Estimating the Capacity of the 2-D Hard Square Constraint Using Generalized Belief Propagation

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1-D RLL Constraints

Let $0 \leq d < k \leq \infty$ be fixed integers ($k$ is allowed to be $\infty$).

Definition

A binary sequence $x_1 x_2 \ldots x_n \in \{0, 1\}^n$ satisfies the (one-dimensional) $(d, k)$-runlength-limited (RLL) constraint if any pair of successive 1s in the sequence is separated by at least $d$ and at most $k$ 0s.
1-D RLL Constraints

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Codes consisting of sequences satisfying a $(d, k)$-RLL constraint are used for writing information on magnetic and optical recording devices such as hard drives and CDs/DVDs.

The maximum rate of such a code is given by the capacity of the $(d, k)$-RLL constraint, defined as

$$C_{d,k} = \lim_{n \to \infty} \frac{1}{n} \log_2 Z_{n}^{(d,k)}$$

where $Z_{n}^{(d,k)}$ denotes the number of binary length-$n$ sequences satisfying the constraint.
The 1-D \((1, \infty)\)-RLL Constraint

A binary sequence satisfies the \((1, \infty)\)-RLL constraint if it does not contain 1s in adjacent (i.e., consecutive) positions.

Some easy facts about \(Z_n := Z_n^{(1, \infty)}\):

- \(Z_n, \ n = 1, 2, 3, \ldots\), forms a Fibonacci sequence

\[
Z_1 = 2, \quad Z_2 = 3, \quad \text{and} \quad Z_n = Z_{n-1} + Z_{n-2} \text{ for all } n \geq 3
\]

- \(C_{1,\infty} := \lim_{n \to \infty} \frac{1}{n} \log_2 Z_n = \log_2 \frac{1 + \sqrt{5}}{2} = 0.6942 \ldots\)
The 2-D Hard Square Constraint

Definition

A binary $m \times n$ array satisfies the 2-D hard square constraint (also called the 2-D $(1, \infty)$-RLL constraint) if no row or column of the array contains 1s in adjacent positions.

Each such array can also be viewed as an independent set in the $m \times n$ grid graph.
The Hard-Square Entropy Constant

Let $Z_{m,n}$ denote the number of such $m \times n$ arrays.

It can be shown (for example, using subadditivity arguments) that the limit

$$\eta = \lim_{m,n \to \infty} \frac{1}{mn} \log_2(Z_{m,n})$$

exists. This limit is called the **hard-square entropy constant**.

Open Problem: Determine the hard-square entropy constant $\eta$. 
The Hard-Square Entropy Constant

Let \( Z_{m,n} \) denote the number of such \( m \times n \) arrays.

It can be shown (for example, using subadditivity arguments) that the limit

\[
\eta = \lim_{m,n \to \infty} \frac{1}{mn} \log_2(Z_{m,n})
\]

exists. This limit is called the hard-square entropy constant.

**Open Problem:** Determine the hard-square entropy constant \( \eta \).

What is known:

- Various upper and lower bounds, resulting in the numerical estimate

\[
\eta = 0.58789116\ldots
\]

[Justesen & Forchhammer, 1999]

- \( \eta \) is computable to within an accuracy of \( \frac{1}{N} \) in time polynomial in \( N \)
[ Pavlov, 2010]
Binary Ising Model on the 2-D Lattice

▶ $m \times n$ grid with $mn$ vertices and $2mn - (m + n)$ edges
▶ Variable $x_i \in \{0, 1\}$ at each vertex $i$
▶ Pairwise function $f : \{0, 1\} \times \{0, 1\} \to \mathbb{R}_+$
▶ Defines a joint distribution on $\{0, 1\}^{m \times n}$:

$$p(x) = \frac{1}{Z} \prod_{i \sim j} f(x_i, x_j)$$
Binary Ising Model on the 2-D Lattice

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- Defines a joint distribution on \( \{0, 1\}^{m \times n} \):

\[
p(x) = \frac{1}{Z} \prod_{i \sim j} f(x_i, x_j)
\]

- Quantity of interest:

\[
Z = \sum_x \prod_{i \sim j} f(x_i, x_j)
\]

Called the partition function.
Special Case: Hard-Square Model

\[ f(a, b) = 1_{(a,b)\neq (1,1)} \]

The partition function here is precisely \( Z_{m,n} \).
Gibbs Free Energy

Define the energy of a configuration $\mathbf{x} \in \{0, 1\}^{m \times n}$ to be

$$E(\mathbf{x}) = - \sum_{i \sim j} \log f(x_i, x_j)$$

so that $p(\mathbf{x}) = \frac{1}{Z} \exp(-E(\mathbf{x}))$. 
Gibbs Free Energy

Define the energy of a configuration \( x \in \{0, 1\}^{m \times n} \) to be

\[
E(x) = - \sum_{i \sim j} \log f(x_i, x_j)
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so that \( p(x) = \frac{1}{Z} \exp(-E(x)) \).

