Index Coding: Old and New

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Joint work with Fatemeh Arbajolfaei (UCSD), Bernd Bandemer (Bosch), Eren Şaşoğlu (Berkeley), and Lele Wang (UCSD)
Index coding (Birk–Kol 1998)
What is the fundamental limit on the number of transmissions?
What is the fundamental limit on the number of transmissions?

Which coding scheme achieves the limit?
Index coding

- Message $x_j \in \{0, 1\}^t, j \in [n]$ 
- Side information $x(A_j), A_j \subseteq [n] \setminus \{j\}$ at receiver $j \in [n]$ 
- Codeword $y \in \{0, 1\}^r$
Index coding

- Optimal broadcast rate

$$\beta^* = \inf_{t} \inf_{C} \frac{r}{t} = \lim_{t \to \infty} \inf_{C} \frac{r}{t}$$

- Zero error probability = small error probability
Representations

- Side information

\[ A_1 = \{2\}, \ A_2 = \{1, 3\}, \ A_3 = \{1\} \]
Representations

- Side information

\[ A_1 = \{2\}, \ A_2 = \{1, 3\}, \ A_3 = \{1\} \]

- Compact form

\[ (1|2), \ (2|1, 3), \ (3|1) \]
Representations

- **Side information**
  \[ A_1 = \{2\}, \ A_2 = \{1, 3\}, \ A_3 = \{1\} \]

- **Compact form**
  \[(1|2), (2|1, 3), (3|1)\]

- **Multiple unicast network coding**

```
x_1
\rightarrow
\rightarrow
↓↓
x_1
x_2
\rightarrow
↓↓
x_2
x_3
\rightarrow
↓↓
x_3
```
Representations

- Side information

$$A_1 = \{2\}, \ A_2 = \{1, 3\}, \ A_3 = \{1\}$$

- Compact form

$$(1|2), (2|1, 3), (3|1)$$

- Multiple unicast network coding

- Side information graph $G$

```
x_1
x_2
x_3

1
2
3
```
Representations

- **Side information**

\[ A_1 = \{2\}, \ A_2 = \{1, 3\}, \ A_3 = \{1\} \]

- **Compact form**

\[(1|2), (2|1, 3), (3|1)\]

- **Multiple unicast network coding**

\[
x_1 \rightarrow x_1
\]
\[
x_2 \rightarrow x_2
\]
\[
x_3 \rightarrow x_3
\]

- **Side information graph** \(\mathcal{G}\)

- **# of \(n\)-message index coding problems = # of \(n\)-node directed graphs**

\[ 1, 3, 16, 218, 9608, 1540944, 88203440, 1793359192848, \ldots \]
Motivations

• Applications
  ▶ Satellite communication
  ▶ Multimedia distribution
  ▶ Distributed caching
  ▶ Topological interference management
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- **Applications**
  - Satellite communication
  - Multimedia distribution
  - Distributed caching
  - Topological interference management

- **Network coding**
  - Index coding $\leq$ network coding
  - Network coding $\leq$ index coding (Effros–El Rouayheb–Langberg 2012)
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• Applications
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• Network coding
  ▶ Index coding $\leq$ network coding
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• Open problems
  ▶ Optimal broadcast rate
  ▶ Optimal coding scheme
Motivations

- **Applications**
  - Satellite communication
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- **Network coding**
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- **Open problems**
  - Optimal broadcast rate
  - Optimal coding scheme

- **Lotus, bamboo, …**
Maslow’s axiom (1966)

If all you have is a hammer, everything looks like a nail.
Approaches

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- Theoretical computer science: graph theory, integer programming, LP
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- Coding theory: algebraic codes (MDS, elastic)
Approaches

Maslow’s axiom (1966)

If all you have is a hammer, everything looks like a nail.

