New Codes and Outer Bounds for Caching Systems: A Computer Aided Investigation

Chao Tian

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August 2016

Based in part on joint work with Jun Chen.
1 Motivation, Preliminaries, and Existing Results

2 Part 1: A New Code Construction

3 Part 2: Symmetry, Demand Types and Outer Bounds

4 Conclusion
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Caching and Its Applications

A natural management strategy when communication is bursty or costly

- Locally storing contents that are anticipated to be useful later;
- Prefetch data into local or faster memory;
- Useful on different time-space scales:
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![Diagram of caching and web communication]
Caching for Content Delivery

- One central server and many users;
- Place contents in users’ local caches during off-peak time;
- Peak time transmission can be reduced.
A Mathematical Model

Proposed by Maddah-Ali and Niesen (IT-14)

- $N$ files, $K$ users, each user has a cache of size $M$;
- Some data is cached during off-peak time: the placement phase;
- A common message to everyone in peak time: the delivery phase.

What is the fundamental limit of memory $M$ vs. transmission rate $R$?
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What is the fundamental limit of memory $M$ vs. transmission rate $R$?
A Tradeoff between Memory and Transmission Rate

There is a tradeoff between $M$ and $R$:

- Cache all content: $(M, R) = (N, 0)$;
- Cache nothing: $(M, R) = (0, K)$;
- Uncoded strategy: cache some parts, and transmit the missing
  - Can we do better? Yes, with coding.

![Diagram showing the tradeoff between memory and transmission rate](image-url)
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The following tradeoff pairs (and the lower convex hull) are achievable

\[(M, R) = \left( \frac{tN}{K}, (K - t) \min\left(\frac{1}{1 + t}, \frac{N}{K}\right) \right), \quad t = 0, 1, \ldots, K. \quad (1)\]

The optimal transmission rate for a given memory \(M\) must satisfy
\[R \geq \max_{s \in \{1, 2, \ldots, \min(n, k)\}} (s - \frac{s}{\lceil N/s \rceil} M).\] As a result, the tradeoff achieved in (1) is within a factor of 12 of the optimum.

An additional result by Chen et al., Arxiv-14

Theorem

When \(N \leq K\), the tradeoff pair \(\left( \frac{1}{K}, \frac{N(K-1)}{K} \right)\) is achievable.
Theorem (A Rough Translation)

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An Example \((N, K) = (3, 3)\)

Main difficulty: in the placement phase, the requests are unknown

- Requests only revealed in the delivery phase.
The Maddah-Ali-Niesen Coding Scheme

Placement strategy:
- Partition each file into \( \binom{K}{t} \) parts of equal size: each part associated with a subset of the users \( \{1, 2, \ldots, K\} \) with \( t \) elements;
- Place each part in the users’s cache of that subset (\( t \) copies in total);

Transmission strategy:
- A group of \( t + 1 \) users: each needs a segment that all other users already have;
- An opportunity to use network coding: send XOR of these segments.

Example: \((N, K) = (3, 3), t = 2\), three files are \((A, B, C)\), \( \binom{3}{2} = 3 \).

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<thead>
<tr>
<th>User 1</th>
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<th>B₁</th>
<th>C₁</th>
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<td>A₁</td>
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Users want \((A, B, C)\): sending \(A₃ + B₂ + C₁\).
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Summary of Existing Results

- Maddah-Ali-Niesen scheme: uncoded placement, coded transmission;
- Cut-set outer bound: not tight in general;
- Approximation: with a constant factor the optimum;
- **Question 1: Inner bound: coded placement and coded transmission?**
  - Maddah-Ali and Niesen gave one for \((N, K) = (2, 2)\);
  - Extended by Chen et al. to \(N \leq K\): only a single tradeoff point;
  - Code constructions of this type very limited
- **Question 2: Outer bounds: tight (or tighter) bounds?**
  - There are a few works on this (three independent papers in ISIT-15);
  - Even for small \((N, K)\) values, no conclusive solutions except \((2, 2)\).