For an arbitrary probability distribution \( b(x) \) on \( \{0, 1\}^{m \times n} \), define

- the average energy
  \[
  U(b) = \sum_x b(x)E(x)
  \]

- the entropy
  \[
  H(b) = - \sum_x b(x) \log b(x)
  \]

- the Gibbs free energy
  \[
  F(b) = U(b) - H(b)
  \]
A Variational Principle

\[ -\log Z = \min_b F(b) \]

**Proof:** Write

\[ F(b) = -\log Z + \sum_x b(x) \log \frac{b(x)}{p(x)} \]

\[ = -\log Z + D(b \parallel p) \]
The Bethe Free Energy

Let \((b_{i,j}(x_i, x_j))\) and \((b_i(x_i))\) be “beliefs” defined for all edges \(i \sim j\) and vertices \(i\), respectively. These must satisfy the following:

- the \(b_{i,j}\)'s and \(b_i\)'s are probability mass functions on \(\{0, 1\}^2\) and \(\{0, 1\}\), respectively
- \(\sum_{x_j} b_{i,j}(x_i, x_j) = b_i(x_i)\) (consistency)
The Bethe Free Energy

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- \(\sum_{x_j} b_{i,j}(x_i, x_j) = b_i(x_i)\) (consistency)

We then define

- the Bethe average energy
  \[
  U_B(\{b_{i,j}\}, \{b_i\}) = \sum_{i \sim j} \sum_{(a,b) \in \{0,1\}^2} b_{i,j}(a, b) \log f(a, b)
  \]

- the Bethe entropy
  \[
  H_B(\{b_{i,j}\}, \{b_i\}) = \sum_{i \sim j} H(b_{i,j}) - \sum_i (d_i - 1) H(b_i)
  \]
  where \(d_i\) denotes the degree of the vertex \(i\).

- the Bethe free energy
  \[
  F_B(\{b_{i,j}\}, \{b_i\}) = U_B(\{b_{i,j}\}, \{b_i\}) - H_B(\{b_{i,j}\}, \{b_i\})
  \]
The Bethe Approximation

\[- \log Z_B := \min_{\{b_{i,j}\}, \{b_i\}} F_B(\{b_{i,j}\}, \{b_i\})\]
The Bethe Approximation

\[- \log Z_B := \min_{\{b_{i,j}\},\{b_i\}} F_B(\{b_{i,j}\}, \{b_i\})\]

Theorem (Yedidia, Freeman and Weiss, 2001)

*Stationary points of the Bethe free energy functional correspond to the beliefs at fixed points of the belief propagation algorithm.*
Belief Propagation (The Sum-Product Algorithm)

Message update rules:

- $m_{i \rightarrow a}(x_i) = \prod_{c \in N(i) \setminus a} m_{c \rightarrow i}(x_i)$
- $m_{a \rightarrow i}(x_i) = \sum_{x_j} f(x_i, x_j) m_{j \rightarrow a}(x_j)$

Beliefs:

- $b_i(x_i) \propto \prod_{a \in N(i)} m_{a \rightarrow i}(x_i)$
- $b_a(x_i, x_j) \propto f(x_i, x_j) m_{i \rightarrow a}(x_i) m_{j \rightarrow a}(x_j)$

(the use of $\propto$ indicates that the beliefs must be normalized to sum to 1)
How Good is the Bethe Approximation?

