Detection of Cooperative Interactions in Logistic Regression Models

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August 2016
Linear Regression Models

- $d$ input variables $X_1, X_2, \ldots, X_d$ and an output variable $Y$
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$$Y = \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_d X_d$$
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- Training samples: observations of $(Y, X_1, X_2, \ldots, X_d)$
  - Sample 1: $(y^{(1)}, x_1^{(1)}, x_2^{(1)}, \ldots, x_d^{(1)})$
  - \vdots
  - Sample $n$: $(y^{(n)}, x_1^{(n)}, x_2^{(n)}, \ldots, x_d^{(n)})$
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- Parameter estimation:
  \[
  (\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_d) = \arg \min_{\beta_1, \beta_2, \ldots, \beta_d} \frac{1}{n} \sum_{t=1}^{n} \left| y^{(t)} - (\beta_1 x_1^{(t)} + \cdots + \beta_d x_d^{(t)}) \right|^2
  \]
Linear Regression Models

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  Sample 1 : $(y^{(1)}, x_1^{(1)}, x_2^{(1)}, \ldots, x_d^{(1)})$
  
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- Parameter estimation:

  $$(\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_d) = \arg\min_{\beta_1, \beta_2, \ldots, \beta_d} \frac{1}{n} \sum_{t=1}^{n} \left| y^{(t)} - (\beta_1 x_1^{(t)} + \cdots + \beta_d x_d^{(t)}) \right|^2$$

- Test sample: $(x_1^{\text{test}}, x_2^{\text{test}}, \ldots, x_d^{\text{test}})$
- Prediction:

  $$y^{\text{test}} = \tilde{\beta}_1 x_1^{\text{test}} + \cdots + \tilde{\beta}_d x_d^{\text{test}}$$
Logistic Regression Models

- Linear regression models

\[ Y = \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_d X_d \triangleq \beta \cdot X \]
Logistic Regression Models

- Linear regression models

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- Logistic regression models

\[ \Pr(Y = +1|X) = \beta \cdot X \]
\[ \Pr(Y = -1|X) = 1 - \Pr(Y = +1|X) \]
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\[ \Pr(Y = +1|X) = \sigma(\beta \cdot X) \]
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\[ Y = \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_d X_d \triangleq \beta \cdot X \]

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\begin{align*}
\Pr(Y = +1|X) &= \beta \cdot X \\
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\end{align*}
\]
Individual Effects and Pairwise Interactions

Logistic regression model with individual effects and pairwise interactions

$$\Pr(Y = +1|X) = \sigma(\beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_d X_d)$$
Individual Effects and Pairwise Interactions

Logistic regression model with individual effects and pairwise interactions

\[ \Pr(Y = +1|X) = \sigma(\beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_d X_d + \beta_{1,2} X_1 X_2 + \beta_{1,3} X_1 X_3 + \cdots + \beta_{d-1,d} X_{d-1} X_d) \]
Logistic regression model with individual effects and pairwise interactions

\[
\Pr(Y = +1|X) = \sigma(\beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_d X_d \\
+ \beta_{1,2} X_1 X_2 + \beta_{1,3} X_1 X_3 + \cdots + \beta_{d-1,d} X_{d-1} X_d)
\]

- \(\beta_i \neq 0\): \(X_i\) has an individual effect.
- \(\beta_i = 0\): \(X_i\) has no individual effect.
- \(\beta_{i,j} \neq 0\): \(X_i\) and \(X_j\) has a pairwise interaction.
- \(\beta_{i,j} = 0\): \(X_i\) and \(X_j\) has no pairwise interaction.
System Model

- $X_1, X_2, \ldots, X_d$ are independent variables with 
  $\Pr\{X_i = +1\} = \Pr\{X_i = -1\} = 1/2$, for $i = 1, 2, \ldots, d$.
- $Y$ is a binary outcome variable

$$
\Pr\{Y = +1|X_1, X_2, \ldots, X_d\} = \sigma\left(\sum_{i=1}^{d} \beta_i X_i + \sum_{1 \leq i < j \leq d} \beta_{i,j} X_i X_j\right) \\
\Pr\{Y = -1|X_1, X_2, \ldots, X_d\} = 1 - \Pr\{Y = +1|X_1, X_2, \ldots, X_d\} \\
= \sigma\left(-\sum_{i=1}^{d} \beta_i X_i - \sum_{1 \leq i < j \leq d} \beta_{i,j} X_i X_j\right)
$$
System Model

