HITCHIN'S CONNECTION AND DIFFERENTIAL OPERATORS WITH VALUES IN THE DETERMINANT BUNDLE

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1. Introduction

For any smooth family $\pi : X \to S$ of curves and a vector bundle E on X, A. Beilinson and V. Schechtman defined in [2] (see also [3]) a so-called trace complex ${}^{tr}\mathcal{A}^{\bullet}_{E}$ on X, together with an algebra structure on it, such that $R^0\pi_*({}^{tr}\mathcal{A}^{\bullet}_{E})$ is canonically isomorphic to the Atiyah algebra $\mathcal{A}_{\lambda_E}$ of the determinant bundle $\lambda_E = \det R\pi_* E$ of the family (See Proposition 3.3 for details). It is generalized by Y.-L. Tong and the second author ([13]) to the families $\pi : \widetilde{X} \to \widetilde{S}$ of stable curves. On the other hand, in his fundamental article [7], N. Hitchin constructed a projective connection on the relative sections of the determinant bundle on the moduli of bundles with trivial determinant on curves of fixed genus. In this paper, we present *a construction of Hitchin's connection and its logarithmic extension within the framework of* [2] *and* [13]. Note that studying the behaviour of spaces of generalized theta functions under degeneration of the curves was already suggested in [1] and [7].

Let $\mathcal{M}$ be the moduli stack of smooth curves of genus $g \geq 2$ and $\mathcal{C} \to \mathcal{M}$ be the universal curve. Let $f : S \to \mathcal{M}$ be the family of moduli spaces of stable bundles of rank $r \geq 2$ with a fixed determinant. Let $\pi : X = \mathcal{C} \times_{\mathcal{M}} S \to S$, E a universal bundle on X and $\mathbb{E} := \mathcal{E}nd^0(E)$. In view of identification of tangent bundle $T_{S/\mathcal{M}}$ with $R^1\pi_*(\mathbb{E})$, it seems natural to ask whether there exists an analogous theory for the side of differential operators, namely for an identification of sheaves of differential operators (with values in the determinant bundle) with certain cohomology groups. The trace complex of [2] provides the identification, but we found unfortunately it hard to work with the trace complex in a direct way. To tackle this difficulty we find that there exist *canonical (locally free) sheaves* $\mathcal{G}_E$, $S(\mathcal{G}_E)$ whose

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relative cohomology coincides with that of the trace complex (at least in the moduli situation). Thus we have canonical identifications

$$\phi : R^1\pi_*(\mathcal{G}_E) \cong \mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\lambda_{\mathbb{E}}), \quad \phi : R^1\pi_* S(\mathcal{G}_E) \cong \mathcal{D}^{\leq 1}_{S}(\lambda_{\mathbb{E}})$$

(cf. Theorem 2.5). This part is of independent interest in itself.

Let Θ be the theta line bundle on S and, for any k, write $\Theta^k = \lambda^{\mu}_{\mathbb{E}}$ for some $\mu \in \mathbb{Q}$. Using the above identifications and taking cohomology for some short exact sequences, we get the commutative diagram

$$\begin{array}{ccc}
\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\Theta^k) & \xrightarrow{\bullet\mu} & \mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\Theta^k) \\
\downarrow & & \downarrow \\
\mathcal{D}^{\leq 1}_{S}(\Theta^k) & \xrightarrow{\tilde{\delta}_{\mathrm{H}}} & S^2\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\Theta^k) \\
{\scriptstyle\sigma}\downarrow & & {\scriptstyle S^2\epsilon_1}\downarrow \\
f^*T_{\mathcal{M}} & \xrightarrow{\delta_{\mathrm{H}}} & S^2T_{S/\mathcal{M}}
\end{array}$$

where the two vertical sequences are short exact sequences and δ_{H}, $\tilde{\delta}_{\mathrm{H}}$ are the connecting maps. δ_{H} is in fact an *alternative of Hitchin's symbol map* (cf. [10], Section 4).

As nicely explained in [6], Hitchin's construction of the connection on the vector bundle $\mathcal{V}_k = f_*(\Theta^k)$ consists in defining, for any $v \in T_{\mathcal{M}}(U)$ $(= f^*T_{\mathcal{M}}(f^{-1}(U)))$, a *heat operator* $\mathrm{H}(v) \in H^0(f^{-1}(U), \mathcal{D}^{\leq 2}_{S}(\Theta^k))$ such that its symbolic part is $\delta_{\mathrm{H}}(v)$. With help of above diagram, we are able to compare the obstruction classes to lift $\delta_{\mathrm{H}}(v)$ to $\mathcal{D}^{\leq 2}_{S/\mathcal{M}}(\Theta^k)$ and to $S^2\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\Theta^k)$ (cf. Proposion 2.7). Then we get a lifting $\mathrm{H}(v)$. The uniqueness of $\mathrm{H}(v)$ follows from the vanishing of $f_*T_{S/\mathcal{M}}$. It is important that we do not use the vanishing of $R^1f_*\mathcal{O}_S$, so that our method can be extended to some stable curves, where such a vanishing may not hold.

Let $\widetilde{\mathcal{M}} \subset \widetilde{\mathcal{M}_g}$ be the open set consisting of irreducible stable curves and reducible curves with only one node. Let $\widetilde{\mathcal{C}} \to \widetilde{\mathcal{M}}$ be the universal curve (extended $\mathcal{C} \to \mathcal{M}$), and $\widetilde{f} : \widetilde{S} \to \widetilde{\mathcal{M}}$ be the family of moduli spaces of stable bundles with fixed determinant (cf. Notation 2.10). Then, in the context of *log-geometry* the similar formulation for $\pi : \widetilde{X} := \widetilde{\mathcal{C}} \times_{\widetilde{\mathcal{M}}} \widetilde{S} \to \widetilde{S}$, via an appropriate generalization of the trace complex (cf. [13]), can be achieved equally well. Thus another result of this paper is that there is a *logarithmic projective connection* on $\widetilde{f}_*(\Theta^k)$. We remark that $R^1\widetilde{f}_*\mathcal{O}_{\widetilde{S}} = 0$ may not hold, but $\widetilde{f}_*T_{\widetilde{S}/\widetilde{\mathcal{M}}} = 0$ remains true since S is dense in $\widetilde{S}$ (cf. Notation 2.10). The family

$\widetilde{f} : \widetilde{S} \to \widetilde{\mathcal{M}}$ is not proper. However, we can prove that $\widetilde{f}_*(\Theta^k)$ is a *coherent* sheaf on $\widetilde{\mathcal{M}}$ (cf. Theorem 4.10). We believe that the coherent property remains true on $\widetilde{M}_g$ of all stable curves; this is not, however, proved in the present paper.

Basically our approach works equally well for both of the general case [7] ($g \geq 3$) and special cases (e.g. $g = 2$ and $r = 2$) [6], with the difference arising from the fact that in $g = r = 2$ case our approach shall work with a Quot scheme $\mathcal{R}^{ss}$ which has a good quotient $\mathcal{R}^{ss} \to S$ where S is the family of moduli spaces of S-equivalence classes of *semistable* vector bundles. As this will introduce additional details, we leave the case $g = r = 2$ to future discussions.

In Section 2, we construct Hitchin's connection and its logarithmic extension under assumption of Theorem 2.5, Lemma 2.4 and Theorem 2.13. It is enough to read this section for a reader who is only interested in Hitchin's connection. Section 3 is devoted to the proof of Theorem 2.5 and Lemma 2.4. In Section 4, we indicate the modifications of arguments in Section 3 to prove Theorem 2.13. Most of Section 4 is devoted to prove the *coherence* of $\widetilde{f}_*(\Theta^k)$ on $\widetilde{\mathcal{M}}$.

Remark that some ideas closely related to Section 3 of this paper were briefly discussed in Sections 9, 10 of [5] where the subject matter is considered for moduli space of (stable) G-bundles, with G being complex semisimple, connected and simply-connected.

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2. Heat operators and cohomology of sheaves

As in Introduction, $\mathcal{M}$ (resp. $\widetilde{\mathcal{M}_g}$) denotes the moduli stack of smooth (resp. stable) curves of genus $g \geq 2$, and $\mathcal{C} \to \mathcal{M}$ (resp. $\widetilde{\mathcal{C}} \to \widetilde{\mathcal{M}_g}$) denotes the universal curve respectively. By *moduli stack* in this article, we mean that we are working on the *fine moduli spaces*. In particular, we assume that both the moduli spaces $\mathcal{M}$, $\widetilde{\mathcal{M}_g}$ and the universal curves $\mathcal{C}$, $\widetilde{\mathcal{C}}$ are smooth. Fix a line bundle $\mathcal{N}$ of relative degree

d on $\widetilde{\mathcal{C}} \to \widetilde{\mathcal{M}_g}$, let $f : S \to \mathcal{M}$ be the family of moduli spaces of stable bundles of rank r with fixed determinant $\mathcal{N}_b := \mathcal{N}|_{\mathcal{C}_b}$ ($b \in \mathcal{M}$) on $\mathcal{C}_b$. Let $\pi : X = \mathcal{C} \times_{\mathcal{M}} S \to S$ be the pull-back of $\mathcal{C} \to \mathcal{M}$ via $f : S \to \mathcal{M}$.

Remark 2.1. If $(d, r) = 1$, there exists a universal bundle E on X. In general S is a good quotient of a Hilbert quotient scheme, denoted by $\mathcal{R}^s$, such that there is a universal bundle E on $X_{\mathcal{R}^s} := \mathcal{C} \times_{\mathcal{M}} \mathcal{R}^s$. E may not descend to X, but objects such as $\mathcal{E}nd^0(E)$, $\mathcal{G}_E$, ${}^{tr}\mathcal{A}^\bullet_E$ (and other relevant constructions that will be discussed later in this section) do descend. Recall that a sheaf $\mathcal{F}$ on $X_{\mathcal{R}^s}$ descends to X if the action, of scalar automorphisms of E (relative to $\mathcal{R}^s$), on $\mathcal{F}$ is trivial, e.g. [8].

Without the danger of confusion we will henceforth be working as if a universal bundle E exists on X.