- Theoretical computer science: graph theory, integer programming, LP
- Network coding: linear coding, matroid theory, information inequalities
- Coding theory: algebraic codes (MDS, elastic)
- Communication theory: interference alignment
Example

(1), (2), (3)

1

2

3
Example

Send $y = (x_1, x_2, x_3)$

$\beta^* = 3$
Example

(1|2, 3), (2|1, 3), (3|1, 2)
Example

\[(1|2, 3), (2|1, 3), (3|1, 2)\]

- Send \( y = x_1 + x_2 + x_3 \)
- \( \beta^* = 1 \)
Example

(1|2), (2|1, 3), (3|1)
Example

Send \( y = (x_1 + x_2, x_3) \)

\( \beta^* = 2 \)
Clique covering

- Side information graph $G$ with the collection $\mathcal{K}$ of all cliques
Clique covering

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- **Clique covering number $cc(\mathcal{G})$:** minimum number of cliques partitioning $\mathcal{G}$
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- Equivalent to the **chromatic number** of the undirected complement of $\mathcal{G}$
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$$cc(\mathcal{G}) = \chi(\bar{\mathcal{G}}) = 3$$
Clique covering

- Side information graph $\mathcal{G}$ with the collection $\mathcal{K}$ of all cliques
- **Clique covering number** $cc(\mathcal{G})$: minimum number of cliques partitioning $\mathcal{G}$
- Equivalent to the **chromatic number** of the undirected complement of $\mathcal{G}$

**Clique covering bound (Birk–Kol 1998)**

$$\beta^* \leq cc(\mathcal{G})$$
Clique covering

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- Equivalent to the chromatic number of the undirected complement of $\mathcal{G}$

Clique covering bound (Birk–Kol 1998)

$$\beta^* \leq cc(\mathcal{G})$$

- Integer programming characterization

$$\text{minimize} \quad \sum_{\mathcal{S} \in \mathcal{K}} \rho_{\mathcal{S}}$$

subject to

$$\sum_{\mathcal{S} \in \mathcal{K} : j \in \mathcal{S}} \rho_{\mathcal{S}} \geq 1, \quad j \in [n],$$

$$\rho_{\mathcal{S}} \in \{0, 1\}, \quad \mathcal{S} \in \mathcal{K}$$
Example

\[(1|2), (2|3), (3|1)\]

- \(cc(\mathcal{G}) = 3\)
Example

\( (1|2), (2|3), (3|1) \)

- \( cc(\mathcal{G}) = 3 \)
- Send \( y = (x_1 + x_2, x_1 + x_3) \)
- \( \beta^* = 2 \)
Example

(1|2), (2|3), (3|1)

- $cc(G) = 3$
- Send $y = (x_1 + x_2, x_1 + x_3)$
- $\beta^* = 2$
- Maximum distance separable (MDS) code for 2 erasures
Example

\[ (1|2), (2|3), (3|1) \]

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- Send \( y = (x_1 + x_2, x_1 + x_3) \)
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- Maximum distance separable (MDS) code for 2 erasures
- \( n \)-node graph is \( k \)-partial clique if the minimum indegree is \( n - k - 1 \)
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  - A clique is a 0-partial clique
Example

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- Maximum distance separable (MDS) code for 2 erasures
- \( n \)-node graph is \( k \)-partial clique if the minimum indegree is \( n - k - 1 \)
  - A clique is a 0-partial clique
  - \( G \) above is a 1-partial clique
Example

(1|2), (2|3), (3|1)

- $\text{cc}(\mathcal{G}) = 3$
- Send $y = (x_1 + x_2, x_1 + x_3)$
- $\beta^* = 2$

- **Maximum distance separable (MDS) code** for 2 erasures
- $n$-node graph is $k$-partial clique if the minimum indegree is $n - k - 1$
  - A clique is a 0-partial clique
  - $\mathcal{G}$ above is a 1-partial clique
- MDS code for $(k + 1)$ erasures suffices
Partial clique covering

- Partial cliques $G_1, \ldots, G_m$ of parameters $k_1, \ldots, k_m$ partitioning $G$
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Partial clique covering bound (Birk–Kol 1998)

$$
\beta^* \leq \min_{G_1, \ldots, G_m} (k_1 + 1) + \cdots + (k_m + 1)
$$
Partial clique covering

- Partial cliques $G_1, \ldots, G_m$ of parameters $k_1, \ldots, k_m$ partitioning $G$

