Network Encoding Complexity: Exact Values, Bounds and Inequalities *†

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Abstract

For an acyclic directed network with multiple pairs of sources and sinks and a set of Menger's paths connecting each pair of source and sink, it is known that the number of mergings among these Menger's paths is closely related to network encoding complexity. In this paper, we focus on networks with two pairs of sources and sinks and we derive bounds on and exact values of two functions relevant to encoding complexity for such networks.

1 Introduction

Let $G = (V, E)$ denote an acyclic directed graph, where V denotes the set of all the vertices (or points) in G and E denotes the set of all the edges (or links) in G. In this paper, a *path* in G is treated as a set of concatenated edges. For k paths $\beta_1, \beta_2, \ldots, \beta_k$ in G, we say these paths *merge* [6] at an edge $e \in E$ if

1. $e \in \bigcap_{i=1}^{k} \beta_i$,
2. there are at least two distinct edges $f, g \in E$ such that f, g are immediate predecessors of e in some β_i, β_j, $i \neq j$, respectively.

We call the maximal subpath that starts with e and is shared by all β_i's (i.e., e together with the subsequent concatenated edges shared by all β_i's until some β_i branches off) *merged subpath* (or simply *merging*) by all β_i's at e; see Figure 1 for a quick example.

For any two vertices $u, v \in V$, we call any set consisting of the maximum number of pairwise edge-disjoint directed paths from u to v a set of *Menger's paths* from u to v. By Menger's theorem [10], the cardinality of Menger's paths from u to v is equal to the size of a minimum cut between u and v. Here, we remark that the Edmonds-Karp algorithm [4] can find a minimum cut and a set of Menger's paths from u to v in polynomial time.

Assume that G has two sources S_1, S_2 and two distinct sinks R_1, R_2. For $i = 1, 2$, let c_i denote the size of a minimum edge-cut between S_i and R_i, and let $\alpha_i = \{\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,c_i}\}$ denote a set of Menger's paths from S_i to R_i, whose elements are often referred to as *α_i-paths*. We are interested in the number of mergings among paths from different α_i's, denoted by $||G||(\alpha_1, \alpha_2)$. In this paper we will count the number of mergings **without** multiplicity: all the mergings at the same edge e will be counted as one merging at e. And we define

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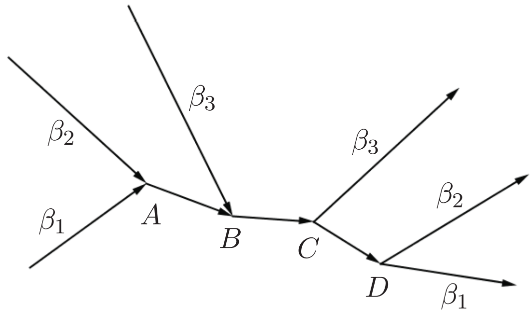

Figure 1: Paths β_1, β_2 merge at edge $A \to B$ and at merged subpath (or merging) $A \to B \to C \to D$, and paths $\beta_1, \beta_2, \beta_3$ merge at edge $B \to C$ and at merged subpath (or merging) $B \to C$ (Here, arrows in the figure represents edges, and the terminal points of arrows should be naturally interpreted as vertices; the same convention applies to other figures in this paper).

$$M(G) \triangleq \min_{\alpha_1, \alpha_2} ||G||(\alpha_1, \alpha_2),$$

where the minimum is taken over all possible Menger's path sets α_i's, $i = 1, 2$. Roughly speaking, $M(G)$ corresponds to the best choice of α_1, α_2 in terms of minimizing the number of mergings.

Let $N^*(c_1, c_2)$ denote the set of all directed networks with one source S, two distinct sinks R_1, R_2, satisfying that the minimum size of edge cut between S and R_i is c_i for $i = 1, 2$. We define

$$\mathcal{M}^*(c_1, c_2) \triangleq \sup_{G \in N^*(c_1, c_2)} M(G).$$

Let $N(c_1, c_2)$ denote the set of all directed networks with two distinct sources S_1, S_2, two distinct sinks R_1, R_2, satisfying that the minimum size of edge cut between S_i and R_i is c_i for $i = 1, 2$. We define

$$\mathcal{M}(c_1, c_2) \triangleq \sup_{G \in N(c_1, c_2)} M(G).$$

The above definitions can be roughly interpreted as follows: for a given G, we try to choose α_1, α_2 to obtain the minimal number of mergings, and $\mathcal{M}$ and $\mathcal{M}^*$ give us the minimum number corresponding to the worst-case scenarios among all possible G.

It is first shown in [8] that $\mathcal{M}^*(c_1, c_2)$ is finite for all c_1, c_2 (see Theorem 22 in [8]). It was first conjectured that $\mathcal{M}(c_1, c_2)$ is finite in [13]. Here, we remark that all the aforementioned work are done primarily in the context of network coding and we have rephrased their results using our notation and terminologies.

In [6], we have shown that for any c_1, c_2, $\mathcal{M}^*(c_1, c_2)$, $\mathcal{M}(c_1, c_2)$ are both finite, and we further studied the behaviors of $\mathcal{M}^*, \mathcal{M}$ as functions of the sizes of minimum cuts. One novel aspect of our approach is that paths, rather than vertices and edges, are treated as "elementary" objects, which can be transformed to different paths through reroutings. The effectiveness of this approach is further evidenced by this work, where the lines of the thoughts in [6] are continued to derive exact values of and tighter bounds on $\mathcal{M}^*$ and $\mathcal{M}$ for certain parameters. The contribution of this paper can be summarized as follows:

- novel methods are used to derive the exact values of some $\mathcal{M}^*$ with two parameters (Theorems 4.1).
- through a non-trivial refinement of the arguments in [6], we obtain tighter upper bounds (Theorems 5.3, 5.8) and a scaling law (Theorem 5.10).

- using the new techniques of "glueing" smaller graphs to obtain larger graphs, we give constructive proofs for tighter lower bounds (Theorems 5.2, 5.6) and some inequality relationships between $\mathcal{M}^*$ and $\mathcal{M}$ (Theorems 6.1, 6.2). These inequalities may serve as a first step to understand the connections between single-source and multiple-source networks.
- our constructive proofs (for the lower bounds on $\mathcal{M}$ and $\mathcal{M}^*$ in Sections 4, 5 and 6) reveal the topological structure of some worse case networks (in terms of the number of encoding nodes required), which may shed some light on the implementation of efficient network coding strategies.

2 Network Encoding Complexity

One of the most fundamental yet challenging problems in the theory of network coding [17] is to determine the encoding complexity [8] for a given network, that is, the minimum number of encoding nodes required for the existence of a network coding solution. It turns out that the combinatorial notions defined in Section 1, such as M, $\mathcal{M}^*$ and $\mathcal{M}$, are closely related to the encoding complexity for a variety of networks, as elaborated below. Roughly speaking, for a network with multiple groups of Menger's paths, each of which is used to transmit a set of messages to a particular sink, network encodings are only needed at mergings by different groups of Menger's paths. As a result, with respect to a given set of Menger's paths, the number of encoding nodes required in the network is always upper bounded by the number of mergings.

Next, we illustrate in greater details the aforementioned connections in multicast networks, two-way channels and multiple unicast networks. Throughout this section, we assume that each link in the considered network is of unit capacity and there is no delay during the message transmission on each link.

Multicast networks. For illustrative purposes, we first consider the famous "butterfly network" [9]. As depicted in Figure 2(a), for the purpose of transmitting messages a, b simultaneously from the source S to the sinks R_1, R_2, network encoding has to be done at node C. Another way to interpret the necessity of network coding at C (for the simultaneous transmission to R_1 and R_2) is as follows: If the transmission to R_2 is ignored, Menger's paths $S \to A \to R_1$ and $S \to B \to C \to D \to R_1$ can be used to transmit messages a, b from S to R_1; if the transmission to R_1 is ignored, Menger's paths $S \to A \to C \to D \to R_2$ and $S \to B \to R_2$ can be used to transmit messages a, b from S to R_2. For the simultaneous transmission to R_1 and R_2, merging by these two groups of Menger's paths at $C \to D$ becomes a "bottleneck", therefore network coding at C is required to avoid the possible congestions.

Generally speaking, consider a network $G \in N^*(c_1, c_2)$ with one source S and two sinks R_1, R_2 and chosen Menger's path sets $\alpha_i = \{\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,c_i}\}$, $i = 1, 2$. Assume that all c_i are equal to c and messages $a_1, a_2, \ldots, a_c$ are to be transmitted to each sink simultaneously. It is well-known that the Jaggi *et al.* algorithm [7] can be applied along Menger's paths to obtain a network coding solution. An examination of the algorithm reveals that as long as the field size is large enough, appropriately chosen network encoding functions at all the tails of the mergings will produce a linear network coding solution. Therefore, for given α_i, $i = 1, 2$, the number of encoding nodes needed in the Jaggi *et al.* algorithm is just $||G||(\alpha_1, \alpha_2)$; and moreover, $M(G)$ gives the minimum number of encoding nodes required for the existence of a network coding solution and $\mathcal{M}^*(c, c)$ is the largest such number among all possible $G \in N^*(c, c)$.

Two-way channels. We first illustrate the idea using a variant of the classical butterfly network (see Example 17.2 of [16]; cf. the two-way channel in [3, Page 519]) with two sources and two sinks, where the source S_1 is attached to the sink R_2 to form a group and the source S_2 is attached to the sink R_1 to form the other group. As depicted in Figure 2(b), the two groups wish

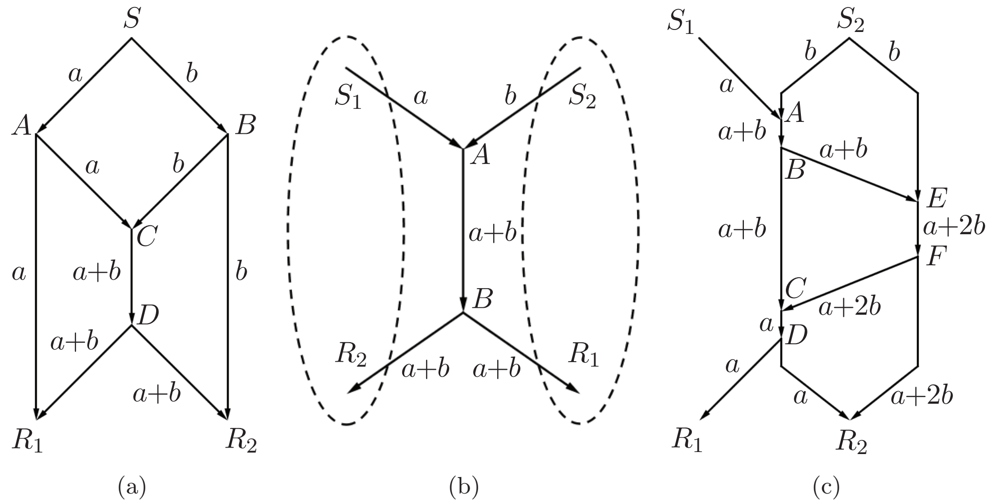

Figure 2: (a) Network coding in the butterfly network (b) Network coding in a two-way channel (c) Network coding in two sessions of unicast

to exchange messages a and b through the network. Similarly as in the previous example, the edge $A \to B$ is where the Menger's paths $S_1 \to A \to B \to R_1$ and $S_2 \to A \to B \to R_2$ merge with each other, which is a bottleneck for the simultaneous transmission of messages a, b. The simultaneous transmission is achievable if upon receiving the messages a and b, network encoding is performed at the node A and the newly derived message $a + b$ is sent over the channel $A \to B$.

More generally, consider a network $G \in N(c_1, c_2)$ with two sources S_1, S_2 and two sinks R_1, R_2 and chosen Menger's path sets $\alpha_i = \{\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,c_i}\}$, $i = 1, 2$, where the source S_1 is attached to the sink R_2 to form a group and the source S_2 is attached to the sink R_1 to form the other group. Assume that messages $a_1, a_2, \ldots, a_{c_1}$ are to be sent from S_1 to R_1, and messages $b_1, b_2, \ldots, b_{c_2}$ from S_2 to R_2. Similarly as in multicast networks, with the field size chosen large enough and appropriately chosen encoding functions the tails of all mergings, the Jaggi *et al.* algorithm can be applied along Menger's paths so that R_1 (or R_2), together with its complete knowledge of messages b_j (or a_i), can decode messages a_i (or b_j) based on what has been transmitted along α_1 (or α_2). Therefore, for given α_i, $i = 1, 2$, the number of encoding nodes needed in the Jaggi *et al.* algorithm is just $||G||(\alpha_1, \alpha_2)$; and moreover, $M(G)$ gives the minimum number of encoding nodes required for the existence of a network coding solution and $\mathcal{M}(c_1, c_2)$ is the largest such number among all possible $G \in N(c_1, c_2)$.

Multiple unicast networks. Our idea can be best illustrated using the following example network with two sessions of unicast [11]. As shown in Figure 2(c), the source S_1 is to transmit message a to the sink R_1 using path $S_1 \to A \to B \to E \to F \to C \to D \to R_1$. And the source S_2 is to transmit message b to the sink R_2 using two Menger's paths $S_2 \to A \to B \to C \to D \to R_2$ and $S_2 \to E \to F \to R_2$. Since mergings $A \to B$, $C \to D$ and $E \to F$ become bottlenecks for the simultaneous transmission of messages a and b, network coding at these bottlenecks, as shown in Figure 2(c), is performed to ensure the simultaneous message transmission.