For beliefs \( \{b_{i,j}\} \) and \( \{b_i\} \) at a fixed point of the BP algorithm, define

\[
Z_{BP}(\{b_{i,j}\}, \{b_i\}) := \exp(-F_B(\{b_{i,j}\}, \{b_i\}))
\]

Theorem (Wainwright, Jaakkola and Willsky, 2003)

At any fixed point of the BP algorithm,

\[
\frac{Z}{Z_{BP}(\{b_{i,j}\}, \{b_i\})} = \sum_x \frac{\prod_{i \sim j} b_{i,j}(x_i, x_j)}{\prod_i b_i(x_i)^{d_i-1}}
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\]

**Theorem (Chertkov and Chernyak, 2006)**

At any fixed point of the BP algorithm,

\[
\frac{Z}{Z_{BP}(\{b_{i,j}\}, \{b_i\})} = 1 + \text{a finite series of correction terms}
\]
How Good is the Bethe Approximation?

Theorem (Ruozzi, 2012)

*For the binary Ising model considered here,*

\[ Z \geq Z_B. \]
Numerical Results for the Hard-Square Model

Recall: $\eta = 0.58789116 \ldots$

[Plot above is from Sabato and Molkaraie, 2012]
$R = R_0 \cup R_1 \cup R_2$, where

- $R_0$: all the 2x2 subgrids
- $R_1$: all the non-boundary edges (intersections of regions in $R_0$)
- $R_2$: all the non-boundary vertices (intersections of regions in $R_1$)
Beliefs $b_R(x_R)$ are defined for all regions $R \in \mathcal{R}$. These must satisfy the following:

- each $b_R$ is a probability mass function on $\{0, 1\}^{|R|}$, where $|R|$ denotes the size (number of vertices) of the region $R$;

- for regions $P \subset R$, we have $\sum_{x_R \setminus P} b_R(x_R) = b_P(x_P)$ (consistency)
Region-Based Free Energy

Given a set of beliefs \( \{ b_R : R \in \mathcal{R} \} \), we define for each region \( R \in \mathcal{R} \):

- the average energy of \( R \)
  \[
  U_R = \sum_{x_{R} \in \{0,1\}^{\text{\#} R}} \sum_{i,j \in R : i \sim j} b_R(x_R) \log f(x_i, x_j)
  \]

- the entropy of \( b_R \)
  \[
  H_R = -\sum_{x_R} b_R(x_R) \log b_R(x_R)
  \]

- the free energy of \( R \)
  \[
  F_R = U_R - H_R
  \]
Region-Based Free Energy

Given a set of beliefs \( \{ b_R : R \in \mathcal{R} \} \), we define for each region \( R \in \mathcal{R} \):

- the average energy of \( R \)
  \[
  U_R = \sum_{x_R \in \{0,1\}^{\lfloor |R|/2 \rfloor}} \sum_{i,j \in R : i \sim j} b_R(x_R) \log f(x_i, x_j)
  \]

- the entropy of \( b_R \)
  \[
  H_R = -\sum_{x_R} b_R(x_R) \log b_R(x_R)
  \]

- the free energy of \( R \)
  \[
  F_R = U_R - H_R
  \]

The region-based free energy of the model \( (\mathcal{R}, \{b_R\}) \) is defined as

\[
F_\mathcal{R}(\{b_R\}) = \sum_{R \in \mathcal{R}_0} F_R - \sum_{R \in \mathcal{R}_1} F_R + \sum_{R \in \mathcal{R}_2} F_R
\]
The Kikuchi Approximation

\[ -\log Z_R = \min_{\{b_R\}} F_R(\{b_R\}) \]
The Kikuchi Approximation

\[- \log Z_R = \min_{\{b_R\}} F_R(\{b_R\})\]

**Theorem (Yedidia, Freeman and Weiss, 2001)**

*Stationary points of the region-based free energy functional correspond to the beliefs at fixed points of a generalized belief propagation (GBP) algorithm.*
The Kikuchi Approximation

\[ -\log Z_{\mathcal{R}} = \min_{\{b_R\}} F_{\mathcal{R}}(\{b_R\}) \]

Theorem (Yedidia, Freeman and Weiss, 2001)