**Partial clique covering bound (Birk–Kol 1998)**

$$\beta^* \leq \min_{G_1, \ldots, G_m} (k_1 + 1) + \cdots + (k_m + 1)$$

**Alternative characterization**

$$\text{minimize} \quad \sum_{S \subseteq [n]} \rho_S (k_S + 1)$$

subject to

$$\sum_{S \subseteq [n]: j \in S} \rho_S \geq 1, \quad j \in [n],$$

$$\rho_S \in \{0, 1\}, \quad S \subseteq [n]$$

where $G|_S$ is a $k_S$-partial clique for $S \subseteq [n]$
Example

\[(1|2, 5), (2|1, 3), (3|2, 4), (4|3, 5), (5|1, 4)\]

- \(cc(G) = 3\)
Example

\[(1|2, 5), (2|1, 3), (3|2, 4), (4|3, 5), (5|1, 4)\]

- \(cc(G) = 3\)
- Split \(x_j\) into \((a_j, b_j)\) and send \(y = (a_1 + a_2, a_3 + a_4, a_5 + b_1, b_2 + b_3, b_4 + b_5)\)
- \(\beta^* = 5/2\)
Example

\[(1|2,5), (2|1,3), (3|2,4), (4|3,5), (5|1,4)\]

- \(cc(\mathcal{G}) = 3\)
- Split \(x_j\) into \((a_j, b_j)\) and send \(y = (a_1 + a_2, a_3 + a_4, a_5 + b_1, b_2 + b_3, b_4 + b_5)\)
- \(\beta^* = 5/2\)
- **Time sharing** over subproblems \(\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}\)
Example

\( (1|2,5), (2|1,3), (3|2,4), (4|3,5), (5|1,4) \)

- \( cc(G) = 3 \)
- Split \( x_j \) into \( (a_j, b_j) \) and send \( y = (a_1 + a_2, a_3 + a_4, a_5 + b_1, b_2 + b_3, b_4 + b_5) \)
- \( \beta^* = 5/2 \)

- Time sharing over subproblems \( \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\} \)
- Fractional partition \( f : 2^{[n]} \setminus \emptyset \rightarrow [0, 1] \) with \( \sum_{S:j \in S} f(S) = 1, j \in [n] \)
Example

\[(1|2, 5), (2|1, 3), (3|2, 4), (4|3, 5), (5|1, 4)\]

- \(cc(\mathcal{G}) = 3\)
- Split \(x_j\) into \((a_j, b_j)\) and send \(y = (a_1 + a_2, a_3 + a_4, a_5 + b_1, b_2 + b_3, b_4 + b_5)\)
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- **Time sharing** over subproblems \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}

- **Fractional partition** \(f : 2^n \setminus \{\emptyset\} \rightarrow [0, 1]\) with \(\sum_{S:j \in S} f(S) = 1, j \in [n]\)
  - Partition: \(f(S) \in \{0, 1\}\)
Example

\[(1|2, 5), (2|1, 3), (3|2, 4), (4|3, 5), (5|1, 4)\]

- \(\text{cc}(\mathcal{G}) = 3\)
- Split \(x_j\) into \((a_j, b_j)\) and send \(y = (a_1 + a_2, a_3 + a_4, a_5 + b_1, b_2 + b_3, b_4 + b_5)\)
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- Time sharing over subproblems \(\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}\)
- Fractional partition \(f : 2^{[n]} \setminus \{\emptyset\} \rightarrow [0, 1] \text{ with } \sum_{S:j \in S} f(S) = 1, j \in [n]\)
  - Partition: \(f(S) \in \{0, 1\}\)
  - \(f(S) = 1/2, S = \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}\)
Fractional clique covering

- Side information graph $G$ with the collection $\mathcal{K}$ of all cliques
Fractional clique covering

- Side information graph $G$ with the collection $K$ of all cliques

- Fractional clique covering number $\text{fcc}(G)$ ($= \text{fractional chromatic number of } \bar{G}$)
Fractional clique covering

- Side information graph $G$ with the collection $\mathcal{K}$ of all cliques
- Fractional clique covering number $\text{fcc}(G)$ (= fractional chromatic number of $\bar{G}$)

**Fractional clique covering bound (Blasiak–Kleinberg–Lubetzky 2010)**

$$\beta^* \leq \text{fcc}(G)$$
Fractional clique covering

- Side information graph $G$ with the collection $\mathcal{K}$ of all cliques
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$$\beta^* \leq \text{fcc}(G)$$

