} = \{0, 1\}, \mathcal{F} = \{11, \begin{array}{c} 1 \\ 1 \end{array} \}$
Let $\mathcal{A}$ be a finite alphabet.
$\mathcal{A}^\mathbb{Z}^d := \{ \text{ all } d\text{-dimensional arrays of symbols from } \mathcal{A} \}$. 

**Shift of finite type (SFT):**
Let $\mathcal{F}$ is a finite list of “forbidden” patterns on finite sets,

$$X = X_{\mathcal{F}} = \{ x \in \mathcal{A}^\mathbb{Z}^d : x \text{ contains no translate of an element of } \mathcal{F} \}$$

SFT’s also known as “finite memory constraints.”

**Nearest neighbor (n.n.) SFT:** an SFT where all forbidden patterns are patterns on edges of $\mathbb{Z}^d$.

**Main Example ($d = 2$):** hard square SFT
$\mathcal{A} = \{0, 1\}, \mathcal{F} = \{11, \begin{array}{c} 1 \\ 1 \end{array} \}$
Shifts of finite type

Let $\mathcal{A}$ be a finite alphabet. 
$\mathcal{A}^{\mathbb{Z}^d} := \{ \text{all } d\text{-dimensional arrays of symbols from } \mathcal{A} \}$.  

**Shift of finite type (SFT):** 
Let $\mathcal{F}$ is a finite list of “forbidden” patterns on finite sets,  
$X = X_{\mathcal{F}} =$ \{ $x \in \mathcal{A}^{\mathbb{Z}^d} : x$ contains no translate of an element of $\mathcal{F}$ \} 

SFT’s also known as “finite memory constraints.”

**Nearest neighbor (n.n.) SFT:** an SFT where all forbidden patterns are patterns on edges of $\mathbb{Z}^d$. 

**Main Example ($d = 2$): hard square SFT**  
$\mathcal{A} = \{0, 1\}, \mathcal{F} = \{11, \begin{array}{c} \uparrow \\ \downarrow \end{array}\}$
Let $\mathcal{A}$ be a finite alphabet. 
$\mathcal{A}^\mathbb{Z}^d := \{ \text{all } d\text{-dimensional arrays of symbols from } \mathcal{A} \}$. 

**Shift of finite type (SFT):**
Let $\mathcal{F}$ is a *finite* list of “forbidden” patterns on *finite* sets,

$$X = X_{\mathcal{F}} = \{ x \in \mathcal{A}^\mathbb{Z}^d : x \text{ contains no translate of an element of } \mathcal{F} \}$$

SFT’s also known as “finite memory constraints.”

**Nearest neighbor (n.n.) SFT:** an SFT where all forbidden patterns are patterns on *edges* of $\mathbb{Z}^d$.

Main Example ($d = 2$): hard square SFT

$\mathcal{A} = \{0, 1\}, \mathcal{F} = \{11, 1\}$
Let $\mathcal{A}$ be a finite alphabet.  
$\mathcal{A}^{\mathbb{Z}^d} := \{ \text{all } d\text{-dimensional arrays of symbols from } \mathcal{A} \}$.  

**Shift of finite type (SFT):**  
Let $\mathcal{F}$ is a *finite* list of “forbidden” patterns on *finite* sets,  
$X = X_{\mathcal{F}} = \{ x \in \mathcal{A}^{\mathbb{Z}^d} : x \text{ contains no translate of an element of } \mathcal{F} \}$  
SFT’s also known as “finite memory constraints.”  

**Nearest neighbor (n.n.) SFT:** an SFT where all forbidden patterns are patterns on *edges* of $\mathbb{Z}^d$.  

**Main Example ($d = 2$): hard square SFT**  
$\mathcal{A} = \{0, 1\}, \mathcal{F} = \{ 11, \begin{array}{c} 1 \\ 1 \end{array} \}$
Let $\mathcal{A}$ be a finite alphabet.
$\mathcal{A}^{\mathbb{Z}^d} := \{ \text{all } d\text{-dimensional arrays of symbols from } \mathcal{A} \}.$

**Shift of finite type (SFT):**
Let $\mathcal{F}$ is a finite list of “forbidden” patterns on finite sets,

$$X = X_{\mathcal{F}} = \{ x \in \mathcal{A}^{\mathbb{Z}^d} : x \text{ contains no translate of an element of } \mathcal{F} \}$$

SFT’s also known as “finite memory constraints.”

**Nearest neighbor (n.n.) SFT:** an SFT where all forbidden patterns are patterns on edges of $\mathbb{Z}^d$.

**Main Example ($d = 2$): hard square SFT**
$\mathcal{A} = \{0, 1\}, \mathcal{F} = \{11, \begin{array}{c} 1 \\ 1 \end{array} \}$
Let $\mathcal{A}$ be a finite alphabet.
$$\mathcal{A}^{\mathbb{Z}^d} := \{ \text{all } d\text{-dimensional arrays of symbols from } \mathcal{A} \}.$$ 

**Shift of finite type (SFT):**
Let $\mathcal{F}$ is a finite list of "forbidden" patterns on finite sets,
$$X = X_\mathcal{F} = \{ x \in \mathcal{A}^{\mathbb{Z}^d} : x \text{ contains no translate of an element of } \mathcal{F} \}$$

SFT's also known as "finite memory constraints."

**Nearest neighbor (n.n.) SFT:** an SFT where all forbidden patterns are patterns on edges of $\mathbb{Z}^d$.

**Main Example ($d = 2$):** hard square SFT
$$\mathcal{A} = \{0, 1\}, \mathcal{F} = \{11, \begin{array}{c} 1 \\ 1 \end{array} \}$$

\[
\begin{array}{cccccccc}
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
\end{array}
\]
Topological entropy

- $d$-dimensional cube: $B_n := [0, n - 1]^d$
- for an SFT $X$,

$$L_n(X) = \{ \text{legal configurations on } B_n \}$$

- Topological entropy (noiseless capacity):

$$h(X) := \lim_{n \to \infty} \frac{\log |L_n(X)|}{n^d}$$

- By subadditivity of $\log |L_n(X)|$,

$$h(X) := \inf_n \frac{\log |L_n(X)|}{n^d}$$
Topological entropy

- \( d \)-dimensional cube: \( B_n := [0, n - 1]^d \)
- for an SFT \( X \),

\[
L_n(X) = \{ \text{legal configurations on } B_n \}
\]

Topological entropy (noiseless capacity):

\[
h(X) := \lim_{n \to \infty} \frac{\log |L_n(X)|}{n^d}
\]

By subadditivity of \( \log |L_n(X)| \),

\[
h(X) := \inf_n \frac{\log |L_n(X)|}{n^d}
\]
- $d$-dimensional cube: $B_n := [0, n - 1]^d$
- for an SFT $X$,

$$L_n(X) = \{ \text{legal configurations on } B_n \}$$

- **Topological entropy (noiseless capacity):**

$$h(X) := \lim_{n \to \infty} \frac{\log |L_n(X)|}{n^d}$$

- By subadditivity of $\log |L_n(X)|$,

$$h(X) := \inf_n \frac{\log |L_n(X)|}{n^d}$$
Topological entropy

- $d$-dimensional cube: $B_n := [0, n - 1]^d$
- for an SFT $X$,

$$L_n(X) = \{ \text{legal configurations on } B_n \}$$

- **Topological entropy (noiseless capacity):**

$$h(X) := \lim_{n \to \infty} \frac{\log |L_n(X)|}{n^d}$$

- By subadditivity of $\log |L_n(X)|$,

$$h(X) := \inf_n \frac{\log |L_n(X)|}{n^d}$$
A one-dimensional n.n. SFT $X = X_F$ is a set of sequences specified by a directed graph $G$ with vertices in $A$ and an edge from $a$ to $b$ iff $ab \notin F$.

**Golden Mean Shift ((1, \infty) constraint):** $F = \{11\}$

- Adjacency matrix $A$ of $G$ is the square matrix indexed by $A$:
  
  $A_{ab} = \begin{cases} 1 & ab \notin F \\ 0 & ab \in F \end{cases}$

- $h(X) = \log \lambda(A)$, where $\lambda(A)$ is the spectral radius of $A$.

Characterization of entropies for $d = 1$ (Lind):

$$\{\log \lambda^{1/q}\}$$

where $\lambda$ is a Perron number and $q \in \mathbb{N}$
A one-dimensional n.n. SFT $X = X_F$ is a set of sequences specified by a directed graph $G$ with vertices in $\mathcal{A}$ and an edge from $a$ to $b$ iff $ab \notin \mathcal{F}$.

**Golden Mean Shift ((1, \infty) constraint):** $\mathcal{F} = \{11\}$

- Adjacency matrix $A$ of $G$ is the square matrix indexed by $\mathcal{A}$:
  $$A_{ab} = \begin{cases} 1 & ab \notin \mathcal{F} \\ 0 & ab \in \mathcal{F} \end{cases}$$

- $h(X) = \log \lambda(A)$, where $\lambda(A)$ is the spectral radius of $A$.
- Characterization of entropies for $d = 1$ (Lind):
  $\{ \log \lambda^{1/q} \}$
  where $\lambda$ is a Perron number and $q \in \mathbb{N}$
A one-dimensional n.n. SFT $X = X_F$ is a set of sequences specified by a directed graph $G$ with vertices in $A$ and an edge from $a$ to $b$ iff $ab \not\in F$.

Golden Mean Shift ($(1, \infty)$ constraint): $F = \{11\}$

Adjacency matrix $A$ of $G$ is the square matrix indexed by $A$:

$$A_{ab} = \begin{cases} 1 & ab \not\in F \\ 0 & ab \in F \end{cases}$$

$h(X) = \log \lambda(A)$, where $\lambda(A)$ is the spectral radius of $A$.

Characterization of entropies for $d = 1$ (Lind):

$$\{\log \lambda^{1/q}\}$$

where $\lambda$ is a Perron number and $q \in \mathbb{N}$
SFT’s, $d = 1$

- A one-dimensional n.n. SFT $X = X_F$ is a set of sequences specified by a directed graph $G$ with vertices in $A$ and an edge from $a$ to $b$ iff $ab \notin F$.

**Golden Mean Shift ($[1, \infty)$ constraint):** $F = \{11\}$

- Adjacency matrix $A$ of $G$ is the square matrix indexed by $A$:
  
  $$A_{ab} = \begin{cases} 
  1 & \text{if } ab \notin F \\
  0 & \text{if } ab \in F
  \end{cases}$$

- $h(X) = \log \lambda(A)$, where $\lambda(A)$ is the spectral radius of $A$.

- Characterization of entropies for $d = 1$ (Lind):
  
  $$\{\log \lambda^{1/q}\}$$

  where $\lambda$ is a Perron number and $q \in \mathbb{N}$
A one-dimensional n.n. SFT $X = X_F$ is a set of sequences specified by a directed graph $G$ with vertices in $A$ and an edge from $a$ to $b$ iff $ab \notin F$.

**Golden Mean Shift ($(1, \infty)$ constraint):** $F = \{11\}$

![Directed graph with vertices 0 and 1 and an edge from 0 to 1, and another edge from 1 back to 0.]

- Adjacency matrix $A$ of $G$ is the square matrix indexed by $A$:
  
  $A_{ab} = \begin{cases} 
  1 & ab \notin F \\
  0 & ab \in F 
  \end{cases}$

- $h(X) = \log \lambda(A)$, where $\lambda(A)$ is the spectral radius of $A$.

Characterization of entropies for $d = 1$ (Lind):

$\{\log \lambda^{1/q}\}$

where $\lambda$ is a Perron number and $q \in \mathbb{N}$.
SFT’s, \( d = 1 \)

- A one-dimensional n.n. SFT \( X = X_F \) is a set of sequences specified by a directed graph \( G \) with vertices in \( \mathcal{A} \) and an edge from \( a \) to \( b \) iff \( ab \notin \mathcal{F} \).

**Golden Mean Shift ((1, \infty) constraint):** \( \mathcal{F} = \{11\} \)

- Adjacency matrix \( A \) of \( G \) is the square matrix indexed by \( \mathcal{A} \):
  
  \[
  A_{ab} = \begin{cases} 
  1 & ab \notin \mathcal{F} \\
  0 & ab \in \mathcal{F}
  \end{cases}
  \]

- \( h(X) = \log \lambda(A) \), where \( \lambda(A) \) is the spectral radius of \( A \).
- Characterization of entropies for \( d = 1 \) (Lind):
  
  \[
  \{\log \lambda^{1/q}\}
  \]

  where \( \lambda \) is a Perron number and \( q \in \mathbb{N} \).
Examples of $\mathbb{Z}^2$ SFTs: hard square

- **hard squares** $\mathcal{A} = \{0, 1\}, \mathcal{F} = \{11, 1\}$

- $h($ hard square SFT $) = ???$

- (Baxter) $h($ hard hexagons $) = \log(\lambda)$ where $\lambda$ is an algebraic integer of degree 24.
Examples of $\mathbb{Z}^2$ SFTs: hard square

- **hard squares** $\mathcal{A} = \{0, 1\}, \mathcal{F} = \{11, \begin{array}{c} 1 \\ 1 \end{array} \}$

  

  \[
  \begin{array}{cccccccc}
  1 & 0 & 1 & 0 & 0 & 1 \\
  0 & 1 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 \\
  1 & 0 & 0 & 1 & 0 & 1 \\
  0 & 1 & 0 & 0 & 1 & 0 \\
  0 & 0 & 1 & 0 & 0 & 1 
  \end{array}
  \]

- $h(\text{hard square SFT}) = \text{???}$

- (Baxter) $h(\text{hard hexagons}) = \log(\lambda)$ where $\lambda$ is an algebraic integer of degree 24.
Examples of $\mathbb{Z}^2$ SFTs: hard square

- **hard squares** $\mathcal{A} = \{0, 1\}, \mathcal{F} = \{11, 1\}$

- $h(\text{hard square SFT}) = ???$

- (Baxter) $h(\text{hard hexagons}) = \log(\lambda)$ where $\lambda$ is an algebraic integer of degree 24.
Examples of $\mathbb{Z}^2$ SFTs: hard square

- **hard squares** $\mathcal{A} = \{0, 1\}, \mathcal{F} = \{11, \begin{array}{c}1 \\ 1 \end{array}\}$

  $\begin{array}{cccccc}
  1 & 0 & 1 & 0 & 1 & 0 \\
  0 & 1 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 \\
  1 & 0 & 0 & 1 & 0 & 1 \\
  0 & 1 & 0 & 0 & 1 & 0 \\
  0 & 0 & 1 & 0 & 0 & 1 \\
  \end{array}$

- $h(\text{hard square SFT}) = ???$

- (Baxter) $h(\text{hard hexagons}) = \log(\lambda)$ where $\lambda$ is an algebraic integer of degree 24.
Examples of $\mathbb{Z}^2$ SFTs: checkerboard (coloring) constraints

- **$q$-checkerboard** $C_q$: $\mathcal{A} = \{1, \ldots, q\}$, $\mathcal{F} = \{ aa, \ a \ a \}$

- $h(C_2) = 0$
- (Lieb): $h(C_3) = (3/2) \log(4/3)$
- $h(C_4) = ???$
Examples of $\mathbb{Z}^2$ SFTs: checkerboard (coloring) constraints

- **$q$-checkerboard** $C_q$: $\mathcal{A} = \{1, \ldots, q\}$, $\mathcal{F} = \{aa, \begin{array}{c} a \\ a \end{array}\}$

- $h(C_2) = 0$
- (Lieb): $h(C_3) = (3/2) \log(4/3)$
- $h(C_4) = ???$
Examples of $\mathbb{Z}^2$ SFTs: checkerboard (coloring) constraints

- **$q$-checkerboard** $C_q$: $A = \{1, \ldots, q\}$, $\mathcal{F} = \{aa, \ a\ a\}$

- $h(C_2) = 0$
- (Lieb): $h(C_3) = (3/2) \log(4/3)$
- $h(C_4) = ???$
Examples of $\mathbb{Z}^2$ SFTs: checkerboard (coloring) constraints

- **$q$-checkerboard** $C_q$: $A = \{1, \ldots, q\}, F = \{aa, \ a \ a\}$

  - $h(C_2) = 0$
  - (Lieb): $h(C_3) = (3/2) \log(4/3)$
  - $h(C_4) = ???$
Examples of $\mathbb{Z}^2$ SFTs: checkerboard (coloring) constraints

- **$q$-checkerboard** $C_q$: $\mathcal{A} = \{1, \ldots, q\}, \mathcal{F} = \{aa, a\}$.

\[ h(C_2) = 0 \]

- (Lieb): $h(C_3) = \left(\frac{3}{2}\right) \log\left(\frac{4}{3}\right)$

- $h(C_4) = ???$
Examples of $\mathbb{Z}^2$ SFTs: checkerboard (coloring) constraints

- **$q$-checkerboard $C_q$:** $\mathcal{A} = \{1, \ldots, q\}$, $\mathcal{F} = \{aa, \ a\}$

- $h(C_2) = 0$
- (Lieb): $h(C_3) = \left(\frac{3}{2}\right) \log\left(\frac{4}{3}\right)$
- $h(C_4) = ???$
Examples of $\mathbb{Z}^2$ SFTs: checkerboard (coloring) constraints

- **q-checkerboard** $C_q$: $A = \{1, \ldots, q\}, \mathcal{F} = \{aa, \begin{array}{c} a \\ a \end{array}\}$

- $h(C_2) = 0$
- (Lieb): $h(C_3) = (3/2) \log(4/3)$
- $h(C_4) = ???$
Examples of $\mathbb{Z}^2$ SFT’s: dimers

- **dimers:**

$$\mathcal{F} = \{ LL, LT, LB, RR, TR, BR, T L, T R, T T, B B, L B, R B \}$$

(Fisher-Kastelyn-Temperley):

$$h(\text{ Dimers } ) = \frac{1}{16\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \log(4 + 2 \cos \theta + 2 \cos \phi) \ d\theta d\phi$$

$$h(\text{ Monomers-Dimers}) = ???$$
Examples of $\mathbb{Z}^2$ SFT’s: dimers

- **dimers:**

$$\mathcal{F} = \{ LL, LT, LB, RR, TR, BR, TL, TR, TB, LB, RB, BR \}$$

(Fisher-Kastelyn-Temperley):

$$h( \text{Dimers} ) = \frac{1}{16\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \log(4 + 2 \cos \theta + 2 \cos \phi) \, d\theta \, d\phi$$

$$h( \text{Monomers-Dimers} ) = ???$$
Examples of $\mathbb{Z}^2$ SFT’s: dimers

- **dimers:**

$\mathcal{F} = \{ LL, LT, LB, RR, TR, BR, TL, TR, TB, LB, RB \}$

(Fisher-Kastelyn-Temperley):

$h(\text{Dimers}) = \frac{1}{16\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \log(4 + 2\cos\theta + 2\cos\phi) \, d\theta \, d\phi$

$h(\text{Monomers-Dimers}) = ????$
Examples of $\mathbb{Z}^2$ SFT’s: dimers

- **dimers:**

![Diagram of dimers with symbols and labels]

\[ \mathcal{F} = \{ LL, LT, LB, RR, TR, BR, TL, TR, TT, B, L, R \} \]

- (Fisher-Kastelyn-Temperley): 
  \[ h(\text{Dimers}) = \frac{1}{16\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \log(4 + 2\cos\theta + 2\cos\phi) \, d\theta d\phi \]

- \[ h(\text{Monomers-Dimers}) = ??? \]
Examples of $\mathbb{Z}^2$ SFT’s: dimers

- **dimers:**

$$\mathcal{F} = \{ LL, LT, LB, RR, TR, BR, T_L, T_R, T_T, B_L, B_R \}$$

(Fisher-Kastelyn-Temperley):

$$h(\text{Dimers}) = \frac{1}{16\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \log(4 + 2 \cos \theta + 2 \cos \phi) \ d\theta d\phi$$

$$h(\text{Monomers-Dimers}) = ???$$
Topological entropy, $d \geq 2$

- Exact formula known only in a few cases.
- Characterization of entropies for $d \geq 2$
  (Hochman-Meyerovitch):

\[
\{ \text{right recursively enumerable (RRE) numbers } h \geq 0 \} 
\]

i.e, there is an algorithm that produces a sequence $r_n \geq h$
s.t. $r_n \to h$.