- \( X_1, X_2, \ldots, X_d \) are independent variables with
  \( \text{Pr}\{X_i = +1\} = \text{Pr}\{X_i = -1\} = 1/2 \), for \( i = 1, 2, \ldots, d \).
- \( Y \) is a binary outcome variable

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\text{Pr}\{Y = +1|X_1, X_2, \ldots, X_d\} &= \sigma\left(\sum_{i=1}^{d} \beta_i X_i + \sum_{1 \leq i < j \leq d} \beta_{i,j} X_i X_j\right) \\
\text{Pr}\{Y = -1|X_1, X_2, \ldots, X_d\} &= 1 - \text{Pr}\{Y = +1|X_1, X_2, \ldots, X_d\} \\
&= \sigma\left(-\sum_{i=1}^{d} \beta_i X_i - \sum_{1 \leq i < j \leq d} \beta_{i,j} X_i X_j\right)
\end{align*}
\]

Target:
Detect all individual effects and pairwise interactions in logistic regression models from a limited number of samples.
Motivation 1: Detection of the Graph Underlying an Ising Model [Bresler (2015)]

- Ising models on a graph $G = (V, E)$ with $|V| = d$:

$$p(X_1, X_2, \ldots, X_d) = \exp \left\{ \sum_{i \in V} \beta_i X_i + \sum_{\{i,j\} \in E} \beta_{i,j} X_i X_j - \Phi(\beta) \right\}$$

- Parameter vector: $\beta = \{\beta_i\}_{i \in V} \cup \{\beta_{i,j}\}_{\{i,j\} \in E}$

- Normalizing constant: $\Phi(\beta)$

- The maximum degree of nodes is $p$ (constant)

- $|\beta_i| \leq h$ and $\lambda \leq |\beta_{i,j}| \leq \mu$. 
Motivation 1: Detection of the Graph Underlying an Ising Model [Bresler (2015)] (Continued)

Theorem (Bresler 2015)

Let $\delta = \frac{1}{2} e^{-2(\mu p + h)}$, $\tau^* = \frac{\lambda^2 \delta^{4p+1}}{16p\mu}$, $\epsilon^* = \frac{\tau^*}{2}$, $\ell^* = \frac{8}{(\tau^*)^2}$. Suppose we observe $n$ samples with

$$n \geq \frac{144(\ell^* + 3)}{(\epsilon^*)^2 \delta^2 \ell^*} \log \frac{d}{\zeta}.$$ 

Then with probability at least $1 - \zeta$, there exists an algorithm to detect the structure of $G$ running in polynomial time $O(\ell^* dn)$. 
Motivation 2: Chow-Liu Tree [Chow & Liu (1968)]

Chow-Liu representation:

\[
p(X_1, X_2, X_3, X_4, X_5) \\
= p(X_1) \cdot p(X_2|X_1) \cdot p(X_3|X_1, X_2) \cdot p(X_4|X_1, X_2, X_3) \cdot p(X_5|X_1, X_2, X_3, X_4) \\
\approx p(X_1) \cdot p(X_2|X_1) \cdot p(X_3|X_2) \cdot p(X_4|X_2) \cdot p(X_5|X_2) \\
\text{(first-order product approximation)} \\
= p'(X_1, X_2, X_3, X_4, X_5)
\]

**Target:** Find \(p'\) to minimize the Kullback-Leibler distance \(D(p||p')\) between \(p\) and \(p'\).
Motivation 2: Chow-Liu Tree [Chow & Liu (1968)] (Continued)