- Linear programming relaxation of clique covering

$$\begin{align*}
\text{minimize} & \quad \sum_{S \in \mathcal{K}} \rho_S \\
\text{subject to} & \quad \sum_{S \in \mathcal{K}: j \in S} \rho_S \geq 1, \quad j \in [n], \\
& \quad \rho_S \in [0, 1], \quad S \in \mathcal{K}
\end{align*}$$
Example

(1|2, 3, 4), (2|1, 3, 4), (3|4, 5, 6), (4|3, 5, 6), (5|6, 1, 2), (6|5, 1, 2)

- \( \text{fcc}(G) = 3 \) (send \( y = (x_1 + x_2, x_3 + x_4, x_5 + x_6) \))
Example

\[(1|2, 3, 4), (2|1, 3, 4), (3|4, 5, 6), (4|3, 5, 6), (5|6, 1, 2), (6|5, 1, 2)\]

- \(fcc(\mathcal{G}) = 3\) \((send\ y = (x_1 + x_2, x_3 + x_4, x_5 + x_6))\)

- Instead, send \(y = ((x_1 + x_2) + (x_3 + x_4), (x_1 + x_2) + (x_5 + x_6))\)

- \(\beta^* = 2\)
Example

(1|2, 3, 4), (2|1, 3, 4), (3|4, 5, 6), (4|3, 5, 6), (5|6, 1, 2), (6|5, 1, 2)

- $\text{fcc}(G) = 3$ (send $y = (x_1 + x_2, x_3 + x_4, x_5 + x_6)$)
- Instead, send $y = ((x_1 + x_2) + (x_3 + x_4), (x_1 + x_2) + (x_5 + x_6))$
- $\beta^* = 2$

- Local time sharing over subproblems \{1, 2\}, \{3, 4\}, \{5, 6\}
Example

\[(1|2, 3, 4), (2|1, 3, 4), (3|4, 5, 6), (4|3, 5, 6), (5|6, 1, 2), (6|5, 1, 2)\]

- \(\text{fcc}(G) = 3\) (send \(y = (x_1 + x_2, x_3 + x_4, x_5 + x_6)\))

- Instead, send \(y = ((x_1 + x_2) + (x_3 + x_4), (x_1 + x_2) + (x_5 + x_6))\)

- \(\beta^* = 2\)

- \textbf{Local time sharing} over subproblems \{1, 2\}, \{3, 4\}, \{5, 6\}

- MDS code of \textit{hyperparity} symbols against 2 erasures
Local clique covering

- Side information graph $G$ with the collection $\mathcal{K}$ of all cliques
Local clique covering

- Side information graph $\mathcal{G}$ with the collection $\mathcal{K}$ of all cliques
- Local clique covering number $lcc(\mathcal{G})$ (= local chromatic number of $\mathcal{G}$)
Local clique covering

- Side information graph $G$ with the collection $\mathcal{K}$ of all cliques
- Local clique covering number $\text{lcc}(G)$ ($= \text{local chromatic number of } \bar{G}$)

Local clique covering bound (Shanmugam–Dimakis–Langberg 2013)

$$\beta^* \leq \text{lcc}(G)$$
Local clique covering

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$$\beta^* \leq \text{lcc}(G)$$

- Integer programming characterization

$$\begin{align*}
\text{minimize} & \quad \max_{j \in [n]} \sum_{S \in \mathcal{K} : S \notin A_j} \rho_S \\
\text{subject to} & \quad \sum_{S \in \mathcal{K} : j \in S} \rho_S \geq 1, \quad j \in [n], \\
& \quad \rho_S \in \{0, 1\}, \quad S \in \mathcal{K}
\end{align*}$$
Local clique covering

- Side information graph $G$ with the collection $\mathcal{K}$ of all cliques
- Local clique covering number $1cc(G)$ (= local chromatic number of $\bar{G}$)

Local clique covering bound (Shanmugam–Dimakis–Langberg 2013)

$$\beta^* \leq 1cc(G)$$

- Integer programming characterization

\[
\begin{align*}
\text{minimize} \quad & \max_{j \in [n]} \sum_{S \in \mathcal{K} : S \notin A_j} \rho_S \\
\text{subject to} \quad & \sum_{S \in \mathcal{K} : j \in S} \rho_S \geq 1, \quad j \in [n], \\
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\end{align*}
\]