Proof:

- Necessity: Let $r_n := \frac{\log |L_n|}{n^d}$.
  $r_n \to h$.
  Since $\lim = \inf$, each $r_n \geq h$.
- Sufficiency (hard): Emulate Turing machine with an SFT.

RRE’s can be arbitrarily poorly computable, or even non-computable.
Topological entropy, $d \geq 2$

- Exact formula known only in a few cases.
- Characterization of entropies for $d \geq 2$
  (Hochman-Meyerovitch):

  \[
  \{ \text{right recursively enumerable (RRE) numbers } h \geq 0 \}
  \]

i.e, there is an algorithm that produces a sequence $r_n \geq h$
  s.t. $r_n \to h$.

Proof:

- Necessity: Let $r_n := \frac{\log |L_n|}{nd}$.
  $r_n \to h$.
  Since lim = inf, each $r_n \geq h$.
- Sufficiency (hard): Emulate Turing machine with an SFT.

  RRE’s can be arbitrarily poorly computable, or even non-computable.
Topological entropy, $d \geq 2$

- Exact formula known only in a few cases.
- Characterization of entropies for $d \geq 2$
  
  (Hochman-Meyerovitch):
  
  \[
  \{ \text{right recursively enumerable (RRE) numbers } h \geq 0 \}
  \]
  
  i.e, there is an algorithm that produces a sequence $r_n \geq h$
  s.t. $r_n \to h$.

Proof:
- Necessity: Let $r_n := \frac{\log |L_n|}{n^d}$.
  $r_n \to h$.
  Since $\lim = \inf$, each $r_n \geq h$.
- Sufficiency (hard): Emulate Turing machine with an SFT.

RRE’s can be arbitrarily poorly computable, or even non-computable.
Topological entropy, $d \geq 2$

- Exact formula known only in a few cases.
- Characterization of entropies for $d \geq 2$
  (Hochman-Meyerovitch):

  \{\text{right recursively enumerable (RRE) numbers } h \geq 0\}

i.e, there is an algorithm that produces a sequence $r_n \geq h$
 s.t. $r_n \to h$.

Proof:

- Necessity: Let $r_n := \frac{\log |L_n|}{n^d}$.
  $r_n \to h$.
  Since $\lim = \inf$, each $r_n \geq h$.
- Sufficiency (hard): Emulate Turing machine with an SFT.

RRE’s can be arbitrarily poorly computable, or even non-computable.
Topological entropy, $d \geq 2$

- Exact formula known only in a few cases.
- Characterization of entropies for $d \geq 2$
  (Hochman-Meyerovitch):

  \[ \{ \text{right recursively enumerable (RRE) numbers } h \geq 0 \} \]

  i.e, there is an algorithm that produces a sequence $r_n \geq h$
  s.t. $r_n \rightarrow h$. 

Proof:

- Necessity: Let $r_n := \frac{\log |L_n|}{n^d}$.  
  $r_n \rightarrow h$. 
  Since $\lim = \inf$, each $r_n \geq h$.
- Sufficiency (hard): Emulate Turing machine with an SFT.

  RRE’s can be arbitrarily poorly computable, or even non-computable.
Topological entropy, $d \geq 2$

- Exact formula known only in a few cases.
- Characterization of entropies for $d \geq 2$
  (Hochman-Meyerovitch):

  \[
  \{ \text{right recursively enumerable (RRE) numbers } h \geq 0 \}\]

  i.e, there is an algorithm that produces a sequence $r_n \geq h$
  s.t. $r_n \to h$.

  Proof:
  
  - Necessity: Let $r_n := \frac{\log |L_n|}{n^d}$.
    
    $r_n \to h$.
    
    Since $\lim = \inf$, each $r_n \geq h$.

  - Sufficiency (hard): Emulate Turing machine with an SFT.

  RRE’s can be arbitrarily poorly computable, or even non-computable.
Topological entropy, $d \geq 2$

- Exact formula known only in a few cases.
- Characterization of entropies for $d \geq 2$ (Hochman-Meyerovitch):

$$\{\text{right recursively enumerable (RRE) numbers } h \geq 0\}$$

i.e, there is an algorithm that produces a sequence $r_n \geq h$ s.t. $r_n \to h$.

Proof:

- Necessity: Let $r_n := \frac{\log |L_n|}{n^d}$.
  
  $r_n \to h$.
  
  Since $\lim = \inf$, each $r_n \geq h$.

- Sufficiency (hard): Emulate Turing machine with an SFT.

RRE’s can be arbitrarily poorly computable, or even non-computable.
Exact formula known only in a few cases.

Characterization of entropies for $d \geq 2$
(Hochman-Meyerovitch):

$$\{\text{right recursively enumerable (RRE) numbers } h \geq 0\}$$

i.e, there is an algorithm that produces a sequence $r_n \geq h$

s.t. $r_n \to h$.

**Proof:**

- **Necessity:** Let $r_n := \frac{\log |L_n|}{n^d}$.
  
  $r_n \to h$.
  
  Since $\lim = \inf$, each $r_n \geq h$.

- **Sufficiency (hard):** Emulate Turing machine with an SFT.

RRE’s can be arbitrarily poorly computable, or even non-computable.
Exact formula known only in a few cases.

Characterization of entropies for $d \geq 2$
(Hochman-Meyerovitch):

$$\{\text{right recursively enumerable (RRE) numbers } h \geq 0\}$$

i.e, there is an algorithm that produces a sequence $r_n \geq h$
s.t. $r_n \to h$.

Proof:

- **Necessity**: Let $r_n := \frac{\log |L_n|}{n^d}$.
  
  $r_n \to h$.

  Since $\lim = \inf$, each $r_n \geq h$.

- **Sufficiency (hard)**: Emulate Turing machine with an SFT.

RRE’s can be arbitrarily poorly computable, or even non-computable.
A polynomial time approximation algorithm: on input $n$, produces $r_n$ s.t. $|r_n - h| < 1/n$ and $r_n$ can be computed in time $\text{poly}(n)$.

Theorem (Gamarnik-Katz, Pavlov): There is a polynomial time approximation algorithm to compute $h$( hard square SFT).
A polynomial time approximation algorithm: on input $n$, produces $r_n$ s.t. $|r_n - h| < 1/n$ and $r_n$ can be computed in time $\text{poly}(n)$.

Theorem (Gamarnik-Katz, Pavlov): There is a polynomial time approximation algorithm to compute $h(\text{hard square SFT})$. 
Measure-theoretic entropy

Given a shift-invariant Borel probability measure \( \mu \) on \( \mathcal{A}^{\mathbb{Z}^d} \),

- For finite \( S \subset \mathbb{Z}^d \),
  \[
  H_{\mu}(S) := \sum_{x \in \mathcal{A}^S} -\mu(x) \log \mu(x) = \int -\log \mu(x) d\mu(x)
  \]

- For finite disjoint \( S, T \),
  \[
  H_{\mu}(S \mid T) := \sum_{x \in \mathcal{A}^S, y \in \mathcal{A}^T: \mu(y)>0} -\mu(x, y) \log \mu(x \mid y)
  \]

- Extend to finite \( S \) and infinite \( T \):
  \[
  H_{\mu}(S \mid T) := \inf_{T' \in T} H_{\mu}(S \mid T')
  \]
Given a shift-invariant Borel probability measure $\mu$ on $\mathcal{A}^{\mathbb{Z}^d}$,

For finite $S \in \mathbb{Z}^d$,

$$H_\mu(S) := \sum_{x \in A^S} -\mu(x) \log \mu(x) = \int -\log \mu(x) d\mu(x)$$

For finite disjoint $S, T$,

$$H_\mu(S \mid T) := \sum_{x \in A^S, y \in A^T: \mu(y)>0} -\mu(x, y) \log \mu(x \mid y)$$

Extend to finite $S$ and infinite $T$:

$$H_\mu(S \mid T) := \inf_{T' \subseteq T} H_\mu(S \mid T')$$
Measure-theoretic entropy

Given a shift-invariant Borel probability measure $\mu$ on $\mathbb{A}^{\mathbb{Z}^d}$,

- For finite $S \subset \mathbb{Z}^d$,

$$H_\mu(S) := \sum_{x \in \mathbb{A}^S} -\mu(x) \log \mu(x) = \int -\log \mu(x) d\mu(x)$$

- For finite disjoint $S, T$,

$$H_\mu(S \mid T) := \sum_{x \in \mathbb{A}^S, y \in \mathbb{A}^T: \mu(y) > 0} -\mu(x, y) \log \mu(x \mid y)$$

- Extend to finite $S$ and infinite $T$:

$$H_\mu(S \mid T) := \inf_{T' \subset T} H_\mu(S \mid T')$$
Measure-theoretic entropy

Given a shift-invariant Borel probability measure $\mu$ on $A^\mathbb{Z}^d$,

- For finite $S \subseteq \mathbb{Z}^d$, 
  $$H_\mu(S) := \sum_{x \in A^S} -\mu(x) \log \mu(x) = \int -\log \mu(x) d\mu(x)$$

- For finite disjoint $S, T$,
  $$H_\mu(S \mid T) := \sum_{x \in A^S, y \in A^T: \mu(y) > 0} -\mu(x, y) \log \mu(x \mid y)$$

- Extend to finite $S$ and infinite $T$:
  $$H_\mu(S \mid T) := \inf_{T' \subseteq T} H_\mu(S \mid T')$$
Entropy (entropy rate) of $\mu$

- $h(\mu) := \lim_{n \to \infty} \frac{H_\mu(B_n)}{n^d}$

- $d = 1$: Theorem: $h(\mu) = H_\mu(0 \mid \{-1, -2, -3, \ldots\})$

- $d = 2$: Let $\prec$ denotes lexicographic order: $(i, j) \prec (i', j')$ iff either $j < j'$ or ($j = j'$ and $i < i'$).

For $\overline{z} \in \mathbb{Z}^2$, let $P(\overline{z}) := \{\overline{z}' \in \mathbb{Z}^2 : \overline{z}' \prec \overline{z}\}$ the lexicographic past of $\overline{z}$, and $P := P(0)$

Theorem: $h(\mu) = H_\mu(0 \mid P)$. 
Entropy (entropy rate) of $\mu$

- $h(\mu) := \lim_{n\to\infty} \frac{H_\mu(B_n)}{n^d}$
- $d = 1$: Theorem: $h(\mu) = H_\mu(0 \mid \{-1, -2, -3, \ldots\})$
- $d = 2$: Let $\prec$ denotes lexicographic order: $(i, j) \prec (i', j')$ iff either $j < j'$ or ($j = j'$ and $i < i'$).

For $\bar{z} \in \mathbb{Z}^2$, let $\mathcal{P}(\bar{z}) := \{\bar{z}' \in \mathbb{Z}^2 : \bar{z}' \prec \bar{z}\}$ the lexicographic past of $\bar{z}$, and $\mathcal{P} := \mathcal{P}(0)$.

Theorem: $h(\mu) = H_\mu(0 \mid \mathcal{P})$. 
Entropy (entropy rate) of $\mu$

- $h(\mu) := \lim_{n \to \infty} \frac{H_\mu(B_n)}{n^d}$
- $d = 1$: Theorem: $h(\mu) = H_\mu(0 \mid \{-1, -2, -3, \ldots\})$
- $d = 2$: Let $\prec$ denotes lexicographic order: $(i, j) \prec (i', j')$ iff either $j < j'$ or ($j = j'$ and $i < i'$).

For $\bar{z} \in \mathbb{Z}^2$, let $\mathcal{P}(\bar{z}) := \{\bar{z}' \in \mathbb{Z}^2 : \bar{z}' \prec \bar{z}\}$ the lexicographic past of $\bar{z}$, and $\mathcal{P} := \mathcal{P}(0)$

Theorem: $h(\mu) = H_\mu(0 \mid \mathcal{P})$. 
Entropy (entropy rate) of \( \mu \)

1. \( h(\mu) := \lim_{n \to \infty} \frac{H_{\mu}(B_n)}{n^d} \)

2. \( d = 1 \): Theorem: \( h(\mu) = H_{\mu}(0 \mid \{-1, -2, -3, \ldots\}) \)

3. \( d = 2 \): Let \( \prec \) denotes lexicographic order: \((i, j) \prec (i', j')\) iff either \( j < j' \) or \((j = j' \text{ and } i < i')\).

For \( \overline{z} \in \mathbb{Z}^2 \), let \( \mathcal{P}(\overline{z}) := \{ \overline{z}' \in \mathbb{Z}^2 : \overline{z}' \prec \overline{z} \} \) the lexicographic past of \( \overline{z} \), and \( \mathcal{P} := \mathcal{P}(0) \)

Theorem: \( h(\mu) = H_{\mu}(0 \mid \mathcal{P}) \).
Entropy (entropy rate) of $\mu$

- $h(\mu) := \lim_{n \to \infty} \frac{H_\mu(B_n)}{n^d}$
- $d = 1$: Theorem: $h(\mu) = H_\mu(0 \mid \{-1, -2, -3, \ldots\})$
- $d = 2$: Let $\prec$ denotes lexicographic order: $(i, j) \prec (i', j')$ iff either $j < j'$ or ($j = j'$ and $i < i'$).

For $\bar{z} \in \mathbb{Z}^2$, let $\mathcal{P}(\bar{z}) := \{\bar{z}' \in \mathbb{Z}^2 : \bar{z}' \prec \bar{z}\}$ the lexicographic past of $\bar{z}$, and $\mathcal{P} := \mathcal{P}(0)$

Theorem: $h(\mu) = H_\mu(0 \mid \mathcal{P})$. 
Entropy (entropy rate) of $\mu$

- $h(\mu) := \lim_{n \to \infty} \frac{H_{\mu}(B_n)}{n^d}$
- $d = 1$: Theorem: $h(\mu) = H_{\mu}(0 \mid \{-1, -2, -3, \ldots\})$
- $d = 2$: Let $\prec$ denotes lexicographic order: $(i, j) \prec (i', j')$ iff either $j < j'$ or $(j = j' \text{ and } i < i')$.

For $\overline{z} \in \mathbb{Z}^2$, let $\mathcal{P}(\overline{z}) := \{\overline{z}' \in \mathbb{Z}^2 : \overline{z}' \prec \overline{z}\}$ the lexicographic past of $\overline{z}$, and $\mathcal{P} := \mathcal{P}(0)$

Theorem: $h(\mu) = H_{\mu}(0 \mid \mathcal{P})$. 
\( \mathcal{P} := \mathcal{P}(0) \)

---

**Theorem:** \( h(\mu) = H_\mu(0 \mid \mathcal{P}) \).

**Defn:** The information function of \( \mu \) is defined as

\[
l_\mu(x) := - \log \mu(x(0) \mid x(\mathcal{P})) \quad (\mu - \text{a.e.})
\]

**Corollary:**

\[
h(\mu) = H_\mu(0 \mid \mathcal{P}) = \int l_\mu(x) d\mu(x).
\]
Theorem: \( h(\mu) = H_\mu(0 \mid \mathcal{P}) \).

Defn: The **information function** of \( \mu \) is defined as

\[
I_\mu(x) := -\log \mu(x(0) \mid x(\mathcal{P})) \quad (\mu - a.e.)
\]

Corollary:

\[
h(\mu) = H_\mu(0 \mid \mathcal{P}) = \int l_\mu(x) \, d\mu(x).
\]
Theorem: $h(\mu) = H_\mu(0 \mid \mathcal{P})$.

Defn: The **information function** of $\mu$ is defined as

$$I_\mu(x) := -\log \mu(x(0) \mid x(\mathcal{P})) \quad (\mu \text{ -- a.e.})$$

Corollary:

$$h(\mu) = H_\mu(0 \mid \mathcal{P}) = \int l_\mu(x) d\mu(x).$$
For an SFT $X$,

$$h(X) = \sup_{\mu} h(\mu)$$

where the sup is taken over all shift-invariant Borel probability measures $\mu$ s.t. $\text{support}(\mu) \subseteq X$.

Fact: The sup is always achieved. A measure which achieves the sup is called a measure of maximal entropy (MME).