More generally, consider a network $G \in N(c_1, c_2)$ with two sources S_1, S_2 and two sinks R_1, R_2 and chosen Menger's path sets $\alpha_i = \{\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,c_i}\}$, $i = 1, 2$. Assume that all c_i are equal to 2 and message a_i is to be transmitted from S_i to R_i through α_i, $i = 1, 2$, simultaneously. Similarly, with appropriately chosen field and encoding functions at the tails of all mergings, the Jaggi *et al.* algorithm can be applied along Menger's paths to obtain a network coding solution.

Therefore, for given α_i, $i = 1, 2$, the number of encoding nodes needed in the Jaggi *et al.* algorithm is just $||G||(\alpha_1, \alpha_2)$; and moreover, $M(G)$ gives the minimum number of encoding nodes required for the existence of a network coding solution and $\mathcal{M}(2, 2)$ is the largest such number among all possible $G \in N(2, 2)$. For closely related work on the network coding solvability in multiple unicast networks, we refer to [2, 11, 12].

3 Notation and Terminology

3.1 Basic Notation

For a path β in G, let $t(\beta), h(\beta)$ denote the *tail* (or *starting point*) and the *head* (or *ending point*) of path β, respectively; let $\beta[u, v]$ denote the subpath of β with the starting point u and the ending point v. For two distinct paths ξ, η in G, we say ξ is *smaller* than η (or, η is *larger* than ξ) if there is a directed path from $h(\xi)$ to $t(\eta)$; if ξ, η and the connecting path from $h(\xi)$ to $t(\eta)$ are subpaths of path β, we say ξ is *smaller* than η on β. Note that the relation "smaller" only imposes a partial order among all paths in G, rather than a total order; and this definition also applies to the case when paths degenerate to vertices/edges (in other words, in the definition, ξ, η or the connecting path from $h(\xi)$ to $t(\eta)$ can be vertices/edges in G, which can be viewed as degenerated paths). If $h(\xi) = t(\eta)$, we use $\xi \circ \eta$ to denote the path obtained by concatenating ξ and η subsequently.

A graph G in $N(c_1, c_2)$ (or $N^*(c_1, c_2)$) is said to be *minimal* if for any $e \in E$, $G \setminus \{e\} \notin N(c_1, c_2)$ (or $G \setminus \{e\} \notin N^*(c_1, c_2)$). It is clear that in order to compute $\mathcal{M}(c_1, c_2)$ (or $\mathcal{M}^*(c_1, c_2)$), it is enough to consider all the minimal graphs with distinct (or identical) sources. Since every graph in $N(c_1, c_2)$ has at least one minimal subgraph in $N(c_1, c_2)$, it follows that

$$\mathcal{M}(c_1, c_2) = \sup_{\substack{G \in N(c_1, c_2) \\ G \text{ is minimal}}} M(G),$$

and similarly

$$\mathcal{M}^*(c_1, c_2) = \sup_{\substack{G \in N^*(c_1, c_2) \\ G \text{ is minimal}}} M(G).$$

A graph G in $N(c_1, c_2)$ (or $N^*(c_1, c_2)$) is said to be a (c_1, c_2)-*graph* if there exists a set α_i of c_i edge-disjoint paths from S_i to R_i, $i = 1, 2$, such that every edge in G belongs to some $\alpha_{i,j}$. For a (c_1, c_2)-graph G, we say α_i is *reroutable* if there exists a different set of Menger's paths α_i' from S_i to R_i, and we say G is *reroutable* (or alternatively, there is a *rerouting* in G), if some α_i, $i = 1, 2$, is reroutable. Note that for a non-reroutable G, the choice of α_i's is unique, so we often write $||G||(\alpha_1, \alpha_2)$ as $||G||$ for notational simplicity.

It follows from the following theorem that to compute $\mathcal{M}(c_1, c_2)$ (or $\mathcal{M}^*(c_1, c_2)$), it is enough to consider all non-reroutable (c_1, c_2)-graphs with distinct (or identical) sources.

Theorem 3.1. *A graph G in $N(c_1, c_2)$ (or $N^*(c_1, c_2)$) is minimal if and only if it is a non-reroutable (c_1, c_2)-graph.*

Proof. Here we prove its equivalent statement: G is not minimal if and only if G is reroutable or G is not a (c_1, c_2)-graph.

Necessity: If G is not minimal, there exists $e \in G$ such that $G \setminus \{e\} \in N(c_1, c_2)$ (or $N^*(c_1, c_2)$). If e belongs to some α_1-path or α_2-path, there exists a different set of Menger's paths α_i' from S_i to R_i (or from S to R_i) for some $i \in \{1, 2\}$ in G, implying G is reroutable. If e does not belong to any α_1-path or α_2-path, G is not a (c_1, c_2)-graph by definition.

Sufficiency: If G is not a (c_1, c_2)-graph, then there exists $e \in G$ that does not belong to any α_1-path or α_2-path. So $G \setminus \{e\}$ in $N(c_1, c_2)$ (or $N^*(c_1, c_2)$), and hence G is not minimal. If G is

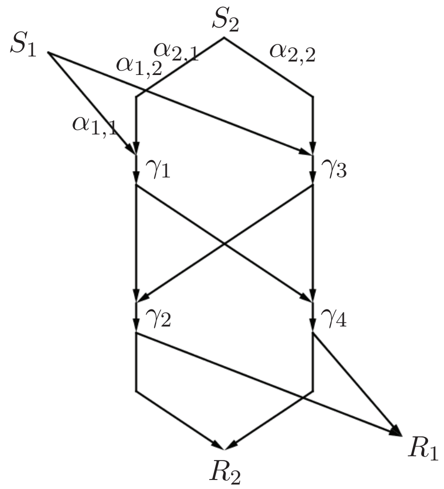

Figure 3: An example of a reroutable graph

a reroutable (c_1, c_2)-graph, by symmetry, assume that there exists another set of Menger's paths $\alpha'_1 = \{\alpha'_{1,1}, \alpha'_{1,2}, \ldots, \alpha'_{1,c_1}\}$ from S_1 to R_1 (or from S to R_1) with $\alpha'_{1,i}$ sharing the same outgoing edge to R_1 as $\alpha_{1,i}$, for $1 \leq i \leq c_1$. Pick an α'-path, say, α'_{1,i_1}, such that $\alpha'_{1,i_1} \neq \alpha_{1,i_1}$, and let v_{i_1} denote the largest vertex on α_{1,i_1} where they leave each other. Assume that, after v_{i_1}, α'_{1,i_1} first meets some α-path, say, α_{1,i_2}, at the vertex u_{i_1}. Denote by v_{i_2} the largest vertex where α'_{1,i_2} and α_{1,i_2} leave each other. Assume that, after v_{i_2}, α'_{1,i_2} first meets some α-path, say, α_{1,i_3} at the vertex u_{i_2}. Continue the procedure in a similar manner to obtain an index sequence $i_1, i_2, \ldots, i_t, \ldots$, and similarly define v_{i_t}'s and u_{i_t}'s. Pick the smallest k such that $i_k = i_j$ for some $j < k$. Notice that each edge of $\alpha'_{1,i_t}[v_{i_t}, u_{i_t}]$, $j \leq t \leq k-1$, belongs to an α_2-path. So, for $j \leq t \leq k-1$, the smallest edge e_{t+1} on $\alpha_{1,i_{t+1}}[v_{i_{t+1}}, u_{i_t}]$ does not belong to any α'_1-path or α_2-path. Thus $G \backslash \{e_{j+1}, e_{j+2}, \ldots, e_k\} \in N(c_1, c_2)$ (or $N^*(c_1, c_2)$) and thereby G is not minimal. □

Now, for a fixed i, reverse the directions of edges that do not belong to any α_i-path to obtain a new graph G'. For any two mergings λ, μ, if there exists a directed path in G' from the head (or tail) of λ to the head (or tail) of μ, we say the head (or tail) of λ *semi-reaches* [6] the head (or tail) of μ along α_i, or simply, λ *semi-reaches* μ along α_i from head (or tail) to head (or tail) [1]. It is easy to check that G is reroutable if and only if there exists i and a merging λ such that λ semi-reaches itself along α_i from head to head, which is equivalent to the condition that there exists i' and a merging λ' such that λ' semi-reaches itself along $\alpha_{i'}$ from tail to tail.

Example 3.2. For the graph depicted in Figure 3, the source S_1 is connected to the sink R_1 by a group of Menger's paths

$$\begin{aligned}\alpha_1 =& \{\alpha_{1,1}, \alpha_{1,2}\} = \{S_1 \to t(\gamma_1) \to h(\gamma_1) \to t(\gamma_4) \to h(\gamma_4) \to R_1,\\ & S_1 \to t(\gamma_3) \to h(\gamma_3) \to t(\gamma_2) \to h(\gamma_2) \to R_1\}\end{aligned}$$

and the source S_2 is connected to the sink R_2 by a group of Menger's paths

$$\begin{aligned}\alpha_2 =& \{\alpha_{2,1}, \alpha_{2,2}\} = \{S_2 \to t(\gamma_1) \to h(\gamma_1) \to t(\gamma_2) \to h(\gamma_2) \to R_2,\\ & S_2 \to t(\gamma_3) \to h(\gamma_3) \to t(\gamma_4) \to h(\gamma_4) \to R_2\}.\end{aligned}$$

[1]Roughly, λ semi-reaches μ along α_1 means that in order to get to μ from λ, one has to traverse along the orientation of α_1 and against that of α_2

Then $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are mergings by α_1-paths and α_2-paths. The mergings γ_1, γ_3 are smaller than γ_2 and γ_4.

The group of Menger's paths α_1 is reroutable, since there exists another group of Menger's paths

$$\begin{aligned}\alpha_1' =& \{\alpha_{1,1}', \alpha_{1,2}'\} = \{S_1 \to h(\gamma_1) \to t(\gamma_1) \to h(\gamma_2) \to t(\gamma_2) \to R_1,\\ & S_1 \to h(\gamma_3) \to t(\gamma_3) \to h(\gamma_4) \to t(\gamma_4) \to R_1\}\end{aligned}$$

from S_1 to R_1. Similarly, α_2 is also reroutable. Hence, G is reroutable. It is easy to check, by definition, that γ_2 semi-reaches γ_4 along α_1 from tail to tail, γ_1 semi-reaches γ_4 along α_1 from head to tail, γ_1 semi-reaches itself along α_1 from head to head, and γ_4 semi-reaches itself along α_2 from tail to tail.

3.2 Merging sequences

Consider a (c_1, c_2)-graph G with sources S_1, S_2, sinks R_1, R_2, a set of c_i disjoint paths $\alpha_i = \{\alpha_{i,1}, \alpha_{i,1}, \ldots, \alpha_{i,c_i}\}$ from S_i to R_i, for $i = 1, 2$. Assume that there are ω mergings in G. It follows from the acyclicity of G that all the mergings in G can be ordered from upstream to downstream, or more precisely, they can be listed as a sequence $(M_1, M_2, \ldots, M_\omega)$ such that as long as M_i and M_j, $i < j$, are comparable (which means there is a directed path between M_i and M_j), then M_i is smaller than M_j. For $i = 1, 2, \ldots, \omega$, suppose that M_i belongs to α_{1,k_i} and $\alpha_{2,\tilde{k}_i}$. The sequence of index pairs $((k_1, \tilde{k}_1), (k_2, \tilde{k}_2), \ldots, (k_\omega, \tilde{k}_\omega))$ is called a *merging sequence* of G. There might be multiple merging sequences associated with the same (c_1, c_2)-graph.

The consideration of merging sequences is motivated by the fact that they can be used to exhaustively "generate" all (c_1, c_2)-graphs (up to some graph isomorphism). Intuitively, consider the following procedure to "draw" a (c_1, c_2)-graph based on the sequence $((k_1, \tilde{k}_1), (k_2, \tilde{k}_2), \ldots, (k_\omega, \tilde{k}_\omega))$: For "fixed" edge-disjoint paths $\alpha_{2,1}, \alpha_{2,2}, \ldots, \alpha_{2,c_2}$ from S_2 to R_2, we draw edge-disjoint paths $\alpha_{1,1}, \alpha_{1,2}, \ldots, \alpha_{1,c_1}$ from S_1 to merge with α_2-paths until we reach R_1. More specifically, the procedure of drawing is done by extending α_1-paths edge by edge, and in the i-th step, we further extend path α_{1,k_i} to merge with path $\alpha_{2,\tilde{k}_i}$, while ensuring the new merging is larger than any existing mergings on path $\alpha_{2,\tilde{k}_i}$. Clearly, the drawing procedure, which is uniquely determined by the merging sequence, yields a (c_1, c_2)-graph.

Example 3.3. Consider the following two graphs in Figure 4 (**here and hereafter, all the mergings in this paper are represented by solid dots instead**). Listing the elements in the merging sequence, Figure 4(a) can be described by $((1, 2), (2, 1))$, or alternatively $((2, 1), (1, 2))$. And Figure 4(b) can be described by a merging sequence $((1, 1), (2, 1), (2, 2), (3, 2))$. Note that it cannot be described by $((1, 1), (2, 1), (3, 2), (2, 2))$, since $(3, 2)$ (or, more precisely, the merging corresponding to $(3, 2)$) is larger than $(2, 2)$ on $\alpha_{2,2}$.

3.3 Alternating sequences

Consider a non-reroutable (c_1, c_2)-graph G with two sources S_1, S_2, two distinct sinks R_1, R_2, a set of Menger's paths $\alpha_i = \{\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,c_i}\}$ from S_i to R_i for $i = 1, 2$. An edge in G is said to be *private* if it is not shared by any pair of α_1-path and α_2-path. An *alternating sequence* in G consists of a set of private edges such that when the orientation of G is ignored, all edges form a path whose intermediate vertices are from $V \setminus \{S_1, S_2, R_1, R_2\}$ and whose pair of terminal vertices is one of the following: (S_1, S_2), (S_1, R_1), (R_2, S_2) and (R_2, R_1). The *length* of an alternating sequence π, denoted by $L(\pi)$, is defined to be the number of intermediate vertices that are either heads or tails of a merging.