Let Θ be the theta line bundle on S. For a sheaf $\mathcal{F}$ on X,

$$\lambda_{\mathcal{F}} = \bigotimes_{q \geq 0} \det R^q \pi_* \mathcal{F}^{(-1)^q}$$

denotes the Knudsen-Mumford determinant bundle on S. As usual T_S, K_S denote tangent bundle, canonical bundle; $T_{S/\mathcal{M}}$, $K_{S/\mathcal{M}}$ denote the relative counterparts. Assume that $K_{S/\mathcal{M}} = \Theta^{-\lambda}$. Let $\mathcal{D}^{\leq i}_S(\Theta^k)$ be the sheaf of differential operators on Θ^k of order $\leq i$, and by ϵ the symbol map of differential operators. Let $\mathcal{W}(\Theta^k) := \mathcal{D}^{\leq 1}_S(\Theta^k) + \mathcal{D}^{\leq 2}_{S/\mathcal{M}}(\Theta^k)$, one has an exact sequence

$$0 \to \mathcal{D}^{\leq 2}_{S/\mathcal{M}}(\Theta^k) \to \mathcal{W}(\Theta^k) \xrightarrow{\sigma} f^* T_{\mathcal{M}} \to 0. \tag{2.1}$$

Definition 2.2. (cf. [6], 2.3.2.) A *heat operator* H on Θ^k is an $\mathcal{O}_S$-map $f^* T_{\mathcal{M}} \xrightarrow{\mathrm{H}} \mathcal{W}(\Theta^k)$ which, while composed with σ above, is the identity. A *projective heat operator* is $\mathrm{H} : T_{\mathcal{M}} \to f_* \mathcal{W}(\Theta^k)/\mathcal{O}_{\mathcal{M}}$ such that any local lifting is a heat operator. The *symbol map* of H is $\epsilon \circ \mathrm{H} : f^* T_{\mathcal{M}} \to S^2 T_{S/\mathcal{M}}$. A heat operator H induces an $\mathcal{O}_{\mathcal{M}}$-map, denoted by the same H, $T_{\mathcal{M}} \to f_* \mathcal{W}(\Theta^k)$ The heat operator and the preceding induced map will be used interchangeably throughout. A (projective) heat operator on Θ^k determines a (projective) connection ∇^{H} on $f_* \Theta^k$ in a natural way [6].

We recall firstly a general result (forget moduli) of G. Faltings (cf. [4]). Let $Z \to S$ be smooth and $\mathcal{K} = K_{Z/S}$. For any line bundle $\mathcal{L}$, consider

$$0 \to \mathcal{D}^{\leq 1}_{Z/S}(\mathcal{L}) \to \mathcal{D}^{\leq 2}_{Z/S}(\mathcal{L}) \xrightarrow{\epsilon_2} S^2 T_{Z/S} \to 0$$

$$0 \to \mathcal{D}^{\leq 1}_{Z/S}(\mathcal{L}) \to S^2 \mathcal{D}^{\leq 1}_{Z/S}(\mathcal{L}) \xrightarrow{S^2 \epsilon_1} S^2 T_{Z/S} \to 0.$$

For any $\rho \in H^0(Z, S^2T_{Z/S})$, let $a(\mathcal{L},\rho)$, $b(\mathcal{L},\rho)$ be the obstruction classes to lift ρ to $H^0(Z,\mathcal{D}^{\leq 2}_{Z/S}(\mathcal{L}))$ and to $H^0(Z,S^2\mathcal{D}^{\leq 1}_{Z/S}(\mathcal{L}))$ respectively. Then

Proposition 2.3. *Under $\mathcal{D}^{\leq 1}_{Z/S}(\mathcal{L}^k) \cong \mathcal{D}^{\leq 1}_{Z/S}(\mathcal{L})$, one has*
i) $b(\mathcal{L}^k,\rho) = kb(\mathcal{L},\rho)$ for any $k \in \mathbb{Q}$.
ii) $a(\mathcal{O}_Z,\rho) \in H^1(\mathcal{O}_Z) \oplus H^1(T_{Z/S})$ has zero projection in $H^1(\mathcal{O}_Z)$.
iii) There is a class $c(\mathcal{K},\rho) \in H^1(\mathcal{O}_Z)$, independent of $\mathcal{L}$, such that

$$2a(\mathcal{L},\rho) = b(\mathcal{L},\rho) + {}^t b(\mathcal{L}^{-1}\otimes\mathcal{K},\rho) + c(\mathcal{K},\rho).$$

We come back to construct the projective connection. Recall that

$$\begin{array}{ccc} X = \mathcal{C}\times_{\mathcal{M}} S & \xrightarrow{\pi} & S \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathcal{C} & \longrightarrow & \mathcal{M}. \end{array}$$

For simplicity, we assume that there exists a universal bundle E on X, let $\mathbb{E} = \mathcal{E}nd^0(E)$. We will arrive at a subsheaf $\mathcal{G}_E \subset {}^{tr}\mathcal{A}^{-1}_E$ (see [2] for details of ${}^{tr}\mathcal{A}^{-1}_E$) fitting into an exact sequence

$$0 \to \omega_{X/S} \to \mathcal{G}_E \xrightarrow{\text{res}} \mathbb{E} \to 0, \tag{2.2}$$

which induces, by taking 2-th symmetric tensor, the exact sequence

$$0 \to \mathcal{G}_E \to S^2(\mathcal{G}_E)\otimes T_{X/S} \xrightarrow{\text{Sym}^2(\text{res})\otimes\text{id}} S^2(\mathbb{E})\otimes T_{X/S} \to 0.$$

Define $S(\mathcal{G}_E) := (\text{Sym}^2(\text{res})\otimes\text{id})^{-1}(\text{id}\otimes T_{X/S})$, which fits into

$$0 \to \mathcal{G}_E \xrightarrow{\iota} S(\mathcal{G}_E) \xrightarrow{q} T_{X/S} \to 0, \tag{2.3}$$

where $q = \text{Sym}^2(\text{res})\otimes\text{id}$, $\iota(\alpha) = \text{Sym}^2(\alpha\otimes dt)\otimes\partial_t$ locally. Let

$$0 \to R^1\pi_*(\omega_{X/S}) \to R^1\pi_*(\mathcal{G}_E) \xrightarrow{res} R^1\pi_*(\mathbb{E}) \to 0 \tag{2.4}$$

$$0 \to R^1\pi_*(\mathcal{G}_E) \xrightarrow{\iota} R^1\pi_*S(\mathcal{G}_E) \xrightarrow{q} R^1\pi_*(T_{X/S}) \to 0 \tag{2.5}$$

be the exact sequences induced by (2.2), (2.3). Let $\Delta \subset X\times_S X$ be the diagonal, consider the induced diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G}_E\boxtimes\mathcal{G}_E & \longrightarrow & \mathcal{G}_E\boxtimes\mathcal{G}_E(\Delta) & \longrightarrow & \mathcal{G}_E\boxtimes\mathcal{G}_E(\Delta)|_\Delta & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \mathbb{E}\boxtimes\mathbb{E} & \longrightarrow & \mathbb{E}\boxtimes\mathbb{E}(\Delta) & \longrightarrow & \mathbb{E}\boxtimes\mathbb{E}(\Delta)|_\Delta & \longrightarrow & 0 \end{array}$$

on $X\times_S X$ ($\mathbb{E}\boxtimes\mathbb{E}$ denotes $p_1^*\mathbb{E}\otimes p_2^*\mathbb{E}$). All vertical maps are induced by $\mathcal{G}_E \xrightarrow{res} \mathbb{E} \to 0$, thus (2.3) is a sub-sequence of the rightmost vertical map above. Let $\mathcal{F}_1 \subset \mathcal{G}_E \boxtimes \mathcal{G}_E(\Delta)$, $\mathcal{F}_2 \subset \mathbb{E} \boxtimes \mathbb{E}(\Delta)$ be subsheaves satisfying

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G}_E \boxtimes \mathcal{G}_E & \longrightarrow & \mathcal{F}_1 & \longrightarrow & S(\mathcal{G}_E) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle q} & & \\ 0 & \longrightarrow & \mathbb{E} \boxtimes \mathbb{E} & \longrightarrow & \mathcal{F}_2 & \longrightarrow & T_{X/S} & \longrightarrow & 0. \end{array}$$

Taking direct images, considering the connecting maps, we have

$$\begin{array}{ccc} R^1\pi_* S(\mathcal{G}_E) & \xrightarrow{\tilde{\delta}} & S^2 R^1\pi_*(\mathcal{G}_E) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle S^2(res)} \\ R^1\pi_*(T_{X/S}) & \xrightarrow{\delta} & S^2 R^1\pi_*(\mathbb{E}) \end{array}$$

which induces the commutative diagram

$$\begin{array}{ccc} R^1\pi_*(\mathcal{G}_E) & \xrightarrow{\delta_1} & R^1\pi_*(\mathcal{G}_E) \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \mathrm{Sym}^2} \\ R^1\pi_* S(\mathcal{G}_E) & \xrightarrow{\tilde{\delta}} & S^2 R^1\pi_*(\mathcal{G}_E) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle S^2(res)} \\ R^1\pi_*(T_{X/S}) & \xrightarrow{\delta} & S^2 R^1\pi_*(\mathbb{E}) \end{array}$$

where δ_1 is defined such that the diagram is commutative, the first vertical exact sequence is (2.5), the second vertical exact sequence is induced by taking 2-th symmetric tensor of (2.4) (note that $\mathcal{O}_S \cong R^1\pi_*(\omega_{X/S})$). For Sym^2 see Remark 2.6.