So for an MME $\mu$, $h(X) = h(\mu) = \int I_\mu(x) d\mu(x)$

Under certain conditions, $h(X) = h(\mu) = \int l_\mu(x) d\nu(x)$ for some other invariant measures $\nu$

If this holds for $\nu = \delta$-measure on a fixed point $s^{\mathbb{Z}^d}$, then

$$h(X) = h(\mu) = l_\mu(s^{\mathbb{Z}^d}) = -\log \mu(x(0) = s \mid x(P) = s^P)$$
Variational Principle for Topological Entropy

For an SFT $X$,

$$h(X) = \sup_{\mu} h(\mu)$$

where the sup is taken over all shift-invariant Borel probability measures $\mu$ s.t. support($\mu$) $\subseteq X$.

Fact: The sup is always achieved. A measure which achieves the sup is called a measure of maximal entropy (MME).

So for an MME $\mu$, $h(X) = h(\mu) = \int I_\mu(x) d\mu(x)$

Under certain conditions, $h(X) = h(\mu) = \int I_\mu(x) d\nu(x)$ for some other invariant measures $\nu$

If this holds for $\nu = \delta$-measure on a fixed point $s^{\mathbb{Z}^d}$, then

$$h(X) = h(\mu) = I_\mu(s^{\mathbb{Z}^d}) = -\log \mu(x(0) = s | x(\mathcal{P}) = s^{\mathcal{P}})$$
Variational Principle for Topological Entropy

For an SFT $X$,

$$h(X) = \sup_{\mu} h(\mu)$$

where the sup is taken over all shift-invariant Borel probability measures $\mu$ s.t. $\text{support}(\mu) \subseteq X$.

Fact: The sup is always achieved. A measure which achieves the sup is called a measure of maximal entropy (MME).

So for an MME $\mu$, $h(X) = h(\mu) = \int I_\mu(x) d\mu(x)$

Under certain conditions, $h(X) = h(\mu) = \int I_\mu(x) d\nu(x)$ for some other invariant measures $\nu$

If this holds for $\nu = \delta$-measure on a fixed point $s^{\mathbb{Z}^d}$, then

$$h(X) = h(\mu) = I_\mu(s^{\mathbb{Z}^d}) = -\log \mu(x(0) = s \mid x(P) = s^P)$$
Variational Principle for Topological Entropy

- For an SFT $X$, $h(X) = \sup_{\mu} h(\mu)$

  where the sup is taken over all shift-invariant Borel probability measures $\mu$ s.t. $\text{support}(\mu) \subseteq X$.

- Fact: The sup is always achieved. A measure which achieves the sup is called a **measure of maximal entropy** (MME).

- So for an MME $\mu$, $h(X) = h(\mu) = \int I_\mu(x) d\mu(x)$

- Under certain conditions, $h(X) = h(\mu) = \int I_\mu(x) d\nu(x)$ for some other invariant measures $\nu$

- If this holds for $\nu = \delta$-measure on a fixed point $s^{\mathbb{Z}^d}$, then

  \[ h(X) = h(\mu) = I_\mu(s^{\mathbb{Z}^d}) = -\log \mu(x(0) = s | x(\mathcal{P}) = s^\mathcal{P}) \]
Variational Principle for Topological Entropy

- For an SFT $X$, 
  \[ h(X) = \sup_{\mu} h(\mu) \]
  where the sup is taken over all shift-invariant Borel probability measures $\mu$ s.t. $\text{support}(\mu) \subseteq X$.

- Fact: The sup is always achieved. A measure which achieves the sup is called a **measure of maximal entropy (MME)**.

- So for an MME $\mu$, 
  \[ h(X) = h(\mu) = \int l_\mu(x) d\mu(x) \]

- Under certain conditions, 
  \[ h(X) = h(\mu) = \int l_\mu(x) d\nu(x) \]
  for some other invariant measures $\nu$.

- If this holds for $\nu = \delta$-measure on a fixed point $s^{\mathbb{Z}^d}$, then
  \[ h(X) = h(\mu) = l_\mu(s^{\mathbb{Z}^d}) = -\log \mu(x(0) = s \mid x(P) = s^P) \]
For an SFT $X$,

$$h(X) = \sup_{\mu} h(\mu)$$

where the sup is taken over all shift-invariant Borel probability measures $\mu$ s.t. support($\mu$) $\subseteq X$.

Fact: The sup is always achieved. A measure which achieves the sup is called a **measure of maximal entropy (MME)**.

So for an MME $\mu$, $h(X) = h(\mu) = \int l_\mu(x) d\mu(x)$

Under certain conditions, $h(X) = h(\mu) = \int l_\mu(x) d\nu(x)$ for some other invariant measures $\nu$

If this holds for $\nu$ = the $\delta$-measure on a fixed point $s^{\mathbb{Z}^d}$, then

$$h(X) = h(\mu) = l_\mu(s^{\mathbb{Z}^d}) = -\log \mu(x(0) = s \mid x(P) = s^P)$$
Rough Idea for showing $h(X) = I_\mu(s^{\mathbb{Z}^d})$

An MME $\mu$ should be “nearly uniform”. So, $\mu$ captures entropy:

If $s^{\mathbb{Z}^d} \in X$, then

$$h(X) = \lim_{n \to \infty} \frac{\log |L_n(X)|}{n^d} = \lim_{n \to \infty} - \frac{\log \mu(x(B_n) = s^{B_n})}{n^d}$$

$$= \lim_{n \to \infty} (1/n^d) \sum_{\overline{z} \in B_n} - \log \mu(x(\overline{z}) = s | x(\mathcal{P}(\overline{z}) \cap B_n) = s^{\mathcal{P}(\overline{z}) \cap B_n})$$

This is an average of $n^d$ terms of two types:

- **Bulk terms**: Terms that are far from the boundary of $B_n$
- **Boundary terms**: Terms that are near the boundary of $B_n$

Bulk terms are close to $I_\mu(s^{\mathbb{Z}^d})$. All terms are uniformly bounded. Most terms are bulk terms. So, $h(X) = I_\mu(s^{\mathbb{Z}^d})$. 
Rough Idea for showing $h(X) = l_\mu(s^{\mathbb{Z}^d})$

An MME $\mu$ should be “nearly uniform”. So, $\mu$ captures entropy: If $s^{\mathbb{Z}^d} \in X$, then

$$h(X) = \lim_{n \to \infty} \frac{\log |L_n(X)|}{n^d} = \lim_{n \to \infty} -\frac{\log \mu(x(B_n) = s^{B_n})}{n^d}$$

$$= \lim_{n \to \infty} \frac{1}{n^d} \sum_{\bar{z} \in B_n} -\log \mu(x(\bar{z}) = s \mid x(\mathcal{P}(\bar{z}) \cap B_n) = s^{\mathcal{P}(\bar{z}) \cap B_n})$$

This is an average of $n^d$ terms of two types:
- **Bulk terms**: Terms that are far from the boundary of $B_n$
- **Boundary terms**: Terms that are near the boundary of $B_n$

Bulk terms are close to $l_\mu(s^{\mathbb{Z}^d})$. All terms are uniformly bounded. Most terms are bulk terms. So, $h(X) = l_\mu(s^{\mathbb{Z}^d})$. 
An MME $\mu$ should be “nearly uniform”. So, $\mu$ captures entropy:

If $s^{\mathbb{Z}^d} \in X$, then

$$h(X) = \lim_{n \to \infty} \frac{\log |L_n(X)|}{n^d} = \lim_{n \to \infty} -\frac{\log \mu(x(B_n) = s^{B_n})}{n^d}$$

$$= \lim_{n \to \infty} \frac{1}{n^d} \sum_{\bar{z} \in B_n} -\log \mu(x(\bar{z}) = s \mid x(\mathcal{P}(\bar{z}) \cap B_n) = s^{\mathcal{P}(\bar{z}) \cap B_n})$$

This is an average of $n^d$ terms of two types:

- **Bulk terms**: Terms that are far from the boundary of $B_n$
- **Boundary terms**: Terms that are near the boundary of $B_n$

Bulk terms are close to $I_\mu(s^{\mathbb{Z}^d})$. All terms are uniformly bounded. Most terms are bulk terms. So, $h(X) = I_\mu(s^{\mathbb{Z}^d})$. 
Rough Idea for showing $h(X) = I_\mu(s^{\mathbb{Z}^d})$

An MME $\mu$ should be “nearly uniform”. So, $\mu$ captures entropy: If $s^{\mathbb{Z}^d} \in X$, then

$$h(X) = \lim_{n \to \infty} \frac{\log |L_n(X)|}{n^d} = \lim_{n \to \infty} -\frac{\log \mu(x(B_n) = s^{B_n})}{n^d}$$

$$= \lim_{n \to \infty} \frac{1}{n^d} \sum_{\bar{z} \in B_n} -\log \mu(x(\bar{z}) = s \mid x(\mathcal{P}(\bar{z}) \cap B_n) = s^{\mathcal{P}(\bar{z}) \cap B_n})$$

This is an average of $n^d$ terms of two types:

- **Bulk terms**: Terms that are far from the boundary of $B_n$
- **Boundary terms**: Terms that are near the boundary of $B_n$

Bulk terms are close to $I_\mu(s^{\mathbb{Z}^d})$. All terms are uniformly bounded. Most terms are bulk terms. So, $h(X) = I_\mu(s^{\mathbb{Z}^d})$. 

Author: 
Short Paper Title:
Rough Idea for showing \( h(X) = I_\mu(s_{\mathbb{Z}^d}) \)

An MME \( \mu \) should be “nearly uniform”. So, \( \mu \) captures entropy: If \( s_{\mathbb{Z}^d} \in X \), then

\[
h(X) = \lim_{n \to \infty} \frac{\log |L_n(X)|}{n^d} = \lim_{n \to \infty} \frac{-\log \mu(x(B_n) = s^{B_n})}{n^d}
\]

\[
= \lim_{n \to \infty} \left( \frac{1}{n^d} \right) \sum_{\overline{z} \in B_n} -\log \mu(x(\overline{z}) = s \mid x(\mathcal{P}(\overline{z}) \cap B_n) = s^{\mathcal{P}(\overline{z}) \cap B_n})
\]

This is an average of \( n^d \) terms of two types:

- **Bulk terms**: Terms that are far from the boundary of \( B_n \)
- **Boundary terms**: Terms that are near the boundary of \( B_n \)

Bulk terms are close to \( I_\mu(s_{\mathbb{Z}^d}) \). All terms are uniformly bounded. Most terms are bulk terms. So, \( h(X) = I_\mu(s_{\mathbb{Z}^d}) \).
Rough Idea for showing $h(X) = l_\mu(s_{\mathbb{Z}^d})$

An MME $\mu$ should be “nearly uniform”. So, $\mu$ captures entropy:

If $s_{\mathbb{Z}^d} \in X$, then

$$h(X) = \lim_{n \to \infty} \frac{\log |L_n(X)|}{n^d} = \lim_{n \to \infty} -\frac{\log \mu(x(B_n) = s_{B_n})}{n^d}$$

$$= \lim_{n \to \infty} \left( \frac{1}{n^d} \right) \sum \limits_{\bar{z} \in B_n} -\log \mu(x(\bar{z}) = s | x(\mathcal{P}(\bar{z}) \cap B_n) = s_{\mathcal{P}(\bar{z}) \cap B_n})$$

This is an average of $n^d$ terms of two types:

- **Bulk terms**: Terms that are far from the boundary of $B_n$
- **Boundary terms**: Terms that are near the boundary of $B_n$

Bulk terms are close to $l_\mu(s_{\mathbb{Z}^d})$. All terms are uniformly bounded. Most terms are bulk terms. So, $h(X) = l_\mu(s_{\mathbb{Z}^d})$. 

Author  
Short Paper Title
Rough Idea for showing $h(X) = I_\mu(s^{\mathbb{Z}^d})$

An MME $\mu$ should be “nearly uniform”. So, $\mu$ captures entropy:

If $s^{\mathbb{Z}^d} \in X$, then

$$h(X) = \lim_{n \to \infty} \frac{\log |L_n(X)|}{n^d} = \lim_{n \to \infty} -\frac{\log \mu(x(B_n) = s^{B_n})}{n^d}$$

$$= \lim_{n \to \infty} \left(1/n^d\right) \sum_{\overline{z} \in B_n} -\log \mu(x(\overline{z}) = s \mid x(\mathcal{P}(\overline{z}) \cap B_n) = s^{\mathcal{P}(\overline{z}) \cap B_n})$$

This is an average of $n^d$ terms of two types:

- **Bulk terms**: Terms that are far from the boundary of $B_n$
- **Boundary terms**: Terms that are near the boundary of $B_n$

Bulk terms are close to $I_\mu(s^{\mathbb{Z}^d})$. All terms are uniformly bounded. Most terms are bulk terms. So, $h(X) = I_\mu(s^{\mathbb{Z}^d})$. 
Rough Idea for showing $h(X) = l_\mu(s^{\mathbb{Z}^d})$

An MME $\mu$ should be “nearly uniform”. So, $\mu$ captures entropy: If $s^{\mathbb{Z}^d} \in X$, then

$$h(X) = \lim_{n \to \infty} \frac{\log |L_n(X)|}{n^d} = \lim_{n \to \infty} -\frac{\log \mu(x(B_n) = s^{B_n})}{n^d}$$

$$= \lim_{n \to \infty} \left(\frac{1}{n^d}\right) \sum_{\overline{z} \in B_n} -\log \mu(x(\overline{z}) = s \mid x(\mathcal{P}(\overline{z}) \cap B_n) = s^{\mathcal{P}(\overline{z}) \cap B_n})$$

This is an average of $n^d$ terms of two types:
- **Bulk terms**: Terms that are far from the boundary of $B_n$
- **Boundary terms**: Terms that are near the boundary of $B_n$

Bulk terms are close to $l_\mu(s^{\mathbb{Z}^d})$. All terms are uniformly bounded. Most terms are bulk terms. So, $h(X) = l_\mu(s^{\mathbb{Z}^d})$. 
Rough Idea for showing $h(X) = I_{\mu}(s^{\mathbb{Z}^d})$

An MME $\mu$ should be "nearly uniform". So, $\mu$ captures entropy:

If $s^{\mathbb{Z}^d} \in X$, then

$$h(X) = \lim_{n \to \infty} \frac{\log |L_n(X)|}{n^d} = \lim_{n \to \infty} -\frac{\log \mu(x(B_n) = s^{B_n})}{n^d}$$

$$= \lim_{n \to \infty} \left( \frac{1}{n^d} \right) \sum_{\overline{z} \in B_n} -\log \mu(x(\overline{z}) = s \mid x(\mathcal{P}(\overline{z}) \cap B_n) = s^{\mathcal{P}(\overline{z}) \cap B_n})$$

This is an average of $n^d$ terms of two types:

- **Bulk terms**: Terms that are far from the boundary of $B_n$
- **Boundary terms**: Terms that are near the boundary of $B_n$

Bulk terms are close to $I_{\mu}(s^{\mathbb{Z}^d})$. All terms are uniformly bounded. Most terms are bulk terms. So, $h(X) = I_{\mu}(s^{\mathbb{Z}^d})$. 

---

**Author**

**Short Paper Title**
Rough Idea for showing $h(X) = I_\mu(s^{Z^d})$

An MME $\mu$ should be “nearly uniform”. So, $\mu$ captures entropy: If $s^{Z^d} \in X$, then

$$h(X) = \lim_{n \to \infty} \frac{\log |L_n(X)|}{n^d} = \lim_{n \to \infty} -\frac{\log \mu(x(B_n) = s^{B_n})}{n^d}$$

$$= \lim_{n \to \infty} \frac{1}{n^d} \sum_{\overline{z} \in B_n} -\log \mu(x(\overline{z}) = s | x(P(\overline{z}) \cap B_n) = s^{P(\overline{z}) \cap B_n})$$

This is an average of $n^d$ terms of two types:

- **Bulk terms**: Terms that are far from the boundary of $B_n$
- **Boundary terms**: Terms that are near the boundary of $B_n$

Bulk terms are close to $I_\mu(s^{Z^d})$. All terms are uniformly bounded. Most terms are bulk terms. So, $h(X) = I_\mu(s^{Z^d})$. 

Author | Short Paper Title
Rough Idea for showing $h(X) = I_\mu(s^{\mathbb{Z}^d})$

An MME $\mu$ should be “nearly uniform”. So, $\mu$ captures entropy: If $s^{\mathbb{Z}^d} \in X$, then

$$h(X) = \lim_{n \to \infty} \frac{\log |L_n(X)|}{n^d} = \lim_{n \to \infty} -\frac{\log \mu(x(B_n) = s^{B_n})}{n^d}$$

$$= \lim_{n \to \infty} \frac{1}{n^d} \sum_{\bar{z} \in B_n} -\log \mu(x(\bar{z}) = s | x(\mathcal{P}(\bar{z}) \cap B_n) = s^{\mathcal{P}(\bar{z}) \cap B_n})$$

This is an average of $n^d$ terms of two types:

- **Bulk terms**: Terms that are far from the boundary of $B_n$
- **Boundary terms**: Terms that are near the boundary of $B_n$

Bulk terms are close to $I_\mu(s^{\mathbb{Z}^d})$. All terms are uniformly bounded. Most terms are bulk terms. So, $h(X) = I_\mu(s^{\mathbb{Z}^d})$. 
Rough Idea for showing $h(X) = I_{\mu}(s_{\mathbb{Z}^d})$

An MME $\mu$ should be “nearly uniform”. So, $\mu$ captures entropy:

If $s_{\mathbb{Z}^d} \in X$, then

$$h(X) = \lim_{n \to \infty} \frac{\log |L_n(X)|}{n^d} = \lim_{n \to \infty} -\frac{\log \mu(x(B_n) = s^{B_n})}{n^d}$$

$$= \lim_{n \to \infty} \left( \frac{1}{n^d} \right) \sum_{\overline{z} \in B_n} -\log \mu(x(\overline{z}) = s \mid x(\mathcal{P}(\overline{z}) \cap B_n) = s^{\mathcal{P}(\overline{z}) \cap B_n})$$

This is an average of $n^d$ terms of two types:

- **Bulk terms**: Terms that are far from the boundary of $B_n$
- **Boundary terms**: Terms that are near the boundary of $B_n$

Bulk terms are close to $I_{\mu}(s_{\mathbb{Z}^d})$. All terms are uniformly bounded. Most terms are bulk terms. So, $h(X) = I_{\mu}(s_{\mathbb{Z}^d})$. 