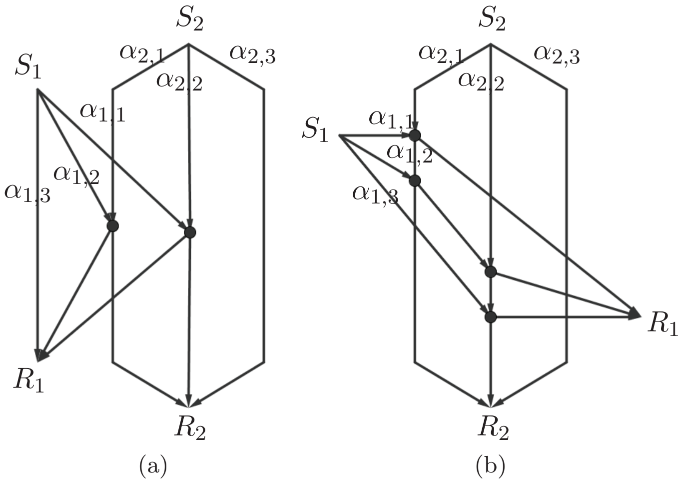

Figure 4: Two examples of merging sequences (as in Remark 3.3, all the mergings in this figure are represented by solid dots instead)

Alternatively, an alternating sequence can be generated through the following procedure on G, where, as an example, S_1 and S_2 are supposed to be distinct. For each i, starting from S_1, traverse along path $\alpha_{1,i}$ until we reach the tail of some merged subpath, we then traverse against the associated α_2-path (corresponding to the merged subpath just visited) until we reach the head of another merged subpath, we then traverse along the associated α_1-path, ... Continue this procedure of alternately traversing along α_1-paths or traversing against α_2-paths until we reach a merged subpath in the same manner as above, then the fact that G is non-reroutable and acyclic guarantees that eventually we will reach R_1 or S_2 (since otherwise, as in the proof of Theorem II.1 in [6], one can find some merging that semi-reaches itself from head to head and thus G is reroutable, a contradiction). Then, an S_1-alternating sequence is produced by sequentially listing all the edges visited during the procedure. Clearly, a similar procedure starting from R_2 will produce an R_2-alternating sequence.

Remark 3.4. For the case when S_1 and S_2 are distinct, to compute $\mathcal{M}(c_1, c_2)$, without loss of generality, we can assume that each Menger's path in G merges at least once, which implies that each alternating sequence is of positive length.

For the case when S_1 and S_2 are identical, by Proposition III.6 in [6], we can restrict our attention to the case when $c_1 = c_2$. For the purpose of computing $\mathcal{M}^*(c_2, c_2)$, without loss of generality, we assume that paths $\alpha_{1,i}$ and $\alpha_{2,i}$ share a *starting subpath* (a maximal shared subpath by $\alpha_{1,i}$ and $\alpha_{2,i}$ starting from the source) for $i = 1, 2, \ldots, c_2$ (since otherwise either some α_1-path or α_2-path would be reroutable, a contradiction to non-reroutability of G). Note that the existence of c_2 starting subpaths implies that any alternating sequence is of positive length and has terminal pair of vertices (R_1, R_2). The *length* of an alternating sequence π, denoted by $L(\pi)$, is defined to be the number of intermediate vertices that are terminal (heads or tails) of a merging or heads of a starting subpath.

It turns out that the lengths of alternating sequences are closely related to the number of mergings in G.

Proposition 3.5. *For a non-reroutable (c_1, c_2)-graph G with distinct sources,*

$$||G|| = \frac{1}{2} \sum_{\pi} L(\pi); \tag{1}$$

for a non-reroutable (c_2, c_2)-graph G with identical sources and c_2 starting subpaths,

$$||G|| = \frac{1}{2}\left(\sum_{\pi} L(\pi) - c_2\right), \tag{2}$$

where the two summations above are over the all the possible alternating sequences π's in G (Here the summations implicitly involve c_1 and/or c_2).

Proof. We first consider the case when G has distinct sources. For an alternating sequence π, let V_π denote the set of intermediate vertices, each of which is the head or tail of some merging in π. Then, (1) immediately follows from the observation that all V_π's are mutually exclusive and any merging is adjacent to some vertex in $\bigcup_\pi V_\pi$.

For the case when G has identical sources, (2) follows from the same observation as above and the fact that each of the c_2 starting paths is also adjacent to some vertex in $\bigcup_\pi V_\pi$. □

Example 3.6. Consider the two graphs in Figure 5. In Graph (a), sequentially listing the edges visited during the above-mentioned procedure, the two S_1-alternating sequences can be represented by $((S_1, t(\gamma_1)), (S_2, t(\gamma_1)))$ and

$$((S_1, t(\gamma_2)), (h(\gamma_1), t(\gamma_2)), (h(\gamma_1), t(\gamma_5)), (h(\gamma_4), t(\gamma_5)), (h(\gamma_4), R_1)).$$

Similarly, the two R_2-alternating sequences can be represented by $((h(\gamma_3), R_2), (h(\gamma_3), R_1))$ and

$$((h(\gamma_5), R_2), (h(\gamma_5), t(\gamma_3)), (h(\gamma_2), t(\gamma_3)), (h(\gamma_2), t(\gamma_4)), (S_2, t(\gamma_4))).$$

It is easy to see that that the number of mergings is 5, which is half of $(1 + 4 + 1 + 4)$, the sum of lengths of all alternating sequences.

In Graph (b), sequentially listing the edges visited during the above-mentioned procedure, the three R_2-alternating sequences can be represented by $((h(\gamma_2), R_2), (h(\gamma_2), R_1))$,

$$((h(\omega_1), R_2), (h(\omega_1), t(\gamma_1)), (h(\omega_2), t(\gamma_1)), (h(\omega_2), t(\gamma_4)), (h(\gamma_3), t(\gamma_4)), (h(\gamma_3), R_1))$$

and

$$((h(\gamma_4), R_2), (h(\gamma_4), t(\gamma_2)), (h(\gamma_1), t(\gamma_2)), (h(\gamma_1), t(\gamma_3)), (h(\omega_3), t(\gamma_3)), (h(\omega_3), R_1)).$$

Clearly, the number of mergings is 4, which is half of $(5 + 1 + 5 - 3)$.

Lemma 3.7. *The shortest S_1-alternating sequence (R_2-alternating sequence) is of length at most 1.*

Proof. Suppose, by contradiction, that the shortest S_1-alternating sequence is of length at least 2. Pick any α_1-path, say, α_{1,i_0}. Assume that α_{1,i_0} first merges with α_{2,j_0} at merging λ_{i_0,j_0}. Since the S_1-alternating sequence associated with α_{1,i_0} is of length at least 2, there exists an α_1-path, say, α_{1,i_1}, such that α_{1,i_1} has a merging, say, μ_{i_1,j_0}, smaller than λ_{i_0,j_0} on α_{2,j_0}. Now assume that α_{1,i_1} first merges with α_{2,j_1} at merging λ_{i_1,j_1}, then similarly there exists an α_1-path, say, α_{1,i_2}, such that α_{1,i_2} has a merging, say, μ_{i_2,j_1}, smaller than λ_{i_1,j_1} on α_{2,j_1}. Continue this procedure in the similar manner to obtain $\alpha_{2,j_2}, \lambda_{i_2,j_2}, \alpha_{1,i_3}, \mu_{i_3,j_2}, \alpha_{2,j_3}, \lambda_{i_3,j_3}, \alpha_{1,i_4}, \mu_{i_4,j_3}, \ldots$ Choose the smallest l such that there exists $k < l$ such that $i_k = i_l$. Note that

$$\alpha_{1,i_k}[t(\lambda_{i_k,j_k}), t(\mu_{i_l,j_{l-1}})] \circ \alpha_{2,j_{l-1}}[t(\mu_{i_l,j_{l-1}}), t(\lambda_{i_{l-1},j_{l-1}})] \circ \alpha_{1,i_{l-1}}[t(\lambda_{i_{l-1},j_{l-1}}), t(\mu_{i_{l-1},j_{l-2}})]$$
$$\circ\, \alpha_{2,j_{l-2}}[t(\mu_{i_{l-1},j_{l-2}}), t(\lambda_{i_{l-2},j_{l-2}})] \circ \cdots \circ \alpha_{1,i_{k+1}}[t(\lambda_{i_{k+1},j_{k+1}}), t(\mu_{i_{k+1},j_k})] \circ \alpha_{2,j_k}[t(\mu_{i_{k+1},j_k}), t(\lambda_{i_k,j_k})]$$

constitutes a cycle, which contradicts the assumption that G is acyclic.

A parallel argument can be applied to the shortest R_2-alternating sequence.

□

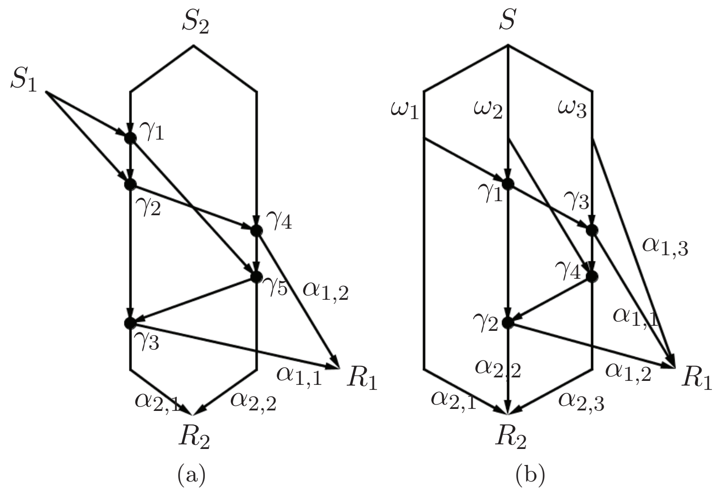

Figure 5: Two examples of alternating sequences

Lemma 3.8. *For a non-reroutable graph G, any path pair occurs at most once in any given alternating sequence.*

Proof. By way of contradiction, suppose that (i, j) occurs in an alternating sequence twice. Let u_1, u_2 be the two vertices where $\alpha_{1,i}$ and $\alpha_{2,j}$ meets. Then, as in the proof of Lemma II.4 in [6], one can prove that u_1 semi-reaches itself from head to head through α_1. So G is reroutable, which is a contradiction. $\square$

Remark 3.9. It then immediately follows from Lemma 3.8 that in a non-reroutable (c_1, c_2)-graph with distinct sources,

- the longest S_1-alternating sequence (R_2-alternating sequence) is of length at most $c_1 c_2$;
- any α_1-path (α_2-path) merges at most $c_1 c_2$ times.

And in a non-reroutable (c_2, c_2)-graph with identical sources,

- the longest R_2-alternating sequence is of length at most c_2^2;
- any α_1-path (α_2-path) merges at most c_2^2 times.

4 Exact Values

In this section, we give exact values of $\mathcal{M}$ and $\mathcal{M}^*$ for certain special parameters.

Theorem 4.1.

$$\mathcal{M}(2, c_2) = 3c_2 - 1.$$

Proof. We first show that $\mathcal{M}(2, c_2) \geq 3c_2 - 1$. Consider the following $(2, c_2)$-graph specified by the merging sequence $(\Omega_k)_{k=1}^{3c_2-1}$ (for a simple example, see Figure 6(a)), where

$$\Omega_k = \begin{cases} ([i]_2, 1) & \text{if } k = 3i - 2 & \text{for } 1 \leq i \leq c_2, \\ ([i]_2, i+1) & \text{if } k = 3i - 1 & \text{for } 1 \leq i \leq c_2 - 1, \\ ([i+1]_2, i+1) & \text{if } k = 3i & \text{for } 1 \leq i \leq c_2 - 1, \\ ([n+1]_2, 1) & \text{if } k = 3c_2 - 1. & \end{cases}$$

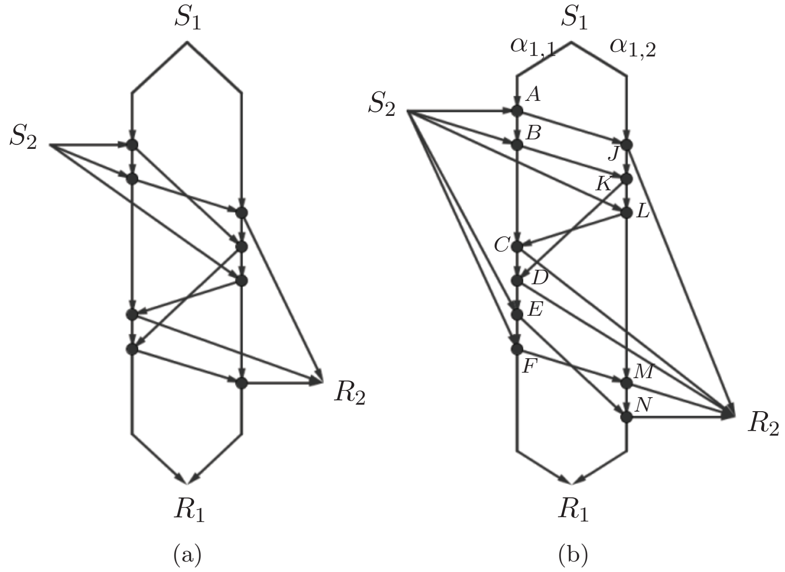

Figure 6: (a) A non-reroutable $(2,3)$-graph with 8 mergings (b) An example of a $(2,5)$-graph

where $[x]_2 = 1$ when x is odd, $[x]_2 = 2$ when x is even. The above graph is non-reroutable with $3c_2 - 1$ mergings, which implies that $\mathcal{M}(2, c_2) \geq 3c_2 - 1$.