Dependency Relationship

\[ p(X_1, X_2, X_3, X_4, X_5) \approx p(X_1)p(X_2|X_1)p(X_3|X_2)p(X_4|X_2)p(X_5|X_2) \]
\[ p(X_1, X_2, X_3, X_4, X_5) \approx p(X_1)p(X_2|X_1)p(X_3|X_1)p(X_4|X_1)p(X_5|X_3) \]
Motivation 2: Chow-Liu Tree [Chow & Liu (1968)] (Continued)

Chow-Liu Algorithm:

- Construct a weighted complete graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_d\}$.
- The weight $w(v_i, v_j)$ of edge $(v_i, v_j)$ is assigned to be $I(X_i; X_j)$.
- Find a maximum spanning tree $T$ of $G$ (by Kruskal's algorithm or Prim's algorithm).
- Set an arbitrarily node $v$ to be the root of $T$, then rank the other nodes by their depths.
Our Work

- Model all individual effects and pairwise interaction by a so-called interaction graph.
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- Establish an algorithm with a similar style as Chow-Liu algorithm to detect the structure of the interaction graph from a limited number of samples.
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- No assumption of the maximum degree of nodes.
Our Work

- Model all individual effects and pairwise interaction by a so-called interaction graph.

- Establish an algorithm with a similar style as Chow-Liu algorithm to detect the structure of the interaction graph from a limited number of samples.

- No assumption of the maximum degree of nodes.

- Sample complexity and running time are both polynomial functions of the number of features.
Model with only Pairwise Interactions

- **Assumption:**
  No individual effects \((\beta_i = 0 \text{ for } 1 \leq i \leq d)\).

- **For example:**
  - 5 variables \(X_1, X_2, X_3, X_4, X_5\)
  - \(\beta_{1,2}, \beta_{2,3}, \beta_{2,4}, \beta_{2,5} \neq 0\) and other \(\beta_{i,j} = 0\)

\[
\Pr\{Y = +1|X_1, X_2, X_3, X_4, X_5\} = \sigma(\beta_{1,2}X_1X_2 + \beta_{2,3}X_2X_3 + \beta_{2,4}X_2X_4 + \beta_{2,5}X_2X_5) \\
\Pr\{Y = -1|X_1, X_2, X_3, X_4, X_5\} = 1 - \Pr\{Y = +1|X_1, X_2, X_3, X_4, X_5\}
\]
Interaction Graph

**Interaction graph:** Let $G = (V, E)$ be the interaction graph with
$V = \{v_1, v_2, \ldots, v_d\}$, and the edge $(v_i, v_j) \in E$ if and only if the
coefficient $\beta_{i,j}$ corresponding to $X_i$ and $X_j$ is nonzero.

For example:

$$\Pr\{Y = +1|X_1, X_2, X_3, X_4, X_5\} = \sigma(\beta_{1,2}X_1X_2 + \beta_{2,3}X_2X_3 + \beta_{2,4}X_2X_4 + \beta_{2,5}X_2X_5)$$

$$\Pr\{Y = -1|X_1, X_2, X_3, X_4, X_5\} = 1 - \Pr\{Y = +1|X_1, X_2, X_3, X_4, X_5\}$$

$\beta_{1,2}, \beta_{2,3}, \beta_{2,4}, \beta_{2,5} \neq 0$
Assumption, Difficulty & Target

- **Assumption:**
  The interaction graph $G = (V, E)$ is acyclic.
  - When the model contains at most two interactions, $G$ is always acyclic.
  - When the number of interactions is far less than the number of features, $G$ is acyclic with a high probability.
  - The model contains at most $d - 1$ interactions.

- **Difficulty:**
  We don’t know which edges this graph has.

- **Target:**
  Detect the structure of the interaction graph from a limited number of samples.
Construction of a Weighted Complete Graph

Construction:

Construct a weighted complete graph $G' = (V', E')$ by

- $V' = (v'_1, v'_2, \ldots, v'_d)$
- The weight of any edge $(v'_i, v'_j) \in E'$ is

\[
    w_{\{i,j\}} = |\Pr\{Y = +1|X_i = +1, X_j = +1\} - \Pr\{Y = -1|X_i = +1, X_j = +1\}|.
\]
Case 1: The third-order joint probability $p(X_i, X_j, Y)$ is known.