- Combine with fractional covering (Shanmugam–Dimakis–Langberg 2013)
Fractional local partial clique covering

- Combine everything we have seen so far
Fractional local partial clique covering

- Combine everything we have seen so far

### Fractional local partial clique covering bound (Arbabjolfaei–K 2013)

\[
\beta^* \leq \text{flpcc}(G)
\]

where \(\text{flpcc}(G)\) is the solution to the linear programming

\[
\begin{align*}
\text{minimize} & \quad \max_{j \in [n]} \sum_{S \subseteq [n]: S \not\subseteq A_j} \rho_S (k_S + 1) \\
\text{subject to} & \quad \sum_{S \subseteq [n]: j \in S} \rho_S \geq 1, \quad j \in [n], \\
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\]

- Strictly tighter than everything we have seen so far
Fractional local partial clique covering

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\]

- Strictly tighter than everything we have seen so far
- Optimal up to \( n = 4 \) (218 problems)
Fractional local partial clique covering

- Combine everything we have seen so far

### Fractional local partial clique covering bound (Arbabjolfaei–K 2013)

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\end{align*}
\]

- Strictly tighter than everything we have seen so far
- Optimal up to \( n = 4 \) (218 problems)
- Can we do better?
Recursive combination

- Replace MDS codes by best known index codes
Recursive combination

- Replace MDS codes by best known index codes

Recursive clique covering bound (Arbabjolfaei–K 2013)

\[ \beta^* \leq \text{rcc}(G) \]

where \( \text{rcc}(G) \) is the solution to the linear programming

\[
\begin{align*}
\text{minimize} & \quad \max_{j \in [n]} \sum_{S \subseteq [n]: S \not\subseteq A_j} \rho_S \text{rcc}(G|_S) \\
\text{subject to} & \quad \sum_{S \subseteq [n]: j \in S} \rho_S \geq 1, \quad j \in [n], \\
& \quad \rho_S \in [0, 1], \quad S \not\subseteq [n]
\end{align*}
\]
Recursive combination

- Replace MDS codes by best known index codes

Recursive clique covering bound (Arbabjolfaei–K 2013)

$$\beta^* \leq \text{rcc}(G)$$

where $\text{rcc}(G)$ is the solution to the linear programming

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& \quad \rho_S \in [0, 1], \quad S \subset [n]
\end{aligned}$$

- Strictly tighter than everything we have seen so far
Recursive combination

- Replace MDS codes by best known index codes

Recursive clique covering bound (Arbabjolfaei–K 2013)

\[ \beta^* \leq rcc(G) \]

where \( rcc(G) \) is the solution to the linear programming

\[
\begin{align*}
\text{minimize} & \quad \max_{j \in [n]} \sum_{S \subseteq [n]: S \not\subseteq A_j} \rho_S \ rcc(G|S) \\
\text{subject to} & \quad \sum_{S \subseteq [n]: j \in S} \rho_S \geq 1, \quad j \in [n], \\
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where \( \text{rcc}(G) \) is the solution to the linear programming

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\text{minimize} & \quad \max_{j \in [n]} \sum_{S \subseteq [n] : S \not\subseteq A_j} \rho_S \text{rcc}(G | S) \\
\text{subject to} & \quad \sum_{S \subseteq [n] : j \in S} \rho_S \geq 1, \quad j \in [n], \\
& \quad \rho_S \in [0, 1], \quad S \not\subseteq [n]
\end{align*}
\]

- Strictly tighter than everything we have seen so far
- Can we do better? *Unfortunately, yes*
Example (revisited)

\[(1|2), (2|3), (3|1)\]

- Let $M$ be a 3-by-3 matrix such that

$$M_{ii} \neq 0 \quad \text{and} \quad M_{ij} = 0 \text{ if } i \notin A_j$$

and send $y = (x_1, x_2, x_3)\bar{M}$, where $\bar{M}$ consists of independent columns of $M$.
Example (revisited)

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and send $y = (x_1, x_2, x_3)\tilde{M}$, where $\tilde{M}$ consists of independent columns of $M$

For example, let

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

and send $y = (x_1 + x_2, x_2 + x_3)$
Example (revisited)