Next, we show that $\mathcal{M}(2, c_2) \leq 3c_2 - 1$. Consider a non-reroutable $(2, c_2)$-graph G with distinct sources S_1, S_2, sinks R_1, R_2, and a set of Menger's paths $\alpha_i = \{\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,c_i}\}$ from S_i to R_i, for $i = 1, 2$. Define

$$\Sigma = \{(\lambda, \mu) : \text{merging } \lambda \text{ is smaller than merging } \mu \text{ on some } \alpha_2\text{-path} \\ \text{and there is no other merging between them on this path}\}.$$

Note that for any $(\lambda, \mu) \in \Sigma$, λ, μ must belong to different α_1-paths. We say $(\lambda, \mu) \in \Sigma$ is of *type I*, if λ belongs to $\alpha_{1,1}$, and $(\lambda, \mu) \in \Sigma$ is of *type II*, if λ belongs to $\alpha_{1,2}$. For any two different elements $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \Sigma$. We say $(\lambda_1, \mu_1) \prec (\lambda_2, \mu_2)$ if either (they are of the same type and λ_1 is smaller than λ_2) or (they are of different types and λ_1 is smaller than μ_2). Clearly, the relationship defined by $\prec$ is a strict total order.

Letting x denote the number of elements in Σ, we define

$$\Theta = (\Theta_1, \Theta_2, \ldots, \Theta_x)$$

to be the sequence of the ordered (by $\prec$) elements in Σ. Now we consecutively partition Θ into z "*medium-blocks*" $B_1, B_2, \ldots, B_z$, and further consecutively partition each B_i into y_i "*mini-blocks*" $B_{i,1}, B_{i,2}, \ldots, B_{i,y_i}$ (see Example 4.3 for an example) such that

- for any i, j, the elements in $B_{i,j}$ are of the same type.
- for any i, j, $B_{i,j}$ is *linked* to $B_{i,j+1}$ in the following sense: let $(\lambda_{i,j}, \mu_{i,j})$ denote the element with the largest second component in $B_{i,j}$ and let $(\lambda_{i,j+1}, \mu_{i,j+1})$ denote the element with the smallest first component in $B_{i,j+1}$, then $\mu_{i,j} = \lambda_{i,j+1}$.
- for any i, B_{i,y_i} is not linked to $B_{i+1,1}$.

A mini-block is said to be a *singleton* if it has only one element. We then have the following lemma.

Lemma 4.2. *Between any two "adjacent" singletons (meaning there is no other singleton between these two singletons) in a medium-block, there must exist a mini-block containing at least three elements.*

Proof. Suppose, by way of contradiction, that there exist two adjacent singletons Θ_i, Θ_j with $i < j$ in a medium-block such that each of $\Theta_{i+1}, \Theta_{i+2}, \ldots, \Theta_{j-1}$ contains exactly two elements. Let $\Theta_j = (\lambda, \mu)$. Then it is easy to see that μ semi-reaches itself along α_2 from head to head, which implies G is reroutable, a contradiction. □

Letting y denote the number of mini-blocks in Θ and x_i denote the number of elements in medium-block B_i for $1 \le i \le z$, we then have

$$x = x_1 + x_2 + \cdots + x_z,$$
$$y = y_1 + y_2 + \cdots + y_z.$$

Suppose there are k singletons in Θ, then by Lemma 4.2, we can find $(k-1)$ mini-blocks, each of which has at least three elements. And any other mini-block has at least two elements. Hence, for $1 \le i \le z$,

$$x_i \ge 1 \cdot k + 3 \cdot (k-1) + 2 \cdot [y_i - k - (k-1)] = 2y_i - 1, \tag{3}$$

which implies

$$x = \sum_{i=1}^{z} x_i \ge \sum_{i=1}^{z} (2y_i - 1) = 2y - z. \tag{4}$$

For any two linked mini-blocks $B_{i,j}$ and $B_{i,j+1}$, let $(\lambda_{i,j}, \mu_{i,j})$ denote the element with the largest second component in $B_{i,j}$, and let $(\lambda_{i,j+1}, \mu_{i,j+1})$ denote the element with the smallest first component in $B_{i,j+1}$. By the definition (of two mini-blocks being linked), we have $\mu_{i,j} = \lambda_{i,j+1}$, which means $B_{i,j}$ and $B_{i,j+1}$ share a common merging. Together with the fact that each element in Σ is a pair of mergings, this further implies that the number of mergings in G is

$$||G|| = 2x - (y - z). \tag{5}$$

Notice that $\lambda_{i,j}, \lambda_{i,j+1}, \mu_{i,j+1}$ belong to the same α_2-path, and furthermore, there exists only one α_1-path passing by both an element (more precisely, passing by both its mergings) in $B_{i,j}$ and an element in $B_{i,j+1}$. So, c_2, the number of α_2-paths in G, can be computed as

$$c_2 = x - (y - z). \tag{6}$$

It then follows from (4), (5), (6) and the fact $z \ge 1$ that

$$c_2 = x - y + z \ge (2y - z) - y + z = y, \tag{7}$$

and furthermore

$$||G|| = 2x - y + z = 2c_2 + y - z \le 2c_2 + c_2 - 1 = 3c_2 - 1, \tag{8}$$

which establishes the theorem. □

Example 4.3. Consider the graph in Figure 6(b) and assume the context is as in the proof of Theorem 4.1. Then we have,

$$\Sigma = \{(A, J), (B, K), (L, C), (K, D), (F, M), (E, N)\}.$$

Among all the elements in Σ, (A, J), (B, K), (F, M) and (E, N) are of type I, and (L, C), (K, D) are of type II. It is easy to see that

$$\Theta = ((A, J), (B, K), (K, D), (L, C), (E, N), (F, M)),$$

which is partitioned into three mini-blocks $((A, J), (B, K))$, $((K, D), (L, C))$ and $((E, N), (F, M))$. The first mini-block is linked to the second one, but the second one is not linked to the third, so Θ is partitioned into two medium-blocks

$$((A, J), (B, K), (K, D), (L, C)) \text{ and } ((E, N), (F, M)).$$

Remark 4.4. The proof of Theorem 4.1 reveals in greater depth the topological structure of non-reroutable $(2, c_2)$-graphs achieving $3c_2 - 1$ mergings, and further helps to determine the number of such graphs.

Assume a non-reroutable $(2, c_2)$-graph G has $3c_2 - 1$ mergings. Then, in the proof of Theorem 4.1, equalities hold for (8). It then follows that

- $z = 1$, namely, there is only one medium-block in Θ;
- equalities hold necessarily for (7), (4) and eventually (3), which further implies that between two adjacent singletons, only one mini-block has three elements and any other mini-block has two elements.

Furthermore, it can be verified that

- for a mini-block with two elements $((\lambda_1, \mu_1), (\lambda_2, \mu_2))$, μ_2 is smaller than μ_1;
- for a mini-block with three elements $((\lambda_1, \mu_1), (\lambda_2, \mu_2), (\lambda_3, \mu_3))$, either ($\mu_2$ is smaller than μ_3 and μ_3 is smaller than μ_1) or (μ_3 is smaller than μ_1 and μ_1 is smaller than μ_2).

Assume that G is "reduced" in the sense that, other than S_1, S_2, R_1, R_2, each vertex in G is a head or tail of some merging. The properties above allow us to count how many reduced non-reroutable $(2, c_2)$-graphs (up to graph isomorphism) can achieve $3c_2 - 1$ mergings: suppose that there are k ($1 \leq k \leq \lfloor \frac{c_2+1}{2} \rfloor$) singletons in G, then necessarily, there are $(k - 1)$ three-element mini-blocks and $(c_2 - 2k + 1)$ two-element mini-blocks in Θ. Obviously, the number of ways for these c_2 mini-blocks to form Θ for some $(2, c_2)$-graph is $\binom{c_2}{2k-1} 2^{k-1}$. This implies that the number of $(2, c_2)$-graph, whose Θ consists of k singletons, $(k-1)$ three element mini-blocks and (c_2-2k+1) two element mini-blocks, is $\binom{c_2}{2k-1} 2^{k-1}$. Through a computation summing over all feasible k, the number of reduced non-reroutable $(2, c_2)$-graphs with $3c_2 - 1$ mergings can be computed as

$$\sum_{k=1}^{\lfloor \frac{c_2+1}{2} \rfloor} \binom{c_2}{2k-1} 2^{k-1} = \frac{(1 + \sqrt{2})^{c_2} - (1 - \sqrt{2})^{c_2}}{2\sqrt{2}} = \mathscr{P}_{c_2},$$

where $\mathscr{P}_{c_2}$ is the c_2-th Pell number [1].

Remark 4.5. The following table lists exact values of $\mathcal{M}(c_1, c_2)$ with some small parameters, obtained via an exhaustive graph searching aided by a personal computer. Note that these results confirm the proven facts that $\mathcal{M}(1, c_2) = c_2$ (see Example II.10 in [6]) $\mathcal{M}(2, c_2) = 3c_2 - 1$ (see Theorem 4.1), and $\mathcal{M}(3, 3) = 13$ (see Theorem 3.1 in [14]). However, the computational power required explodes drastically as parameters increase, and we are not able to fill in the blanks in the table.

$\mathbf{c_1, c_2}$	**1**	**2**	**3**	**4**	**5**	**6**
1	1	2	3	4	5	6
2	2	5	8	11	14	17
3	3	8	13	18	23	28
4	4	11	18	27		
5	5	14	23			
6	6	17	28			

And we also find the following results via computer searching: $\mathcal{M}^*(4, 4) = 9$ (which has also been proven in Theorem III.1 in [15]), $\mathcal{M}^*(5, 5) = 16$, $\mathcal{M}^*(6, 6) = 27$.

5 Bounds

5.1 Bounds on $\mathcal{M}^*(c_2, c_2)$

In this section, for any positive integer c_2, we will construct a non-reroutable (c_2, c_2)-graph $\mathcal{E}(c_2, c_2)$ with one source S, two sinks R_1, R_2, a set of Menger's paths $\alpha_i = \{\alpha_{i,0}, \alpha_{i,1}, \dots, \alpha_{i,c_2-1}\}$ from S to R_i for $i = 1, 2$, and $(c_2 - 1)^2$ mergings, thus giving a lower bound on $\mathcal{M}^*(c_2, c_2)$.

The graph $\mathcal{E}(c_2, c_2)$ can be described as follows: for each $0 \le i \le c_2 - 1$, paths $\alpha_{1,i}$ and $\alpha_{2,i}$ share a starting subpath ω_i. After ω_{c_2-1}, path α_{1,c_2-1} does not merge any more, directly "flowing" to R_1; after ω_0, path $\alpha_{2,0}$ does not merge any more, directly "flowing" to R_2. The rest of the graph can be determined how paths $\alpha_{1,0}, \alpha_{1,1}, \dots, \alpha_{1,c_2-2}$ merge with $\alpha_{2,1}, \alpha_{2,2}, \dots, \alpha_{2,c_2-1}$. In more detail, for a given c_2, we define

$$X = \{x_{i,j} = i(2c_2 - i - 2) + j : 0 \le i \le c_2 - 2, 1 \le j \le c_2 - i - 1\}$$

and

$$Y = \{y_{i,j} = i(2c_2 - i - 3) + (c_2 - 1) + j : 0 \le i \le c_2 - 3, 1 \le j \le c_2 - i - 2\}.$$

Clearly, $x_{i,j}$'s, $y_{i,j}$'s are distinct and

$$X \cup Y = \{1, 2, \dots, (c_2 - 1)^2\}.$$

Now we define a mapping $f : \{1, 2, \dots, (c_2 - 1)^2\} \mapsto \{(i, j) : 0 \le i, j \le c_2 - 1\}$ by

$$f(k) = \begin{cases} (i, j) & \text{if } k = x_{i,j}, \\ (c_2 - 1 - j, c_2 - 1 - i) & \text{if } k = y_{i,j}. \end{cases}$$

Then the merging sequence of the rest of the graph can be defined as $(f(k))_{k=1}^{(c_2-1)^2}$. For example, $\mathcal{E}(4, 4)$, as illustrated in Figure 7, is determined by the merging sequence

$$((0, 1), (0, 2), (0, 3), (2, 3), (1, 3), (1, 1), (1, 2), (2, 2), (2, 1)).$$

Now, we prove that

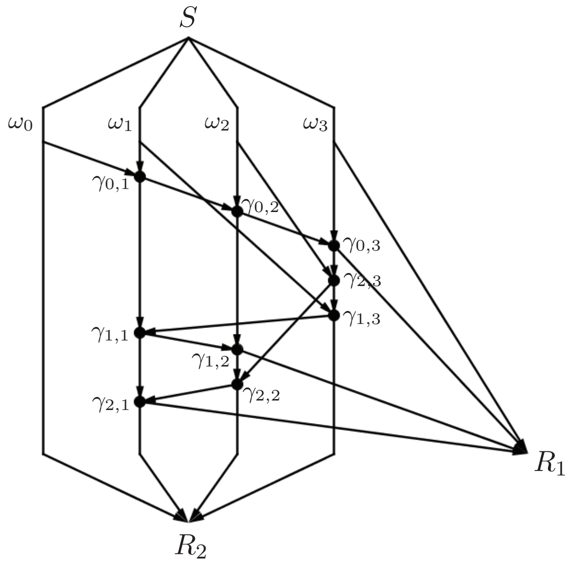

Figure 7: Graph $\mathcal{E}(4,4)$ with 9 mergings

Lemma 5.1. *$\mathcal{E}(c_2, c_2)$ is non-reroutable.*

Proof. Let $z = c_2 - 1$. For each $i, j = 0, 1, \ldots, z$, label each merging (i, j) in the merging sequence as $\gamma_{i,j}$ (Here, note that no two mergings share the same label).