Lemma 2.4. *The map $\delta_1 : R^1\pi_*(\mathcal{G}_E) \to R^1\pi_*(\mathcal{G}_E)$ is the identity map.*

Proof. See Lemma 3.18. □

Theorem 2.5. *i) There is an isomorhism $\phi : R^1\pi_*(\mathcal{G}_E) \cong \mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\lambda_{\mathbb{E}})$ such that the following diagram is commutative*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_S = R^1\pi_*\omega_{X/S} & \longrightarrow & R^1\pi_*(\mathcal{G}_E) & \xrightarrow{\mathrm{res}} & R^1\pi_*(\mathbb{E}) & \longrightarrow & 0 \\ & & {\scriptstyle (2r)\cdot \mathrm{id}}\downarrow & & {\scriptstyle \phi}\downarrow & & {\scriptstyle \vartheta}\downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_S = R^0\pi_*\omega_{X/S}[1] & \longrightarrow & \mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\lambda_{\mathbb{E}}) & \xrightarrow{\epsilon_1} & T_{S/\mathcal{M}} & \longrightarrow & 0 \end{array}$$

where ϑ is Kodaira-Spencer type identification.

ii) For any affine covering $\{\mathcal{M}_i\}_{i\in I}$ of $\mathcal{M}$, on each $S_i := f^{-1}(\mathcal{M}_i)$, there is an isomorphism $\phi_i : R^1\pi_*S(\mathcal{G}_E) \cong \mathcal{D}^{\leq 1}_S(\lambda_{\mathbb{E}})$ such that

$$\begin{tikzcd}
0 \arrow[r] & R^1\pi_*(\mathcal{G}_E) \arrow[r, "2\cdot\iota"] \arrow[d, "\phi"'] & R^1\pi_*S(\mathcal{G}_E) \arrow[r, "q"] \arrow[d, "\phi_i"'] & R^1\pi_*(T_{X/S}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\lambda_{\mathbb{E}}) \arrow[r] & \mathcal{D}^{\leq 1}_S(\lambda_{\mathbb{E}}) \arrow[r, "\sigma"] & f^*T_{\mathcal{M}} \arrow[r] & 0
\end{tikzcd}$$

is commutative on S_i. Moreover, $\{\phi_i - \phi_j\}$ define a class in $\mathrm{H}^1(\Omega^1_{\mathcal{M}})$.

Proof. See Theorem 3.7, Corollary 3.12 and Remark 3.17. □

By the above theorem, on each S_i, we get commutative diagram

$$\begin{tikzcd}
\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\lambda_{\mathbb{E}}) \arrow[r, "\phi^{-1}"] \arrow[d] & R^1\pi_*(\mathcal{G}_E) \arrow[r, "\delta_1"] \arrow[d, "2\cdot\iota"'] & R^1\pi_*(\mathcal{G}_E) \arrow[r, "2r\cdot\phi"] \arrow[d, "\mathrm{Sym}^2"'] & \mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\lambda_{\mathbb{E}}) \arrow[d] \\
\mathcal{D}^{\leq 1}_S(\lambda_{\mathbb{E}}) \arrow[r, "\phi_i^{-1}"] \arrow[d, "\sigma"'] & R^1\pi_*S(\mathcal{G}_E) \arrow[r, "\tilde{\delta}"] \arrow[d] & S^2R^1\pi_*(\mathcal{G}_E) \arrow[r, "S^2(\phi)"] \arrow[d, "q"'] & S^2\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\lambda_{\mathbb{E}}) \arrow[d, "S^2\epsilon_1"'] \\
f^*T_{\mathcal{M}} \arrow[r, "\cong"] & R^1\pi_*(T_{X/S}) \arrow[r, "\delta"] & S^2R^1\pi_*(\mathbb{E}) \arrow[r, "\cong"] & S^2T_{S/\mathcal{M}}.
\end{tikzcd}$$

Remark 2.6. i) For a precise definition of Sym^2 above we refer to proof of Lemma 3.18; ii) The insertion of $2r$ in $2r\cdot\phi$ in the 1st row is due to Theorem 2.5 i) combined with the definition of Sym^2.

The above diagram gives the following commutaive diagram on S_i

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\lambda_{\mathbb{E}}) \arrow[r] \arrow[d, equal] & \mathcal{D}^{\leq 1}_S(\lambda_{\mathbb{E}}) \arrow[r, "\sigma"] \arrow[d, "\tilde{\delta}^i_{\mathrm{H}}"'] & f^*T_{\mathcal{M}} \arrow[r] \arrow[d, "\delta_{\mathrm{H}}"'] & 0 \\
0 \arrow[r] & \mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\lambda_{\mathbb{E}}) \arrow[r] & S^2\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\lambda_{\mathbb{E}}) \arrow[r, "S^2\epsilon_1"] & S^2T_{S/\mathcal{M}} \arrow[r] & 0
\end{tikzcd}$$

where δ_{H}, $\tilde{\delta}^i_{\mathrm{H}}$ are defined in a clear way such that each $\tilde{\delta}^i_{\mathrm{H}}$ induces an identity map on $\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\lambda_{\mathbb{E}})$.

Proposition 2.7. *For any $\rho = \delta_{\mathrm{H}}(v) \in H^0(S, S^2T_{S/\mathcal{M}})$, where $v \in H^0(S, f^*T_{\mathcal{M}})$ and $\mathcal{M}$ is replaces by its affine open set, one has*
i) $2a(\mathcal{L},\rho) = b(\mathcal{L},\rho) + {}^tb(\mathcal{L}^{-1}\otimes K_{S/\mathcal{M}},\rho)$.
ii) When $\mathcal{L} = K^{\mu}_{S/\mathcal{M}}$, where $\mu\in\mathbb{Q}$ and $\mu\neq 1$, one has

$$a(K^{\mu}_{S/\mathcal{M}},\rho) = \frac{2\mu-1}{2\mu}b(K^{\mu}_{S/\mathcal{M}},\rho).$$

Proof. To prove i), by Proposition 2.3 iii), it is enough to show that $c(\mathcal{K},\rho)=0$. The class is independent of $\mathcal{L}$. Thus, by taking $\mathcal{L}=\mathcal{O}_S$ and using Proposition 2.3 ii), it is enough to show that

$$^{t}b(\mathcal{K},\rho)\in H^1(\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\mathcal{O}_S))=H^1(\mathcal{O}_S)\oplus H^1(T_{S/\mathcal{M}})$$

has trivial projection in $H^1(\mathcal{O}_S)$ where $\mathcal{K}=K_{S/\mathcal{M}}$. We have (noting $\mathcal{K}=\lambda_{\mathbb{E}}$)

$$\begin{tikzcd}
\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\mathcal{K}) \arrow[r, equal] \arrow[d] & \mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\mathcal{K}) \arrow[d] \\
\mathcal{D}^{\leq 1}_{S}(\mathcal{K}) \arrow[r, "\tilde{\delta}_{\mathrm{H}}"] \arrow[d, "\sigma"'] & S^2\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\mathcal{K}) \arrow[d, "S^2\epsilon_1"'] \\
f^*T_{\mathcal{M}} \arrow[r, "\delta_{\mathrm{H}}"] & S^2T_{S/\mathcal{M}}.
\end{tikzcd}$$

Let $\{\mathcal{U}_i\}_{i\in I}$ be an affine covering of S and $v_i\in T_S(\mathcal{U}_i)$ be local liftings of $v\in f^*T_{\mathcal{M}}(S)$. Let $\{d_i\in\mathcal{D}^{\leq 1}_{S}(\mathcal{K})(\mathcal{U}_i)\}_{i\in I}$ be such that $\epsilon_1(d_i)=v_i$ ($i\in I$). Then, by above diagram, $b(\mathcal{K},\rho)=\{d_i-d_j\in\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\mathcal{K})(\mathcal{U}_i\cap\mathcal{U}_j)\}$. Thus the class $^{t}b(\mathcal{K},\rho)\in H^1(\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\mathcal{O}_S))=H^1(\mathcal{O}_S)\oplus H^1(T_{S/\mathcal{M}})$ is defined by the cocycle $\{\,^{t}d_i-\,^{t}d_j\}$, where $^{t}d_i:=A_i-v_i\in\mathcal{D}^{\leq 1}_{S}(\mathcal{O}_S)(\mathcal{U}_i)$. Thus the projection of $^{t}b(\mathcal{K},\rho)$ in $H^1(\mathcal{O}_S)$ is defined by $\{A_i-A_j\}$, which is a trivial class.

To show ii), we remark that for any nonzero $\mu\in\mathbb{Q}$, through canonical isomorphisms $\psi_\mu:\mathcal{D}^{\leq 1}(\mathcal{K})\cong\mathcal{D}^{\leq 1}(\mathcal{K}^{\mu})$, the above diagram induces

$$\begin{tikzcd}
\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\mathcal{K}^{\mu}) \arrow[r, "\bullet\mu"] \arrow[d] & \mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\mathcal{K}^{\mu}) \arrow[d] \\
\mathcal{D}^{\leq 1}_{S}(\mathcal{K}^{\mu}) \arrow[r, "\tilde{\delta}^{\mu}_{\mathrm{H}}"] \arrow[d, "\sigma"'] & S^2\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\mathcal{K}^{\mu}) \arrow[d, "S^2\epsilon_1"'] \\
f^*T_{\mathcal{M}} \arrow[r, "\delta_{\mathrm{H}}"] & S^2T_{S/\mathcal{M}}
\end{tikzcd}$$

where $\tilde{\delta}^{\mu}_{\mathrm{H}}=S^2\psi_\mu\circ\tilde{\delta}_{\mathrm{H}}\circ\psi_\mu^{-1}$. Using the above diagram, we can compute $^{t}b(\mathcal{K}^{1-\mu},\rho)$. Let $\{d_i\in\mathcal{D}^{\leq 1}_{S}(\mathcal{K}^{1-\mu})(\mathcal{U}_i)\}_{i\in I}$ be such that $\epsilon_1(d_i)=v_i$ ($i\in I$). Then $b(\mathcal{K}^{1-\mu},\rho)=(1-\mu)\{d_i-d_j\}$, which implies that

$$^{t}b(\mathcal{K}^{1-\mu},\rho)=(1-\mu)\{\,^{t}d_i-\,^{t}d_j\}.$$

On the other hand, $\{-{}^t d_i \in \mathcal{D}_S^{\leq 1}(\mathcal{K}^\mu)(\mathcal{U}_i)\}_{i\in I}$ are local liftings of v, which means that $b(\mathcal{K}^\mu, \rho) = -\mu\{{}^t d_i - {}^t d_j\} = \frac{\mu}{\mu-1}\,{}^t b(\mathcal{K}^{1-\mu}, \rho)$. Thus

$$a(\mathcal{K}^\mu, \rho) = \frac{2\mu - 1}{2\mu} b(\mathcal{K}^\mu, \rho).$$

□

Theorem 2.8. *Replace $\mathcal{M}$ by its affine open sets, let $\{\mathcal{U}_i\}_{i\in I}$ be an affine open covering of S. Then, for any $v \in f^*T_{\mathcal{M}}(S)$, there are*

$$d_S^i \in \mathcal{D}_S^{\leq 1}(\mathcal{K}^\mu)(\mathcal{U}_i), \quad d_{S/\mathcal{M}}^i \in \mathcal{D}_{S/\mathcal{M}}^{\leq 1}(\mathcal{K}^\mu)(\mathcal{U}_i), \quad D_{S/\mathcal{M}}^i \in \mathcal{D}_{S/\mathcal{M}}^{\leq 2}(\mathcal{K}^\mu)(\mathcal{U}_i),$$

where $\mathcal{K} := K_{S/\mathcal{M}}$, such that

$$\left\{ \mathrm{H}(v)_i := d_S^i - d_{S/\mathcal{M}}^i + \frac{2}{1-2\mu} D_{S/\mathcal{M}}^i \in \mathcal{D}_S^{\leq 2}(\mathcal{K}^\mu)(\mathcal{U}_i) \right\}_{i\in I}$$

form a global section $\mathrm{H}(v) \in H^0(S, \mathcal{D}_S^{\leq 2}(\mathcal{K}^\mu))$ with

$$\sigma(\mathrm{H}(v)) = v, \quad \epsilon_2(\mathrm{H}(v)) = \frac{2}{1-2\mu}\delta_{\mathrm{H}}(v).$$