Let $M$ be a $3$-by-$3$ matrix such that

\[ M_{ii} \neq 0 \quad \text{and} \quad M_{ij} = 0 \text{ if } i \notin A_j \]

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For example, let

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1 & 0 & 1 \\
1 & 1 & 0 \\
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\end{pmatrix}
\]

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Minimum rank of all such $M$: $\text{minrk}_2(G)$
Linear coding

Minimum rank bound (Bar-Yossef–Birk–Jayram–Kol 2006)

\[ \beta^* \leq \text{minrk}_2(G) \]
Linear coding

Minimum rank bound (Bar-Yossef–Birk–Jayram–Kol 2006)

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- Extension to general finite fields (Lubetzky–Stav 2007)
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- **Interference alignment**: extension to vector linear codes \( M_{ij} \in \mathbb{F}^{t \times t} \)
  (Maleki–Cadambe–Jafar 2012)
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- Local time sharing and multiple fields
Minimum rank bound (Bar-Yossef–Birk–Jayram–Kol 2006)

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- Extension to general finite fields (Lubetzky–Stav 2007)
- **Interference alignment**: extension to vector linear codes \( M_{ij} \in \mathbb{F}^{t \times t} \) (Maleki–Cadambe–Jafar 2012)
- Local time sharing and multiple fields
- **Multiletter characterization** (no code, no bound)
Alternative approach

Maslow’s axiom (1966)

If all you have is a hammer,
everything looks like a nail.
Alternative approach

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If all you have is a hammer, everything looks like a nail.

- Our hammer: Shannon’s random coding (Cover’s random binning)
Alternative approach

Maslow’s axiom (1966)
If all you have is a hammer, everything looks like a nail.

- Our hammer: Shannon’s random coding (Cover’s random binning)
- Flat coding (= partial clique covering)
- Dual index coding
- Composite coding
Flat coding

(1|2), (2|1, 3), (3|1)
Flat coding

(1|2), (2|1, 3), (3|1)

- **Codebook generation:**
  - For each \((x_1, x_2, x_3)\), generate a Bern(1/2) sequence \(y(x_1, x_2, x_3)\)

- **Encoding:**
  - To send \((x_1, x_2, x_3)\), transmit \(y(x_1, x_2, x_3)\)
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- **Decoding:**
  - Each receiver uniquely decodes for all the messages that it does not have
    - Receiver 1 finds the unique \((\hat{x}_1, \hat{x}_3)\) such that \(y(\hat{x}_1, x_2, \hat{x}_3) = y\)
    - Number of wrong tuples: \(2^{2t} - 1\)
    - Probability that two codewords are identical: \(1/2^r\)
    - Thus, by the union of events bound, \(P(\mathcal{E}_1) \rightarrow 0\) if \(r/t > 2\)
Flat coding

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  ▶ Similarly, we obtain \(r/t > 1\) and \(r/t > 2\)

• Can be combined with local time sharing (not optimal in general)
Interlude: Dual index coding

- \((2^n - 1)\) senders cooperatively communicate \((x_1, \ldots, x_n)\)

- Sender \(S \subseteq [n]\) encodes \(x(S) = (x_j : j \in S)\) into an index \(w_S \in [2^{|S|}]\)
Interlude: Dual index coding

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- Sender \(S \subseteq [n]\) encodes \(x(S) = (x_j: j \in S)\) into an index \(w_S \in [2^{y_S t}]\)

- A special case of the general multiple access channel with correlated messages (Slepian–Wolf 1973, Han 1979)
Interlude: Dual index coding

Optimal condition for reliable communication

\[ |S| \leq \sum_{T \cap S \neq \emptyset} y_S, \quad S \subseteq [n] \]
Interlude: Dual index coding

Optimal condition for reliable communication

\[ |S| \leq \sum_{\mathcal{T}: \mathcal{T} \cap S \neq \emptyset} \gamma_{\mathcal{T}}, \quad S \subseteq [n] \]

- Achieved by random coding and simultaneous decoding
Interlude: Dual index coding

Optimal condition for reliable communication

\[ |S| \leq \sum_{T: T \cap S \neq \emptyset} \gamma_S, \quad S \subseteq [n] \]