Author | Short Paper Title
Rough Idea for showing \( h(X) = I_\mu(s_Z^d) \)

An MME \( \mu \) should be “nearly uniform”. So, \( \mu \) captures entropy: If \( s_Z^d \in X \), then

\[
h(X) = \lim_{n \to \infty} \frac{\log |L_n(X)|}{n^d} = \lim_{n \to \infty} \frac{-\log \mu(x(B_n) = s^{B_n})}{n^d}
\]

\[
= \lim_{n \to \infty} (1/n^d) \sum_{\bar{z} \in B_n} -\log \mu(x(\bar{z}) = s \mid x(\mathcal{P}(\bar{z}) \cap B_n) = s^{\mathcal{P}(\bar{z}) \cap B_n})
\]

This is an average of \( n^d \) terms of two types:
- **Bulk terms**: Terms that are far from the boundary of \( B_n \)
- **Boundary terms**: Terms that are near the boundary of \( B_n \)

Bulk terms are close to \( I_\mu(s_Z^d) \). All terms are uniformly bounded. Most terms are bulk terms. So, \( h(X) = I_\mu(s_Z^d) \).
Rough Idea for showing $h(X) = I_\mu(s^{\mathbb{Z}^d})$

An MME $\mu$ should be "nearly uniform". So, $\mu$ captures entropy: If $s^{\mathbb{Z}^d} \in X$, then

$$h(X) = \lim_{n \to \infty} \frac{\log |L_n(X)|}{n^d} = \lim_{n \to \infty} -\frac{\log \mu(x(B_n) = s^{B_n})}{n^d}$$

$$= \lim_{n \to \infty} \frac{1}{n^d} \sum_{\bar{z} \in B_n} -\log \mu(x(\bar{z}) = s | x(\mathcal{P}(\bar{z}) \cap B_n) = s^{\mathcal{P}(\bar{z}) \cap B_n})$$

This is an average of $n^d$ terms of two types:
- **Bulk terms**: Terms that are far from the boundary of $B_n$
- **Boundary terms**: Terms that are near the boundary of $B_n$

Bulk terms are close to $I_\mu(s^{\mathbb{Z}^d})$. All terms are uniformly bounded. Most terms are bulk terms. So, $h(X) = I_\mu(s^{\mathbb{Z}^d})$. 
Rough Idea for showing \( h(X) = I_\mu(s^{\mathbb{Z}^d}) \)

An MME \( \mu \) should be “nearly uniform”. So, \( \mu \) captures entropy:

If \( s^{\mathbb{Z}^d} \in X \), then

\[
h(X) = \lim_{n \to \infty} \frac{\log |L_n(X)|}{n^d} = \lim_{n \to \infty} \frac{-\log \mu(x(B_n) = s^{B_n})}{n^d}
\]

\[
= \lim_{n \to \infty} \left( \frac{1}{n^d} \right) \sum_{\bar{z} \in B_n} -\log \mu(x(\bar{z}) = s \mid x(\mathcal{P}(\bar{z}) \cap B_n) = s^{\mathcal{P}(\bar{z}) \cap B_n})
\]

This is an average of \( n^d \) terms of two types:

- **Bulk terms**: Terms that are far from the boundary of \( B_n \)
- **Boundary terms**: Terms that are near the boundary of \( B_n \)

Bulk terms are close to \( I_\mu(s^{\mathbb{Z}^d}) \). All terms are uniformly bounded. Most terms are bulk terms. So, \( h(X) = I_\mu(s^{\mathbb{Z}^d}) \).
An MME $\mu$ should be “nearly uniform”. So, $\mu$ captures entropy: If $s^{\mathbb{Z}^d} \in X$, then

$$h(X) = \lim_{n \to \infty} \frac{\log |L_n(X)|}{n^d} = \lim_{n \to \infty} - \frac{\log \mu(x(B_n) = s^{B_n})}{n^d}$$

$$= \lim_{n \to \infty} (1/n^d) \sum_{\bar{z} \in B_n} - \log \mu(x(\bar{z}) = s \mid x(\mathcal{P}(\bar{z}) \cap B_n) = s^{\mathcal{P}(\bar{z}) \cap B_n})$$

This is an average of $n^d$ terms of two types:

- **Bulk terms**: Terms that are far from the boundary of $B_n$.
- **Boundary terms**: Terms that are near the boundary of $B_n$.

Bulk terms are close to $I_{\mu}(s^{\mathbb{Z}^d})$. All terms are uniformly bounded. Most terms are bulk terms. So, $h(X) = I_{\mu}(s^{\mathbb{Z}^d})$. 

Author | Short Paper Title
Theorem (Lanford-Ruelle): Every MME on a n.n. SFT is a uniform Markov random field i.e., conditioned on boundary of a finite set, interior and exterior are independent, and the conditional distribution on interior is uniform.

Under a mild topological (combinatorial) assumption on a n.n. SFT $X$, we get:

- Even if you condition on a boundary condition, $\mu$ still captures topological entropy.
- All terms are uniformly bounded.

Also need a convergence condition to get bulk terms close to their limit $I_\mu(a^{\mathbb{Z}^d})$. Obtained by coupling and Peirels arguments.
Theorem (Lanford-Ruelle): Every MME on a n.n. SFT is a uniform Markov random field i.e., conditioned on boundary of a finite set, interior and exterior are independent, and the conditional distribution on interior is uniform.

Under a mild topological (combinatorial) assumption on a n.n. SFT $X$, we get:

- Even if you condition on a boundary condition, $\mu$ still captures topological entropy.
- All terms are uniformly bounded.

Also need a convergence condition to get bulk terms close to their limit $I_\mu(a^{\mathbb{Z}^d})$. Obtained by coupling and Peirels arguments.
Theorem (Lanford-Ruelle): Every MME on a n.n. SFT is a uniform Markov random field i.e., conditioned on boundary of a finite set, interior and exterior are independent, and the conditional distribution on interior is uniform.

Under a mild topological (combinatorial) assumption on a n.n. SFT $X$, we get:

- Even if you condition on a boundary condition, $\mu$ still captures topological entropy.
- All terms are uniformly bounded.

Also need a convergence condition to get bulk terms close to their limit $I_\mu(a^{\mathbb{Z}^d})$. Obtained by coupling and Peirels arguments.
Theorem (Lanford-Ruelle): Every MME on a n.n. SFT is a uniform Markov random field i.e., conditioned on boundary of a finite set, interior and exterior are independent, and the conditional distribution on interior is uniform.

Under a mild topological (combinatorial) assumption on a n.n. SFT $X$, we get:

- Even if you condition on a boundary condition, $\mu$ still captures topological entropy.
- All terms are uniformly bounded.

Also need a convergence condition to get bulk terms close to their limit $I_\mu(a_{\mathbb{Z}^d})$. Obtained by coupling and Peirels arguments.
Theorem (Lanford-Ruelle): Every MME on a n.n. SFT is a uniform Markov random field i.e., conditioned on boundary of a finite set, interior and exterior are independent, and the conditional distribution on interior is uniform.

Under a mild topological (combinatorial) assumption on a n.n. SFT $X$, we get:

- Even if you condition on a boundary condition, $\mu$ still captures topological entropy.
- All terms are uniformly bounded.

Also need a convergence condition to get bulk terms close to their limit $I_\mu(a^{\mathbb{Z}^d})$. Obtained by coupling and Peierls arguments.
Theorem (Lanford-Ruelle): Every MME on a n.n. SFT is a uniform Markov random field i.e., conditioned on boundary of a finite set, interior and exterior are independent, and the conditional distribution on interior is uniform.

Under a mild topological (combinatorial) assumption on a n.n. SFT $X$, we get:

- Even if you condition on a boundary condition, $\mu$ still captures topological entropy.
- All terms are uniformly bounded.

Also need a convergence condition to get bulk terms close to their limit $I_\mu(aZ^d)$. Obtained by coupling and Peirels arguments.
Theorem (Lanford-Ruelle): Every MME on a n.n. SFT is a uniform Markov random field i.e., conditioned on boundary of a finite set, interior and exterior are independent, and the conditional distribution on interior is uniform.

Under a mild topological (combinatorial) assumption on a n.n. SFT $X$, we get:

- Even if you condition on a boundary condition, $\mu$ still captures topological entropy.
- All terms are uniformly bounded.

Also need a convergence condition to get bulk terms close to their limit $I_\mu(a^{\mathbb{Z}^d})$. Obtained by coupling and Peirels arguments.
Theorem (Lanford-Ruelle): Every MME on a n.n. SFT is a uniform Markov random field i.e., conditioned on boundary of a finite set, interior and exterior are independent, and the conditional distribution on interior is uniform.

Under a mild topological (combinatorial) assumption on a n.n. SFT $X$, we get:

- Even if you condition on a boundary condition, $\mu$ still captures topological entropy.
- All terms are uniformly bounded.

Also need a convergence condition to get bulk terms close to their limit $I_\mu(a^{\mathbb{Z}^d})$. Obtained by coupling and Peirels arguments.
A n.n. SFT $X$ has a **safe symbol** $s$ if it is legal with every configuration of nearest neighbours:

![Diagram](image)

Examples: Yes: Hard squares ($s = 0$)
No: Checkerboard shifts, Dimers, Monomer-dimers
A n.n. SFT $X$ has a **safe symbol** $s$ if it is legal with every configuration of nearest neighbours:

\[
\begin{array}{c}
\ast \\
\ast & S & \ast \\
\ast
\end{array}
\]

Examples: Yes: Hard squares ($s = 0$)
No: Checkerboard shifts, Dimers, Monomer-dimers
A n.n. SFT $X$ has a **safe symbol** $s$ if it is legal with every configuration of nearest neighbours:

\[
\begin{array}{c|c|c}

\star & \star & \star \\
\hline
\star & s & \star \\
\star & \star & \\
\end{array}
\]

Examples: Yes: Hard squares ($s = 0$)
No: Checkerboard shifts, Dimers, Monomer-dimers
A n.n. SFT $X$ has a **safe symbol** $s$ if it is legal with every configuration of nearest neighbours:

```
*   
*   S   
*   
```

Examples: Yes: Hard squares ($s = 0$)
No: Checkerboard shifts, Dimers, Monomer-dimers
Entropy Representation

Let $R_{a,b,c} := [-a, -1] \times [1, c] \cup [0, b] \times [0, c]$

Example: $R_{3,4,3}$:

\[
\begin{array}{cccccccccc}
& & & & & & & & & \\
| & & & & & & & & & \\
c & & & & & & & & & \\
| & & & & & & & & & \\
& & & & & & & & & \\
- a & - & - & b & -
\end{array}
\]

Theorem: Let $X$ be a n.n. $\mathbb{Z}^d$ SFT and $\mu$ an MME on $X$. If

1. $X$ has a safe symbol $s$ — and —

2. (for $d = 2$)

\[
L := \lim_{a,b,c \to \infty} \mu(s^0 \mid s^{\partial R_{a,b,c}}) \text{ exists}
\]

Then

\[
h(X) = - \log L
\]
Let $R_{a,b,c} := [-a, -1] \times [1, c] \cup [0, b] \times [0, c]$

Example: $R_{3,4,3}$:

```
| · · · · · · · ·
| · · · · · · · ·
| · · · · · · · ·
| · · · · · · · ·
| · · · · · · · ·
| - a - - b -
```

Theorem: Let $X$ be a n.n. $\mathbb{Z}^d$ SFT and $\mu$ an MME on $X$. If

1. $X$ has a safe symbol $s$ – and –
2. (for $d = 2$)

$$L := \lim_{a,b,c \to \infty} \mu(s^0 | s^{\partial R_{a,b,c}})$$ exists

Then

$$h(X) = - \log L$$
Let $R_{a,b,c} := [-a, -1] \times [1, c] \cup [0, b] \times [0, c]$

Example: $R_{3,4,3}$:

$$
\begin{array}{cccccccc}
& & & & & & & \\
& c & & & & & & \\
& & & & & & & \\
& & & & & & & \\
- & a - & & & & & - & b -
\end{array}
$$

Theorem: Let $X$ be a n.n. $\mathbb{Z}^d$ SFT and $\mu$ an MME on $X$. If

1. $X$ has a safe symbol $s$ — and —
2. (for $d = 2$)

$$
L := \lim_{a,b,c \to \infty} \mu(s^0 \mid s^{\partial R_{a,b,c}}) \text{ exists}
$$

Then

$$
h(X) = - \log L
$$
Entropy Representation

Let \( R_{a,b,c} := [-a, -1] \times [1, c] \cup [0, b] \times [0, c] \)

Example: \( R_{3,4,3} : \)

\[
\begin{array}{cccccccc}
| & . & . & . & . & . & . & . & . \\
\wedge & . & . & . & . & . & . & . & . \\
\wedge & . & . & . & . & . & . & . & . \\
\wedge & . & . & . & . & . & . & . & . \\
\wedge & . & . & . & . & . & . & . & . \\
\wedge & a & - & - & b & - & - & - & - \\
\end{array}
\]

Theorem: Let \( X \) be a n.n. \( \mathbb{Z}^d \) SFT and \( \mu \) an MME on \( X \). If

1. \( X \) has a safe symbol \( s \) – and –
2. (for \( d = 2 \))

\[
L := \lim_{a,b,c \to \infty} \mu(s^0 | s^{\partial R_{a,b,c}}) \text{ exists}
\]

Then

\[
h(X) = - \log L
\]
Let $R_{a,b,c} := [-a, -1] \times [1, c] \cup [0, b] \times [0, c]$.

Example: $R_{3,4,3}$:

```
| . . . . . . . . . . .     |
| c . . . . . . . . . . .  |
| . . . . . . . . . . . .  |
- a - - b -
```

Theorem: Let $X$ be a n.n. $\mathbb{Z}^d$ SFT and $\mu$ an MME on $X$. If

1. $X$ has a safe symbol $s$ — and —
2. (for $d = 2$)

$$L := \lim_{a,b,c \to \infty} \mu(s^0 | s^{\partial R_{a,b,c}})$$ exists

Then

$$h(X) = - \log L$$
Entropy Representation

Let \( R_{a,b,c} := [-a, -1] \times [1, c] \cup [0, b] \times [0, c] \)

Example: \( R_{3,4,3} : \)

\[
\begin{array}{ccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
Moreover, if \( d = 2 \) and convergence in hypothesis 2 is exponential, then there is a polynomial time algorithm to compute \( h(X) \).

Proof of Moreover: Approximate \( L \) by \( \mu(s^0 \mid s^\partial R_{n,n}, n) \).

- Accuracy is \( e^{-\Omega(n)} \)
- Claim: Computation time is \( e^{O(n)} \)
- Trade exponential accuracy in exponential time for linear accuracy \((1/n)\) in polynomial time.  \( \square \)
Moreover, if $d = 2$ and convergence in hypothesis 2 is exponential, then there is a polynomial time algorithm to compute $h(X)$.

Proof of Moreover: Approximate $L$ by $\mu(s^0 \mid s^{\partial R_{n,n,n}})$.

- Accuracy is $e^{-\Omega(n)}$
- Claim: Computation time is $e^{O(n)}$
- Trade exponential accuracy in exponential time for linear accuracy $(1/n)$ in polynomial time.
Moreover, if $d = 2$ and convergence in hypothesis 2 is exponential, then there is a polynomial time algorithm to compute $h(X)$.

Proof of Moreover: Approximate $L$ by $\mu(s^0 \mid s^{\partial R_{n,n,n}})$.

- Accuracy is $e^{-\Omega(n)}$
- Claim: Computation time is $e^{O(n)}$
- Trade exponential accuracy in exponential time for linear accuracy $(1/n)$ in polynomial time.
Moreover, if $d = 2$ and convergence in hypothesis 2 is exponential, then there is a polynomial time algorithm to compute $h(X)$.

Proof of Moreover: Approximate $L$ by $\mu(s^0 \mid s^{\partial R_{n,n,n}})$.

- Accuracy is $e^{-\Omega(n)}$
- Claim: Computation time is $e^{O(n)}$

Trade exponential accuracy in exponential time for linear accuracy $(1/n)$ in polynomial time. □
Moreover, if \( d = 2 \) and convergence in hypothesis 2 is exponential, then there is a polynomial time algorithm to compute \( h(X) \).

Proof of Moreover: Approximate \( L \) by \( \mu(s^0 \mid s^{\partial R_{n,n,n}}) \).