Proof. Let $\{d_S^i \in \mathcal{D}_S^{\leq 1}(\mathcal{K}^\mu)(\mathcal{U}_i)\}_{i\in I}$ be such that $\sigma(d_S^i) = v|_{\mathcal{U}_i}$. Then

$$\{\mu(d_S^i - d_S^j) \in \mathcal{D}_{S/\mathcal{M}}^{\leq 1}(\mathcal{K}^\mu)(\mathcal{U}_i \cap \mathcal{U}_j)\}$$

defines the class $b(\mathcal{K}^\mu, \delta_{\mathrm{H}}(v)) \in H^1(S, \mathcal{D}_{S/\mathcal{M}}^{\leq 1}(\mathcal{K}^\mu))$, which is the obstruction for lifting $\delta_{\mathrm{H}}(v) \in H^0(S, S^2 T_{S/\mathcal{M}})$ to $H^0(S, S^2\mathcal{D}_{S/\mathcal{M}}^{\leq 1}(\mathcal{K}^\mu))$. Let

$$\{D_{S/\mathcal{M}}^i \in \mathcal{D}_{S/\mathcal{M}}^{\leq 2}(\mathcal{K}^\mu)(\mathcal{U}_i)\}_{i\in I}$$

be local liftings of $\delta_{\mathrm{H}}(v)$. Then, by Proposition 2.7,

$$\{d_S^i - d_S^j\} = \frac{2}{2\mu - 1}\{D_{S/\mathcal{M}}^i - D_{S/\mathcal{M}}^j\}$$

as cohomology classes. Thus there are $\{d_{S/\mathcal{M}}^i \in \mathcal{D}_{S/\mathcal{M}}^{\leq 1}(\mathcal{K}^\mu)(\mathcal{U}_i)\}_{i\in I}$ satisfying the requirements in the theorem. □

Corollary 2.9. *There exists uniquely a projective heat operator,*

$$\mathrm{H} : T_{\mathcal{M}} \to f_*\mathcal{W}(\Theta^k)/\mathcal{O}_{\mathcal{M}}$$

such that $(f_*\epsilon_2) \cdot \mathrm{H} : T_{\mathcal{M}} \to f_* S^2 T_{S/\mathcal{M}}$ coincides with $f_*\delta_{\mathrm{H}}$.

Proof. For any open set $U \subset \mathcal{M}$ and $v \in T_{\mathcal{M}}(U)$, by Theorem 2.8, we can construct a $\mathrm{H}(v) \in f_*\mathcal{W}(\Theta^k)(U)$. If $\mathrm{H}(v)'$ is another such operator, $\mathrm{H}(v) - \mathrm{H}(v)'$ must have symbol in $H^0(f^{-1}(U), T_{S/\mathcal{M}}) = 0$, so

$$\mathrm{H}(v) - \mathrm{H}(v)' \in H^0(f^{-1}(U), \mathcal{O}_S) = f_*\mathcal{O}_S(U) = \mathcal{O}_{\mathcal{M}}(U).$$

Hence a unique map $T_{\mathcal{M}} \to f_*\mathcal{W}(\Theta^k)/\mathcal{O}_{\mathcal{M}}$. □

Now we construct the logarithmic extension of above operator. Let $\widetilde{\mathcal{C}} \to \widetilde{\mathcal{M}}$ be the family in Introduction. Let $\mathcal{U}(r,d) \to \widetilde{\mathcal{M}}$ be the family of moduli spaces of semistable (for canonical polarization) torsion free sheaves of rank r and degree d. Fix a line bundle $\mathcal{N}$ on $\widetilde{\mathcal{C}}$ of relative degree d, let $f : S \to \mathcal{M}$ be the family of moduli spaces of stable bundles of rank r with fixed determinant $\mathcal{N}|_C$ ($[C] \in \mathcal{M}$).

Notation 2.10. Let $f_Z : Z \to \widetilde{\mathcal{M}}$ be defined as the Zariski closure of S in $\mathcal{U}(r,d)$ and $f_T : T \to \widetilde{\mathcal{M}}$ be the open set of Z consisting of locally free sheaves. Let $\widetilde{f} : \widetilde{S} \subset T \to \widetilde{\mathcal{M}}$ be the open set of stable bundles. Then $\widetilde{f} : \widetilde{S} \to \widetilde{\mathcal{M}}$ is smooth (cf. Lemma 4.4).

Let E be the unversal bundle on $\widetilde{X}$, where $\widetilde{X}$ is defined by

$$\begin{array}{ccc} \widetilde{X} = \widetilde{\mathcal{C}} \times_{\widetilde{\mathcal{M}}} \widetilde{S} & \xrightarrow{\pi} & \widetilde{S} \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \widetilde{f}} \\ \widetilde{\mathcal{C}} & \longrightarrow & \widetilde{\mathcal{M}}. \end{array}$$

Let $D \subset \widetilde{X}$ be the divisor of singular curves. Then we have

$$\mathcal{G}_E \subset {}^{tr}\mathcal{A}_E^{-1}(\log D)$$

fitting into the exact sequence

$$0 \to \omega_{\widetilde{X}/\widetilde{S}} \to \mathcal{G}_E \xrightarrow{res} \mathbb{E} := \mathcal{E}nd^0(E) \to 0. \tag{2.6}$$

Similarly, there is a sheaf $S(\mathcal{G}_E) \subset \mathrm{Sym}^2(\mathcal{G}_E) \otimes \omega^{-1}_{\widetilde{X}/\widetilde{S}}$ fitting into

$$0 \to \mathcal{G}_E \xrightarrow{\iota} S(\mathcal{G}_E) \xrightarrow{q} \omega^{-1}_{\widetilde{X}/\widetilde{S}} \to 0. \tag{2.7}$$

They induce the following exact sequences

$$0 \to R^1\pi_*(\omega_{\widetilde{X}/\widetilde{S}}) \to R^1\pi_*(\mathcal{G}_E) \xrightarrow{res} R^1\pi_*(\mathbb{E}) \to 0, \tag{2.8}$$

$$0 \to R^1\pi_*(\mathcal{G}_E) \xrightarrow{\iota} R^1\pi_*S(\mathcal{G}_E) \xrightarrow{q} R^1\pi_*(\omega^{-1}_{\widetilde{X}/\widetilde{S}}) \to 0. \tag{2.9}$$

Let $\widetilde{\Delta} \subset \widetilde{X} \times_{\widetilde{S}} \widetilde{X} := P$ be the diagonal and $\mathcal{O}(\widetilde{\Delta})$ be the dual of its ideal sheaf. Then $0 \to \omega_{P/\widetilde{S}} \to \omega_{P/\widetilde{S}}(\widetilde{\Delta}) \to \mathcal{E}xt^1_P(\mathcal{O}_{\widetilde{\Delta}}, \omega_{P/\widetilde{S}}) \to 0$, one checks that the relative dualizing sheaf $\omega_{P/\widetilde{S}}$ is $\omega_{\widetilde{X}/\widetilde{S}} \boxtimes \omega_{\widetilde{X}/\widetilde{S}}$ and $\mathcal{E}xt^1_P(\mathcal{O}_{\widetilde{\Delta}}, \omega_{P/\widetilde{S}})$ is the relative dualizing sheaf of $\widetilde{\Delta}/\widetilde{S}$. Thus we have

$$0 \to \mathcal{O} \to \mathcal{O}(\widetilde{\Delta}) \to \omega^{-1}_{\widetilde{X}/\widetilde{S}} \to 0, \tag{2.10}$$

which similarly induces the commutative diagram

$$\begin{array}{ccc}
R^1\pi_*(\mathcal{G}_E) & \xrightarrow{\delta_1} & R^1\pi_*(\mathcal{G}_E) \\
{\scriptstyle\iota}\big\downarrow & & {\scriptstyle\mathrm{Sym}^2}\big\downarrow \\
R^1\pi_*S(\mathcal{G}_E) & \xrightarrow{\tilde{\delta}} & S^2R^1\pi_*(\mathcal{G}_E) \\
{\scriptstyle q}\big\downarrow & & {\scriptstyle S^2(res)}\big\downarrow \\
R^1\pi_*(\omega^{-1}_{\widetilde{X}/\widetilde{S}}) & \xrightarrow{\delta} & S^2R^1\pi_*(\mathbb{E})
\end{array}$$

where δ_1 denote the map induced by $\tilde{\delta}$, the first vertical exact sequence is (2.9), the second vertical exact sequence is induced by taking 2-th symmetric tensor of (2.8) (note that $\mathcal{O}_{\widetilde{S}} \cong R^1\pi_*(\omega_{\widetilde{X}/\widetilde{S}})$).