$w_{\{i,j\}}$ can be calculated from the third-order joint distribution of $X_i, X_j, Y$

$$w_{\{i,j\}} = \left| \Pr\{Y = +1 | X_i = +1, X_j = +1\} - \Pr\{Y = -1 | X_i = +1, X_j = +1\} \right|$$

$$= \left| 8 \Pr\{X_i = +1, X_j = +1, Y = +1\} - 1 \right|$$
Theorem on Detection (Case 1)

Let $T = (V', E_T)$ be a maximum spanning tree of $G'$. Then

$$(v_i, v_j) \in E \text{ if and only if } (v'_i, v'_j) \in E_T \text{ and } w_{\{i, j\}} > 0.$$
Theorem on Detection (Case 1)

**Theorem**

Let $T = (V', E_T)$ be a maximum spanning tree of $G'$. Then

$$(v_i, v_j) \in E \text{ if and only if } (v'_i, v'_j) \in E_T \text{ and } w_{\{i,j\}} > 0.$$
Algorithm (Detecting the interaction graph)

- Construct a weighted graph $G' = (V', E')$ with $V' = \{v'_1, v'_2, \ldots, v'_d\}$.
- The weight $w_{\{i,j\}}$ of edge $(v'_i, v'_j)$ is assigned to be
  $|\Pr\{Y = +1 | X_i = +1, X_j = +1\} - \Pr\{Y = -1 | X_i = +1, X_j = +1\}|$.
- Find a maximum spanning tree $T' = (V', E_T)$ of $G'$ (by Kruskal’s algorithm or Prim’s algorithm).
- Then the set of the edges in $G$ is
  $\{(v_i, v_j) : (v'_i, v'_j) \in E_T \text{ and } w_{\{i,j\}} > 0\}$. 
Detection Algorithm (Case 1)

**Algorithm (Detecting the interaction graph)**

- **Construct a weighted graph** $G' = (V', E')$ with $V' = \{v'_1, v'_2, \ldots, v'_d\}$.
- **The weight** $w_{\{i, j\}}$ of edge $(v'_i, v'_j)$ **is assigned to be** $|\Pr\{Y = +1 | X_i = +1, X_j = +1\} - \Pr\{Y = -1 | X_i = +1, X_j = +1\}|$.
- **Find a maximum spanning tree** $T' = (V', E_T)$ of $G'$ **(by Kruskal's algorithm or Prim's algorithm)**.
- **Then the set of the edges in** $G$ **is** $\{ (v_i, v_j) : (v'_i, v'_j) \in E_T \text{ and } w_{\{i,j\}} > 0 \}$.

The algorithm is executed in polynomial time $O(d^2)$. 
Structure Detection of the Interaction Graph (Case 2)

- **Case 2:**
  - The third-order joint probability $p(X_i, X_j, Y)$ is unknown.
  - Any non-zero parameter $\beta_{i,j}$ satisfies that
    \[ \lambda \leq |\beta_{i,j}| \leq \mu. \]

- **Weight Assignment:** With $n$ samples $(Y^{(t)}, X_1^{(t)}, X_2^{(t)}, \ldots, X_d^{(t)})$ for $1 \leq t \leq n$, we estimate
  \[ w_{\{i,j\}} = |8 \Pr\{X_i = +1, X_j = +1, Y = +1\} - 1| \]
  
  by
  \[ \hat{w}_{\{i,j\}} = \left| \frac{8}{n} \sum_{t=1}^{n} \mathbf{1}(X_i^{(t)}, X_j^{(t)}, Y^{(t)}) = (+1, +1, +1) - 1 \right|. \]
Theorem on Detection (Case 2)

Let

\[ \gamma = \sqrt{\frac{2}{\pi d}} \left[ \sigma(\lambda + 3\mu) - \sigma(-\lambda + 3\mu) \right]. \]