We only prove that there is only one possible set of Menger's paths from S to R_1. The uniqueness of Menger's path sets from S to R_2 can be established using a parallel argument.

Let α_1 be an arbitrary yet fixed set of Menger's paths from S to R_1. It suffices to prove that α_1 is non-reroutable. Note that each path in α_1 must end with either $\omega_z \to R_1$ or $\gamma_{i,z-i} \to R_1$, $i = 0, 1, \ldots, z-1$ (here and hereafter, slightly abusing the notations "$\to$" and "$\leftarrow$", for paths (or vertices) $A_1, A_2, \ldots, A_k$, we use $A_1 \to A_2 \to \cdots \to A_k$ or $A_k \leftarrow \cdots \leftarrow A_2 \leftarrow A_1$ to denote the path which sequentially passes through $A_1, A_2, \ldots, A_k$; note that such an expression uniquely determines a path in this proof). In α_1, label the Menger's path ending with $\gamma_{i,z-i} \to R_1$ as the i-th Menger's path for $0 \le i \le z-1$, and the Menger's path ending with $\omega_z \to R_1$ as the z-th one.

It is obvious that in $\mathcal{E}(c_2, c_2)$, there is only one Menger's path ending at $\omega_z \to R_1$, which implies that the z-th Menger's path in α_1 is "fixed" (as $S \to \omega_z \to R_1$); or, more rigorously, for any set of Menger's paths α_1', the z-th Menger's path in α_1' is the same as the z-th one in α_1. So, for the purpose of choosing other Menger's paths, all the edges on $S \to \omega_z \to R_1$ are "occupied". It then follows that, in α_1, $\gamma_{0,z}$ must "come" from $\gamma_{0,z-1}$; more precisely, in α_1, $\gamma_{0,z-1}$ is smaller than $\gamma_{0,z}$ on the 0-th path and there is no other merging between them on this path. Now, all the edges on $\gamma_{0,z-1} \to \gamma_{0,z} \to R_1$ are occupied.

Inductively, only considering unoccupied edges, one can verify that for $0 \le i \le z-2$, $\gamma_{i,z-i}$ must come from $\gamma_{i,z-i-1}$; in other words, for $0 \le i \le z-2$, the i-th Menger's path must end with $\gamma_{i,z-i-1} \to \gamma_{i,z-i} \to R_1$. It then follows that the $(z-1)$-th Menger's path must come from $\gamma_{z-1,2} \leftarrow \gamma_{z-1,3} \leftarrow \cdots \leftarrow \gamma_{z-1,z} \leftarrow \omega_{z-1}$; so, the $(z-1)$-th Menger's path is fixed as $S \to \omega_{z-1} \to \gamma_{z-1,z} \to \gamma_{z-1,z-1} \to \cdots \to \gamma_{z-1,2} \to \gamma_{z-1,1} \to R_1$.

We now proceed by induction on j, $j = z-2, z-3, \ldots, 1$. Suppose that, for $j+1 \le i \le z$, the i-th Menger's path is already fixed (and hence the edges on these paths are all occupied), and for $0 \le i \le j$, the i-th Menger's path ends with $\gamma_{i,j-i+1} \to \gamma_{i,j-i+2} \to \cdots \to \gamma_{i,z-i} \to R_1$ (so, the edges on these paths are all occupied). Only considering the unoccupied edges, similarly, for $0 \le i \le j-1$, $\gamma_{i,j-i+1}$ must come from $\gamma_{i,j-i}$. It then follows that the j-th Menger's path, which ends with $\gamma_{j,1} \to \gamma_{j,2} \to \cdots \to \gamma_{j,z-j} \to R_1$, must come from $\gamma_{j,z-j+1} \leftarrow \gamma_{j,z-j+2} \leftarrow \cdots \leftarrow \gamma_{j,z} \leftarrow \omega_j$. So, the j-th Menger's path can now be fixed as $S \to \omega_j \to \gamma_{j,z} \to \gamma_{j,z-1} \to \cdots \to \gamma_{j,z-j+1} \to \gamma_{j,1} \to \gamma_{j,2} \to \cdots \to \gamma_{j,z-j} \to R_1$. Now, for $j \leq i \leq z$, the i-th Menger's path is fixed, and for $0 \leq i \leq j-1$, the i-th Menger's path must end with $\gamma_{i,j-i} \to \gamma_{i,j-i+1} \to \cdots \to \gamma_{i,z-i} \to R_1$.

It follows from the above inductive argument that for $1 \leq i \leq z$, the i-th Menger's path is fixed, and the 0-th Menger's path must end with $\gamma_{0,1} \to \gamma_{0,2} \to \cdots \to \gamma_{0,z} \to R_1$. It turns out that $\gamma_{0,1}$ must come from ω_0, which implies that the 0-th Menger's path is fixed as $S \to \omega_0 \to \gamma_{0,1} \to \gamma_{0,2} \to \cdots \to \gamma_{0,z} \to R_1$. The proof of uniqueness of Menger's path set from S to R_1 is then complete. $\square$

The above lemma then immediately implies a lower bound on $\mathcal{M}^*(c_2, c_2)$.

Theorem 5.2.

$$\mathcal{M}^*(c_2, c_2) \geq (c_2 - 1)^2.$$

The following theorem gives an upper bound on $\mathcal{M}^*(c_2, c_2)$. First, we remind the reader that, by Proposition III.6 in [6], $\mathcal{M}^*(c_1, c_2) = \mathcal{M}^*(c_2, c_2)$ for any $c_1 \geq c_2$.

Theorem 5.3.

$$\mathcal{M}^*(c_2, c_2) \leq \left\lceil \frac{c_2}{2} \right\rceil (c_2^2 - 4c_2 + 5).$$

Proof. Consider any (c_2, c_2)-graph G with one source S, sinks R_1, R_2, a set of Menger's paths $\alpha_i = \{\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,c_2}\}$ from S to R_i for $i = 1, 2$.

As discussed in Section 3.3, we assume that, for $1 \leq i \leq n$, paths $\alpha_{1,i}$ and $\alpha_{2,i}$ share a starting subpath, and paths α_{1,c_2} and $\alpha_{2,1}$ do not merge with any other paths, directly flowing to the sinks (then, necessarily, each R_2-alternating sequence is of positive length, and by Lemma 3.7, the shortest R_2-alternating sequence is of length 1). We say that the path pair $(\alpha_{1,i}, \alpha_{2,j})$ is *matched* if $i = j$, otherwise, *unmatched*. Clearly, each starting subpath corresponds to a matched path pair; and among the set of all path pairs, each of which corresponds some merging in G, there are at most $(c_2 - 2)$ matched and at most $(c_2^2 - 3c_2 + 3)$ unmatched.

We then consider the following two cases (note that they may not be mutually exclusive):

Case 1: there exists a shortest R_2-alternating sequence associated with a matched path pair. By Lemma 3.8 and the fact that each starting subpath corresponds to a matched path pair, there are at most $\lfloor \frac{c_2-1}{2} \rfloor$ mergings corresponding to this path pair, at most $\lfloor \frac{c_2-2}{2} \rfloor$ corresponding to any other matched path pair, and at most $\lfloor \frac{c_2-1}{2} \rfloor$ mergings corresponding to any unmatched. So, the number of mergings is upper bounded by

$$\left\lfloor \frac{c_2 - 1}{2} \right\rfloor + (c_2 - 3) \left\lfloor \frac{c_2 - 2}{2} \right\rfloor + (c_2^2 - 3c_2 + 3) \left\lfloor \frac{c_2 - 1}{2} \right\rfloor. \tag{9}$$

Case 2: there exists a shortest R_2-alternating sequence associated with an unmatched path pair. Again, by Lemma 3.8 and the fact that each starting subpath corresponds to a matched path pair, there are at most $\lfloor \frac{c_2}{2} \rfloor$ mergings corresponding to this path pair, at most $\lfloor \frac{c_2-1}{2} \rfloor$ mergings corresponding to any other unmatched path pair, and at most $\lfloor \frac{c_2-2}{2} \rfloor$ mergings corresponding to any matched. So, the number of mergings is upper bounded by

$$\left\lfloor \frac{c_2}{2} \right\rfloor + (c_2 - 2) \left\lfloor \frac{c_2 - 2}{2} \right\rfloor + (c_2^2 - 3c_2 + 2) \left\lfloor \frac{c_2 - 1}{2} \right\rfloor. \tag{10}$$

Then $\mathcal{M}^*(c_2, c_2) \leq \max\{(9), (10)\}$. For odd c_2, (9) is larger than (10), so we have

$$\begin{aligned} \mathcal{M}^*(c_2, c_2) &\leq \left(\frac{c_2 - 1}{2} \right) + (c_2 - 3) \left(\frac{c_2 - 3}{2} \right) + (c_2^2 - 3c_2 + 3) \left(\frac{c_2 - 1}{2} \right) \\ &= (c_2^2 - 4c_2 + 5) \left(\frac{c_2 + 1}{2} \right). \end{aligned}$$

For even c_2, (10) is larger than (9), so we have

$$\begin{aligned}\mathcal{M}^*(c_2, c_2) &\leq \binom{c_2}{2} + (c_2 - 2)\binom{c_2 - 2}{2} + (c_2^2 - 3c_2 + 2)\binom{c_2 - 2}{2} \\ &= (c_2^2 - 4c_2 + 5)\binom{c_2}{2}.\end{aligned}$$

The proof is then complete. □

5.2 Bounds on $\mathcal{M}(c_1, c_2)$

Consider the following (c_2, c_2)-graph $\mathcal{F}(c_2, c_2)$ with distinct sources S_1, S_2, distinct sinks R_1, R_2, a set of Menger's paths $\alpha_1 = \{\alpha_{1,1}, \alpha_{1,2}, \ldots, \alpha_{1,c_2}\}$ from S_1 to R_1, a set of Menger's paths $\alpha_2 = \{\alpha_{2,1}, \alpha_{2,2}, \ldots, \alpha_{2,c_2}\}$ from S_2 to R_2, and a merging sequence $(\Omega_k)_{k=1}^{2c_2^2-3c_2+2}$, where

$$\Omega_k = \begin{cases} ([j-i]_{c_2}, i+1) & \text{if } k = 2i(c_2-1)+j \\ & \text{for } (0 \leq i \leq c_2-1, 1 \leq j \leq c_2-1) \text{ or } (i = c_2-1, j = c_2), \\ (c_2 - i, [i-j+2]_{c_2}) & \text{if } k = (2i+1)(c_2-1)+j \quad \text{for } 0 \leq i \leq c_2-2, 1 \leq j \leq c_2-1, \end{cases}$$

where, for any integer x, $[x]_{c_2}$ denotes the least strictly positive residue of x modulo c_2. For a quick example, see $\mathcal{F}(3,3)$ in Figure 8(a), whose merging sequence is

$$((1,1),(2,1),(3,1),(3,3),(3,2),(1,2),(2,2),(2,1),(2,3),(3,3),(1,3)).$$

Then, as in the proof of Lemma 5.1, through verifying the uniqueness of the set of Menger's paths from S_i to R_i, we have

Lemma 5.4. $\mathcal{F}(c_2, c_2)$ *is non-reroutable.*

Consider a non-reroutable (k, c_2)-graph $\mathcal{G}(k, c_2)$ with distinct sources $\widehat{S}_1, \widehat{S}_2$, sinks $\widehat{R}_1, \widehat{R}_2$, a set of Menger's paths $\hat{\alpha}_1 = \{\hat{\alpha}_{1,1}, \hat{\alpha}_{1,2}, \ldots, \hat{\alpha}_{1,k}\}$ from $\widehat{S}_1$ to $\widehat{R}_1$, a set of Menger's paths $\hat{\alpha}_2 = \{\hat{\alpha}_{2,1}, \hat{\alpha}_{2,2}, \ldots, \hat{\alpha}_{2,c_2}\}$ from $\widehat{S}_2$ to $\widehat{R}_2$. For a fixed merging sequence of $\mathcal{G}(k, c_2)$, assume, without loss of generality, that the first element is $(1, c_2)$. Now, we consider the following procedure of *concatenating* graphs $\mathcal{F}(c_2, c_2)$ and $\mathcal{G}(k, c_2)$ to obtain a new graph:

1. split R_1 into c_2 copies $R_1^{(1)}, R_1^{(2)}, \ldots, R_1^{(c_2)}$ such that path $\alpha_{1,i}$ has the ending point $R_1^{(i)}$; split R_2 into c_2 copies $R_2^{(1)}, R_2^{(2)}, \ldots, R_2^{(c_2)}$ such that path $\alpha_{2,i}$ has the ending point $R_2^{(i)}$;

2. split $\widehat{S}_1$ into k copies $\widehat{S}_1^{(1)}, \widehat{S}_1^{(2)}, \ldots, \widehat{S}_1^{(k)}$ such that path $\hat{\alpha}_{1,i}$ has the starting point $\widehat{S}_1^{(i)}$; split $\widehat{S}_2$ into c_2 copies $\widehat{S}_2^{(1)}, \widehat{S}_2^{(2)}, \ldots, \widehat{S}_2^{(c_2)}$ such that path $\hat{\alpha}_{2,i}$ has the starting point $\widehat{S}_2^{(i)}$;

3. delete all edges on $\alpha_{1,1}$ and all edges on α_{2,c_2}, each of which is larger than merging $(\alpha_{1,1}, \alpha_{2,c_2})$ to obtain new $\alpha_{1,1}$ and α_{2,c_2};

4. delete all edges on $\hat{\alpha}_{1,1}$ and all edges on $\hat{\alpha}_{2,c_2}$, each of which is smaller than merging $(\hat{\alpha}_{1,1}, \hat{\alpha}_{2,c_2})$ to obtain new $\hat{\alpha}_{1,1}$ and $\hat{\alpha}_{2,c_2}$;

5. concatenate $\alpha_{1,1}$ and $\hat{\alpha_{1,1}}$ to obtain $\alpha_{1,1} \circ \hat{\alpha}_{1,1}$ (so, necessarily, α_{2,c_2} and $\hat{\alpha}_{2,c_2}$ are concatenated simultaneously and we obtain $\alpha_{2,c_2} \circ \hat{\alpha}_{2,c_2}$);

6. identify $S_1, \widehat{S}_1^{(2)}, \widehat{S}_1^{(3)}, \ldots, \widehat{S}_1^{(k)}$; identify $\widehat{R}_1, R_1^{(2)}, R_1^{(3)}, \ldots, R_1^{(k)}$; identify $R_2^{(i)}$ and $\widehat{S}_2^{(i)}$ for $1 \leq i \leq n-1$.