Let $B = \widetilde{\mathcal{M}} \setminus \mathcal{M}$ and $W = \widetilde{f}^{-1}(B) \subset \widetilde{S}$. Consider

$$0 \to T_{\widetilde{S}/\widetilde{\mathcal{M}}} \to T_{\widetilde{S}} \xrightarrow{d\widetilde{f}} \widetilde{f}^*T_{\widetilde{\mathcal{M}}} \to 0,$$

$$0 \to \mathcal{D}^{\leq 1}_{\widetilde{S}/\widetilde{\mathcal{M}}}(\mathcal{L}) \to \mathcal{D}^{\leq 1}_{\widetilde{S}}(\mathcal{L}) \xrightarrow{\sigma} \widetilde{f}^*T_{\widetilde{\mathcal{M}}} \to 0.$$

Notation 2.11. Let $T_{\widetilde{\mathcal{M}}}(B) \subset T_{\mathcal{M}}$ be the subsheaf of vector fields that preserve B. Let $T_{\widetilde{S}}(\log W) \subset T_{\widetilde{S}}$, $\mathcal{D}^{\leq 1}_{\widetilde{S}}(\mathcal{L})(\log W) \subset \mathcal{D}^{\leq 1}_{\widetilde{S}}(\mathcal{L})$ be the subsheaves such that the following are exact sequences

$$0 \to T_{\widetilde{S}/\widetilde{\mathcal{M}}} \to T_{\widetilde{S}}(\log W) \xrightarrow{d\widetilde{f}} \widetilde{f}^*T_{\widetilde{\mathcal{M}}}(B) \to 0, \tag{2.11}$$

$$0 \to \mathcal{D}^{\leq 1}_{\widetilde{S}/\widetilde{\mathcal{M}}}(\mathcal{L}) \to \mathcal{D}^{\leq 1}_{\widetilde{S}}(\mathcal{L})(\log W) \xrightarrow{\sigma} \widetilde{f}^*T_{\widetilde{\mathcal{M}}}(B) \to 0. \tag{2.12}$$

Lemma 2.12. *The map $\delta_1 : R^1\pi_*(\mathcal{G}_E) \to R^1\pi_*(\mathcal{G}_E)$ is identity.*

Proof. Since $S \subset \widetilde{S}$ is dense in $\widetilde{S}$ by definition (cf. Notation 2.10), the lemma follows from Lemma 2.4. □

Theorem 2.13. *i) There is an isomorhism $\phi : R^1\pi_*(\mathcal{G}_E) \cong \mathcal{D}^{\leq 1}_{\widetilde{S}/\widetilde{\mathcal{M}}}(\lambda_{\mathbb{E}})$ such that the following diagram is commutative*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{\widetilde{S}} = R^1\pi_*\omega_{\widetilde{X}/\widetilde{S}} & \longrightarrow & R^1\pi_*(\mathcal{G}_E) & \xrightarrow{\mathrm{res}} & R^1\pi_*(\mathbb{E}) & \longrightarrow & 0 \\
 & & {\scriptstyle (2r)\cdot\mathrm{id}}\big\downarrow & & {\scriptstyle\phi}\big\downarrow & & {\scriptstyle\vartheta}\big\downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\widetilde{S}} = R^0\pi_*\omega_{\widetilde{X}/\widetilde{S}}[1] & \longrightarrow & \mathcal{D}^{\leq 1}_{\widetilde{S}/\widetilde{\mathcal{M}}}(\lambda_{\mathbb{E}}) & \xrightarrow{\epsilon_1} & T_{\widetilde{S}/\widetilde{\mathcal{M}}} & \longrightarrow & 0.
\end{array}$$

ii) For any affine covering $\{\widetilde{\mathcal{M}}_i\}_{i\in I}$ of $\widetilde{\mathcal{M}}$, on each $\widetilde{S}_i := \widetilde{f}^{-1}(\widetilde{\mathcal{M}}_i)$, there is an isomorphism $\phi_i : R^1\pi_* S(\mathcal{G}_E) \cong \mathcal{D}^{\leq 1}_{\widetilde{S}}(\lambda_{\mathbb{E}})(\log W)$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^1\pi_*(\mathcal{G}_E) & \xrightarrow{2\cdot\iota} & R^1\pi_*S(\mathcal{G}_E) & \xrightarrow{q} & R^1\pi_*(\omega^{-1}_{\widetilde{X}/\widetilde{S}}) & \longrightarrow & 0 \\ & & \phi\big\downarrow & & \phi_i\big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \mathcal{D}^{\leq 1}_{\widetilde{S}/\widetilde{\mathcal{M}}}(\lambda_{\mathbb{E}}) & \longrightarrow & \mathcal{D}^{\leq 1}_{\widetilde{S}}(\lambda_{\mathbb{E}})(\log W) & \xrightarrow{\sigma} & \widetilde{f}^*T_{\widetilde{\mathcal{M}}}(B) & \longrightarrow & 0 \end{array}$$

is commutative on $\widetilde{S}_i$, and $\{\phi_i - \phi_j\}$ define a class in $\mathrm{H}^1(\Omega^1_{\widetilde{\mathcal{M}}}(\log B))$.

Proof. See Proposition 4.5 and Theorem 4.8. $\square$

Similarly, we have the commutative diagram on each $\widetilde{S}_i$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{D}^{\leq 1}_{\widetilde{S}/\widetilde{\mathcal{M}}}(\lambda_{\mathbb{E}}) & \longrightarrow & \mathcal{D}^{\leq 1}_{\widetilde{S}}(\lambda_{\mathbb{E}})(\log W) & \xrightarrow{\sigma} & \widetilde{f}^*T_{\widetilde{\mathcal{M}}}(B) & \longrightarrow & 0 \\ & & \Big\| & & \tilde{\delta}^i_{\mathrm{H}}\big\downarrow & & \delta_{\mathrm{H}}\big\downarrow & & \\ 0 & \longrightarrow & \mathcal{D}^{\leq 1}_{\widetilde{S}/\widetilde{\mathcal{M}}}(\lambda_{\mathbb{E}}) & \longrightarrow & S^2\mathcal{D}^{\leq 1}_{\widetilde{S}/\widetilde{\mathcal{M}}}(\lambda_{\mathbb{E}}) & \xrightarrow{S^2\epsilon_1} & S^2T_{\widetilde{S}/\widetilde{\mathcal{M}}} & \longrightarrow & 0. \end{array}$$

Proposition 2.14. *For any $\rho = \delta_{\mathrm{H}}(v) \in H^0(\widetilde{S}, S^2T_{\widetilde{S}/\widetilde{\mathcal{M}}})$, where $\widetilde{\mathcal{M}}$ is replaced by its affine open sets and $v \in H^0(\widetilde{S}, \widetilde{f}^*T_{\widetilde{\mathcal{M}}}(B))$, one has*
i) $2a(\mathcal{L},\rho) = b(\mathcal{L},\rho) + {}^t b(\mathcal{L}^{-1}\otimes K_{\widetilde{S}/\widetilde{\mathcal{M}}},\rho)$.
ii) When $\mathcal{L} = K^{\mu}_{\widetilde{S}/\widetilde{\mathcal{M}}}$, where $\mu\in\mathbb{Q}$ and $\mu\neq 1$, one has

$$a(K^{\mu}_{\widetilde{S}/\widetilde{\mathcal{M}}},\rho) = \frac{2\mu-1}{2\mu} b(K^{\mu}_{\widetilde{S}/\widetilde{\mathcal{M}}},\rho).$$

Proof. Note that we still have $\mathcal{K} = K_{\widetilde{S}/\widetilde{\mathcal{M}}} = \lambda_{\mathbb{E}}$, the proof is the same as that of Proposition 2.7. We just remark two points: (1) for any operator d of $\mathcal{D}^{\leq 1}_{\widetilde{S}}(\mathcal{L})(\log W) \subset \mathcal{D}^{\leq 1}_{\widetilde{S}}(\mathcal{L})$, its adjoint operator ${}^t d$ is still in $\mathcal{D}^{\leq 1}_{\widetilde{S}}(\mathcal{L}^{-1}\otimes\mathcal{K})(\log W)$. (2) for any nonzero $\mu\in\mathbb{Q}$, the canonical isomorphism $\psi_\mu : \mathcal{D}^{\leq 1}_{\widetilde{S}}(\mathcal{K}) \cong \mathcal{D}^{\leq}_{\widetilde{S}}(\mathcal{K}^{\mu})$ induces an isomorphism $\mathcal{D}^{\leq 1}_{\widetilde{S}}(\mathcal{K})(\log W) \cong \mathcal{D}^{\leq 1}_{\widetilde{S}}(\mathcal{K}^{\mu})(\log W)$. $\square$

Theorem 2.15. *Replace $\widetilde{\mathcal{M}}$ by its affine open set, let $\{\mathcal{U}_i\}_{i\in I}$ be an affine open covering of $\widetilde{S}$. Then, for any $v \in \widetilde{f}^*T_{\widetilde{\mathcal{M}}}(B)(\widetilde{S})$, there are*

$$d^i_{\widetilde{S}} \in \mathcal{D}^{\leq 1}_{\widetilde{S}}(\mathcal{K}^{\mu})(\log W)(\mathcal{U}_i), \quad d^i_{\widetilde{S}/\widetilde{\mathcal{M}}} \in \mathcal{D}^{\leq 1}_{\widetilde{S}/\widetilde{\mathcal{M}}}(\mathcal{K}^{\mu})(\mathcal{U}_i),$$

and $D^i_{\widetilde{S}/\widetilde{\mathcal{M}}} \in \mathcal{D}^{\leq 2}_{\widetilde{S}/\widetilde{\mathcal{M}}}(\mathcal{K}^{\mu})(\mathcal{U}_i)$ such that

$$\left\{\mathrm{H}(v)_i := d^i_{\widetilde{S}} - d^i_{\widetilde{S}/\widetilde{\mathcal{M}}} + \frac{2}{1-2\mu} D^i_{\widetilde{S}/\widetilde{\mathcal{M}}} \in \mathcal{D}^{\leq 2}_{\widetilde{S}}(\mathcal{K}^{\mu})(\mathcal{U}_i)\right\}_{i\in I}$$

form a global section $\mathrm{H}(v) \in H^0(\widetilde{S}, \mathcal{D}^{\leq 2}_{\widetilde{S}}(\mathcal{K}^{\mu}))$ *with*

$$\sigma(\mathrm{H}(v)) = v, \quad \epsilon_2(\mathrm{H}(v)) = \frac{2}{1-2\mu}\delta_{\mathrm{H}}(v).$$