In this talk: results presented at ISIT 2016

- **Part 1:** A novel scheme with coded placement and transmission.
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Outline

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A New Code: An Example for \((N, K) = (2, 4)\)

- Two files \((A, B)\);
- Each partitioned into \(\binom{4}{2} = 6\) segments (symbols);
- **Linear combinations** are cached;
- Delivery phase: send 6 symbols.

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Requests are $(A, A, A, B)$, send

Step 1: $B_1, B_2, B_4$;
Step 2: $A_3 + 2A_5 + 3A_6$, $A_3 + 3A_5 + 4A_6$;
Step 3: $A_1 + A_2 + A_4$.

- **User 1:**
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- User 1: after step 3: has \((A_1, A_2, A_3, A_4, A_5, A_6)\)
- User 4:
A New Code: An Example for \((N, K) = (2, 4)\)

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Requests are \((A, A, A, B)\), send

\[
\begin{align*}
\text{Step 1:} & \quad B_1, B_2, B_4; \\
\text{Step 2:} & \quad A_3 + 2A_5 + 3A_6, A_3 + 3A_5 + 4A_6; \\
\text{Step 3:} & \quad A_1 + A_2 + A_4.
\end{align*}
\]

- User 1: after step 3: has \((A_1, A_2, A_3, A_4, A_5, A_6)\)
- User 4: after step 1: has \((B_1, B_2, B_4)\), and has \(A_3 + A_5 + A_6\)
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Requests are \((A, A, B, B)\), send

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Requests are \((A, A, B, B)\), send

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Some Simple Rules

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- Each file is partitioned into $\binom{K}{t}$ segments;
- A segment is cached at a subset of users, but as a component of **linear combinations**;
- When a user requests a file, other components in his cached linear combinations are **interferences**;
- Need to eliminate the interferences and recover the wanted segments;
- What are the rules for the transmission steps?
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- Step 1 is uncoded;
- Only transmit when this segment is not present at any users requesting this file.
Some Simple Rules

<table>
<thead>
<tr>
<th>User 1</th>
<th>$A_1 + B_1$</th>
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Requests are $(A, A, A, B)$, send

Step 1: $B_1, B_2, B_4$;
Step 2: $A_3 + 2A_5 + 3A_6, A_3 + 3A_5 + 4A_6$;
Step 3: $A_1 + A_2 + A_4$.

- Step 2 is coded;
- Linear combinations of segments of a single file: maintain linear independence, then each transmission can provide rank reduction.
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• The first two steps together need to guarantee: with enough linear combinations, all the symbols at a user can be resolved.
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- Step 1 is uncoded, Step 2 is coded: eliminate interferences.
- Step 3 transmission then completes the missing pieces among users requesting the same file.
The first two step transmissions guarantee elimination of interferences:

- For small \((N, K)\): reasonably straightforward, as in the example;
- When \((N, K)\) are large: a complication.
An Example for \((N, K) = (3, 6)\)

Example \((N, K) = (3, 6), \ t = 3\)

- Three files \((A, B, C)\), each partitioned into \(\binom{6}{3} = 20\) segments;
- Label a segment of a file by the corresponding subset: e.g., \(A_{1,2,4}\)
- Each user caches 18 linear combinations of the appropriate segments;
- Consider the requests \((A, A, A, B, B, C)\);
- After step 1, the following interferences are present at users \((4, 5, 6)\)

<table>
<thead>
<tr>
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To eliminate these interferences, we send linear combinations of them
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- **Strategy 1:** transmit linear combinations of interferences of each user
  - 1 transmission = 1 dimension reduction at one user.
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To eliminate these interferences, we send linear combinations of them

- **Strategy 1:** transmit linear combinations of interferences of each user
  - 1 transmission\(\Rightarrow\)1 dimension reduction at one user.
- **Strategy 2:** transmit the common subspace, e.g., linear combinations of \(A_{1,4,5}, A_{2,4,5}, A_{3,4,5}\)
  - 1 transmission\(\Rightarrow\)1 dimension reduction at two users.
The General Scheme

Placement strategy:
1. Partition each file into \( \binom{K}{t} \) segments;
2. A fixed number of linear combinations of these segments at each user.