- Accuracy is \( e^{-\Omega(n)} \)
- Claim: Computation time is \( e^{O(n)} \)
- Trade exponential accuracy in exponential time for linear accuracy \((1/n)\) in polynomial time.  □
Proof of Claim, via transfer matrices

\[
\mu(s^0 \mid s^{\partial R_{n,n,n}}) = \frac{(\prod_{i=-n}^{-1} M_i)\hat{M}_0 (\prod_{i=1}^{n-1} M_i)}{(\prod_{i=-n}^{-1} M_i)M_0 (\prod_{i=1}^{n-1} M_i)}
\]

\(M_i\) is transition matrix from column \(i\) to column \(i + 1\) compatible with \(s^{\partial R_{n,n,n}}\) and \(\hat{M}_0\) is matrix obtained from \(M_0\) by forcing \(s\) at origin.
Proof of Claim, via transfer matrices

\[ \mu\left(S^0 \mid S^{\partial R_{n,n,n}}\right) = \frac{S S S S S S}{S S S S S S} \]

\[ = \frac{\left(\prod_{i=-n}^{1-n} M_i\right) \hat{M}_0 \left(\prod_{i=1}^{n-1} M_i\right)}{\left(\prod_{i=-n}^{1-n} M_i\right) M_0 \left(\prod_{i=1}^{n-1} M_i\right)} \]

\( M_i \) is transition matrix from column \( i \) to column \( i + 1 \) compatible with \( S^{\partial R_{n,n,n}} \) and 
\( \hat{M}_0 \) is matrix obtained from \( M_0 \) by forcing \( S \) at origin. \( \square \)
Proof of Claim, via transfer matrices

\[ \mu(s^0 \mid s^{\partial R_{n,n,n}}) = \frac{\prod^{n-1}_{i=-n} M_i \hat{M}_0 \prod^{n-1}_{i=1} M_i}{\prod^{n-1}_{i=-n} M_i M_0 \prod^{n-1}_{i=1} M_i} \]

\( M_i \) is transition matrix from column \( i \) to column \( i + 1 \) compatible with \( s^{\partial R_{n,n,n}} \) and

\( \hat{M}_0 \) is matrix obtained from \( M_0 \) by forcing \( s \) at origin.
Proof of Claim, via transfer matrices

\[ \mu(s^0 \mid s^{\partial R_{n,n,n}}) = \begin{array}{cccccc}
  s & s & s & s & s & s \\
  s & \cdot & \cdot & \cdot & \cdot & s \\
  \# & s & \cdot & \cdot & \cdot & s \\
  s & s & s & s & s \\
  \cdot & s & s & s & s \\
  s & \cdot & \cdot & \cdot & \cdot & s \\
  \# & s & \cdot & \cdot & \cdot & s \\
  s & s & s & \cdot & \cdot & s \\
  \cdot & s & s & s & s \\
\end{array} \]

\[
= \frac{(\prod_{i=-n}^{-1} M_i) \hat{M}_0 (\prod_{i=1}^{n-1} M_i)}{(\prod_{i=-n}^{-1} M_i) M_0 (\prod_{i=1}^{n-1} M_i)}
\]

\(M_i\) is transition matrix from column \(i\) to column \(i + 1\) compatible with \(s^{\partial R_{n,n,n}}\) and
\(\hat{M}_0\) is matrix obtained from \(M_0\) by forcing \(s\) at origin. \(\square\)
Extensions

- Weaken fixed point $s^{\mathbb{Z}^d}$ to periodic orbit
- Weaken safe symbol to topological strong spatial mixing
- Applies to
  - hard squares
  - monomer-dimers
  - $q$-checkerboard SFT with $q \geq 6$

- Generalize results from entropy to pressure of n n. interactions on n.n. SFT's
  - Applies to large sets of temperature regions for classical models in statistical physics, in both subcritical and supercritical regions:
    - Hard square
    - Ising
    - Potts
    - Widom-Rowlinson
Extensions

- Weaken fixed point $s^d$ to periodic orbit
- Weaken safe symbol to topological strong spatial mixing

 Applies to
  - hard squares
  - monomer-dimers
  - $q$-checkerboard SFT with $q \geq 6$

 Generalize results from entropy to pressure of n.n. interactions on n.n. SFT’s

 Applies to large sets of temperature regions for classical models in statistical physics, in both subcritical and supercritical regions:
  - Hard square
  - Ising
  - Potts
  - Widom-Rowlinson
Extensions

- Weaken fixed point $s^d$ to periodic orbit
- Weaken safe symbol to topological strong spatial mixing
- Applies to
  - hard squares
  - monomer-dimers
  - $q$-checkerboard SFT with $q \geq 6$
- Generalize results from entropy to pressure of n.n. interactions on n.n. SFT's
  - Applies to large sets of temperature regions for classical models in statistical physics, in both subcritical and supercritical regions:
    - Hard square
    - Ising
    - Potts
    - Widom-Rowlinson
Extensions

- Weaken fixed point $s^\mathbb{Z}^d$ to periodic orbit
- Weaken safe symbol to topological strong spatial mixing
- Applies to
  - hard squares
  - monomer-dimers
  - $q$-checkerboard SFT with $q \geq 6$
- Generalize results from entropy to pressure of n.n. interactions on n.n. SFT’s

- Applies to large sets of temperature regions for classical models in statistical physics, in both subcritical and supercritical regions:
  - Hard square
  - Ising
  - Potts
  - Widom-Rowlinson
Extensions

- Weaken fixed point $s^{\mathbb{Z}^d}$ to periodic orbit
- Weaken safe symbol to topological strong spatial mixing
- Applies to
  - hard squares
  - monomer-dimers
  - $q$-checkerboard SFT with $q \geq 6$
- Generalize results from entropy to pressure of n.n. interactions on n.n. SFT's
- Applies to large sets of temperature regions for classical models in statistical physics, in both subcritical and supercritical regions:
  - Hard square
  - Ising
  - Potts
  - Widom-Rowlinson
Extensions

- Weaken fixed point $s^{\mathbb{Z}^d}$ to periodic orbit
- Weaken safe symbol to topological strong spatial mixing
- Applies to
  - hard squares
  - monomer-dimers
  - $q$-checkerboard SFT with $q \geq 6$
- Generalize results from entropy to pressure of n n. interactions on n.n. SFT’s
- Applies to large sets of temperature regions for classical models in statistical physics, in both subcritical and supercritical regions:
  - Hard square
  - Ising
  - Potts
  - Widom-Rowlinson
The following slides form a hodge-podge of topics that were not included in the talk.
Defn of TSSM with gap $g$:
for any disjoint $U, S, V \subseteq Z^d$ s.t. $d(U, V) \geq g$, if $u \in A^U$, $s \in A^S$, $v \in A^V$, s.t. $us$ and $sv$ are allowed, then so is $usv$. 
Defn of TSSM with gap $g$:
for any disjoint $U, S, V \subseteq Z^d$ s.t. $d(U, V) \geq g$,
if $u \in A^U$, $s \in A^S$, $v \in A^V$, s.t. $us$ and $sv$ are allowed, then so is $usv.$
Defn of TSSM with gap $g$:
for any disjoint $U, S, V \subseteq \mathbb{Z}^d$ s.t. $d(U, V) \geq g$,
if $u \in A^U$, $s \in A^S$, $v \in A^V$, s.t. $us$ and $sv$ are allowed, then so is $usv$. 
Defn of TSSM with gap $g$:
for any disjoint $U, S, V \subseteq \mathbb{Z}^d$ s.t. $d(U, V) \geq g$,
if $u \in A^U$, $s \in A^S$, $v \in A^V$, s.t. $us$ and $sv$ are allowed, then so is $usv$. 
Defn of TSSM with gap $g$:
for any disjoint $U, S, V \subseteq Z^d$ s.t. $d(U, V) \geq g$,
if $u \in A^U$, $s \in A^S$, $v \in A^V$, s.t. $us$ and $sv$ are allowed, then so is $usv$. 
Defn of TSSM with gap $g$: for any disjoint $U, S, V \in \mathbb{Z}^d$ s.t. $d(U, V) \geq g$, if $u \in A^U$, $s \in A^S$, $v \in A^V$, s.t. $us$ and $sv$ are allowed, then so is $usv$. 
Defn of TSSM with gap $g$: for any disjoint $U, S, V \subseteq \mathbb{Z}^d$ s.t. $d(U, V) \geq g$, if $u \in A^U$, $s \in A^S$, $v \in A^V$, s.t. $us$ and $sv$ are allowed, then so is $usv$. 
Let $R_{a,b,c} := [-a, -1] \times [1, c] \cup [0, b] \times [0, c]$

Example: $R_{3,4,3}$:

```
<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```

- $a$  -  - $b$  -

Theorem: Let $X$ be a $\mathbb{Z}^d$ n.n. SFT and $\mu$ an MME on $X$. If

1. $X$ satisfies TSSM
2. (for $d = 2$) For some periodic orbit $O$ in $X$ and all $\omega \in O$

$$L(\omega) := \lim_{a,b,c \to \infty} \mu(\omega(0) | \omega(\partial R_{a,b,c}))$$ exists

Then

$$h(X) = -\frac{1}{|O|} \sum_{\omega \in O} \log L(\omega)$$
Let $R_{a,b,c} := [-a, -1] \times [1, c] \cup [0, b] \times [0, c]$

Example: $R_{3,4,3}$:

```
  . . . . . . . . . . . . .
  | . . . . . . . . . . .
  c . . . . . . . . . . .
  | . . . . . . . . . . .
  | . . . . . . . . . . .
  . . . . . . . . . . . .
  - a -  - b -
```

Theorem: Let $X$ be a $\mathbb{Z}^d$ n.n. SFT and $\mu$ an MME on $X$. If

1. $X$ satisfies TSSM
2. (for $d = 2$) For some periodic orbit $O$ in $X$ and all $\omega \in O$

$$L(\omega) := \lim_{a,b,c \to \infty} \mu(\omega(0) \mid \omega(\partial R_{a,b,c}))$$ exists

Then

$$h(X) = -\frac{1}{|O|} \sum_{\omega \in O} \log L(\omega)$$
Let $R_{a,b,c} := [-a, -1] \times [1, c] \cup [0, b] \times [0, c]$

Example: $R_{3,4,3}$:

```

  . . . . . . . . . .
  | . . . . . . . . .
  c . . . . . . . . .
  | . . . . . . . . .

  - a - - - b -
```

Theorem: Let $X$ be a $\mathbb{Z}^d$ n.n. SFT and $\mu$ an MME on $X$. If

1. $X$ satisfies TSSM
2. (for $d = 2$) For some periodic orbit $O$ in $X$ and all $\omega \in O$

$$L(\omega) := \lim_{a,b,c \to \infty} \mu(\omega(0) \mid \omega(\partial R_{a,b,c}))$$ exists

Then

$$h(X) = - \frac{1}{|O|} \sum_{\omega \in O} \log L(\omega)$$
Let \( R_{a,b,c} := [-a, -1] \times [1, c] \cup [0, b] \times [0, c] \)

Example: \( R_{3,4,3} : \)

\[
\begin{array}{cccccccccc}
& & & & & & & & & & \\
& & & & & & & & & & \\
& & & & & & & & & & \\
& & & & & & & & & & \\
& & & & & & & & & & \\
c & & & & & & & & & \\
& & & & & & & & & & \\
& & & & & & & & & & \\
& & & & & & & & & & \\
\end{array}
\]

\(-a\) \hspace{1cm} \(-b\)

Theorem: Let \( X \) be a \( \mathbb{Z}^d \) n.n. SFT and \( \mu \) an MME on \( X \). If

1. \( X \) satisfies TSSM
2. (for \( d = 2 \)) For some periodic orbit \( O \) in \( X \) and all \( \omega \in O \)

\[
L(\omega) := \lim_{a,b,c \to \infty} \mu(\omega(0) | \omega(\partial R_{a,b,c})) \text{ exists}
\]

Then

\[
h(X) = -\frac{1}{|O|} \sum_{\omega \in O} \log L(\omega)
\]
Let $R_{a,b,c} := [-a, -1] \times [1, c] \cup [0, b] \times [0, c]$

Example: $R_{3,4,3}$:

\[
\begin{array}{cccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
| & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
| & c & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
| & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
- & a & - & - & b & - & \vdots & \vdots \\
\end{array}
\]

Theorem: Let $X$ be a $\mathbb{Z}^d$ n.n. SFT and $\mu$ an MME on $X$. If

1. $X$ satisfies TSSM
2. (for $d = 2$) For some periodic orbit $O$ in $X$ and all $\omega \in O$

\[L(\omega) := \lim_{a,b,c \to \infty} \mu(\omega(0) \mid \omega(\partial R_{a,b,c})) \text{ exists}\]

Then

\[h(X) = -\frac{1}{|O|} \sum_{\omega \in O} \log L(\omega)\]
Let $R_{a,b,c} := [-a, -1] \times [1, c] \cup [0, b] \times [0, c]$

Example: $R_{3,4,3}$:

```
     . . . . . . . . . .
     | . . . . . . . . .
     c . . . . . . . . .
     | . . . . . . . . .
     | . . . . . . . . .
     . . . . . . . . .
     - a - - - b -
```

Theorem: Let $X$ be a $\mathbb{Z}^d$ n.n. SFT and $\mu$ an MME on $X$. If

1. $X$ satisfies TSSM
2. (for $d = 2$) For some periodic orbit $O$ in $X$ and all $\omega \in O$

   $$L(\omega) := \lim_{a,b,c \to \infty} \mu(\omega(0) | \omega(\partial R_{a,b,c}))$$ exists

Then

$$h(X) = -\frac{1}{|O|} \sum_{\omega \in O} \log L(\omega)$$
MME is characterized by as much:

- Site-to-site independence -and-
- Uniformity of distribution

as possible.
MME is characterized by as much:

- Site-to-site independence -and-
- Uniformity of distribution

as possible.
MME is characterized by as much:
- Site-to-site independence -and-
- Uniformity of distribution

as possible.
MME is characterized by as much:
- Site-to-site independence -and-
- Uniformity of distribution
as possible.
A Markov random field (MRF) is a shift-invariant Borel probability measure $\mu$ on $\mathcal{A}^{\mathbb{Z}^d}$ such that for any choice of:

- $S \subseteq \mathbb{Z}^d$,
- $T \subseteq \mathbb{Z}^d$ s.t. $\partial S \subseteq T \subseteq \mathbb{Z}^d \setminus S$
- configuration $x$ on $S$
- configuration $y$ on $T$ s.t. $\mu(y) > 0$,

we have:

$$\mu(x \mid y) = \mu(x \mid y(\partial S))$$
A Markov random field (MRF) is a shift-invariant Borel probability measure $\mu$ on $\mathbb{A}^\mathbb{Z}^d$ such that for any choice of:

- $S \in \mathbb{Z}^d$,
- $T \in \mathbb{Z}^d$ s.t. $\partial S \subseteq T \subseteq \mathbb{Z}^d \setminus S$
- configuration $x$ on $S$
- configuration $y$ on $T$ s.t. $\mu(y) > 0$,

we have:

$$\mu(x | y) = \mu(x | y(\partial S))$$
A Markov random field (MRF) is a shift-invariant Borel probability measure $\mu$ on $\mathbb{A}^{\mathbb{Z}^d}$ such that for any choice of:

- $S \subseteq \mathbb{Z}^d$,
- $T \subseteq \mathbb{Z}^d$ s.t. $\partial S \subseteq T \subseteq \mathbb{Z}^d \setminus S$

configuration $x$ on $S$
configuration $y$ on $T$ s.t. $\mu(y) > 0$,

we have:

$$\mu(x \mid y) = \mu(x \mid y(\partial S))$$
A Markov random field (MRF) is a shift-invariant Borel probability measure $\mu$ on $\mathcal{A}^\mathbb{Z}^d$ such that for any choice of:

- $S \in \mathbb{Z}^d$,
- $T \in \mathbb{Z}^d$ s.t. $\partial S \subseteq T \subseteq \mathbb{Z}^d \setminus S$
- configuration $x$ on $S$
- configuration $y$ on $T$ s.t. $\mu(y) > 0$,

we have:

$$\mu(x \mid y) = \mu(x \mid y(\partial S))$$
A Markov random field (MRF) is a shift-invariant Borel probability measure $\mu$ on $\mathbb{A}^{\mathbb{Z}^d}$ such that for any choice of:
- $S \subseteq \mathbb{Z}^d$,
- $T \subseteq \mathbb{Z}^d$ s.t. $\partial S \subseteq T \subseteq \mathbb{Z}^d \setminus S$
- configuration $x$ on $S$
- configuration $y$ on $T$ s.t. $\mu(y) > 0$,
we have:

$$\mu(x \mid y) = \mu(x \mid y(\partial S))$$
A Markov random field (MRF) is a shift-invariant Borel probability measure $\mu$ on $\mathbb{Z}^d$ such that for any choice of:

- $S \subseteq \mathbb{Z}^d$,
- $T \subseteq \mathbb{Z}^d$ s.t. $\partial S \subseteq T \subseteq \mathbb{Z}^d \setminus S$
- configuration $x$ on $S$
- configuration $y$ on $T$ s.t. $\mu(y) > 0$,

we have:

$$\mu(x \mid y) = \mu(x \mid y(\partial S))$$
A Markov random field (MRF) is a shift-invariant Borel probability measure $\mu$ on $\mathbb{Z}^d$ such that for any choice of:
- $S \subseteq \mathbb{Z}^d$,
- $T \subseteq \mathbb{Z}^d$ s.t. $\partial S \subseteq T \subseteq \mathbb{Z}^d \setminus S$
- configuration $x$ on $S$
- configuration $y$ on $T$ s.t. $\mu(y) > 0$,
we have:

$$\mu(x \mid y) = \mu(x \mid y(\partial S))$$
Let $X$ be a n.n. SFT. For $S \subseteq \mathbb{Z}^d$ and $y \in A^{\partial S}$, let

$$L_y^S(X) := \{ x \in A^S : xy \text{ is legal} \}$$

An MRF on $X$ is **uniform** if whenever $\mu(y) > 0$, then for $x \in L_y^S(X)$

$$\mu(x \mid y) = \frac{1}{|L_y^S(X)|}$$

Theorem (Lanford/Ruelle, Burton/Steif): Every MME on a n.n. SFT is a uniform MRF.
Let $X$ be a n.n. SFT. For $S \subseteq \mathbb{Z}^d$ and $y \in A^d$, let

$$L^y_S(X) := \{ x \in A^S : xy \text{ is legal} \}$$

An MRF on $X$ is **uniform** if whenever $\mu(y) > 0$, then for $x \in L^y_S(X)$

$$\mu(x \mid y) = \frac{1}{|L^y_S(X)|}$$

Theorem (Lanford/Ruelle, Burton/Steif): Every MME on a n.n. SFT is a uniform MRF.
Let $X$ be a n.n. SFT. For $S \subseteq \mathbb{Z}^d$ and $y \in A^{\partial S}$, let

$$L^y_S(X) := \{ x \in A^S : xy \text{ is legal} \}$$

An MRF on $X$ is **uniform** if whenever $\mu(y) > 0$, then for $x \in L^y_S(X)$

$$\mu(x \mid y) = \frac{1}{|L^y_S(X)|}$$

Theorem (Lanford/Ruelle, Burton/Steif): Every MME on a n.n. SFT is a uniform MRF.
Let $X$ be a n.n. SFT. For $S \subseteq \mathbb{Z}^d$ and $y \in A^{\partial S}$, let

$$L^y_S(X) := \{ x \in A^S : xy \text{ is legal} \}$$

An MRF on $X$ is **uniform** if whenever $\mu(y) > 0$, then for $x \in L^y_S(X)$

$$\mu(x \mid y) = \frac{1}{|L^y_S(X)|}$$

Theorem (Lanford/Ruelle, Burton/Steif): Every MME on a n.n. SFT is a uniform MRF.
Let $X$ be a n.n. SFT. For $S \subseteq \mathbb{Z}^d$ and $y \in \mathcal{A}^{\partial S}$, let

$$L^y_S(X) := \{ x \in \mathcal{A}^S : xy \text{ is legal} \}$$

An MRF on $X$ is uniform if whenever $\mu(y) > 0$, then for $x \in L^y_S(X)$

$$\mu(x \mid y) = \frac{1}{|L^y_S(X)|}$$

Theorem (Lanford/Ruelle, Burton/Steif): Every MME on a n.n. SFT is a uniform MRF.
Let $X$ be a n.n. SFT. For $S \subseteq \mathbb{Z}^d$ and $y \in A^{\partial S}$, let

$$L^y_S(X) := \{ x \in A^S : xy \text{ is legal} \}$$

An MRF on $X$ is \textbf{uniform} if whenever $\mu(y) > 0$, then for $x \in L^y_S(X)$

$$\mu(x \mid y) = \frac{1}{|L^y_S(X)|}$$

Theorem (Lanford/Ruelle, Burton/Steif): Every MME on a n.n. SFT is a uniform MRF.
Proof

Since $\mu$ is an MME, $\mu$ must be a uniform MRF.

Since $s$ is a safe symbol,

1. For all $T \subseteq \mathbb{Z}^d$ containing 0,

$$\mu(s_0 | s^{\partial T}) \geq \frac{1}{|A|}.$$  

2. $h(X) = \lim_{n \to \infty} -\log \mu(s_{B_n} | s^{\partial B_n})$ divided by $n^d$.

Proof:

$$\mu(s_{B_n} | s^{\partial B_n}) = \frac{1}{\left| GA_n(X) \right|}.$$
Proof

Since $\mu$ is an MME, $\mu$ must be a uniform MRF.

Since $s$ is a safe symbol,

1. For all $T \subseteq \mathbb{Z}^d$ containing 0,

\[
\mu(s^0 \mid s^{\partial T}) \geq \frac{1}{|A|}.
\]

2. $h(X) = \lim_{n \to \infty} \frac{- \log \mu(s^{B_n} \mid s^{\partial B_n})}{n^d}$

Proof:

\[
\mu(s^{B_n} \mid s^{\partial B_n}) = \frac{1}{|GA_n(X)|}.
\]
Proof

Since $\mu$ is an MME, $\mu$ must be a uniform MRF.

Since $s$ is a safe symbol,

1. For all $T \subseteq \mathbb{Z}^d$ containing 0,
   \[ \mu(s^0 | s^{\partial T}) \geq \frac{1}{|A|}. \]

2. \[ h(X) = \lim_{n \to \infty} \frac{-\log \mu(s^{B_n} | s^{\partial B_n})}{n^d} \]

Proof:

\[ \mu(s^{B_n} | s^{\partial B_n}) = \frac{1}{|GA_n(X)|} \]
Proof

Since $\mu$ is an MME, $\mu$ must be a uniform MRF.

Since $s$ is a safe symbol,

1. For all $T \subseteq \mathbb{Z}^d$ containing 0,

$$\mu(s^0 \mid s^{\partial T}) \geq \frac{1}{|A|}.$$ 

2. $h(X) = \lim_{n \to \infty} -\log \frac{\mu(s^{B_n} \mid s^{\partial B_n})}{n^d}$

Proof:

$$\mu(s^{B_n} \mid s^{\partial B_n}) = \frac{1}{|GA_n(X)|}$$
Since $\mu$ is an MME, $\mu$ must be a uniform MRF.

Since $s$ is a safe symbol,

1. For all $T \subseteq \mathbb{Z}^d$ containing 0,

$$\mu(s^0 | s^{\partial T}) \geq \frac{1}{|A|}.$$

2. $h(X) = \lim_{n \to \infty} \frac{-\log \mu(s_{B_n} | s^{\partial B_n})}{n^d}$

Proof:

$$\mu(s_{B_n} | s^{\partial B_n}) = \frac{1}{|GA_n(X)|}.$$
Decomposition

\[ h(X) = \lim_{n \to \infty} -\log \frac{\mu(s^{B_n} | s^{\partial B_n})}{n^d} \]

\[ \mu(s^{B_n} | s^{\partial B_n}) = \prod_{\overline{z} \in B_n} \mu(s^{\overline{z}} | s^{P(\overline{z}) \cap B_n} s^{\partial B_n}) \]
Decomposition

\[ h(X) = \lim_{n \to \infty} \frac{-\log \mu(s^{B_n} | s^{\partial B_n})}{n^d} \]

\[ \mu(s^{B_n} | s^{\partial B_n}) = \prod_{\bar{z} \in B_n} \mu(s^{\bar{z}} | s^{P(\bar{z}) \cap B_n} s^{\partial B_n}) \]
Decomposition

\[ h(X) = \lim_{n \to \infty} -\log \frac{\mu(s_B^n | s_{\partial B^n})}{n^d} \]

\[ \mu(s_B^n | s_{\partial B^n}) = \prod_{z \in B_n} \mu(s_{\bar{z}} | s_{\mathcal{P}(\bar{z}) \cap B_n} s_{\partial B^n}) \]
Decomposition

\[ h(X) = \lim_{n \to \infty} -\log \frac{\mu(s^{B_n} \mid s^{\partial B_n})}{n^d} \]

\[ \mu(s^{B_n} \mid s^{\partial B_n}) = \prod_{\overline{z} \in B_n} \mu(s^{\overline{z}} \mid s^{P(\overline{z}) \cap B_n} s^{\partial B_n}) \]
Decomposition

$$h(X) = \lim_{n \to \infty} -\log \frac{\mu(s^{B_n} | s^\partial B_n)}{n^d}$$

$$\mu(s^{B_n} | s^\partial B_n) = \prod_{\bar{z} \in B_n} \mu(s^{\bar{z}} | s^{\mathcal{P}(\bar{z}) \cap B_n} s^\partial B_n)$$
Decomposition

\[ h(X) = \lim_{n \to \infty} - \frac{\log \mu(s^{B_n} | s^{\partial B_n})}{n^d} \]

\[ \mu(s^{B_n} | s^{\partial B_n}) = \prod_{\overline{z} \in B_n} \mu(s^{\overline{z}} | s^{P(\overline{z}) \cap B_n} s^{\partial B_n}) \]
Decomposition

\[
h(X) = \lim_{n \to \infty} - \log \frac{\mu(s^{B_n} | s^{\partial B_n})}{n^d}
\]

\[
\mu(s^{B_n} | s^{\partial B_n}) = \prod_{\bar{z} \in B_n} \mu(s^{\bar{z}} | s^{P(\bar{z}) \cap B_n} s^{\partial B_n})
\]
Decomposition

\[ h(X) = \lim_{n \to \infty} - \log \frac{\mu(s^{B_n} | s^\partial B_n)}{n^d} \]

\[ \mu(s^{B_n} | s^\partial B_n) = \prod_{\bar{z} \in B_n} \mu(s^{\bar{z}} | s^{P(\bar{z}) \cap B_n} s^\partial B_n) \]
Decomposition

\[ h(X) = \lim_{n \to \infty} \frac{-\log \mu(s^{B_n} \mid s^{\partial B_n})}{n^d} \]

\[
\mu(s^{B_n} \mid s^{\partial B_n}) = \prod_{\bar{z} \in B_n} \mu(s^{\bar{z}} \mid s^{P(\bar{z}) \cap B_n} s^{\partial B_n}) = \prod_{\bar{z} \in B_n} \mu(s^0 \mid s^{\partial R_{a(\bar{z}), b(\bar{z}), c(\bar{z})}}) 
\]
Proof

So,

$$\log \mu(s^{B_n} | s^{\partial B_n}) = \sum_{\bar{z} \in B_n} \log \mu(s^{\bar{z}} | s^{\partial R_{a(z)}, b(z), c(z)})$$

By the convergence assumption, for “most” $\bar{z} \in B_n$

$$\log \mu(s^{\bar{z}} | s^{\partial R_{a(z)}, b(z), c(z)}) \approx \log L$$

By safe symbol assumption, for the remaining $\bar{z} \in B_n$,

$$0 \geq \log \mu(s^{\bar{z}} | s^{\partial R_{a(z)}, b(z), c(z)}) \geq -\log |A|$$

Thus, $h(X) = \lim_{n \to \infty} -\frac{\log \mu(s^{B_n} | s^{\partial B_n})}{n^d} = -\log L$. □
Proof

So,

\[ \log \mu(s^{B_n} \mid s^{\partial B_n}) = \sum_{\bar{z} \in B_n} \log \mu(s^{\bar{z}} \mid s^{\partial R_{a(z)}, b(z), c(z)}) \]

By the convergence assumption, for “most” \( \bar{z} \in B_n \)

\[ \log \mu(s^{\bar{z}} \mid s^{\partial R_{a(z)}, b(z), c(z)}) \approx \log L \]

By safe symbol assumption, for the remaining \( \bar{z} \in B_n \),

\[ 0 \geq \log \mu(s^{\bar{z}} \mid s^{\partial R_{a(z)}, b(z), c(z)}) \geq -\log |A| \]

Thus, \( h(X) = \lim_{n \to \infty} \frac{-\log \mu(s^{B_n} \mid s^{\partial B_n})}{n^d} = -\log L. \)

□
Proof

- So,
  \[ \log \mu(s^{B_n} \mid s^{\partial B_n}) = \sum_{\bar{z} \in B_n} \log \mu(s^{\bar{z}} \mid s^{\partial R_{a(\bar{z}),b(\bar{z}),c(\bar{z})}}) \]

- By the convergence assumption, for “most” \( \bar{z} \in B_n \)
  \[ \log \mu(s^{\bar{z}} \mid s^{\partial R_{a(\bar{z}),b(\bar{z}),c(\bar{z})}}) \approx \log L \]

- By safe symbol assumption, for the remaining \( \bar{z} \in B_n \),
  \[ 0 \geq \log \mu(s^{\bar{z}} \mid s^{\partial R_{a(\bar{z}),b(\bar{z}),c(\bar{z})}}) \geq - \log |A| \]

Thus, \( h(X) = \lim_{n \to \infty} - \frac{\log \mu(s^{B_n} \mid s^{\partial B_n})}{n^d} = - \log L \). \( \square \)
Proof

So,

$$\log \mu(s^{B_n} | s^{\partial B_n}) = \sum_{\bar{z} \in B_n} \log \mu(s^{\bar{z}} | s^{\partial R_{a(\bar{z}), b(\bar{z}), c(\bar{z})}}$$

By the convergence assumption, for “most” $\bar{z} \in B_n$

$$\log \mu(s^{\bar{z}} | s^{\partial R_{a(\bar{z}), b(\bar{z}), c(\bar{z})}} \approx \log L$$

By safe symbol assumption, for the remaining $\bar{z} \in B_n$,

$$0 \geq \log \mu(s^{\bar{z}} | s^{\partial R_{a(\bar{z}), b(\bar{z}), c(\bar{z})}} \geq - \log |A|$$

Thus, $h(X) = \lim_{n \to \infty} -\frac{\log \mu(s^{B_n} | s^{\partial B_n})}{nd} = - \log L$.
Theorem: Let $X$ be a n.n. $\mathbb{Z}^2$ SFT and $\mu$ an MME on $X$. If

1. $X$ has a safe symbol $s$ — and —
2. $$L := \lim_{a,b,c \to \infty} \mu(s^0 | s^{\partial R_{a,b,c}})$$ exists

and convergence is exponential

Then there is a polynomial time algorithm to compute $h(X) = -\log L$. 
Theorem: Let $X$ be a n.n. $\mathbb{Z}^2$ SFT and $\mu$ an MME on $X$. If $X$ has a safe symbol $s$ and $L := \lim_{a, b, c \to \infty} \mu(s^0 \mid s^{\partial R_{a, b, c}})$ exists, and convergence is exponential, then there is a polynomial time algorithm to compute $h(X) = -\log L$. 
Theorem: Let $X$ be a n.n. $\mathbb{Z}^2$ SFT and $\mu$ an MME on $X$. If

1. $X$ has a safe symbol $s$ – and –
2. $L := \lim_{a,b,c \to \infty} \mu(s^0 \mid s^{\partial R_{a,b,c}})$ exists

and convergence is exponential

Then there is a polynomial time algorithm to compute $h(X) = -\log L$. 
Theorem: Let $X$ be a n.n. $\mathbb{Z}^2$ SFT and $\mu$ an MME on $X$. If

1. $X$ has a safe symbol $s$ and
2. $L := \lim_{a,b,c \to \infty} \mu(s^0 | s^\partial R_{a,b,c})$ exists

and convergence is exponential

Then there is a polynomial time algorithm to compute $h(X) = -\log L$. 
Theorem: Let $X$ be a n.n. $\mathbb{Z}^2$ SFT and $\mu$ an MME on $X$. If

1. $X$ has a safe symbol $s$ – and –
2. $L := \lim_{a,b,c \to \infty} \mu(s^0 \mid s^\partial R_{a,b,c})$ exists

and convergence is exponential

Then there is a polynomial time algorithm to compute $h(X) = -\log L$. 

Author
Short Paper Title
An MRF $\mu$ satisfies **strong spatial mixing (SSM)** at rate $f(n)$ if for all $V \subseteq \mathbb{Z}^d$, $U \subseteq V$ all $u \in A^U$, and $v, v' \in A^\partial V$ satisfying $\mu(v), \mu(v') > 0$, we have $|\mu(u \mid v) - \mu(u \mid v')| \leq |U| f(d(U, \Sigma_{\partial V}(v, v')))$. Where $\Sigma_{\partial V}(v, v') = \{ t \in \partial V : v(t) \neq v(t') \}$. 

SSM $\Rightarrow$ convergence condition in theorem.
An MRF $\mu$ satisfies **strong spatial mixing (SSM)** at rate $f(n)$

if for all $V \subseteq Z^d$, $U \subset V$

all $u \in A^U$, and $v, v' \in A^{\partial V}$ satisfying $\mu(v), \mu(v') > 0$,

we have $|\mu(u \mid v) - \mu(u \mid v')| \leq |U|f(d(U, \Sigma_{\partial V}(v, v')))$.

where $\Sigma_{\partial V}(v, v') = \{ t \in \partial V : v(t) \neq v(t') \}$.

SSM $\Rightarrow$ convergence condition in theorem.
An MRF $\mu$ satisfies **strong spatial mixing (SSM)** at rate $f(n)$ if for all $V \subseteq \mathbb{Z}^d$, $U \subseteq V$

all $u \in A^U$, and $v, v' \in A^{\partial V}$ satisfying $\mu(v), \mu(v') > 0$,