Theorem

Assume for \(1 \leq i < j \leq d\),

\[ |\hat{w}_{\{i,j\}} - w_{\{i,j\}}| < \gamma/2. \]

Let \( T = (V', E_T) \) be a maximum spanning tree of \( G' \). Then

\( (v_i, v_j) \in E \) if and only if \((v'_i, v'_j) \in E_T \) and \( \hat{w}_{\{i,j\}} > \gamma/2 \).
Theorem on Detection (Case 2)

Let

\[ \gamma = \sqrt{\frac{2}{\pi d}} \left[ \sigma(\lambda + 3\mu) - \sigma(-\lambda + 3\mu) \right]. \]

Theorem

Assume for \( 1 \leq i < j \leq d \),

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Let \( T = (V', E_T) \) be a maximum spanning tree of \( G' \). Then

\( (v_i, v_j) \in E \) if and only if \( (v'_i, v'_j) \in E_T \) and \( \hat{w}_{\{i,j\}} > \gamma/2 \).

edges in the interaction graph

\[ \Leftrightarrow \]

large weighted edges in the maximum spanning tree
Detection Algorithm (Case 2)

Algorithm (Detecting the interaction graph)

- Construct a weighted graph $G'(V', E')$ with $V' = \{v'_1, v'_2, \ldots, v'_d\}$.
- The weight $w_{\{i,j\}}$ of edge $(v'_i, v'_j)$ is assigned to be
  \[
  \left| \frac{8}{n} \sum_{t=1}^{n} 1(X_i[t], X_j[t], Y[t]) = (+1,+1,+1) - 1 \right|.
  \]
- Find a maximum spanning tree $T'(V', E_T)$ of $G'$ (by Kruskal's algorithm or Prim's algorithm).
- Then the set of the edges in $G$ is
  \[
  \{(v_i, v_j) : (v'_i, v'_j) \in E_T \text{ and } w_{\{i,j\}} > \gamma/2\}.
  \]

The algorithm is executed in polynomial time $O(nd^2)$. 
Theorem

Fix $0 < \epsilon < 1$ and let $n$ be a positive integer such that

$$n \geq \frac{128}{\gamma^2} \log \frac{d^2}{\epsilon} = \frac{64\pi d}{\left[\sigma(\lambda + 3\mu) - \sigma(-\lambda + 3\mu)\right]^2} \log \frac{d^2}{\epsilon}.$$  (1)

Then with probability at least $1 - \epsilon$, the algorithm can successfully detect the graph $G$ from $n$ i.i.d. samples of $(X_1, X_2, \ldots, X_d, Y)$.

The order of sample complexity: $\Theta(d \log \frac{d}{\epsilon})$

Running time: $O(d^3 \log \frac{d}{\epsilon})$
Models with both Individual Effects and Pairwise Interactions

- For example:
  - 4 variables $X_1, X_2, X_3, X_4$
  - $\beta_2, \beta_{1,2}, \beta_{2,3}, \beta_{2,4} \neq 0$ and other $\beta_i, \beta_{i,j} = 0$

$$
\Pr\{Y = +1|X_1, X_2, X_3, X_4\} = \sigma(\beta_2 X_2 + \beta_{1,2} X_1 X_2 \\
+ \beta_{2,3} X_2 X_3 + \beta_{2,4} X_2 X_4)
$$

$$
\Pr\{Y = -1|X_1, X_2, X_3, X_4\} = 1 - \Pr\{Y = +1|X_1, X_2, X_3, X_4\}
$$
Extended Interaction Graph

For extended interaction graph $G = (V, E)$,

- $V = \{v_0 \text{(virtual vertex)}, v_1, v_2, \ldots, v_d\}$
- $(v_0, v_i) \in E$ if and only if $X_i$ has an individual effect
- $(v_i, v_j) \in E$ if and only if $X_i$ and $X_j$ have a cooperative interaction

With the help of the virtual vertex $v_0$, $G$ can capture all individual effects and pairwise interactions.