Delivery strategy:
1. For the users requesting the same file, transmit uncoded segments that none of them have;
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Revisiting the Example

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The Main Theorem

Key difficulty:
- Choose the numbers of combinations nicely (placement, 1st and 2nd step transmissions): guarantee interference elimination;
  - Linear combination coefficients not critical: full rank.
- Correctness and performance are tied to these numbers.

**Theorem**

For $N \in \mathbb{N}$ files and $K \in \mathbb{N}$ users each with a cache of size $M$, where $\mathbb{N}$ is the set of natural numbers and $N \leq K$, the following $(M, R)$ pair is achievable

$$\left( \frac{t[(N - 1)t + K - N]}{K(K - 1)}, \frac{N(K - t)}{K} \right), \quad t = 0, 1, \ldots, K.$$
Performance Example \((N, K) = (2, 4)\)

- One new corner point on the inner bound for this case;
- Optimal tradeoff now known for \(M \in [0, 1/4] \cup [2/3, 2]\).
Performance Example \((N, K) = (4, 20)\)
We present a new code construction

- The caching strategy and transmission strategies (mysteriously) work;
- Its performance can be analyzed with nice closed form formulas;
- Some simple rules are provided as guiding principles;
- Where did this come from?

In fact some key insights came from the investigation of outer bounds.
Recap: What Just Happened?

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Outline

1 Motivation, Preliminaries, and Existing Results
2 Part 1: A New Code Construction
3 Part 2: Symmetry, Demand Types and Outer Bounds
4 Conclusion
Fundamental Limits: The Conventional Approach

An art more than a science:

1. Develop a good understanding of the engineering problem;

Often heard comments:
- Need a smarter student!
- He really needs to spend more time on it!

Heavy reliance on humans: human ingenuity and effort
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Derivation of the chains of inequalities as an optimization procedure:
- Many possible information inequalities: choose the right combination.

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Yeung’s Linear Program to Prove Information Inequalities

Is a certain information inequality true? “Yes or can’t-determine”

- Use all inequalities from the basic properties (Shannon-type);
- Linear inequalities: one joint entropy represented by one LP variable;
  - Example LP variables: \( v_8 = H(S_1X_1X_7) \), \( v_{28} = H(S_2\hat{S}_1X_3X_5) \);...
- Example LP constraints:
  \[
  H(S_1X_1X_6) + H(X_1X_3X_6) \geq H(X_1X_6) + H(S_1X_1S_3X_6)
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- Note: there are indeed non-Shannon-type inequalities.
- Shannon-type inequalities sufficient to prove most results in the literature for “practical” coding problems!

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- Investigation of entropic region;
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3. LP dual to generate human-readable proofs.

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- First time the entropy LP approach used on an engineering problem;
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- Applied on MLD coding with regeneration (Tian-Liu, Allerton-14);
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A more domain-specific LP approach:

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From Table to Chain (to a Research Paper)

<table>
<thead>
<tr>
<th>B</th>
<th>α</th>
<th>β</th>
<th>y1</th>
<th>y2</th>
<th>y3</th>
<th>y4</th>
<th>y5</th>
<th>y6</th>
<th>y7</th>
<th>y8</th>
</tr>
</thead>
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<tr>
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<td>7</td>
<td>-7</td>
<td>1</td>
<td>-1</td>
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<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

For notational simplicity, from here on we shall write (a), (7), (8), (9) and (11) on top of the equalities in the derivation to signal the reasons for the equalities, i.e., by the symmetry of the entropy vectors, or by equations (7), (8), (9) and (11), respectively. We next write the following chain of inequalities:

\[ 2H(W_1, S_{1,4}) = 2H(W_1, S_{1,3} S_{2,4}) \]
\[ \geq H(W_1, W_2 S_{1,4}) + H(W_1, S_{1,3} S_{2,4}) \]
\[ \geq H(W_1, W_2 S_{1,4}) + H(W_1, S_{1,3} S_{2,4}) + 2H(W_2 S_{3,5}) \]