we have $|\mu(u \mid v) - \mu(u \mid v')| \leq |U| f(d(U, \Sigma_{\partial V}(v, v')))$. where $\Sigma_{\partial V}(v, v') = \{t \in \partial V : v(t) \neq v(t')\}$.

SSM $\Rightarrow$ convergence condition in theorem.
Strong Spatial Mixing

- An MRF $\mu$ satisfies **strong spatial mixing (SSM)** at rate $f(n)$ if for all $V \subseteq Z^d$, $U \subset V$

  all $u \in A^U$, and $v, v' \in A^{\partial V}$ satisfying $\mu(v), \mu(v') > 0$,

  we have $|\mu(u | v) - \mu(u | v')| \leq |U| f(d(U, \Sigma_{\partial V}(v, v')))$. 

  where $\Sigma_{\partial V}(v, v') = \{t \in \partial V : v(t) \neq v(t')\}$.

- SSM $\Rightarrow$ convergence condition in theorem.
An MRF $\mu$ satisfies **strong spatial mixing (SSM)** at rate $f(n)$ if for all $V \subseteq \mathbb{Z}^d$, $U \subset V$

all $u \in A^U$, and $v, v' \in A^{\partial V}$ satisfying $\mu(v), \mu(v') > 0$, we have $|\mu(u | v) - \mu(u | v')| \leq |U|f(d(U, \Sigma_{\partial V}(v, v')))$. Where $\Sigma_{\partial V}(v, v') = \{t \in \partial V : v(t) \neq v(t')\}$.

**SSM** $\Rightarrow$ convergence condition in theorem.
An MRF $\mu$ satisfies **strong spatial mixing (SSM)** at rate $f(n)$ if for all $V \subset Z^d$, $U \subset V$

all $u \in A^U$, and $v, v' \in A^{\partial V}$ satisfying $\mu(v), \mu(v') > 0$,

we have $|\mu(u | v) - \mu(u | v')| \leq |U| f(d(U, \Sigma_{\partial V}(v, v')))$.  
where $\Sigma_{\partial V}(v, v') = \{ t \in \partial V : v(t) \neq v(t') \}$.

**SSM $\Rightarrow$ convergence condition in theorem.**
Theorem (Briceno): Let $X$ be a $\mathbb{Z}^d$ n.n. SFT and $\mu$ an MME on $X$. If

1. $X$ satisfies TSSM
2. $\mu$ satisfies SSM

Then for all invariant measures $\nu$ s.t. $\text{support}(\nu) \subseteq X$,

$$h(X) = \int l_\mu(x) \, d\nu(x)$$

Applies to:
- hard squares
- $q$-checkerboard with $q \geq 6$
Theorem (Briceno): Let $X$ be a $\mathbb{Z}^d$ n.n. SFT and $\mu$ an MME on $X$. If

1. $X$ satisfies TSSM
2. $\mu$ satisfies SSM

Then for all invariant measures $\nu$ s.t. $\text{support}(\nu) \subseteq X$,

$$h(X) = \int l_\mu(x) \, d\nu(x)$$

Applies to:

- hard squares
- $q$-checkerboard with $q \geq 6$
Theorem (Briceno): Let $X$ be a $\mathbb{Z}^d$ n.n. SFT and $\mu$ an MME on $X$. If

1. $X$ satisfies TSSM
2. $\mu$ satisfies SSM

Then for all invariant measures $\nu$ s.t. $\text{support}(\nu) \subseteq X$,

$$h(X) = \int l_{\mu}(x) \, d\nu(x)$$

Applies to:

- hard squares
- $q$-checkerboard with $q \geq 6$
Theorem (Briceno): Let $X$ be a $\mathbb{Z}^d$ n.n. SFT and $\mu$ an MME on $X$. If

1. $X$ satisfies TSSM
2. $\mu$ satisfies SSM

Then for all invariant measures $\nu$ s.t. $\text{support}(\nu) \subseteq X$,

$$h(X) = \int l_\mu(x) \, d\nu(x)$$

Applies to:
- hard squares
- $q$-checkerboard with $q \geq 6$
Theorem (Briceno): Let $X$ be a $\mathbb{Z}^d$ n.n. SFT and $\mu$ an MME on $X$. If

1. $X$ satisfies TSSM
2. $\mu$ satisfies SSM

Then for all invariant measures $\nu$ s.t. $\text{support}(\nu) \subseteq X$,

$$h(X) = \int l_{\mu}(x) \, d\nu(x)$$

Applies to:

- hard squares
- $q$-checkerboard with $q \geq 6$
Theorem (Briceno): Let $X$ be a $\mathbb{Z}^d$ n.n. SFT and $\mu$ an MME on $X$. If

1. $X$ satisfies TSSM
2. $\mu$ satisfies SSM

Then for all invariant measures $\nu$ s.t. $\text{support}(\nu) \subseteq X$,

$$h(X) = \int l_\mu(x) \, d\nu(x)$$

Applies to:
- hard squares
- $q$-checkerboard with $q \geq 6$
Let $X$ be a shift space and $f : X \to \mathbb{R}$ a continuous function. 

**Topological Pressure** (defined by Variational Principle):

$$P_X(f) := \sup_{\mu} h(\mu) + \int f d\mu$$

where the sup is taken over all shift-invariant Borel probability measures $\mu$ such that support$(\mu) \subseteq X$.

Fact: The sup is always achieved.

A measure which achieves the sup is called an **equilibrium state**.

Note: $P_X(0) = h(X)$. 
Let $X$ be a shift space and $f : X \to \mathbb{R}$ a continuous function.

**Topological Pressure** (defined by Variational Principle):

$$P_X(f) := \sup_{\mu} h(\mu) + \int f d\mu$$

where the sup is taken over all shift-invariant Borel probability measures $\mu$ such that $\text{support}(\mu) \subseteq X$.

Fact: The sup is always achieved.

A measure which achieves the sup is called an equilibrium state.

Note: $P_X(0) = h(X)$. 
Let $X$ be a shift space and $f : X \to \mathbb{R}$ a continuous function.

**Topological Pressure** (defined by Variational Principle):

$$P_X(f) := \sup_{\mu} h(\mu) + \int f d\mu$$

where the sup is taken over all shift-invariant Borel probability measures $\mu$ such that $\text{support}(\mu) \subseteq X$.

**Fact:** The sup is always achieved.

- A measure which achieves the sup is called an **equilibrium state**.
- **Note:** $P_X(0) = h(X)$. 
Let $X$ be a shift space and $f : X \to \mathbb{R}$ a continuous function.

**Topological Pressure** (defined by Variational Principle):

$$P_X(f) := \sup_{\mu} h(\mu) + \int f d\mu$$

where the sup is taken over all shift-invariant Borel probability measures $\mu$ such that support($\mu$) $\subseteq X$.

- Fact: The sup is always achieved.
- A measure which achieves the sup is called an **equilibrium state**.
- Note: $P_X(0) = h(X)$. 
Let $X$ be a shift space and $f : X \to \mathbb{R}$ a continuous function.

**Topological Pressure** (defined by Variational Principle):

$$P_X(f) := \sup_{\mu} h(\mu) + \int f d\mu$$

where the sup is taken over all shift-invariant Borel probability measures $\mu$ such that support($\mu$) $\subseteq X$.

Fact: The sup is always achieved.

A measure which achieves the sup is called an **equilibrium state**.

Note: $P_X(0) = h(X)$. 
Let $X$ be a shift space and $f : X \to \mathbb{R}$ a continuous function.

**Topological Pressure** (defined by Variational Principle):

$$P_X(f) := \sup_{\mu} h(\mu) + \int f d\mu$$

where the sup is taken over all shift-invariant Borel probability measures $\mu$ such that $\text{support}(\mu) \subseteq X$.

- Fact: The sup is always achieved.
- A measure which achieves the sup is called an **equilibrium state**.
- Note: $P_X(0) = h(X)$. 
A nearest-neighbor interaction is a shift-invariant function $\Phi$ from a set of configurations on vertices and edges in $\mathbb{Z}^d$ to $\mathbb{R} \cup \infty$.

For a nearest-neighbor interaction $\Phi$, the underlying SFT:

$$X = X_\Phi := \{ x \in A^{\mathbb{Z}^d} : \Phi(x(\{v, v\'})) \neq \infty, \text{ for all } v \sim v' \}.$$

A nearest neighbour (n.n.) Gibbs measure $\mu$ corresponding to $\Phi$ is an MRF on $X$ such that for $S \subset \mathbb{Z}^d$, $\delta \in A^{\partial S}$, $\mu(\delta) > 0$, $w \in A^S$:

$$\mu(w|\delta) = \frac{e^{-U_{\Phi}(w\delta)}}{Z_{\Phi,\delta}(S)}.$$

where

- $U_{\Phi}(w\delta)$ is the sum of all $\Phi$-values of $w\delta$ for vertices, edges in $S \cup \partial S$
- $Z_{\Phi,\delta}(S)$ is the normalization factor.
A nearest-neighbor interaction is a shift-invariant function \( \Phi \) from a set of configurations on vertices and edges in \( \mathbb{Z}^d \) to \( \mathbb{R} \cup \infty \).

For a nearest-neighbor interaction \( \Phi \), the underlying SFT:

\[
X = X_{\Phi} := \{ x \in A^{\mathbb{Z}^d} : \Phi(x(\{v, v'\})) \neq \infty, \text{ for all } v \sim v' \}.
\]

A nearest neighbour (n.n.) Gibbs measure \( \mu \) corresponding to \( \Phi \) is an MRF on \( X \) such that for \( S \subset \mathbb{Z}^d \), \( \delta \in A^{\partial S} \), \( \mu(\delta) > 0 \), \( w \in A^{S} \):

\[
\mu(w|\delta) = \frac{e^{-U_{\Phi}(w\delta)}}{Z_{\Phi,\delta}(S)}.
\]

where

- \( U_{\Phi}(w\delta) \) is the sum of all \( \Phi \)-values of \( w\delta \) for vertices, edges in \( S \cup \partial S \)
- \( Z_{\Phi,\delta}(S) \) is the normalization factor.
A nearest-neighbor interaction is a shift-invariant function \( \Phi \) from a set of configurations on vertices and edges in \( \mathbb{Z}^d \) to \( \mathbb{R} \cup \infty \).

For a nearest-neighbor interaction \( \Phi \), the underlying SFT:

\[
X = X_\Phi := \{ x \in A^{\mathbb{Z}^d} : \Phi(x(\{v, v'\})) \neq \infty, \text{ for all } v \sim v' \}.
\]

A nearest neighbour (n.n.) Gibbs measure \( \mu \) corresponding to \( \Phi \) is an MRF on \( X \) such that for \( S \in \mathbb{Z}^d \), \( \delta \in A^{\partial S} \), \( \mu(\delta) > 0 \), \( w \in A^S \):

\[
\mu(w|\delta) = \frac{e^{-U_\Phi(w\delta)}}{Z_{\Phi,\delta}(S)}.
\]

where
- \( U_\Phi(w\delta) \) is the sum of all \( \Phi \)-values of \( w\delta \) for vertices, edges in \( S \cup \partial S \)
- \( Z_{\Phi,\delta}(S) \) is the normalization factor.
A nearest-neighbor interaction is a shift-invariant function \( \Phi \) from a set of configurations on vertices and edges in \( \mathbb{Z}^d \) to \( \mathbb{R} \cup \infty \).

For a nearest-neighbor interaction \( \Phi \), the underlying SFT:

\[
X = X_\Phi := \{ x \in \mathcal{A}^{\mathbb{Z}^d} : \Phi( x(\{v, v'\})) \neq \infty, \text{ for all } v \sim v' \}.
\]

A nearest neighbour (n.n.) Gibbs measure \( \mu \) corresponding to \( \Phi \) is an MRF on \( X \) such that for \( S \in \mathbb{Z}^d \), \( \delta \in \mathcal{A}^{\partial S} \), \( \mu(\delta) > 0 \), \( w \in \mathcal{A}^S \):

\[
\mu(w|\delta) = \frac{e^{-U_\Phi(w\delta)}}{Z_{\Phi,\delta}(S)}.
\]

where

- \( U_\Phi(w\delta) \) is the sum of all \( \Phi \)-values of \( w\delta \) for vertices, edges in \( S \cup \partial S \)
- \( Z_{\Phi,\delta}(S) \) is the normalization factor.
A nearest-neighbor interaction is a shift-invariant function $\Phi$ from a set of configurations on vertices and edges in $\mathbb{Z}^d$ to $\mathbb{R} \cup \infty$.

For a nearest-neighbor interaction $\Phi$, the underlying SFT:

$$X = X_\Phi := \{ x \in A^{\mathbb{Z}^d} : \Phi(x(\{v, v\}')) \neq \infty, \text{ for all } v \sim v' \}.$$

A nearest neighbour (n.n.) Gibbs measure $\mu$ corresponding to $\Phi$ is an MRF on $X$ such that for $S \subseteq \mathbb{Z}^d$, $\delta \in A^{\partial S}$, $\mu(\delta) > 0$, $w \in A^S$:

$$\mu(w|\delta) = \frac{e^{-U_\Phi(w\delta)}}{Z_{\Phi,\delta}(S)}.$$ 

where

- $U_\Phi(w\delta)$ is the sum of all $\Phi$-values of $w\delta$ for vertices, edges in $S \cup \partial S$
- $Z_{\Phi,\delta}(S)$ is the normalization factor.
A nearest-neighbor interaction is a shift-invariant function $\Phi$ from a set of configurations on vertices and edges in $\mathbb{Z}^d$ to $\mathbb{R} \cup \infty$.

For a nearest-neighbor interaction $\Phi$, the underlying SFT:

$$X = X_\Phi := \{ x \in \mathcal{A}^{\mathbb{Z}^d} : \Phi(x(\{v, v\}')) \neq \infty, \text{ for all } v \sim v' \}.$$ 

A nearest neighbour (n.n.) Gibbs measure $\mu$ corresponding to $\Phi$ is an MRF on $X$ such that for $S \subset \mathbb{Z}^d$, $\delta \in \mathcal{A}^{\partial S}$, $\mu(\delta) > 0$, $w \in \mathcal{A}^S$:

$$\mu(w|\delta) = \frac{e^{-U^\Phi(w\delta)}}{Z^\Phi,\delta(S)}.$$

where

- $U^\Phi(w\delta)$ is the sum of all $\Phi$-values of $w\delta$ for vertices, edges in $S \cup \partial S$
- $Z^\Phi,\delta(S)$ is the normalization factor.
Examples of n.n. Gibbs measures

- uniform MME on n.n. SFT
- hard square model with activities
- ferromagnetic Ising model with no external field.
Equilibrium states versus n.n. Gibbs measures

Pressure of n.n. interaction $\Phi$:

$$P(\Phi) := \lim_{n \to \infty} \frac{\log Z^\Phi(B_n)}{n^d}$$

where $Z^\Phi(B_n)$ is the “free boundary” normalization.

Let $A_\Phi(x) := -\Phi(x(0)) - \sum_{i=1}^d \Phi(x(0), x(e_i))$.

Fact: $P_{X_\Phi}(A_\Phi) = P(\Phi)$.

Lanford-Ruelle Theorem: Every equilibrium state for $A_\Phi$ is a Gibbs measure for $\Phi$.

Dobrushin Theorem: If $X_\Phi$ is strongly irreducible, then every Gibbs measure for $\Phi$ is an equilibrium state for $A_\Phi$.

These theorems hold in much greater generality.
Pressure of n.n. interaction $\Phi$:

$$P(\Phi) := \lim_{n \to \infty} \frac{\log Z^\Phi(B_n)}{n^d}$$

where $Z^\Phi(B_n)$ is the “free boundary” normalization.

- Let $A(\Phi) := -\Phi(x(0)) - \sum_{i=1}^{d} \Phi(x(0), x(e_i))$.
- Fact: $P_{X^\Phi}(A(\Phi)) = P(\Phi)$.
- Lanford-Ruelle Theorem: Every equilibrium state for $A(\Phi)$ is a Gibbs measure for $\Phi$.
- Dobrushin Theorem: If $X^\Phi$ is strongly irreducible, then every Gibbs measure for $\Phi$ is an equilibrium state for $A(\Phi)$.
- These theorems hold in much greater generality.
Pressure of n.n. interaction $\Phi$:

$$P(\Phi) := \lim_{n \to \infty} \frac{\log Z^\Phi(B_n)}{n^d}$$

where $Z^\Phi(B_n)$ is the “free boundary” normalization.

Let $A_\Phi(x) := -\Phi(x(0)) - \sum_{i=1}^d \Phi(x(0), x(e_i))$.

Fact: $P_{X_\Phi}(A_\Phi) = P(\Phi)$.

Lanford-Ruelle Theorem: Every equilibrium state for $A_\Phi$ is a Gibbs measure for $\Phi$.

Dobrushin Theorem: If $X_\Phi$ is strongly irreducible, then every Gibbs measure for $\Phi$ is an equilibrium state for $A_\Phi$.

These theorems hold in much greater generality.
Pressure of n.n. interaction $\Phi$:

$$P(\Phi) := \lim_{n \to \infty} \frac{\log Z^\Phi(B_n)}{n^d}$$

where $Z^\Phi(B_n)$ is the “free boundary” normalization.

Let $A_\Phi(x) := -\Phi(x(0)) - \sum_{i=1}^{d} \Phi(x(0), x(e_i))$.

Fact: $P_{X_\Phi}(A_\Phi) = P(\Phi)$.

Lanford-Ruelle Theorem: Every equilibrium state for $A_\Phi$ is a Gibbs measure for $\Phi$.

Dobrushin Theorem: If $X_\Phi$ is strongly irreducible, then every Gibbs measure for $\Phi$ is an equilibrium state for $A_\Phi$.

These theorems hold in much greater generality.
Pressure of n.n. interaction $\Phi$:

$$P(\Phi) := \lim_{n \to \infty} \frac{\log Z^\Phi(B_n)}{n^d}$$

where $Z^\Phi(B_n)$ is the “free boundary” normalization.

Let $A_\Phi(x) := -\Phi(x(0)) - \sum_{i=1}^d \Phi(x(0), x(e_i))$.

Fact: $P_{X_\Phi}(A_\Phi) = P(\Phi)$.

Lanford-Ruelle Theorem: Every equilibrium state for $A_\Phi$ is a Gibbs measure for $\Phi$.

Dobrushin Theorem: If $X_\Phi$ is strongly irreducible, then every Gibbs measure for $\Phi$ is an equilibrium state for $A_\Phi$.

These theorems hold in much greater generality.
Pressure of n.n. interaction $\Phi$:

$$P(\Phi) := \lim_{n \to \infty} \frac{\log Z^\Phi(B_n)}{n^d}$$

where $Z^\Phi(B_n)$ is the “free boundary” normalization.

Let $A_\Phi(x) := -\Phi(x(0)) - \sum_{i=1}^d \Phi(x(0), x(e_i))$.

Fact: $P_{X_\Phi}(A_\Phi) = P(\Phi)$.

Lanford-Ruelle Theorem: Every equilibrium state for $A_\Phi$ is a Gibbs measure for $\Phi$.

Dobrushin Theorem: If $X_\Phi$ is strongly irreducible, then every Gibbs measure for $\Phi$ is an equilibrium state for $A_\Phi$.

These theorems hold in much greater generality.
Pressure of n.n. interaction $\Phi$:

$$P(\Phi) := \lim_{n \to \infty} \frac{\log Z^\Phi(B_n)}{n^d}$$

where $Z^\Phi(B_n)$ is the “free boundary” normalization.

Let $A_\Phi(x) := -\Phi(x(0)) - \sum_{i=1}^d \Phi(x(0), x(e_i))$.

Fact: $P_{X_\Phi}(A_\Phi) = P(\Phi)$.