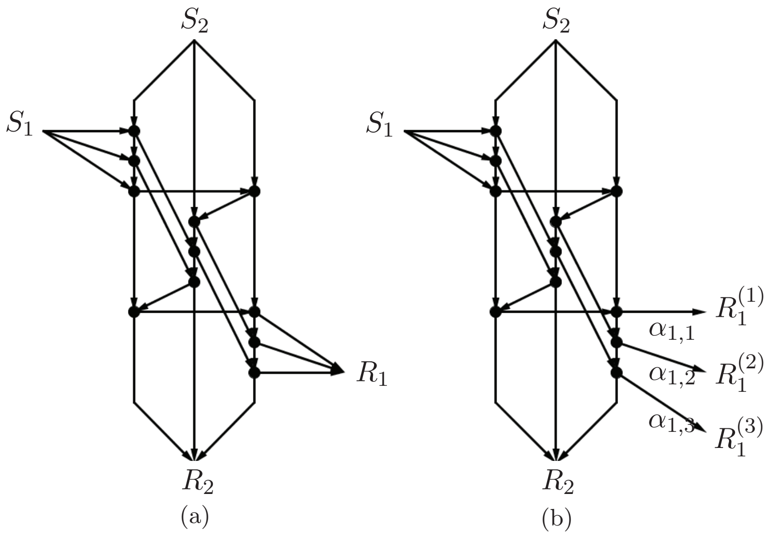

Figure 8: (a) Graph $\mathcal{F}(3,3)$ with 11 mergings (b) Splitting of R_1 in $\mathcal{F}(3,3)$

Obviously, such procedure produces a $(k+c_2-1, c_2)$-graph with distinct sources S_1, S_2 and distinct sinks $\widehat{R}_1$ and $\widehat{R}_2$, a set of $(k+c_2-1)$ Menger's paths $\{\alpha_{1,1}\circ\hat{\alpha}_{1,1}, \alpha_{1,2}, \alpha_{1,3}, \ldots, \alpha_{1,c_2}, \hat{\alpha}_{1,2}, \hat{\alpha}_{1,3}, \ldots, \hat{\alpha}_{1,k}\}$ from S_1 to $\widehat{R}_1$ and a set of c_2 Menger's paths $\{\alpha_{2,1}\circ\hat{\alpha}_{2,1}, \alpha_{2,2}\circ\hat{\alpha}_{2,2}, \ldots, \alpha_{2,c_2}\circ\hat{\alpha}_{2,c_2}\}$ from S_2 to $\widehat{R}_2$. For example, in Figure 9, we concatenate $\mathcal{F}(2,2)$ and a non-reroutable $(2,2)$-graph to obtain a $(3,2)$-graph.

We then have the following lemma, whose proof is similar to Lemma 5.1 and thus omitted.

Lemma 5.5. *The concatenated graph as above is a non-reroutable $(k+c_2-1, c_2)$-graph with the number of mergings equal to $||\mathcal{F}(c_2,c_2)|| + ||\mathcal{G}(k,c_2)|| - 1$.*

We are now ready for the following theorem, which gives us a lower bound on $\mathcal{M}(c_1,c_2)$.

Theorem 5.6.

$$\mathcal{M}(c_1,c_2) \geq 2c_1c_2 - c_1 - c_2 + 1.$$

Proof. Without loss of generality, assume that $c_1 \leq c_2$. For $1 \leq c_1' \leq c_1$ and $1 \leq c_2' \leq c_2$, we will iteratively construct a sequence of non-reroutable (c_1', c_2')-graphs with $2c_1'c_2' - c_1' - c_2' + 1$ mergings, which immediately implies the theorem.

First, for any k, $\mathcal{H}(1,k)$, a non-reroutable $(1,k)$-graph can be given by specifying its merging sequence

$$((1,1),(1,2),\ldots,(1,k)).$$

Next, consider the case $2 \leq c_1 \leq c_2$. Assume that for any c_1', c_2' such that $c_1' \leq c_2'$, $c_1' \leq c_1$, $c_2' \leq c_2$, however $(c_1', c_2') \neq (c_1, c_2)$, we have constructed a non-reroutable (c_1', c_2')-graph, which is effectively a non-reroutable (c_2', c_1')-graph as well. We obtain a new (c_1, c_2)-graph through the following procedure:

1. if $c_1 = c_2$, concatenate $\mathcal{F}(c_1, c_1)$ and an already constructed non-reroutable $(1, c_1)$-graph $\mathcal{H}(1, c_1)$;

2. if $c_1 < c_2$, concatenate $\mathcal{F}(c_1, c_1)$ and an already constructed non-reroutable $(c_2 - c_1 + 1, c_1)$-graph.

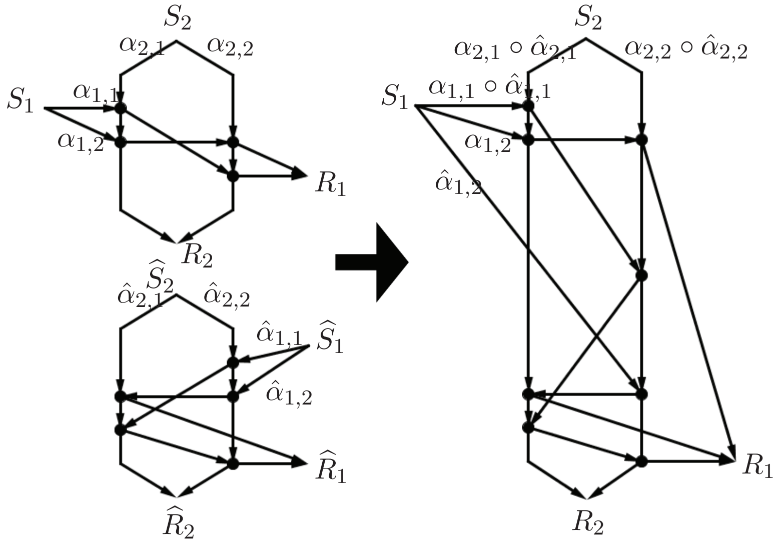

Figure 9: Concatenation of $\mathcal{F}(2,2)$ and a non-reroutable $(2,2)$-graph

For the first case, according to Lemma 5.5, the obtained graph is a non-reroutable (c_1, c_1)-graph with the number of mergings

$$(2c_1^2 - 3c_1 + 2) + c_1 - 1 = 2c_1^2 - 2c_1 + 1.$$

Similarly, for the second case, the obtained graph is a non-reroutable (c_1, c_2)-graph with the number of mergings

$$(2c_1^2 - 3c_1 + 2) + [2(c_2 - c_1 + 1)c_1 - (c_2 - c_1 + 1) - c_1 + 1] - 1 = 2c_1c_2 - c_1 - c_2 + 1.$$

We then have established the theorem. □

Example 5.7. To construct a non-reroutable $(4,6)$-graph with 39 mergings, one can concatenate $\mathcal{F}(4,4)$ and a non-reroutable $(3,4)$-graph, which can be obtained by concatenating $\mathcal{F}(3,3)$ and a non-reroutable $(2,3)$-graph. The latter can be obtained by concatenating $\mathcal{F}(2,2)$ and a non-reroutable $(2,2)$-graph. Finally, a non-reroutable $(2,2)$-graph can be obtained by concatenating $\mathcal{F}(2,2)$ and $\mathcal{H}(1,2)$. One readily verifies that the number of mergings in the eventually obtained graph is

$$||\mathcal{F}(4,4)|| + ||\mathcal{F}(3,3)|| + ||\mathcal{F}(2,2)|| + ||\mathcal{F}(2,2)|| + ||\mathcal{H}(1,2)|| - 4 = 22 + 11 + 4 + 4 + 2 - 4 = 39.$$

Theorem 5.8.

$$\mathcal{M}(c_1, c_2) \le (c_1 + c_2 - 1) + (c_1c_2 - 2)\left\lfloor \frac{c_1 + c_2 - 2}{2} \right\rfloor.$$

Proof. Consider any (c_1, c_2)-graph G with distinct sources S_1, S_2, distinct sinks R_1, R_2, and a set of Menger's paths $\alpha_i = \{\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,c_i}\}$ from S_i to R_i for $i = 1, 2$. As discussed in Section 3.3, we assume that all the alternating sequences are of positive lengths. By Lemma 3.7, the shortest S_1-alternating sequence and R_2-alternating sequence are both of length 1. We then consider the following two cases (note that they may not be mutually exclusive):

Case 1: there exists a shortest S_1-alternating sequence and a shortest R_2-alternating sequence, which are associated with the same path pair. By Lemma 3.8, there are at most $\lfloor \frac{c_1+c_2}{2} \rfloor$ mergings corresponding to this path pair, and at most $\lfloor \frac{c_1+c_2-2}{2} \rfloor$ mergings corresponding to any other path pair. So, the number of mergings is upper bounded by

$$\left\lfloor \frac{c_1+c_2}{2} \right\rfloor + (c_1c_2-1)\left\lfloor \frac{c_1+c_2-2}{2} \right\rfloor. \tag{11}$$

Case 2: there exists a shortest S_1-alternating sequence and a shortest R_2-alternating sequence, which are associated with two distinct path pairs. Again, by Lemma 3.8, there are at most $\lfloor \frac{c_1+c_2-1}{2} \rfloor$ mergings corresponding to each of these two path pairs, and at most $\lfloor \frac{c_1+c_2-2}{2} \rfloor$ mergings corresponding to any other path pair. So, the number of mergings is upper bounded by

$$2\left\lfloor \frac{c_1+c_2-1}{2} \right\rfloor + (c_1c_2-2)\left\lfloor \frac{c_1+c_2-2}{2} \right\rfloor. \tag{12}$$

Then, $\mathcal{M}(c_1, c_2) \leq \max\{(11), (12)\}$. Straightforward computations then lead to the claimed result. □

Remark 5.9. It has been established in [8] that

$$c_2(c_2-1)/2 \leq \mathcal{M}^*(c_2, c_2) \leq c_2^3.$$

By summarizing all the four bounds, we obtain

$$(c_2-1)^2 \leq \mathcal{M}^*(c_2, c_2) \leq \left\lceil \frac{c_2}{2} \right\rceil (c_2^2 - 4c_2 + 5),$$
$$2c_1c_2 - c_1 - c_2 + 1 \leq \mathcal{M}(c_1, c_2) \leq (c_1+c_2-1) + (c_1c_2-2)\left\lfloor \frac{c_1+c_2-2}{2} \right\rfloor.$$

5.3 Bounds on $\mathcal{M}(3, c_2)$

In this section, we give the following scaling law for $\mathcal{M}$ when $c_1 = 3$.

Theorem 5.10.

$$\mathcal{M}(3, c_2) \leq 14c_2.$$

Proof. Consider any non-reroutable $(3, c_2)$-graph G with distinct sources S_1, S_2, sinks R_1, R_2, a set of Menger's paths $\alpha_1 = \{\alpha_{1,1}, \alpha_{1,2}, \alpha_{1,3}\}$ from S_1 to R_1 and a set of Menger's paths $\alpha_2 = \{\alpha_{2,1}, \alpha_{2,2}, \ldots, \alpha_{2,c_2}\}$ from S_2 to R_2. If a merging is the smallest (the largest) on an α_2-path, we say it is an *x-terminal* (*y-terminal*) merging on the α_2-path, or simply a α_2*-terminal* merging. For a set of vertices $v_1, v_2, \ldots, v_k$ in G, define $G\langle v_1, \ldots, v_k \rangle$ to be the subgraph of G induced on the set of vertices, each of which is smaller or equal to some v_i, $i = 1, 2, \ldots, k$.