We first write

\[ 4\alpha + 6\beta \geq 4H(W_1) + 6H(S_{2,4}) \]
\[ = 4H(W_1) + 3H(S_{3,4}) + 3H(S_{4,4}) \]
\[ \geq 2H(W_1) + 3H(W_2 S_{3,4}) + 3H(W_2 S_{3,4}) \]

where the first inequality is by (22) and (23), the equality is by the symmetry of the solution

\[ H(S_{2,4}) = H(S_{3,4}). \]

and the second inequality is because summation of individual entropy is greater than or equal to the joint entropy.

However, notice that

\[ H(W_1 S_{2,4} S_{3,4}) + H(W_3 S_{2,4} S_{3,4}) \]
\[ H(W_4 S_{1,4} S_{3,4} S_{2,4}) \]
\[ H(W_4 W_5 S_{1,4} S_{3,4} S_{2,4}) \]
\[ = H(W_1 S_{1,4} S_{3,4} S_{2,4}) + H(S_{2,4} W_3 S_{3,4}) + H(S_{3,4} W_4 S_{2,4}) \]
\[ \geq H(W_1, W_4 S_{1,4} S_{3,4} S_{2,4}) + 2H(W_3 S_{2,4}) \]
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This implies that

\[ 4\alpha + 6\beta \geq 2B + H(W_1) + H(W_2 S_{1,4}) \]
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It follows that

\[ 4\alpha + 6\beta \geq B + H(W_1) + H(W_1, S_{1,4}) + H(W_2 S_{3,4} S_{1,4}) \]
Quantities in the problem: \( n = N + K + N^K \)

- \( N \) files: \( \mathcal{W} = \{ W_1, W_2, \ldots, W_N \} \);
- Cached contents at \( K \) users: \( \mathcal{Z} = \{ Z_1, Z_2, \ldots, Z_K \} \);
- Transmission for demands \( (d_1, d_2, \ldots, d_K) \): \( \mathcal{X} = \{ X_{d_1,d_2,...,d_K} \} \).
Symmetry in the Caching Problem

User index symmetry $\tilde{\pi}$: permute the cached contents $Z_i$ at users

- Delivery: need to transmit the corresponding $X_{d_1,\ldots,d_K}$.

File index symmetry $\hat{\pi}$: permute the files before encoding

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The Existence of Optimal Symmetric Codes

What are symmetric codes?

- For all permutation-induced mappings, joint entropies the same.

Example: \((N, K) = (3, 4)\)

- User-index: \(\bar{\pi} = \begin{pmatrix} 1234 \\ 2314 \end{pmatrix}, \ H(W_2, Z_2, X_{1,2,3,2}) = H(W_2, Z_3, X_{3,1,2,2})\)

- File-index: \(\hat{\pi} = \begin{pmatrix} 123 \\ 231 \end{pmatrix}, \ H(W_3, Z_3, X_{1,2,3,2}) = H(W_1, Z_3, X_{2,3,1,3})\)

Proposition

For any caching code, there is a code with the same or smaller caching memory and transmission rate, which is both user-index-symmetric and file-index-symmetric.
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More about the Symmetry

A “simple” question: after the symmetry reduction, how many unique joint entropy values do we have?

- Estimate: \(2^{N+K+N^K}/N!K!\);
- More accurate: Polya’s theory for counting (generating function and cycle index).

Symmetry induced by permutation groups

- The base symmetric groups: \(S_N\) and \(S_K\);
- First induced permutation group: \(W \cup Z \cup X \rightarrow W \cup Z \cup X\);
  - Compositions of any induced permutations by \(\tilde{\pi} \in S_K\) and \(\hat{\pi} \in S_N\);
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Demand Types

Some demands are equivalent, but not all

- E.g., \( N = 3, K = 5 \): \((2, 2, 1, 1, 3)\) is equivalent to \((1, 3, 3, 2, 2)\), but not \((1, 1, 1, 2, 3)\);
- Symmetric optimal solutions exist, but only up to such symmetry;
- Demand type: represented as an \( N \)-dimensional non-negative integer vector, in decreasing order, that sums to \( K \).