Lanford-Ruelle Theorem: Every equilibrium state for $A_\Phi$ is a Gibbs measure for $\Phi$.

Dobrushin Theorem: If $X_\Phi$ is strongly irreducible, then every Gibbs measure for $\Phi$ is an equilibrium state for $A_\Phi$.

These theorems hold in much greater generality.
Theorem (Adams, Briceno, Marcus, Pavlov): Let $\mu$ a Gibbs measure for a n.n. interaction $\Phi$ with underlying $\mathbb{Z}^d$ n.n. SFT $X$. If

1. $X$ satisfies TSSM
2. For some periodic orbit $O$ in $X$ and all $\omega \in O$

\[
L(\omega) := \lim_{a,b,c \to \infty} \mu(\omega(0) \mid \omega(\partial R_{a,b,c})) \text{ exists}
\]

Then

\[
P(\Phi) = \frac{1}{|O|} \sum_{\omega \in O} - \log L(\omega) + A_{\Phi}(\omega)
\]

Moreover, if $d = 2$ and convergence in hypothesis 2 is exponential, then there is a polynomial time algorithm to compute $P(\Phi)$. 

Theorem (Adams, Briceno, Marcus, Pavlov): Let $\mu$ a Gibbs measure for a n.n. interaction $\Phi$ with underlying $\mathbb{Z}^d$ n.n. SFT $X$. If

1. $X$ satisfies TSSM
2. For some periodic orbit $O$ in $X$ and all $\omega \in O$

$$L(\omega) := \lim_{a,b,c \to \infty} \mu(\omega(0) | \omega(\partial R_{a,b,c}))$$ exists

Then

$$P(\Phi) = \frac{1}{|O|} \sum_{\omega \in O} -\log L(\omega) + A_\Phi(\omega)$$

Moreover, if $d = 2$ and convergence in hypothesis 2 is exponential, then there is a polynomial time algorithm to compute $P(\Phi)$. 
Pressure representation and approximation

Theorem (Adams, Briceno, Marcus, Pavlov): Let $\mu$ a Gibbs measure for a n.n. interaction $\Phi$ with underlying $\mathbb{Z}^d$ n.n. SFT $X$. If

1. $X$ satisfies TSSM
2. For some periodic orbit $O$ in $X$ and all $\omega \in O$

\[ L(\omega) := \lim_{a,b,c \to \infty} \mu(\omega(0) \mid \omega(\partial R_{a,b,c})) \] exists

Then

\[ P(\Phi) = \frac{1}{|O|} \sum_{\omega \in O} - \log L(\omega) + A_\Phi(\omega) \]

Moreover, if $d = 2$ and convergence in hypothesis 2 is exponential, then there is a polynomial time algorithm to compute $P(\Phi)$. 
Theorem (Adams, Briceno, Marcus, Pavlov): Let $\mu$ a Gibbs measure for a n.n. interaction $\Phi$ with underlying $\mathbb{Z}^d$ n.n. SFT $X$. If

1. $X$ satisfies TSSM
2. For some periodic orbit $O$ in $X$ and all $\omega \in O$

$$L(\omega) := \lim_{a,b,c \to \infty} \mu(\omega(0) \mid \omega(\partial R_{a,b,c}))$$ exists

Then

$$P(\Phi) = \frac{1}{|O|} \sum_{\omega \in O} -\log L(\omega) + A_\Phi(\omega)$$

Moreover, if $d = 2$ and convergence in hypothesis 2 is exponential, then there is a polynomial time algorithm to compute $P(\Phi)$.
Theorem (Adams, Briceno, Marcus, Pavlov): Let $\mu$ a Gibbs measure for a n.n. interaction $\Phi$ with underlying $\mathbb{Z}^d$ n.n. SFT $X$. If

1. $X$ satisfies TSSM
2. For some periodic orbit $O$ in $X$ and all $\omega \in O$

$$L(\omega) := \lim_{a,b,c \to \infty} \mu(\omega(0) \mid \omega(\partial R_{a,b,c}))$$

exists

Then

$$P(\Phi) = \frac{1}{|O|} \sum_{\omega \in O} - \log L(\omega) + A_\Phi(\omega)$$

Moreover, if $d = 2$ and convergence in hypothesis 2 is exponential, then there is a polynomial time algorithm to compute $P(\Phi)$. 
Theorem (Adams, Briceno, Marcus, Pavlov): Let $\mu$ a Gibbs measure for a n.n. interaction $\Phi$ with underlying $\mathbb{Z}^d$ n.n. SFT $X$. If

1. $X$ satisfies TSSM
2. For some periodic orbit $O$ in $X$ and all $\omega \in O$

$$L(\omega) := \lim_{a,b,c \to \infty} \mu(\omega(0) \mid \omega(\partial R_{a,b,c}))$$

exists

Then

$$P(\Phi) = \frac{1}{|O|} \sum_{\omega \in O} -\log L(\omega) + A_\Phi(\omega)$$

Moreover, if $d = 2$ and convergence in hypothesis 2 is exponential, then there is a polynomial time algorithm to compute $P(\Phi)$. 
Theorem (Briceno): Let $\mu$ a Gibbs measure for a n.n. interaction $\Phi$ with underlying $\mathbb{Z}^d$ n.n. SFT $X$. If

- $X$ satisfies TSSM
- $\mu$ satisfies SSM.

Then for all shift-invariant measures $\nu$ such that $\text{support}(\nu) \subseteq X$,

$$P(\Phi) = \int (l_\mu(x) + A_\Phi(x))d\nu$$
Theorem (Briceno): Let $\mu$ a Gibbs measure for a n.n. interaction $\Phi$ with underlying $\mathbb{Z}^d$ n.n. SFT $X$. If

- $X$ satisfies TSSM
- $\mu$ satisfies SSM.

Then for all shift-invariant measures $\nu$ such that $\text{support}(\nu) \subseteq X$,

$$P(\Phi) = \int (I_\mu(x) + A_\Phi(x)) d\nu$$
Theorem (Briceno): Let $\mu$ a Gibbs measure for a n.n. interaction $\Phi$ with underlying $\mathbb{Z}^d$ n.n. SFT $X$. If
- $X$ satisfies TSSM
- $\mu$ satisfies SSM.

Then for all shift-invariant measures $\nu$ such that $\text{support}(\nu) \subseteq X$,

$$P(\Phi) = \int (I_\mu(x) + A_\Phi(x)) d\nu$$
Theorem (Briceno): Let $\mu$ a Gibbs measure for a n.n. interaction $\Phi$ with underlying $\mathbb{Z}^d$ n.n. SFT $X$. If
- $X$ satisfies TSSM
- $\mu$ satisfies SSM.
Then for all shift-invariant measures $\nu$ such that $\text{support}(\nu) \subseteq X$,

$$P(\Phi) = \int (I_\mu(x) + A_\Phi(x)) d\nu$$
Theorem (Briceno): Let $\mu$ a Gibbs measure for a n.n. interaction $\Phi$ with underlying $\mathbb{Z}^d$ n.n. SFT $X$. If

- $X$ satisfies TSSM
- $\mu$ satisfies SSM.

Then for all shift-invariant measures $\nu$ such that $\text{support}(\nu) \subseteq X$,

$$P(\Phi) = \int (I_\mu(x) + A_\Phi(x))d\nu$$
An SFT $X$ satisfies the **D-condition** if

- there exist sequences of finite subsets $(\Lambda_n), (M_n)$ of $\mathbb{Z}^d$ such that $\Lambda_n \nearrow \infty$, $\Lambda_n \subseteq M_n$, $\frac{|M_n|}{|\Lambda_n|} \to 1$, such that

- for any globally admissible $v \in A^{\Lambda_n}$ and finite $S \subseteq M_n^c$ and globally admissible $w \in A^S$, we have that $vw$ is globally admissible.

Safe symbol $\Rightarrow$ TSSM $\Rightarrow$ D-condition
An SFT $X$ satisfies the **D-condition** if

- there exist sequences of finite subsets $(\Lambda_n), (M_n)$ of $\mathbb{Z}^d$ such that $\Lambda_n \nearrow \infty$, $\Lambda_n \subseteq M_n$, $\frac{|M_n|}{|\Lambda_n|} \rightarrow 1$, such that

- for any globally admissible $v \in A^{\Lambda_n}$ and finite $S \subseteq M_n^c$ and globally admissible $w \in A^S$, we have that $vw$ is globally admissible.

Safe symbol $\Rightarrow$ TSSM $\Rightarrow$ D-condition
An SFT $X$ satisfies the **D-condition** if

- there exist sequences of finite subsets $(\Lambda_n), (M_n)$ of $\mathbb{Z}^d$ such that $\Lambda_n \nearrow \infty$, $\Lambda_n \subseteq M_n$, $\frac{|M_n|}{|\Lambda_n|} \to 1$, such that
- for any globally admissible $v \in A^{\Lambda_n}$ and finite $S \subseteq M_n^c$ and globally admissible $w \in A^S$, we have that $vw$ is globally admissible.

Safe symbol $\Rightarrow$ TSSM $\Rightarrow$ D-condition
An SFT $X$ satisfies the **D-condition** if

- there exist sequences of finite subsets $(\Lambda_n), (M_n)$ of $\mathbb{Z}^d$ such that $\Lambda_n \nearrow \infty$, $\Lambda_n \subseteq M_n$, $\frac{|M_n|}{|\Lambda_n|} \to 1$, such that

- for any globally admissible $v \in \mathcal{A}^{\Lambda_n}$ and finite $S \subseteq M_n^c$ and globally admissible $w \in \mathcal{A}^S$, we have that $vw$ is globally admissible.

Safe symbol $\Rightarrow$ TSSM $\Rightarrow$ D-condition
Theorem: Let $\mu$ a Gibbs measure for a n.n. interaction $\Phi$ with underlying $\mathbb{Z}^d$ n.n. SFT $X$. If

- $X$ satisfies the D-condition
- $l_\mu = A_\Psi$ for some \textit{absolutely summable} interaction $\Psi$ s.t. $X_\Psi = X$,

Then

$$P(\Phi) = \int l_\mu(x) + A_\Phi(x) \, d\nu(x)$$

for every shift-invariant measure $\nu$ with support($\nu$) $\subseteq X$. 
Theorem: Let $\mu$ a Gibbs measure for a n.n. interaction $\Phi$ with underlying $\mathbb{Z}^d$ n.n. SFT $X$. If

- $X$ satisfies the D-condition
- $l_\mu = A_\Psi$ for some absolutely summable interaction $\Psi$ s.t. $X_\Psi = X$,

Then

$$P(\Phi) = \int l_\mu(x) + A_\Phi(x) \, d\nu(x)$$

for every shift-invariant measure $\nu$ with support($\nu$) $\subseteq X$. 
Theorem: Let $\mu$ a Gibbs measure for a n.n. interaction $\Phi$ with underlying $\mathbb{Z}^d$ n.n. SFT $X$. If

- $X$ satisfies the D-condition
- $I_\mu = A_\Psi$ for some absolutely summable interaction $\Psi$ s.t. $X_\Psi = X$,

Then

$$P(\Phi) = \int l_\mu(x) + A_\Phi(x) \ d\nu(x)$$

for every shift-invariant measure $\nu$ with support($\nu$) $\subseteq X$. 
Theorem: Let $\mu$ a Gibbs measure for a n.n. interaction $\Phi$ with underlying $\mathbb{Z}^d$ n.n. SFT $X$. If

- $X$ satisfies the D-condition
- $l_\mu = A_\Psi$ for some *absolutely summable* interaction $\Psi$ s.t. $X_\Psi = X$,

Then

$$P(\Phi) = \int l_\mu(x) + A_\Phi(x) \, d\nu(x)$$

for every shift-invariant measure $\nu$ with support($\nu$) $\subseteq X$. 
Assuming adjacency matrix $A$ is irreducible and aperiodic, there is a unique MME $\mu_{\text{max}}$, which is a Markov chain given by transition matrix

$$P_{ij} = \begin{cases} \frac{r_j}{\lambda r_i} & ij \not\in \mathcal{F} \\ 0 & ij \in \mathcal{F} \end{cases}$$

where $\lambda = \lambda(A)$ and $r$ is a right eigenvector for $\lambda$, and stationary vector $r_i \ell_i$ where $\ell$ is a left eigenvector for $\lambda$ (suitably normalized).

Thus, if $\mu(w_1 w_2 \ldots w_{n-1} w_n) > 0$, then

$$\mu(w_1 w_2 \ldots w_{n-1} w_n) = \frac{\ell w_1 r_{w_n}}{\lambda^{n-1}}$$

Thus, fixing $w_1, w_n$,

$$\mu(w_2 \ldots w_{n-1} \mid w_1, w_n)$$

is uniform
Assuming adjacency matrix $A$ is irreducible and aperiodic, there is a unique MME $\mu_{\text{max}}$, which is a Markov chain given by transition matrix

$$P_{ij} = \begin{cases} \frac{r_j}{\lambda r_i} & ij \not\in \mathcal{F} \\ 0 & ij \in \mathcal{F} \end{cases}$$

where $\lambda = \lambda(A)$ and $r$ is a right eigenvector for $\lambda$, and stationary vector $r_i \ell_i$ where $\ell$ is a left eigenvector for $\lambda$ (suitably normalized).

Thus, if $\mu(w_1 w_2 \ldots w_{n-1} w_n) > 0$, then

$$\mu(w_1 w_2 \ldots w_{n-1} w_n) = \frac{\ell w_1 r_w n}{\lambda^{n-1}}$$

Thus, fixing $w_1, w_n$,

$$\mu(w_2 \ldots w_{n-1} | w_1, w_n)$$

is uniform.
Assuming adjacency matrix $A$ is irreducible and aperiodic, there is a unique MME $\mu_{\text{max}}$, which is a Markov chain given by transition matrix

$$P_{ij} = \begin{cases} \frac{r_j}{\lambda r_i} & ij \notin \mathcal{F} \\ 0 & ij \in \mathcal{F} \end{cases}$$

where $\lambda = \lambda(A)$ and $r$ is a right eigenvector for $\lambda$, and stationary vector $r_i \ell_i$ where $\ell$ is a left eigenvector for $\lambda$ (suitably normalized).

Thus, if $\mu(w_1 w_2 \ldots w_{n-1} w_n) > 0$, then

$$\mu(w_1 w_2 \ldots w_{n-1} w_n) = \frac{\ell w_1 r_w}{\lambda^{n-1}}$$

Thus, fixing $w_1, w_n$,

$$\mu(w_2 \ldots w_{n-1} | w_1, w_n)$$ is uniform
Assuming adjacency matrix $A$ is irreducible and aperiodic, there is a unique MME $\mu_{\text{max}}$, which is a Markov chain given by transition matrix

$$P_{ij} = \begin{cases} \frac{r_i}{\lambda r_i} & ij \notin \mathcal{F} \\ 0 & ij \in \mathcal{F} \end{cases}$$

where $\lambda = \lambda(A)$ and $r$ is a right eigenvector for $\lambda$, and stationary vector $r_i \ell_i$ where $\ell$ is a left eigenvector for $\lambda$ (suitably normalized)

Thus, if $\mu(w_1 w_2 \ldots w_{n-1} w_n) > 0$, then

$$\mu(w_1 w_2 \ldots w_{n-1} w_n) = \frac{\ell w_1 r_{w_n}}{\lambda^{n-1}}$$

Thus, fixing $w_1, w_n$, $\mu(w_2 \ldots w_{n-1} | w_1, w_n)$ is uniform
MME, \( d = 1 \)

- Assuming adjacency matrix \( A \) is irreducible and aperiodic, there is a unique MME \( \mu_{\text{max}} \), which is a Markov chain given by transition matrix

\[
P_{ij} = \begin{cases} \frac{r_i}{\lambda r_i} & \text{if } ij \not\in \mathcal{F} \\ 0 & \text{if } ij \in \mathcal{F} \end{cases}
\]

where \( \lambda = \lambda(A) \) and \( r \) is a right eigenvector for \( \lambda \), and stationary vector \( r_i \ell_i \) where \( \ell \) is a left eigenvector for \( \lambda \) (suitably normalized).

- Thus, if \( \mu(w_1 w_2 \ldots w_{n-1} w_n) > 0 \), then

\[
\mu(w_1 w_2 \ldots w_{n-1} w_n) = \frac{\ell w_1 r_{w_n}}{\lambda^{n-1}}
\]

- Thus, fixing \( w_1, w_n \),

\[
\mu(w_2 \ldots w_{n-1} \mid w_1, w_n) \text{ is uniform}
\]
Entropy representation for MME, $d = 1$

\[ I_\mu(x) = -\log \mu(x(0)|x(P)) \]
\[ = -\log P_{x_0x_{-1}} \]
\[ = \log \lambda + \log r_{x_{-1}} - \log r_{x_0} \]

So, for all invariant measures $\nu$,

\[
\int I_\mu(x) d\nu(x) = \int (\log \lambda + \log r_{x_{-1}} - \log r_{x_0}) d\nu(x) \\
= \log \lambda \\
= h(X)
\]

In particular, if the SFT has a fixed point $x^* := a^\mathbb{Z}$ and $\nu$ is the delta measure on $x^*$, then on

\[ h(X) = \int I_\mu(x) d\nu(x) = I_\mu(x^*) = -\log \mu(x^*) \]

and so $h(X)$ can be computed from the value of the information function at only one point.

In this case, $I_\mu(x)$ is defined everywhere.
Entropy representation for MME, \( d = 1 \)

\[
I_\mu(x) = -\log \mu(x(0)|x(\mathcal{P}))
\]
\[
= -\log P_{x_0x_{-1}}
\]
\[
= \log \lambda + \log r_{x_{-1}} - \log r_{x_0}
\]

So, for all invariant measures \( \nu \),

\[
\int I_\mu(x) d\nu(x) = \int (\log \lambda + \log r_{x_{-1}} - \log r_{x_0}) d\nu(x)
\]
\[
= \log \lambda
\]
\[
= h(X)
\]

In particular, if the SFT has a fixed point \( x^* := a^\mathbb{Z} \) and \( \nu \) is the delta measure on \( x^* \), then on

\[
h(X) = \int I_\mu(x) d\nu(x) = I_\mu(x^*) = -\log \mu(x^*)
\]

and so \( h(X) \) can be computed from the value of the information function at only one point.

In this case, \( I_\mu(x) \) is defined everywhere.
Entropy representation for MME, $d = 1$

\[ I_\mu(x) = - \log \mu(x(0)| x(\mathcal{P})) \]
\[ = - \log P_{x_0 x_{-1}} \]
\[ = \log \lambda + \log r_{x_{-1}} - \log r_{x_0} \]

So, for all invariant measures $\nu$,

\[ \int I_\mu(x) d\nu(x) = \int (\log \lambda + \log r_{x_{-1}} - \log r_{x_0}) d\nu(x) \]
\[ = \log \lambda \]
\[ = h(X) \]

In particular, if the SFT has a fixed point $x^* := a^\mathbb{Z}$ and $\nu$ is the delta measure on $x^*$, then on

\[ h(X) = \int I_\mu(x) d\nu(x) = I_\mu(x^*) = - \log \mu(x^*) \]

and so $h(X)$ can be computed from the value of the information function at only one point.

In this case, $I_\mu(x)$ is defined everywhere.
Entropy representation for MME, \( d = 1 \)

\[
I_\mu(x) = - \log \mu(x(0) | x(P)) = - \log P_{x_0x_{-1}} = \log \lambda + \log r_{x_{-1}} - \log r_{x_0}
\]

So, for all invariant measures \( \nu \),

\[
\int I_\mu(x) d\nu(x) = \int (\log \lambda + \log r_{x_{-1}} - \log r_{x_0}) d\nu(x) = \log \lambda = h(X)
\]

In particular, if the SFT has a fixed point \( x^* := a^Z \) and \( \nu \) is the delta measure on \( x^* \), then on

\[
h(X) = \int I_\mu(x) d\nu(x) = I_\mu(x^*) = - \log \mu(x^*)
\]

and so \( h(X) \) can be computed from the value of the information function at only one point.

In this case, \( I_\mu(x) \) is defined everywhere.