For example:

$$
\Pr\{Y = +1|X_1, X_2, X_3, X_4\} = \sigma(\beta_2 X_2 + \beta_{1,2} X_1 X_2 + \beta_{2,3} X_2 X_3 + \beta_{2,4} X_2 X_4)
$$

$$
\Pr\{Y = -1|X_1, X_2, X_3, X_4\} = 1 - \Pr\{Y = +1|X_1, X_2, X_3, X_4\}
$$
Auxiliary Model

- **Assumption:**
  The extended interaction graph $G = (V, E)$ is acyclic.

- **Auxiliary model:**
  $\Pr\{\tilde{X}_i = +1\} = \Pr\{\tilde{X}_i = -1\} = 1/2$ for $0 \leq i \leq d$.
  $(\tilde{X}_0$: the virtual feature corresponding to the virtual node $v_0)$

$$
\begin{align*}
\Pr\{\tilde{Y} = +1|\tilde{X}_0, \tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_d\} &= \sigma\left(\sum_{i=1}^{d} \beta_i \tilde{X}_0 \tilde{X}_i + \sum_{1 \leq i < j \leq d} \beta_{i,j} \tilde{X}_i \tilde{X}_j\right) \\
\Pr\{\tilde{Y} = -1|\tilde{X}_0, \tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_d\} &= 1 - \Pr\{\tilde{Y} = +1|\tilde{X}_0, \tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_d\} \\
&= \sigma\left(-\sum_{i=1}^{d} \beta_i \tilde{X}_0 \tilde{X}_i - \sum_{1 \leq i < j \leq d} \beta_{i,j} \tilde{X}_i \tilde{X}_j\right)
\end{align*}
$$
Relationship between Original Model and its Auxiliary Model

- **Original model:**

\[
\begin{align*}
    w_{\{0,i\}} & := |\Pr(Y = +1|X_i = +1) - \Pr(Y = -1|X_i = +1)| \\
    w_{\{i,j\}} & := |\Pr(Y = +1|X_i = +1, X_j = +1) \\
    & \quad + \Pr(Y = +1|X_i = -1, X_j = -1) - 1|
\end{align*}
\]

- **Auxiliary model:**

\[
\begin{align*}
    \tilde{w}_{\{i,j\}} & := |\Pr(\tilde{Y} = +1|\tilde{X}_i = +1, \tilde{X}_j = +1) - \Pr(\tilde{Y} = -1|\tilde{X}_i = +1, \tilde{X}_j = +1)|
\end{align*}
\]

**Theorem**

For \(0 \leq i < j \leq d\),

\[w_{i,j} = \tilde{w}_{i,j}\]
Idea of Converting

- Original model and auxiliary model share the same interaction graph.
- Auxiliary model contains only pairwise interactions.
- Assign the empirical weight of the original model into each edge of the auxiliary model.
Detection Algorithm of Extended Interaction Graphs

Algorithm

- **Construct a weighted complete graph** $G' = (V', E')$ with $V' = \{v'_0, v'_1, v'_2, \ldots, v'_d\}$.
- For $1 \leq i \leq d$, the weight $w_{\{0,i\}}$ of edge $(v'_0, v'_i)$ is assigned to be

$$\frac{4}{n} \sum_{t=1}^{n} \mathbf{1}((x_i[t], y[t]) = (+1, +1)) - 1$$

for $1 \leq i < j \leq d$, the weight $w_{\{i,j\}}$ of edge $(v'_i, v'_j)$ is assigned to be

$$\frac{4}{n} \sum_{t=1}^{n} \mathbf{1}((x_i[t], x_j[t], y[t]) = (+1, +1, +1)) + \frac{4}{n} \sum_{t=1}^{n} \mathbf{1}((x_i[t], x_j[t], y[t]) = (-1, -1, +1)) - 1$$
Detection Algorithm of Extended Interaction Graphs (Continued)

Algorithm

- **Find a maximum spanning tree** $T' = (V', E_T)$ of $G'$ (by Kruskal’s algorithm or Prim’s algorithm).