Stationary points of the region-based free energy functional correspond to the beliefs at fixed points of a generalized belief propagation (GBP) algorithm.
GBP Message Updates (The Parent-to-Child Algorithm)

\[
m_{(4,5,8,9)\rightarrow(5,9)}(x_5, x_9) = \frac{\sum f(x_4, x_5) f(x_4, x_8) f(x_8, x_9) \prod (\text{red messages})}{\prod (\text{green messages})}
\]
GBP Message Updates (The Parent-to-Child Algorithm)

\[ m_{(4,5)\to(5)}(x_5) = \sum_{x_4} f(x_4, x_5) \prod (\text{red messages}) \]
GBP Beliefs (The Parent-to-Child Algorithm)

\[ b_{(4,5,8,9)}(x_4, x_5, x_8, x_9) \propto f(x_4, x_5)f(x_4, x_8)f(x_5, x_9)f(x_8, x_9) \prod \text{(red messages)} \]
GBP Beliefs (The Parent-to-Child Algorithm)

\[ b_{(4,5)}(x_4, x_5) \propto f(x_4, x_5) \prod \text{(red messages)} \]
GBP Beliefs (The Parent-to-Child Algorithm)

\[ b(5)(x_5) \propto \prod \text{(red messages)} \]
How Good is the Kikuchi Approximation?

Looking at the hard-square model again . . .

Recall: $\eta = 0.58789116 . . .$. 

[Plot above is from Sabato and Molkaraie, 2012]
Conjecture (Chan et al., ISIT’14)

For the binary Ising model considered here,

$$\frac{1}{mn} \log Z - \frac{1}{mn} \log Z_\mathcal{R} = o(1)$$

where $o(1)$ is a positive term that goes to 0 as $m, n \to \infty$.

In other words, we conjecture that

$$\frac{Z}{Z_\mathcal{R}} = \exp(mn o(1)).$$
Attacking the Conjecture: The Opening Gambit

For beliefs \( \{ b_R \} \) at a fixed point of the GBP algorithm, define

\[
Z_{\text{GBP}}(\{ b_R \}) := \exp(-F_R(\{ b_R \}))
\]

**Theorem (Chan et al., ISIT'14)**

At a fixed point of the GBP algorithm,

\[
\frac{Z}{Z_{\text{GBP}}(\{ b_R \})} = \sum_x \frac{\prod_{R \in \mathcal{R}_0 \cup \mathcal{R}_2} b_R(x_R)}{\prod_{R \in \mathcal{R}_1} b_R(x_R)}
\]
Attacking the Conjecture: The Opening Gambit

For beliefs \( \{b_R\} \) at a fixed point of the GBP algorithm, define

\[
Z_{GBP}(\{b_R\}) := \exp(-F_R(\{b_R\}))
\]

**Theorem (Chan et al., ISIT’14)**

At a fixed point of the GBP algorithm,

\[
\frac{Z}{Z_{GBP}(\{b_R\})} = \sum_x \frac{\prod_{R \in R_0 \cup R_2} b_R(x_R)}{\prod_{R \in R_1} b_R(x_R)}
\]

Compare this with

**Theorem (Wainwright, Jaakkola and Willsky, 2003)**

At any fixed point of the BP algorithm,

\[
\frac{Z}{Z_{BP}(\{b_{i,j}\}, \{b_i\})} = \sum_x \frac{\prod_{i \sim j} b_{i,j}(x_i, x_j)}{\prod_i b_i(x_i)^{d_i-1}}
\]
Question to be Addressed

Question

For the binary Ising model considered here, is it true that the beliefs \( \{b_R\} \) at a fixed point of GBP satisfy

\[
\frac{1}{mn} \log \sum_x \frac{\prod_{R \in \mathcal{R}_0 \cup \mathcal{R}_2} b_R(x_R)}{\prod_{R \in \mathcal{R}_1} b_R(x_R)} = o(1)
\]

where \( o(1) \) is a positive term that goes to 0 as \( m, n \to \infty \)?
Theorem (Chan et al., ISIT’14)

For a binary Ising model of size at most $5 \times 5$, or size equal to $3 \times n$ (or $n \times 3$) we have

$$\sum_{x} \frac{\prod_{R \in R_0 \cup R_2} b_R(x_R)}{\prod_{R \in R_1} b_R(x_R)} \geq 1$$

at any fixed point of GBP. Consequently,

$$Z \geq Z_{GBP}(\{b_R\})$$

at any fixed point of GBP.
A function \( g : \{0, 1\}^k \rightarrow \mathbb{R}_+ \) is called log-supermodular if

\[
g(x)g(y) \leq g(x \lor y)g(x \land y)
\]

for all \( x, y \in \{0, 1\}^k \).
A Key Tool: Log-Supermodularity

A function $g : \{0, 1\}^k \rightarrow \mathbb{R}_+$ is called log-supermodular if

$$g(x)g(y) \leq g(x \lor y)g(x \land y)$$

for all $x, y \in \{0, 1\}^k$.