Proof. Let $\{d^i_{\widetilde{S}} \in \mathcal{D}^{\leq 1}_{\widetilde{S}}(\mathcal{K}^{\mu})(\log W)(\mathcal{U}_i)\}_{i\in I}$ be such that $\sigma(d^i_{\widetilde{S}}) = v|_{\mathcal{U}_i}$. Then

$$\{\mu(d^i_{\widetilde{S}} - d^j_{\widetilde{S}}) \in \mathcal{D}^{\leq 1}_{\widetilde{S}/\widetilde{\mathcal{M}}}(\mathcal{K}^{\mu})(\mathcal{U}_i \cap \mathcal{U}_j)\}$$

defines the class $b(\mathcal{K}^{\mu}, \delta_{\mathrm{H}}(v)) \in H^1(\widetilde{S}, \mathcal{D}^{\leq 1}_{\widetilde{S}/\widetilde{\mathcal{M}}}(\mathcal{K}^{\mu}))$, which is the obstruction for lifting $\delta_{\mathrm{H}}(v) \in H^0(\widetilde{S}, S^2T_{\widetilde{S}/\widetilde{\mathcal{M}}})$ to $H^0(\widetilde{S}, S^2\mathcal{D}^{\leq 1}_{\widetilde{S}/\widetilde{\mathcal{M}}}(\mathcal{K}^{\mu})$. Let

$$\{D^i_{\widetilde{S}/\widetilde{\mathcal{M}}} \in \mathcal{D}^{\leq 2}_{\widetilde{S}/\widetilde{\mathcal{M}}}(\mathcal{K}^{\mu})(\mathcal{U}_i)\}_{i\in I}$$

be local liftings of $\delta_{\mathrm{H}}(v)$. Then, by Proposition 2.5,

$$\{d^i_{\widetilde{S}} - d^j_{\widetilde{S}}\} = \frac{2}{2\mu - 1}\{D^i_{\widetilde{S}/\widetilde{\mathcal{M}}} - D^j_{\widetilde{S}/\widetilde{\mathcal{M}}}\}$$

as cohomology classes. Thus there are $\{d^i_{\widetilde{S}/\widetilde{\mathcal{M}}} \in \mathcal{D}^{\leq 1}_{\widetilde{S}/\widetilde{\mathcal{M}}}(\mathcal{K}^{\mu})(\mathcal{U}_i)\}_{i\in I}$ satisfying the requirements in the theorem. □

Corollary 2.16. *There exists uniquely a projective heat operator*

$$\widetilde{\mathrm{H}} : T_{\widetilde{\mathcal{M}}}(B) \to \widetilde{f}_*\mathcal{D}^{\leq 2}_{\widetilde{S}}(\Theta^k)_B/\mathcal{O}_{\widetilde{\mathcal{M}}}$$

such that $(\widetilde{f}_*\epsilon_2) \cdot \widetilde{\mathrm{H}} : T_{\widetilde{\mathcal{M}}}(B) \to f_*S^2T_{S/\mathcal{M}}$ *coincides with* $\widetilde{f}_*\delta_{\mathrm{H}}$. *Moreover,* $\widetilde{f}_*(\Theta^k)$ *is a coherent sheaf on* $\widetilde{\mathcal{M}}$.

Proof. The coherence of $\widetilde{f}_*(\Theta^k)$ follows from Theorem 4.10. Since $\widetilde{f}_*T_{\widetilde{S}/\widetilde{\mathcal{M}}} = 0$ is still true, the rest follows the same proof of Corollary 2.9 if $\widetilde{f}_*\mathcal{O}_{\widetilde{S}} = \mathcal{O}_{\widetilde{\mathcal{M}}}$. We will prove (cf. Proposition 4.9) that the fibres of $\widetilde{f} : \widetilde{S} \to \widetilde{\mathcal{M}}$ is dense in the fibres of $f_Z : Z \to \widetilde{\mathcal{M}}$. Thus $Z \setminus \widetilde{S}$ has codimension at least 2. By passing to the normalization $\iota : \widetilde{Z} \to Z$, $\iota^{-1}(\widetilde{S}) \cong \widetilde{S}$ and $\mathrm{codim}(\widetilde{Z}\setminus\iota^{-1}(\widetilde{S})) \geq 2$ since $\widetilde{S}$ is the open set of smooth points of Z and ι is finite. We have $\widetilde{f}_*\mathcal{O}_{\widetilde{S}} = (f_Z \cdot \iota)_*\mathcal{O}_{\widetilde{Z}} = \mathcal{O}_{\widetilde{\mathcal{M}}}$. □

3. First Order Differential Operators of the Determinant Bundle

We give at first a short review of what we need from [2]. Let $\pi : X \to S$ be a smooth proper morphism of relative dimension 1 between smooth varieties in characteristic 0. We write $K_{X/S}$ or $\omega_{X/S}$ interchangeably for the dualizing sheaf. One has an exact sequence

$$0 \to T_{X/S} \to T_X \xrightarrow{d\pi} \pi^*T_S \to 0. \tag{3.1}$$

As in [2], one defines the subsheaf $\pi^{-1}T_S \subset \pi^*T_S$ and its preimage $T_\pi = d\pi^{-1}T_\pi \subset T_X$, defining the exact sequence

$$0 \to T_{X/S} \to T_\pi \xrightarrow{d\pi} \pi^{-1}T_S \to 0. \tag{3.2}$$

Let E be a vector bundle on X, and $\lambda_E = \det R\pi_*E$ be its determinant bundle. The Atiyah algebra $\mathcal{A}_E$ is the subalgebra of the sheaf of first order differential operators on E with symbolic part in $(\mathrm{id}\otimes T_X) \cong T_X$. The relative Atiyah algebra $\mathcal{A}_{E/S} \subset \mathcal{A}_E$ consists of those differential operators with symbolic part in $T_{X/S}$, and $\mathcal{A}_{E,\pi} \subset \mathcal{A}_E$ with symbolic part in T_π. Let ${}^{tr}\mathcal{A}_E^{-1}$ be the subquotient of the sheaf defined in [2]

$$E \boxtimes_{\mathcal{O}_S} (E^* \otimes \omega_{X/S})(2\Delta)/E \boxtimes_{\mathcal{O}_S} (E^* \otimes \omega_{X/S})(-\Delta)$$

where $\Delta \subset X \times_S X$ denotes the diagonal, which fits into an exact sequence

$$0 \to \omega_{X/S} \to {}^{tr}\mathcal{A}_E^{-1} \xrightarrow{\mathrm{res}} \mathcal{A}_{E/S} \to 0. \tag{3.3}$$

The trace complex is defined by

$${}^{tr}\mathcal{A}_E^{\bullet} : \mathcal{O}_X \xrightarrow{d_{X/S}} {}^{tr}\mathcal{A}_E^{-1} \xrightarrow{\mathrm{res}} \mathcal{A}_{E,\pi} \tag{3.4}$$

with $\mathcal{A}_{E,\pi}$ in degree 0. One has

Proposition 3.1. *${}^{tr}\mathcal{A}_E^{\bullet}$ carries an algebra structure for which $R^0\pi_*({}^{tr}\mathcal{A}_E^{\bullet})$ is canonically isomorphic to $\mathcal{A}_{\lambda_E}$ ([2], 2.3.1, see also [3]).*

For the purpose of this paper it is more convenient to define the trace complex concentrated only on $i = -1$ and $i = 0$ of the original trace complex. This modified trace complex is still denoted by ${}^{tr}\mathcal{A}_E^{\bullet}$ whose 0-th direct image is easily seen to be the same as that of the original one. With the modified trace complex, one has now an exact sequence

$$0 \to \omega_{X/S}[1] \to {}^{tr}\mathcal{A}_E^{\bullet} \to \mathcal{A}_{E,\pi}^{\bullet} \to 0, \tag{3.5}$$

where the complex $\mathcal{A}_{E,\pi}^{\bullet}$ is defined by

$$\mathcal{A}_{E,\pi}^{\bullet} : \mathcal{A}_{E/S} \to \mathcal{A}_{E,\pi}, \tag{3.6}$$

and thus is quasi-isomorphic to $\pi^{-1}T_S$.

Notation 3.2. Let $\pi : X \to S$ be as before, and $f : S \to M$ be a smooth morphism where M is a smooth variety. Denote by

$$\mathcal{A}_{E,\pi/\mathcal{M}} \subset \mathcal{A}_{E,\pi}, \quad T_{\pi/\mathcal{M}} \subset T_{X/\mathcal{M}} \subset T_\pi$$

the pullback of $\pi^{-1}T_{S/\mathcal{M}} \subset \pi^{-1}T_S$. Let

$${}^{tr}\mathcal{A}_{E/\mathcal{M}}^{\bullet} := ({}^{tr}\mathcal{A}_E^{-1} \to \mathcal{A}_{E,\pi/\mathcal{M}}), \quad \mathcal{A}_{E,\pi/\mathcal{M}}^{\bullet} := (\mathcal{A}_{E/S} \to \mathcal{A}_{E,\pi/\mathcal{M}}).$$

Proposition 3.3. *The exact sequences*

$$0 \to \omega_{X/S}[1] \to {}^{tr}\mathcal{A}_E^\bullet \to \mathcal{A}_{E,\pi}^\bullet \to 0$$
$$0 \to \omega_{X/S}[1] \to {}^{tr}\mathcal{A}_{E/M}^\bullet \to \mathcal{A}_{E,\pi/M}^\bullet \to 0$$

have 0*-th direct images (via* π*) isomorphic to*

$$0 \to \mathcal{O}_S \to \mathcal{D}_S^{\leq 1}(\lambda_E) \to T_S \to 0$$
$$0 \to \mathcal{O}_S \to \mathcal{D}_{S/M}^{\leq 1}(\lambda_E) \to T_{S/M} \to 0.$$