- Then the set of the edges in $G$ is
  \[
  \{(v_i, v_j) : (v'_i, v'_j) \in E_T \text{ and } w_{\{i,j\}} > \gamma'/2\},
  \]
  with
  \[
  \gamma' = \sqrt{\frac{2}{\pi(d+1)}} \left[ \sigma(\lambda + 3\mu) - \sigma(-\lambda + 3\mu) \right].
  \]

The algorithm is also executed in polynomial time.
Non-Uniform Case

- **Assumption:**
  - \( X_1, X_2, \ldots, X_d \) are independent variables with
    \( \Pr\{X_i = +1\} = p_i, \Pr\{X_i = -1\} = q_i \) with \( p_i + q_i = 1 \), for \( i = 1, 2, \ldots, d \) (non-uniform features)
  - The interaction graph \( G = (V, E) \) is simply a path of length at most 4.

- **Target:**
  - Reconstruct the graph from the samples of \((Y, X_1, X_2, \ldots, X_d)\).

- **Construction:**
  - Construct a weighted complete graph \( G' = (V', E') \) by
    - \( V' = (v_1', v_2', \ldots, v_d') \)
    - The weight of any edge \((v_i', v_j') \in E'\) is assigned to be
      \[
      w_{\{i,j\}} = \left| Q_{i+1, j+1, +1}^i + Q_{i-1, j-1, +1}^i + Q_{i+1, j-1, -1}^i + Q_{i-1, j+1, -1}^i
      \right.
      \]
      \[
      - Q_{i+1, j-1, -1} - Q_{i-1, j+1, -1} - Q_{i+1, j+1, +1} - Q_{i-1, j-1, +1}
      \]
      \[
      \right).
      
      \[
      (Q_{i_1, i_2, i_3}^i := \Pr\{Y = i_3|X_i = i_1, X_j = i_2\})
      \]
Theorem on Detection (Non-uniform Case)

Let $T = (V', E_T)$ be a maximum spanning tree of $G'$. Then

$$(v_i, v_j) \in E \text{ if and only if } (v'_i, v'_j) \in E_T \text{ and } w_{\{i,j\}} > 0.$$
Hardness of Detection (Non-uniform Case)

**Theorem**

Assume that the interaction graph is simply a path of length 5. If the weight of edge \((v_i', v_j')\) in \(G'\) is assigned to be

\[
w_{\{i,j\}} = \left| \sum_{i_1, i_2, i_3 \in \{+1, -1\}} \alpha_{i_1, i_2, i_3} Q_{i_1, i_2, i_3}^{i,j} \right|
\]

for any constants \(\{\alpha_{i_1, i_2, i_3} : i_1, i_2, i_3 \in \{+1, -1\}\}\), then there exists a counterexample where we cannot correctly detect the structure of the interaction graph by finding a maximum spanning tree of \(G'\).

The theorem for the uniform cases cannot be extended into the generic non-uniform cases.
Simulation Experiments

- 1000 logistic regression models
- 15 features, 5 individual effects, 10 pairwise interactions
- 400, 800, 1,200, 1,600, 2,000 samples
- Detection of the interaction graphs
Comparison of false positive rates for detection between $L_1$-penalized logistic regression [Park & Hastie (2007)] and our Algorithm.
Summary

- Logistic regression models:

  \[
  \Pr\{Y = +1|X_1, X_2, \ldots, X_d\} = \sigma\left( \sum_{1 \leq i \leq d} \beta_i X_i + \sum_{1 \leq i < j \leq d} \beta_{i,j} X_i X_j \right)
  \]

  \[
  \Pr\{Y = -1|X_1, X_2, \ldots, X_d\} = 1 - \Pr\{Y = +1|X_1, X_2, \ldots, X_d\}
  \]

- Interaction graph \( G = (V, E) \):

  \[
  (v_i, v_j) \in E \iff \beta_{i,j} \neq 0.
  \]

- Detection of the interaction graph:
  - Construct a weighted complete graph.
  - Find its maximum spanning tree.
  - Pick the edges with large weights.

- Extended to the models with both individual effects and pairwise interactions
Key References

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Key References (Continued)

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Thank you!