<table>
<thead>
<tr>
<th>((N, K))</th>
<th>Demand types</th>
</tr>
</thead>
<tbody>
<tr>
<td>((2, 3))</td>
<td>((3, 0), (2, 1))</td>
</tr>
<tr>
<td>((2, 4))</td>
<td>((4, 0), (3, 1), (2, 2))</td>
</tr>
<tr>
<td>((3, 2))</td>
<td>((2, 0, 0), (1, 1, 0))</td>
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</tr>
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A Complete Characterization for $K = 2$

**Theorem**

For any integer $N \geq 3$, any memory-rate tradeoff pair for the $(N, K) = (N, 2)$ caching problem must satisfy

$$3M + NR \geq 2N, \quad M + NR \geq N.$$  \hspace{1cm} (2)

Conversely, for any integer $N \geq 3$, there exist codes for any nonnegative $(M, R)$ pair satisfying (2).

- The first slice of cases to have a complete solution;
- First investigate $N = 3, 4$ using the computational approach, then use the proofs to deduce a general pattern;
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How Did We Form This Hypothesis?

- \((N, K) = (2, 2)\) previously known: tradeoff has two corner points;
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- For \(N \geq 3\), has only one corner point (surprise!);
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![Diagram](image-url)
A Partial Characterization for $N = 2$

**Theorem**

When $K \geq 3$ and $N = 2$, any $(M, R)$ pair must satisfy

$$K(K + 1)M + 2(K - 1)KR \geq 2(K - 1)(K + 2).$$

(3)

As a consequence, the Maddah-Ali-Niesen scheme is optimal when $M \geq \frac{2(K-2)}{K}$, for the cases with $K > 3$ and $N = 2$. 
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Bounds tight for $M \in [0, 1/4] \cup [1, 2]$;
Investigate the bounds, identify a corner point not achievable;
ASSUMING it achievable: attempt to design code (success 😊).
Be Nice If We Know More?

Assuming each file has 6 symbols in some finite field:

| Joint entropy               | Value*6 | $H(\cdot |A)$ |
|-----------------------------|---------|-----------|
| $H(A, Z_1)$                 | 9       | 3         |
| $H(A, Z_1, Z_2)$            | 11      | 5         |
| $H(A, Z_1, Z_2, Z_3)$       | 12      | 6         |
| $H(A, X_{1,2,2,2})$         | 9       | 3         |

Target: find a linear code with the given joint entropy structure

- Each user cache 3, combination of any two gives 5, any three gives 6;
- Delivery and cached are linear independent.
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Results for $N = K = 3$

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Outline

1. Motivation, Preliminaries, and Existing Results
2. Part 1: A New Code Construction
3. Part 2: Symmetry, Demand Types and Outer Bounds
4. Conclusion
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A new code construction for the caching problem

- Coded placement and coded transmission;
- Based on interference elimination;
- Roughly a dual of the Maddhuh-Ali-Niesen scheme.

New outer bound results

- Computer aided approach can provide important clues;
- The notion of demand types;
- Complete characterizations for $K = 2$;
- Partial characterizations for $N = 2$;
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- Reduce the human factors by introducing more machine intelligence;
- A more domain specific LP approach;
- Application on several research problems proves its effectiveness.
- More than proofs for simple inequalities: new insights, for both fundamental limits and code constructions.

The main challenge:
- High complexity: how much power can we squeeze out?
- Incorporating more domain knowledge into the approach?
- Computerized proof checking?
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How about AlphaIT?

“The system could process much larger volumes of data and surface the structural insight to the human expert in a way that is much more efficient—or maybe not possible for the human expert...”–Demis Hassabis, Google Deepmind Leader.