Furthermore, we need to review the description of ${}^{tr}\mathcal{A}_E^\bullet$ in terms of local coordinates, [2], p. 660. Let t be a local coordinate (along the fiber), and a trivilization $\mathcal{O}_X^n \cong E$; s a local coordinate on S. Note t naturally induces local coordinates (t_1, t_2) around the diagonal of $X \times_S X$. One has isomorphisms

$$\mathcal{O}_X \oplus \mathrm{Mat}_n(\mathcal{O}_X) \oplus \mathcal{O}_X \cong {}^{tr}\mathcal{A}_E^{-1}; \tag{3.7}$$
$$(\chi, B, \nu) \to \left[\frac{\chi(t_1)}{(t_2 - t_1)^2} + \frac{B(t_1)}{(t_2 - t_1)} + \nu(t_1)\right] dt_2;$$
$$T_\pi \oplus \mathrm{Mat}_n(\mathcal{O}_X) \cong {}^{tr}\mathcal{A}_E^0 = \mathcal{A}_{E,\pi}; (\tau, A) \to \tau(t,s)\partial_t + \mu(s)\partial_s + A.$$

For different choices of coordinates and trivializations, there are formulas for transition functions, namely the gauge change and coordinate change formulas, where $g \in \mathrm{GL}_r(\mathcal{O}_X)$ and $y = y(t)$,

$$(\tau, A) \xrightarrow{g} (\tau, -\tau(g)g^{-1} + gAg^{-1}); \tag{3.8}$$
$$(\chi, B, \nu) \xrightarrow{g}$$
$$\left(\chi, -\chi g' g^{-1} + gBg^{-1}, \mathrm{Tr}\big(-\frac{1}{2}\chi g'' g^{-1} + \chi(g'g^{-1})^2 - Bg^{-1}g'\big) + \nu\right);$$
$$(\chi, B, \nu) \xrightarrow{y(t)}$$
$$\left(\chi {y'}^{-1}, B, r\chi\left(\frac{1}{6}\frac{y'''}{y'^2} - \frac{1}{4}\frac{y''^2}{y'^3}\right) + \frac{1}{2}\frac{y''}{y'}\mathrm{Tr}B + \nu y'\right).$$

The main result of this section is Theorem 3.7 which enables us to take care of Theorem 2.5. Follow the notation of Section 2, let

$$p : X = \mathcal{C} \times_{\mathcal{M}} S \to \mathcal{C}$$

be the projection.

Definition 3.4. $\mathcal{E}nd(E)^{-1} := \mathrm{res}^{-1}(\mathcal{E}nd(E)) \subset {}^{tr}\mathcal{A}_E^{-1}$, and

$$\mathcal{G}_E := \mathrm{res}^{-1}(\mathbb{E}) \subset \mathcal{E}nd(E)^{-1}$$

where $\mathcal{E}nd(E) = \mathbb{E} \oplus \mathcal{O}_X$ with its trace free part $\mathbb{E} := \mathcal{E}nd^0(E)$ and the trivial bundle $\mathcal{O}_X$. There are exact sequences

$$0 \to \omega_{X/S} \to \mathcal{G}_E \xrightarrow{\text{res}} \mathbb{E} \to 0;$$
$$0 \to \omega_{X/S} \to \mathcal{E}nd(E)^{-1} \xrightarrow{\text{res}} \mathcal{E}nd(E) \to 0.$$

Consider the natural morphism

$$S^2(\mathcal{G}_E) \otimes_{\mathcal{O}_X} T_{X/S} \xrightarrow{q} S^2(\mathbb{E}) \otimes_{\mathcal{O}_X} T_{X/S} \to 0$$

induced by $\mathcal{G}_E \xrightarrow{\text{res}} \mathbb{E}$. Denote the kernel of q by $\mathcal{K}$. There is a canonical isomorphism $\iota : \mathcal{G}_E \cong \mathcal{K}$, that is $\iota(s) = \text{Sym}^2(dt \otimes s) \otimes \partial_t$ locally.

Definition 3.5. $q^{-1}(\text{id} \otimes T_{X/S}) := S(\mathcal{G}_E)$ where $\text{id} \in S^2(\mathbb{E})$ is the identity element. It follows that we have the exact sequence

$$0 \to \mathcal{G}_E \xrightarrow{\iota} S(\mathcal{G}_E) \to T_{X/S} \to 0. \tag{3.9}$$

Locally, for choosen coordinate and trivilization (cf. (3.7)), any local section $s \in S(\mathcal{G}_E)$ is of the form

$$s = \begin{pmatrix} \chi \sum_a (0, J_a, 0) \otimes (0, J_a, 0) + \sum_a \mu_a(0, J_a, 0) \otimes (0, 0, 1) \\ + \sum_a \nu_a(0, 0, 1) \otimes (0, J_a, 0) + w(0, 0, 1) \otimes (0, 0, 1) \end{pmatrix} \otimes \partial_t$$

where J_a is a (local) basis of $\mathbb{E}$ (which we assume to be orthonormal under $\text{Trace}(\,\cdot\,)$) such that $\sum_a J_a \otimes J_a = \text{id}$.

We will need the Kodaira-Spencer maps

$$\begin{aligned} \text{KS}_S : T_S \to R^1\pi_*\mathcal{A}^0_{E/S}, \quad \mathcal{A}^0_{E/S} = \mathcal{A}_{E/S}/\mathcal{O}_X; \\ \text{KS}_{\mathcal{M}} : f^*T_{\mathcal{M}} \to R^1\pi_*T_{X/S} \end{aligned} \tag{3.10}$$

They fit into the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & R^1\pi_*(\mathbb{E}) \arrow[r] & R^1\pi_*\mathcal{A}^0_{E/S} \arrow[r] & R^1\pi_*T_{X/S} \arrow[r] & 0 \\ 0 \arrow[r] & T_{S/\mathcal{M}} \arrow[r] \arrow[u, "\text{KS}"] & T_S \arrow[r] \arrow[u, "\text{KS}_\text{S}"] & f^*T_{\mathcal{M}} \arrow[r] \arrow[u, "\text{KS}_{\mathcal{M}}"] & 0. \end{tikzcd}$$

Remark 3.6. One way to see KS_S of (3.10) is via the natural map (with cohomology) $\mathcal{A}^\bullet_{E,\pi} \to \mathcal{A}^0_{E/S}[1]$ from (3.6); similarly $\text{KS}_{\mathcal{M}}$ is via $(T_{\mathcal{C}/\mathcal{M}} \to T_\pi) \to T_{\mathcal{C}/\mathcal{M}}[1]$ (with cohomology) combined with its pullback via $f : S \to \mathcal{M}$. The diagram (3.10) via natural maps $T_S \to f^*T_{\mathcal{M}}$ and $R^1\pi_*\mathcal{A}^0_{E/S} \to R^1\pi_*T_{X/S}$, for Kodaira-Spencer maps commute.

Theorem 3.7. *i) If the Kodaira-Spencer map $KS : T_{S/\mathcal{M}} \to R^1\pi_*(\mathbb{E})$ is an isomorphism, then there exists a canonical isomorphism*

$$\phi : R^1\pi_*(\mathcal{G}_E) \cong R^0\pi_*(\mathcal{E}nd(\mathbb{E})^{-1} \to \mathcal{A}_{\mathbb{E},\pi/\mathcal{M}}) \cong R^0\pi_* {}^{tr}\mathcal{A}^{\bullet}_{\mathbb{E}/\mathcal{M}}.$$

ii) If KS_S is an isomorphism and shrink $\mathcal{M}$ enough, then the above ϕ extends to an isomorphism

$$\phi = \phi_{-2r} : R^1\pi_* S(\mathcal{G}_E) \cong R^0\pi_*({}^{tr}\mathcal{A}^{-1}_{\mathbb{E}} \to \mathcal{A}_{\mathbb{E},\pi}).$$

(The R.H.S. of i), ii) are canonically identified with $\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\lambda_{\mathbb{E}})$, $\mathcal{D}^{\leq 1}_{S}(\lambda_{\mathbb{E}})$ respectively, cf. Proposition 3.3.)

Remark 3.8. $R^0\pi_*(\mathcal{E}nd(\mathbb{E})^{-1} \to \mathcal{A}_{\mathbb{E},\pi/\mathcal{M}}) \cong R^0\pi_*(\mathcal{E}nd(\mathbb{E})^{-1} \to \mathcal{A}_{\mathbb{E},\pi})$ holds under $\mathcal{A}_{\mathbb{E},\pi/\mathcal{M}} \to \mathcal{A}_{\mathbb{E},\pi}$, cf. the proof of i) of Proposition 4.5.

Corollary 3.9. *Assumptions being as in ii) of Theorem 3.7, suppose $\lambda_{\mathbb{E}} \cong \Theta^{-\lambda}$ ($\lambda = 2r$). For $k \in \mathbb{Z}$ one has an isomorphism denoted by*

$$\phi_k : R^1\pi_* S(\mathcal{G}_E) \cong \mathcal{D}^{\leq 1}_{S}(\Theta^k),$$

extending Theorem 3.7. (If $k = -\lambda$, write ϕ for $\phi_{-\lambda}$.)