- Achieved by random coding and simultaneous decoding
- Extension \( R(D|A) \): Demand \( D \) and side information \( A \) at the receiver
Composite coding

(1|4), (2|3, 4), (3|1, 2), (4|2, 3)
Composite coding

\[(1|4), (2|3, 4), (3|1, 2), (4|2, 3)\]

- **Encoding (step 1):** Introduce 2 “virtual” senders (cf. dual index coding)
  - Random coding of \((x_1, x_4)\) into \(w_{1,4}\) at rate \(\gamma_{1,4} > 1\)
  - Random coding of \((x_2, x_3, x_4)\) into \(w_{2,3,4}\) at rate \(\gamma_{2,3,4} > 1\)
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- **Encoding (step 2):** Send the “composite” indices \(y = (w_{1,4}, w_{2,3,4}) \in \{0, 1\}^{\beta t}\)
Composite coding

\[(1|4), (2|3, 4), (3|1, 2), (4|2, 3)\]

- **Decoding (step 1):** Recover the composite indices \( 2 < y_{1,4} + y_{2,3,4} < \beta \)
Composite coding

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- **Decoding (step 1):** Recover the composite indices \(2 < \gamma_{1,4} + \gamma_{2,3,4} < \beta\)
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  - **Simultaneous decoding** of the message and some interference
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  - Decoder 1 uses \( w_{1,4} \) to recover \( x_1 \)

\[
1 < \gamma_{1,4}
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  - Decoder 3 uses \( (w_{1,4}, w_{2,3,4}) \) to recover \( (x_3, x_4) \)
    \[ 2 < \gamma_{1,4} + \gamma_{2,3,4}, \]
    \[ 1 < \gamma_{2,3,4}, \]
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  - Decoder 3 uses \((w_{1,4}, w_{2,3,4})\) to recover \((x_3, x_4)\)
    \[2 < \gamma_{1,4} + \gamma_{2,3,4}, \quad 1 < \gamma_{2,3,4}, \quad 1 < \gamma_{1,4} + \gamma_{2,3,4}\]
  - Decoder 4 uses \(w_{2,3,4}\) to recover \(x_4\)
    \[1 < \gamma_{2,3,4}\]

\[w_{1,4}, w_{2,3,4} \rightarrow \text{Dec 4} \rightarrow x_2, x_3, x_4\]
• $(2^n - 1)$ virtual senders to encode $n$ messages
(2^n – 1) virtual senders to encode n messages

Flat coding of the composite indices using side information
(2^n – 1) virtual senders to encode n messages

Flat coding of the composite indices using side information

Optimal (simultaneous nonunique) decoding of the desired message
Composite coding

**Composite coding bound**

\[ \beta^* \leq \text{comp}(G) \]

where \( \text{comp}(G) \) is the solution to the optimization problem

\[
\begin{align*}
\text{minimize} & \quad \max_{j \in [n]} \sum_{S \subseteq [n]: S \notin A_j} \gamma_S \\
\text{subject to} & \quad \min_{T \subseteq D_j \setminus A_j} \frac{1}{|T|} \sum_{S \subseteq D_j \cup A_j: S \cap T \neq \emptyset} \gamma_S \geq 1, \quad j \in [n], \\
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& \quad \gamma_S \geq 0, \quad S \subseteq [n], \\
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- Similar, but richer structure than clique covering bounds
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\]

- Similar, but richer structure than clique covering bounds
- Decoding spanned over multiple subproblems (time slots)
More on composite coding

😊 Optimal up to $n = 5$ (9608 problems)
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😊 Simultaneous decoding: random coding over interference networks
  (Bandemer–El Gamal–Kim 2012)
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😊 **Recursive composite coding**: difficult to evaluate
Concluding remarks

- **Random coding** is a powerful tool
  - Original network coding theorem (Ahlswede–Cai–Li–Yeung 2000)
  - Quick and dirty
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- Index coding
  - One of the most fundamental network information theory problems (cf. 2-DMBC)
  - Down the rabbit hole (full of exciting adventures)
    - Lower bounds (Sun–Jafar 2013)
    - Capacity region vs. optimal broadcast rate


Sun, H. and Jafar, S. A. (2013). Index coding capacity: How far can one go with only Shannon inequalities?