Consider the following iterative procedure (Figures 10, 11 and 12 roughly illustrate the procedure), where, for notational simplicity, we treat a graph as the union of its vertex set and edge set. Initially set $\mathcal{S}^{(0)} = \emptyset$, and $\mathcal{R}^{(0)} = G$. Now for each $j = 1, 2, 3$, pick a merging $\gamma_{0,j}$ such that $\gamma_{0,j}$ belongs to path $\alpha_{1,j}$ and

$$||\mathcal{R}^{(0)}\langle t(\gamma_{0,1}), t(\gamma_{0,2}), t(\gamma_{0,3}) \rangle|| = 14,$$

where one can choose $\gamma_{0,j}$ to be S_1 if such merging does not exist on $\alpha_{1,j}$. Now set

$$\mathcal{L}_1 = \mathcal{R}^{(0)}\langle t(\gamma_{0,1}), t(\gamma_{0,2}), t(\gamma_{0,3}) \rangle,$$

and $\mathcal{S}^{(1)} = \mathcal{S}^{(0)} \cup \mathcal{L}_1$, $\mathcal{R}^{(1)} = G \setminus \mathcal{S}^{(1)}$. Suppose that we already obtain

$$\mathcal{L}_i = \mathcal{R}^{(i-1)}\langle t(\gamma_{i-1,1}), t(\gamma_{i-1,2}), t(\gamma_{i-1,3}) \rangle,$$

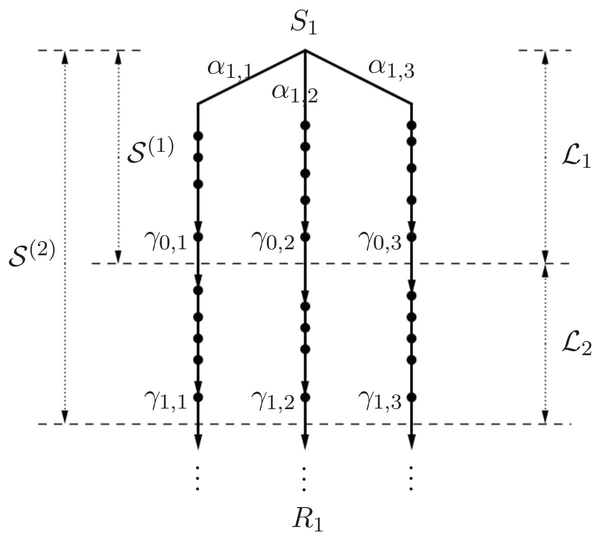

Figure 10: Partition a $(3, c_2)$-graph into blocks

and $\mathcal{S}^{(i)} = \mathcal{S}^{(i-1)} \cup \mathcal{L}_i$, $\mathcal{R}^{(i)} = G \setminus \mathcal{S}^{(i)}$, where $\mathcal{L}_i$ contains exactly 14 mergings and at least two α_2-terminal mergings. We then continue to pick merging $\gamma_{i,j}$ on $\alpha_{1,j}$ from $\mathcal{R}^{(i)}$ such that

$$||\mathcal{R}^{(i)}\langle t(\gamma_{i,1}), t(\gamma_{i,2}), t(\gamma_{i,3})\rangle|| = 14$$

and there are at least two α_2-terminal mergings in $\mathcal{R}^{(i)}\langle t(\gamma_{i,1}), t(\gamma_{i,2}), t(\gamma_{i,3})\rangle$. If such $\gamma_{i,j}$'s exist, set

$$\mathcal{L}_{i+1} = \mathcal{R}^{(i)}\langle t(\gamma_{i,1}), t(\gamma_{i,2}), t(\gamma_{i,3})\rangle,$$

and if $|\mathcal{R}^{(i)}| < 14$, set $\mathcal{L}_{i+1} = \mathcal{R}^{(i)}$ and terminate the iterative procedure. So far, for any obtained "block" $\mathcal{L}_{i+1}$, either we have $||\mathcal{L}_{i+1}|| < 14$ or ($||\mathcal{L}_{i+1}|| = 14$ and there are at least two α_2-terminal mergings in $\mathcal{L}_{i+1}$); such block $\mathcal{L}_{i+1}$ is said to be *normal*. If $|\mathcal{R}^{(i)}| \geq 14$, however, we cannot find a normal block, we continue the procedure and define a *singular* $\mathcal{L}_{i+1}$ in the following.

Note that $\mathcal{S}^{(i)} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \cdots \cup \mathcal{L}_i$. Let $z_i = \sum_{j=1}^{i}(x_j - y_j)$, where x_i and y_i denote the numbers of x-terminal and y-terminal mergings in the α_2-paths in $\mathcal{L}_i$, respectively; then z_i is the number of α_2-paths which can continue to merge within $\mathcal{R}^{(i)}$. If a normal block does not exist after i iterations, necessarily we will have $z_i \geq 3$ (suppose $z_i \leq 2$, by the fact that $\mathcal{M}(3,3) = 13$ (see Theorem 3.1 in [14]), we would be able to obtain a normal block $\mathcal{L}_{i+1}$, which contains two x-terminal mergings or (an x-terminal merging and a y-terminal merging)). We say that a merging is *critical* within a subgraph of G if the corresponding α_2-path, does not merge any more after this merging within this subgraph. It then follows that the number of the critical mergings within $\mathcal{S}^{(i)}$ is z_i.

Now, let $\mathcal{K}_i$ denote the set of all the mergings within $\mathcal{R}^{(i)}$, each of which can semi-reach the tail of some critical merging within $\mathcal{S}^{(i)}$ along ϕ. Note that at least one of those α_1-paths, each of which contains at least one critical merging within $\mathcal{S}^{(i)}$, does not contain any merging within $\mathcal{K}_i$. Without loss of generality, we assume that $\alpha_{1,3} \cap \mathcal{K}_i = \emptyset$. Now we consider the following two cases:

Case 1: $\alpha_{1,1} \cap \mathcal{K}_i \neq \emptyset$ and $\alpha_{1,2} \cap \mathcal{K}_i \neq \emptyset$. As shown in Figure 11, assume that within $\mathcal{K}_i$, $\lambda_{i,1}, \lambda_{i,2}$ are the largest mergings on $\alpha_{1,1}, \alpha_{1,2}$, respectively. Now, set

$$\mathcal{L}_{i+1} = \mathcal{R}^{(i)}\langle t(\lambda_{i,1}), t(\lambda_{i,2})\rangle, \qquad \mathcal{Q}_i = \alpha_{1,1}[t(\gamma_{i-1,1}), t(\lambda_{i,1})] \cup \alpha_{1,2}[t(\gamma_{i-1,2}), t(\lambda_{i,2})].$$

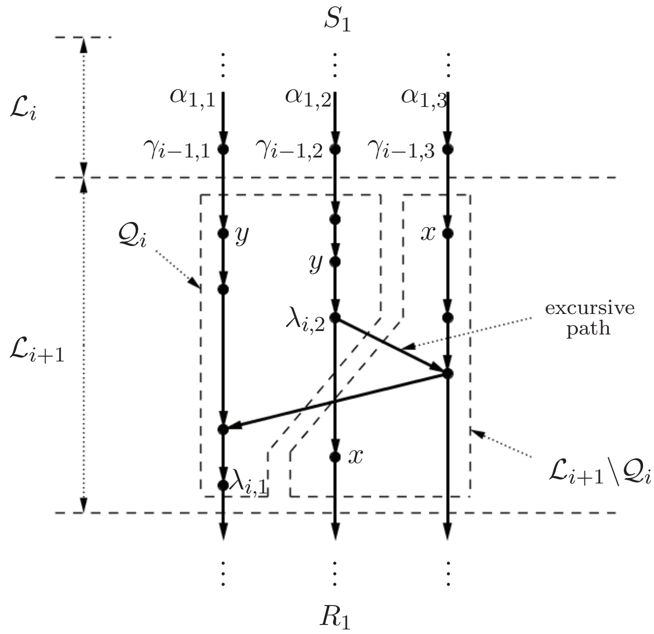

Figure 11: Case 1

Note that for $\lambda_{i,j}$, $j = 1, 2$, the associated α_2-path, from $\lambda_{i,j}$, may merge outside $\mathcal{Q}_i$ next time. If this α_2-path merges within $\mathcal{Q}_i$ again after a number of mergings outside $\mathcal{Q}_i$, we call it an *excursive* α_2-path. Note that there are at most one excursive α_2-path (since, otherwise, we can find a cycle in G, which is a contradiction). On the other hand, for any merging from $\mathcal{K}_i$ other than $\lambda_{i,1}, \lambda_{i,2}$, say, μ, the associated α_2-path, from μ, can only merge within $\mathcal{Q}_i$ and will not merge outside $\mathcal{Q}_i$. So, the number of connected α_2-paths that contain at least one merging within $\mathcal{L}_{i+1} \cap \mathcal{Q}_i$ is upper bounded by $y_{i+1} + 2$. Then, by the fact that $\mathcal{M}(2, c_2) = 3c_2 - 1$ (see Theorem 4.1), we have

$$||\mathcal{L}_{i+1} \cap \mathcal{Q}_i|| \leq 3(y_{i+1} + 2) - 1. \tag{13}$$

It is clear that all non-excursive α_2-paths that contain at least one merging within $\mathcal{L}_{i+1} \setminus \mathcal{Q}_i$ must have x-terminal mergings in $\mathcal{L}_{i+1}$. Thus, again by the fact that $\mathcal{M}(2, c_2) = 3c_2 - 1$, we have

$$||\mathcal{L}_{i+1} \setminus \mathcal{Q}_i|| \leq 3(x_{i+1} + 1) - 1. \tag{14}$$

It then immediately follows from (13) and (14) that

$$||\mathcal{L}_{i+1}|| = ||\mathcal{L}_{i+1} \cap \mathcal{Q}_i|| + ||\mathcal{L}_{i+1} \setminus \mathcal{Q}_i|| \leq 3(x_{i+1} + y_{i+1}) + 7.$$

Next, we claim that $x_{i+1} + y_{i+1} \geq 3$. To see this, suppose, by contradiction, that $x_{i+1} + y_{i+1} \leq 2$. Observing that $y_{i+1} \geq z_i - 2 \geq 1$, we then consider the following two cases:

If $x_{i+1} + y_{i+1} = 2$, we have

$$||\mathcal{L}_{i+1}|| \leq 3(x_{i+1} + y_{i+1}) + 7 = 13,$$

which implies that we can continue to choose a normal block (with two α_2-terminal mergings), a contradiction.

If $x_{i+1} + y_{i+1} = 1$, we have $x_{i+1} = 0$, $y_{i+1} = 1$. Note that if there is no excursive α_2-path, we have $z_i \leq y_{i+1} + 2$; if there is one excursive α_2-path, then $z_i \leq y_{i+1} + 1$. This, together with $z_i \geq 3$, implies that $z_i = 3$ and there is no excursive α_2-path. Consequently, we have

$$||\mathcal{L}_{i+1} \cap \mathcal{Q}_i|| \leq \mathcal{M}(2, 3) = 8, \qquad ||\mathcal{L}_{i+1} \setminus \mathcal{Q}_i|| = 0.$$

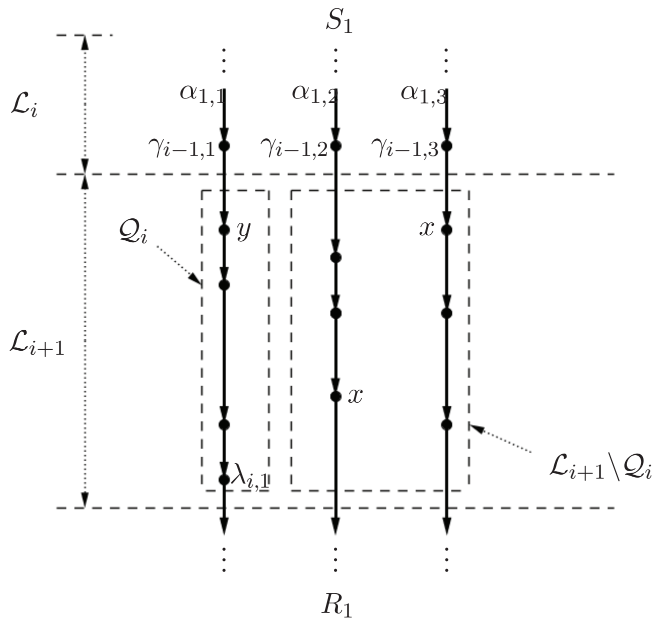

Figure 12: Case 2

But this, together with $\mathcal{M}(3,3)=13$, implies that we can continue to choose a normal block with an x-terminal merging and a y-terminal merging, which is a contradiction.

Case 2: $\alpha_{1,1}\cap\mathcal{K}_i\neq\emptyset$ and $\alpha_{1,2}\cap\mathcal{K}_i=\emptyset$. As shown in Figure 12, assume that within $\mathcal{K}_i$, $\lambda_{i,1}$ is the largest merging on $\alpha_{1,1}$. Clearly, there is no excursive α_2-path. By the fact that $\mathcal{M}(1,c_2)=c_2$ (see Example II.10 of [6]) and $\mathcal{M}(2,c_2)=3c_2-1$, we have

$$||\mathcal{L}_{i+1}\cap\mathcal{Q}_i||\leq y_{i+1}+1, \qquad ||\mathcal{L}_{i+1}\setminus\mathcal{Q}_i||\leq 3x_{i+1}-1.$$

It then immediately follows that $||\mathcal{L}_{i+1}||\leq 3x_{i+1}+y_{i+1}$.

Similarly as before, we claim that $x_{i+1}+y_{i+1}\geq 3$. To see this, suppose, by contradiction, that $x_{i+1}+y_{i+1}\leq 2$. From $y_{i+1}+1\geq z_i\geq 3$, we infer that $y_{i+1}=2$ and $x_{i+1}=0$, and further $||\mathcal{L}_{i+1}||\leq 3x_{i+1}+y_{i+1}=2$, which implies that we can in fact obtain a normal block with two y-terminal mergings, a contradiction.

Combining the above two cases, we conclude that the number of mergings within the singular block $\mathcal{L}_{i+1}$ is upper bounded by $3(x_{i+1}+y_{i+1})+7$, where $x_{i+1}+y_{i+1}\geq 3$.