Log-supermodularity is preserved under

- multiplication: $g_1, g_2$ log-supermodular $\implies g_1g_2$ log-supermodular
- marginalization: $g$ log-supermodular $\implies \sum_{x_1} g(x)$ log-supermod

(this follows from the Ahlswede-Daykin four functions theorem)
Log-Supermodularity of Functions with Binary Inputs

▶ A function $f : \{0, 1\}^2 \to \mathbb{R}_+$ is log-supermodular iff

\[ f(01)f(10) \leq f(00)f(11) \]

▶ If $f : \{0, 1\}^2 \to \mathbb{R}_+$ is not log-supermodular, then the function

\[ \tilde{f}(a, b) = f(a, 1 - b) \]

is log-supermodular.
A binary Ising model defined by local functions $f$ is equivalent to a binary Ising model defined by $\bar{f}$:

- The partition functions are equal: $Z(f) = Z(\bar{f})$
- For each set of beliefs $\{b_R\}$, there exists a corresponding $\{\bar{b}_R\}$ such that

$$
\sum_x \frac{\prod_{R \in \mathcal{R}_0 \cup \mathcal{R}_2} b_R(x_R)}{\prod_{R \in \mathcal{R}_1} b_R(x_R)} = \sum_x \frac{\prod_{R \in \mathcal{R}_0 \cup \mathcal{R}_2} \bar{b}_R(x_R)}{\prod_{R \in \mathcal{R}_1} \bar{b}_R(x_R)}
$$
Lemma

In a binary Ising model with log-supermodular local functions, the BP and GBP message update rules preserve log-supermodularity of messages.

Thus, if BP and GBP are initialized with log-supermodular messages, then

- the messages in subsequent iterations of BP and GBP remain log-supermodular, and
- the BP-based beliefs $b_{i,j}$, $b_i$ and the GBP-based beliefs $b_R$ are all log-supermodular.
The Kikuchi approximation is exact: $Z_R = Z$. 

The 2 × 2 Grid
The 2 × 2 Grid

The Kikuchi approximation is exact: \( Z_\mathcal{R} = Z \).

**Theorem**

*For the 2 × 2 grid, at any fixed point of BP, we have*

\[
\frac{Z}{Z_{BP}(\{b_{i,j}\}, \{b_i\})} = 1 + \frac{\Delta_{1,2} \Delta_{2,3} \Delta_{3,4} \Delta_{4,1}}{\prod_i b_i(0)b_i(1)},
\]

*where \( \Delta_{i,j} = b_{i,j}(00)b_{i,j}(11) - b_{i,j}(01)b_{i,j}(10) \).*
Proof of $2 \times 2$ Theorem

- Start with Wainwright-Jaakkola-Willsky:

$$\frac{Z}{Z_{BP}(\{b_{i,j}\}, \{b_i\})} = \sum_{x_1, \ldots, x_4} \prod_{i \sim j} b_{i,j}(x_i, x_j) \prod_i b_i(x_i)$$

$$= \sum_{x_1, \ldots, x_4} \prod_i b_i(x_i) \prod_{i \sim j} \frac{b_{i,j}(x_i, x_j)}{b_i(x_i) b_j(x_j)}$$

- Verify that

$$\frac{b_{i,j}(x_i, x_j)}{b_i(x_i) b_j(x_j)} = 1 + \frac{s(x_i)s(x_j)\Delta_{ij}}{b_i(x_i) b_j(x_j)},$$

where $s(0) = -1$ and $s(1) = +1$.

- Hence,

$$\frac{Z}{Z_{BP}(\{b_{i,j}\}, \{b_i\})} = \sum_{x_1, \ldots, x_4} \prod_i b_i(x_i) \prod_{i \sim j} \left(1 + \frac{s(x_i)s(x_j)\Delta_{ij}}{b_i(x_i) b_j(x_j)}\right)$$
Proof (cont’d)

▶ Expand out the product $\prod_{i \sim j} \left(1 + \frac{s(x_i)s(x_j)\Delta_{ij}}{b_i(x_i)b_j(x_j)}\right)$:

$$\frac{Z}{Z_{BP}} = \sum_{x_1, \ldots, x_4} \prod_i b_i(x_i) \left(1 + \cdots + \frac{\prod_{i \sim j} \Delta_{i,j}}{\prod_i b_i(x_i)^2}\right)$$