We continue these operations in an iterative fashion to further obtain normal blocks and singular blocks until there are no mergings left in the graph. Suppose there are q_1 singular blocks $\mathcal{L}_{j_1},\mathcal{L}_{j_2},\ldots,\mathcal{L}_{j_{q_1}}$ and q_2 normal blocks. Note that each singular block has at least three α_2-terminal mergings and each normal block except the last one has at least two α_2-terminal mergings. If the last normal block has at least two α_2-terminal mergings, we then have

$$3q_1\leq\sum_{i=1}^{q_1}(x_{j_i}+y_{j_i})\leq 2c_2-2q_2.$$

It then follows that

$$||G||\leq 14q_2+\sum_{i=1}^{q_1}[3(x_{j_i}+y_{j_i})+7]\leq 14q_2+3(2c_2-2q_2)+7q_1=6c_2+7q_1+8q_2\leq 14c_2. \tag{15}$$

If the last normal block has only one α_2-terminal merging, necessarily, there are at most three mergings in the last normal block, we then have

$$3q_1 \leq \sum_{i=1}^{q_1}(x_{j_i} + y_{j_i}) \leq 2c_2 - 2(q_2 - 1) - 1.$$

It then follows that

$$||G|| \leq 14(q_2 - 1) + 3 + \sum_{i=1}^{q_1}[3(x_{j_i} + y_{j_i}) + 7] \leq 6c_2 + 7q_1 + 8q_2 - 8 \leq 14c_2. \quad (16)$$

Combining (15) and (16), we then have established the theorem.

□

6 Inequalities

Consider two non-reroutable (c_2, c_2)-graph $G^{(1)}, G^{(2)}$. For $j = 1, 2$, assume that $G^{(j)}$ has one source $S^{(j)}$, two sinks $R_1^{(j)}, R_2^{(j)}$. Let $\alpha_1^{(j)} = \{\alpha_{1,1}^{(j)}, \alpha_{1,2}^{(j)}, \ldots, \alpha_{1,c_2}^{(j)}\}$ denote the set of Menger's paths from $S^{(j)}$ to $R_1^{(j)}$ and $\alpha_2^{(j)} = \{\alpha_{2,1}^{(j)}, \alpha_{2,2}^{(j)}, \ldots, \alpha_{2,c_2}^{(j)}\}$ denote the set of Menger's paths from $S^{(j)}$ to $R_2^{(j)}$. As before, we assume that, for $1 \leq i \leq c_2$, paths $\alpha_{1,i}^{(j)}$ and $\alpha_{2,i}^{(j)}$ share a starting subpath.

Now, consider the following procedure of *concatenating* graphs $G^{(1)}$ and $G^{(2)}$:

1. reverse the direction of each edge in $G^{(2)}$ to obtain a new graph $\widehat{G}^{(2)}$ (for $1 \leq i \leq c_2$, path $\alpha_{1,i}^{(2)}$ in $G^{(2)}$ becomes path $\hat{\alpha}_{1,i}^{(2)}$ in $\widehat{G}^{(2)}$ and path $\alpha_{2,i}^{(2)}$ in $G^{(2)}$ becomes path $\hat{\alpha}_{2,i}^{(2)}$ in $\widehat{G}^{(2)}$);
2. split $S^{(1)}$ into c_2 copies $S_1^{(1)}, S_2^{(1)}, \ldots, S_{c_2}^{(1)}$ in $G^{(1)}$ such that paths $\alpha_{1,i}^{(1)}$ and $\alpha_{2,i}^{(1)}$ have the same starting point $S_i^{(1)}$; split $S^{(2)}$ into c_2 copies $S_1^{(2)}, S_2^{(2)}, \ldots, S_{c_2}^{(2)}$ in $\widehat{G}^{(2)}$ such that paths $\hat{\alpha}_{1,i}^{(2)}$ and $\hat{\alpha}_{2,i}^{(2)}$ have the same ending point $S_i^{(2)}$;
3. for $1 \leq i \leq c_2$, identify $S_i^{(1)}$ and $S_i^{(2)}$.

Obviously, such procedure produces a (c_2, c_2)-graph with two distinct sources $R_1^{(2)}, R_2^{(2)}$, two sinks $R_1^{(1)}, R_2^{(1)}$, a set of Menger's paths $\{\hat{\alpha}_{1,1}^{(2)} \circ \alpha_{1,1}^{(1)}, \hat{\alpha}_{1,2}^{(2)} \circ \alpha_{1,2}^{(1)}, \ldots, \hat{\alpha}_{1,c_2}^{(2)} \circ \alpha_{1,c_2}^{(1)}\}$ from $R_1^{(2)}$ to $R_1^{(1)}$ and a set of Menger's paths $\{\hat{\alpha}_{2,1}^{(2)} \circ \alpha_{2,1}^{(1)}, \hat{\alpha}_{2,2}^{(2)} \circ \alpha_{2,2}^{(1)}, \ldots, \hat{\alpha}_{2,c_2}^{(2)} \circ \alpha_{2,c_2}^{(1)}\}$ from $R_2^{(2)}$ to $R_2^{(1)}$. See Figure 13 for an example where we concatenate two $(3, 3)$-graphs. The following theorem then follows from the fact that the concatenated graph as above is a non-reroutable (c_2, c_2)-graph with $M(G^{(1)}) + M(G^{(2)}) + c_2$ mergings. So we have the following theorem

Theorem 6.1.

$$\mathcal{M}(c_2, c_2) \geq 2\mathcal{M}^*(c_2, c_2) + c_2.$$

Consider a non-reroutable $(c_2 + 1, c_2 + 1)$-graph $G^{(1)}$ and a non-reroutable $(c_2 - 1, c_2 - 1)$-graph $G^{(2)}$. The graph $G^{(1)}$ has one source $S^{(1)}$, two sinks $R_1^{(1)}, R_2^{(1)}$, a set of Menger's paths $\alpha_1 = \{\alpha_{1,0}, \alpha_{1,1}, \ldots, \alpha_{1,c_2}\}$ from $S^{(1)}$ to $R_1^{(1)}$ and a set of Menger's paths $\alpha_2 = \{\alpha_{2,0}, \alpha_{2,1}, \ldots, \alpha_{2,c_2}\}$ from $S^{(1)}$ to $R_2^{(1)}$. As discussed in Section 3.3, we assume paths $\alpha_{1,i}$ and $\alpha_{2,i}$ share a starting subpath ω_i, and paths α_{1,c_2}, $\alpha_{2,0}$ do not merge with any other paths in $G^{(1)}$, directly flowing to the sinks. The graph $G^{(2)}$ has one source $S^{(2)}$, two sinks $R_1^{(2)}, R_2^{(2)}$, and a set of Menger's paths

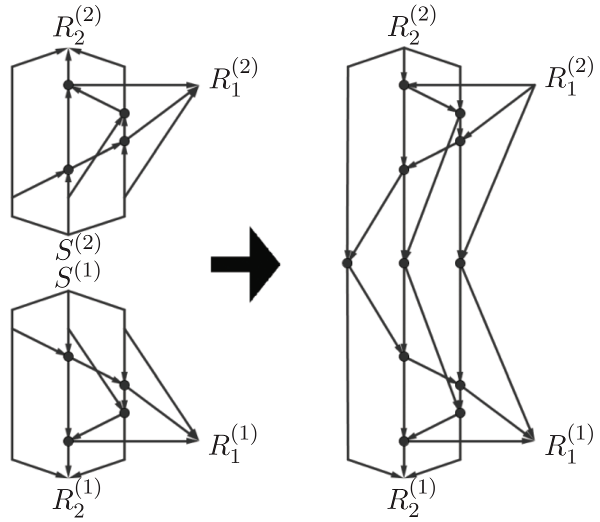

Figure 13: Concatenation of two $(3,3)$-graphs

$\zeta_i = \{\zeta_{i,1}, \zeta_{i,2}, \ldots, \zeta_{i,c_2-1}\}$ from $S^{(2)}$ to $R_i^{(2)}$, for $i = 1, 2$. Again, assume paths $\zeta_{1,i}$ and $\zeta_{2,i}$ share a starting subpath.

Now, we consider the following procedure of *concatenating* graphs $G^{(1)}$ and $G^{(2)}$:

1. reverse the direction of each edge in $G^{(2)}$ to obtain a new graph $\widehat{G}^{(2)}$ (for $1 \leq i \leq c_2 - 1$, path $\zeta_{1,i}$ in $G^{(2)}$ becomes path $\hat{\zeta}_{1,i}$ in $\widehat{G}^{(2)}$ and path $\zeta_{2,i}$ in $G^{(2)}$ becomes path $\hat{\zeta}_{2,i}$ in $\widehat{G}^{(2)}$);

2. split $S^{(1)}$ into $c_2 + 1$ copies $S_0^{(1)}, S_1^{(1)}, \ldots, S_{c_2}^{(1)}$ in $G^{(1)}$ such that paths $\alpha_{1,i}$ and $\alpha_{2,i}$ have the same starting point $S_i^{(1)}$; split $S^{(2)}$ into $c_2 - 1$ copies $S_1^{(2)}, S_2^{(2)}, \ldots, S_{c_2-1}^{(2)}$ in $\widehat{G}^{(2)}$ such that paths $\hat{\zeta}_{1i}$ and $\hat{\zeta}_{2i}$ have the same ending point $S_i^{(2)}$;

3. delete all edges on α_{1,c_2}, each of which is larger than ω_{c_2}; delete all edges on $\alpha_{2,0}$, each of which is larger than ω_0;

4. identify $R_1^{(2)}$ and $S_0^{(1)}$; for $1 \leq i \leq c_2 - 1$, identify $S_i^{(2)}$ and $S_i^{(1)}$; identify $R_2^{(2)}$ and $S_{c_2}^{(1)}$.

Obviously, such procedure produces a (c_2, c_2)-graph with two distinct sources $R_1^{(2)}, R_2^{(2)}$, two sinks $R_1^{(1)}, R_2^{(1)}$, a set of Menger's paths $\{\alpha_{1,0}, \hat{\zeta}_{1,1} \circ \alpha_{1,1}, \hat{\zeta}_{1,2} \circ \alpha_{1,2}, \ldots, \hat{\zeta}_{1,c_2-1} \circ \alpha_{1,c_2-1},\}$ from $R_1^{(2)}$ to $R_1^{(1)}$ and a set of Menger's paths $\{\hat{\zeta}_{2,1} \circ \alpha_{2,1}, \hat{\zeta}_{2,2} \circ \alpha_{2,2}, \ldots, \hat{\zeta}_{2,c_2-1} \circ \alpha_{2,c_2-1}, \alpha_{2,c_2}\}$ from $R_2^{(2)}$ to $R_2^{(1)}$. For example, in Figure 14, we concatenate a $(2,2)$-graph and a $(4,4)$-graph to obtain a $(3,3)$-graph. It turns out that the concatenated graph as above is a non-reroutable (c_2, c_2)-graph with $M(G^{(1)}) + M(G^{(2)}) + (c_2 - 1)$ mergings. Then we establish the following theorem, which improves the result in Proposition III.8 of [6].

Theorem 6.2.

$$\mathcal{M}(c_2, c_2) \geq \mathcal{M}^*(c_2 + 1, c_2 + 1) + \mathcal{M}^*(c_2 - 1, c_2 - 1) + (c_2 - 1).$$

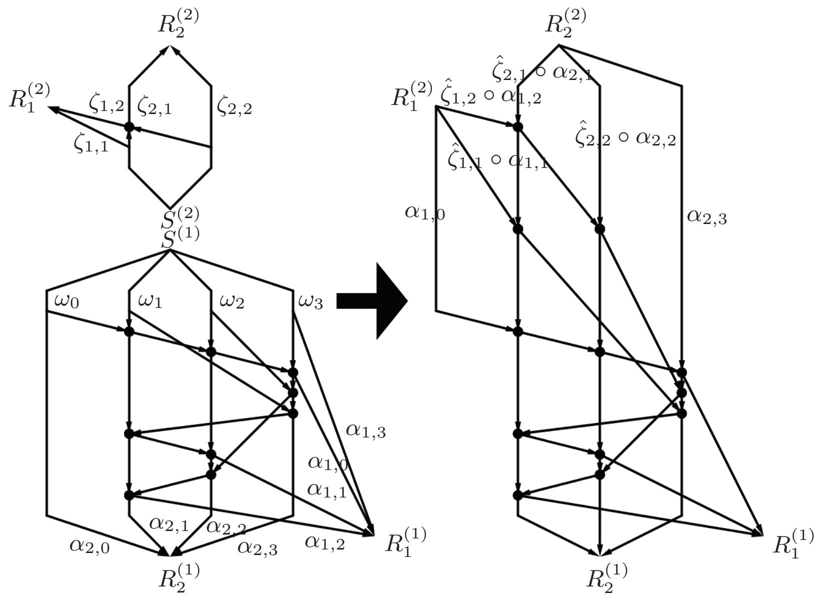

Figure 14: Concatenation of a $(2, 2)$-graph and a $(4, 4)$-graph

7 Conclusions and Future Work

The two functions $\mathcal{M}$ and $\mathcal{M}^*$, originally defined in [6], are of great relevance to network encoding complexity for a variety of networks, as they indicate the numbers of encoding operations needed for some worst case networks. One natural and fundamental question in network encoding complexity is to compute the exact values of these two functions, which, however, appears to be extremely difficult. It turns out that, even for small parameters, computations of these two functions are far from straightforward, and these two functions for general parameters remain largely unknown to date. In this paper, we have further developed the graph theoretical approach proposed in [6] to derive exact values and bounds for these two functions for certain special parameters. A natural follow-up problem is to further explore the power of our approach to derive exact values and tighter bounds for more general parameters.

A central idea that runs through all the proofs (for the upper bound on $\mathcal{M}$ and $\mathcal{M}^*$) in this paper is as follows: for any (c_1, c_2)-graph G, if the number of mergings is sufficiently large, then there must exist some "reroutable patterns". In this sense, the results obtained in this work are of Ramsey theory [5] flavor.

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