▶ Note that

$$\sum_{x_1, \ldots, x_4} \prod_i b_i(x_i) = \prod_i \sum_{x_i} b_i(x_i) = 1.$$

and

$$\sum_{x_1, \ldots, x_4} \prod_i b_i(x_i) \frac{\prod_{i \sim j} \Delta_{i,j}}{\prod_i b_i(x_i)^2} = \prod_{i \sim j} \Delta_{i,j} \sum_{x_1, \ldots, x_4} \frac{1}{\prod_i b_i(x_i)}$$

$$= \prod_{i \sim j} \Delta_{i,j} \prod_i \sum_{x_i} \frac{1}{b_i(x_i)}$$

$$= \prod_{i \sim j} \Delta_{i,j} \prod_i \frac{1}{b_i(0)b_i(1)}$$

All other terms $\sum_{x_1, \ldots, x_4} \prod_i b_i(x_i)(\cdots)$ vanish.
Theorem
For the $3 \times 3$ grid, at any fixed point of GBP, the ratio $Z/Z_{GBP}(\{b_R\})$ is given by

$$1 + b_5(0)\left(\frac{\Delta(0)}{b_{51}(00)b_{51}(01)}\right)^4 + b_5(1)\left(\frac{\Delta(1)}{b_{51}(10)b_{51}(11)}\right)^4,$$

where $\Delta(x) = b_{512}(x00)b_{512}(x11) - b_{512}(x01)b_{512}(x10)$. 
Proof of $3 \times 3$ Theorem

Start with

\[
\frac{Z}{Z_{GBP}(\{b_R\})} = \sum_{x_1, \ldots, x_9} \frac{b(x_1) b(x_{152}) b(x_{153}) b(x_{254}) b(x_{354}) b(x_5)}{b(x_{15}) b(x_{25}) b(x_{35}) b(x_{45})}
\]

Marginalize out $x_6, x_7, x_8, x_9$:

\[
\frac{Z}{Z_{GBP}(\{b_R\})} = \sum_{x_1, \ldots, x_5} \frac{b(x_1) b(x_{152}) b(x_{153}) b(x_{254}) b(x_{354}) b(x_5)}{b(x_{15}) b(x_{25}) b(x_{35}) b(x_{45})}
\]
Proof (cont’d)

► Now, define

\[ B(x) = \frac{b(x_{15})b(x_{25})b(x_{35})b(x_{45})}{b(x_5)^3} \]

and write \( \frac{Z}{Z_{\text{GBP}(\{b_R\})}} \) as

\[ \sum_{x_1, \ldots, x_5} B(x) \cdot \frac{b(x_{152})b(x_5)}{b(x_{15})b(x_{25})} \cdot \frac{b(x_{153})b(x_5)}{b(x_{15})b(x_{35})} \cdot \frac{b(x_{254})b(x_5)}{b(x_{25})b(x_{45})} \cdot \frac{b(x_{354})b(x_5)}{b(x_{35})b(x_{45})} \]

► Next, verify that

\[ \frac{b(x_{i5j})b(x_5)}{b(x_{i5})b(x_{j5})} = 1 + \frac{s(x_i)s(x_j)\Delta_{5ij}(x_5)}{b(x_{i5})b(x_{j5})} \]

where \( s(0) = -1 \) and \( s(1) = +1 \).

Plug this back into the expression for \( \frac{Z}{Z_{\text{GBP}(\{b_R\})}} \) and simplify.
Proof (cont’d)

- Upon simplification, we obtain

\[
\frac{Z}{Z_{\text{GBP}}(\{b_R\})} = 1 + \sum_{x_5} \Delta(x_5)^4 \sum_{x_1,\ldots,x_4} \prod_{i=1}^{4} \frac{1}{b(x_{i5})}
\]

- This further simplifies to

\[
\frac{Z}{Z_{\text{GBP}}(\{b_R\})} = 1 + \sum_{x_5} \Delta(x_5)^4 \left( \frac{1}{b(x_50)} + \frac{1}{b(x_51)} \right)^4
\]

\[
= 1 + \sum_{x_5} b(x_5) \left( \frac{\Delta(x_5)}{b(x_50)b(x_51)} \right)^4
\]
References