Proposition 3.10. *i) The morphism $\mathbb{E} \to \mathcal{E}nd^0(\mathbb{E})$ induced by the adjoint representation extends naturally to a canonical morphism* ad : $\mathcal{A}_E \to \mathcal{A}_{\mathbb{E}}$ *(preserving algebra structures). ii) The morphism* ad *has a natural lifting $\widetilde{\mathrm{ad}} : \mathcal{G}_E \to \mathcal{G}_{\mathbb{E}}$, which induces $(2r)\cdot \mathrm{id}$ on $\omega_{X/S}$.*

Proof. i) For any $D \in \mathcal{A}_E$, $L \in \mathbb{E}$, $\mathrm{ad}(D)(L) := D \circ L - L \circ D$ is a section of $\mathbb{E}$ (note that $\mathrm{Tr}(D \circ L - L \circ D) = \epsilon_1(D)\mathrm{Tr}(L)$). Thus $\mathrm{ad}(D)$ defines a map $\mathbb{E} \to \mathbb{E}$, which is a differential operator since $\mathrm{ad}(D)(\lambda \cdot L) = \lambda \cdot \mathrm{ad}(D)(L) + \epsilon_1(D)(\lambda)\cdot L$.

ii) This can be proved via the local formulas given in (3.7) Namely, a local element of $\mathcal{G}_E$ is expressed as $(0, B, \nu)$ (with $B \in \mathrm{Mat}_r(\mathcal{O}_X)$ and $\nu \in \mathcal{O}_X$). Define a lifting by sending $(0, B, \nu)$ to $(0, \mathrm{ad}B, 2r\nu)$. We will show by using formulas in (3.8) that the above lifting is in fact globally defined. Using $\chi = 0$, $\mathrm{Tr}B = 0$ and $\mathrm{Tr}(\mathrm{ad}B) = 0$ in (3.8), we have

$$(0, B, \nu) \xrightarrow{g} \big(0, gBg^{-1}, \mathrm{Tr}(-Bg^{-1}g') + \nu\big);$$

$$(0, B, \nu) \xrightarrow{y(t)} \big(0, B, \nu y'\big);$$

$$(0, \mathrm{ad}B, \nu) \xrightarrow{g} \Big(0, \mathrm{ad}(gBg^{-1}), \mathrm{Tr}\big(-\mathrm{ad}B\,\mathrm{ad}(g^{-1}g')\big) + \nu\Big);$$

$$(0, \mathrm{ad}B, \nu) \xrightarrow{y(t)} (0, \mathrm{ad}B, \nu y').$$

If change the trivilization of E by g, the induced trivilization of $\mathcal{E}nd(E)$ will be changed by $e_g : \mathbb{M}_r(\mathcal{O}_X) \to \mathbb{M}_r(\mathcal{O}_X)$, where $e_g(B) = gBg^{-1}$.

It is easy to check that $e_g^{-1}e_g' = \mathrm{ad}(g^{-1}g')$, thus we obtain the term $\mathrm{Tr}\big(-\mathrm{ad}B\,\mathrm{ad}(g^{-1}g')\big)$ in the 3rd row above. One knows that

$$\mathrm{Tr}(\mathrm{ad}M\,\mathrm{ad}N) = 2r\mathrm{Tr}(MN) \tag{3.11}$$

for traceless matrices M, N of rank r. Let $(g^{-1}g')_0$ be the traceless compoment of $g^{-1}g'$. Note $\mathrm{Tr}B = 0$, $\mathrm{ad}(g^{-1}g') = \mathrm{ad}((g^{-1}g')_0)$, and $\mathrm{Tr}(Bg^{-1}g') = \mathrm{Tr}(B(g^{-1}g')_0)$ etc. It follows that the morphism $(0, B, \nu) \to (0, \mathrm{ad}B, 2r\nu)$ as given is well-defined (globally). □

Lemma 3.11. *$R^0\pi_*\mathcal{A}_{E,\pi} \cong \mathcal{O}_S$ and $R^0\pi_*\mathcal{A}^0_{E,\pi} = 0$ provided KS_S being injective, where $\mathcal{A}^0_{E,\pi} := \mathcal{A}_{E,\pi}/\mathcal{O}_X$.*

Proof. It suffices to prove, by using an exact sequence similar to (3.2) (with $T_\bullet$ replaced by $\mathcal{A}_\bullet$), that i) $\pi_*\pi^{-1}T_S \to R^1\pi_*\mathcal{A}_{E/S}$ is injective and ii) $\pi_*\mathcal{A}_{E/S} \cong \mathcal{O}_S$. But the map in i) composed with $R^1\pi_*\mathcal{A}_{E/S} \to R^1\pi_*\mathcal{A}^0_{E/S}$ is nothing but KS_S, hence i). ii) is from $\pi_*T_{X/S} = 0$ (genus ≥ 2) and $\pi_*\mathcal{E}nd(E) = \mathcal{O}_S$ (E is fiberwise stable). □

We are ready to give a proof of i) of Theorem 3.7.

Proof. Firstly, we remark that morphisms in Proposition 3.10 make the following diagram of complexes commutative

$$\begin{array}{ccc}
({}^{tr}\mathcal{A}_{\mathbb{E}}^{-1} \xrightarrow{\mathrm{res}} \mathcal{A}_{\mathbb{E},\pi/\mathcal{M}}) & \longrightarrow & (\mathcal{A}_{\mathbb{E}/S} \to \mathcal{A}_{\mathbb{E},\pi/\mathcal{M}}) \\
\uparrow & & \uparrow \\
(\mathcal{G}_E \xrightarrow{\mathrm{res}} \mathcal{A}_{E,\pi/\mathcal{M}}) & \longrightarrow & (\mathcal{A}^0_{E/S} \to \mathcal{A}^0_{E,\pi/\mathcal{M}}).
\end{array}$$

Secondly, we observe that the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{E} & \longrightarrow & \mathcal{A}^0_{E,\pi/\mathcal{M}} & \longrightarrow & T_{\pi/\mathcal{M}} & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbb{E} & \longrightarrow & \mathcal{A}^0_{E/S} & \longrightarrow & T_{X/S} & \longrightarrow & 0
\end{array}$$

induces a commutative diagram

$$\begin{array}{ccccccc}
T_{S/\mathcal{M}} & = & \pi_*T_{\pi/\mathcal{M}} & \xrightarrow{\mathrm{KS}} & R^1\pi_*(\mathbb{E}) & \longrightarrow & R^1\pi_*\mathcal{A}^0_{E,\pi/\mathcal{M}} \\
 & & \uparrow & & \| & & \uparrow \\
 & & 0 & \longrightarrow & R^1\pi_*(\mathbb{E}) & \longrightarrow & R^1\pi_*\mathcal{A}^0_{E/S}.
\end{array}$$

Thus $R^1\pi_*(\mathbb{E})$ vanishes in $R^1\pi_*\mathcal{A}^0_{E,\pi/\mathcal{M}}$ since KS is an isomorphism.

We construct ϕ for any affine open set $U^i \subset S$. Let $\{U_i, \dot{X}_i\}$ be an affine covering of $\pi^{-1}(U^i)$, let $\dot{U}_i = U_i \cap \dot{X}_i$. For any Čech cocycle $r_{\dot{U}_i} \in \mathcal{G}_E(\dot{U}_i)$ in $C^1(\mathcal{G}_E)$, the class $[\mathrm{res}(r_{\dot{U}_i})] \in R^1\pi_*(\mathbb{E})(U^i)$

vanishes in $R^1\pi_*\mathcal{A}^0_{E,\pi/\mathcal{M}}(U^i)$. Thus there exists $\tau_{\dot{X}_i} \in \mathcal{A}^0_{E,\pi/\mathcal{M}}(\dot{X}_i)$, $\tau_{U_i} \in \mathcal{A}^0_{E,\pi/\mathcal{M}}(U_i)$ such that $\tau_{\dot{X}_i} - \tau_{U_i} = \operatorname{res} r_{\dot{U}_i}$ on $\dot{U}_i$. For given $r_{\dot{U}_i}$, the choice of $\tau_{\dot{X}_i}$ and τ_{U_i} is unique (by Lemma 3.11). Then

$$\{\mathrm{ad}(\tau_{\dot{X}_i}), \mathrm{ad}(\tau_{U_i}); \widetilde{\mathrm{ad}}(r_{\dot{U}_i})\} \tag{3.12}$$

is a Čech cocycle in $C^0(\mathcal{G}_{\mathbb{E}} \to \mathcal{A}_{\mathbb{E},\pi/\mathcal{M}})$ (cf. [2], p. 673). It is easily checked that the assignment

$$\tilde{\phi} : r_{\dot{U}_i} \to \{\mathrm{ad}(\tau_{\dot{X}_i}), \mathrm{ad}(\tau_{U_i}); \widetilde{\mathrm{ad}}(r_{\dot{U}_i})\} \quad \in C^0({}^{tr}\mathcal{A}^\bullet_{\mathbb{E}/\mathcal{M}}) \tag{3.13}$$

preserves the respective coboundaries. Hence it descends to a map

$$\phi : R^1\pi_*(\mathcal{G}_E) \to R^0\pi_*\,{}^{tr}\mathcal{A}^\bullet_{\mathbb{E}/\mathcal{M}}.$$

By the same way, we construct $\vartheta : R^1\pi_*(\mathbb{E}) \to R^0\pi_*\mathcal{A}^\bullet_{\mathbb{E},\pi/\mathcal{M}}$ such that

$$\begin{tikzcd}
0 \arrow[r] & R^1\pi_*\omega_{X/S} \arrow[r] \arrow[d, "(2r)\cdot\mathrm{id}"'] & R^1\pi_*(\mathcal{G}_E) \arrow[r, "\mathrm{res}"] \arrow[d, "\phi"'] & R^1\pi_*(\mathbb{E}) \arrow[r] \arrow[d, "\vartheta"'] & 0 \\
0 \arrow[r] & R^0\pi_*\omega_{X/S}[1] \arrow[r] & R^0\pi_*\,{}^{tr}\mathcal{A}^\bullet_{\mathbb{E}/\mathcal{M}} \arrow[r, "\mathrm{res}"] & R^0\pi_*\mathcal{A}^\bullet_{\mathbb{E},\pi/\mathcal{M}} \arrow[r] & 0
\end{tikzcd}$$

is commutative. The map ϑ is the composition of

$$\mathrm{KS}^{-1} : R^1\pi_*(\mathbb{E}) \to T_{S/\mathcal{M}} = R^0\pi_*(\mathcal{A}^0_{E/S} \to \mathcal{A}^0_{E,\pi/\mathcal{M}})$$

and the map $R^0\pi_*(\mathcal{A}^0_{E/S} \to \mathcal{A}^0_{E,\pi/\mathcal{M}}) \to R^0\pi_*\mathcal{A}^\bullet_{\mathbb{E},\pi/\mathcal{M}}$, which is induced by the qusi-isomorphism of complexes at the begining of our proof. Thus ϑ is an isomorphism, then ϕ has to be an isomorphism. $\square$

Both $R^1\pi_*(\mathcal{G}_E)$ and $R^0\pi_*(\mathcal{E}nd(\mathbb{E})^{-1} \to \mathcal{A}_{\mathbb{E},\pi/\mathcal{M}}) = R^0\pi_*\,{}^{tr}\mathcal{A}^\bullet_{\mathbb{E}/\mathcal{M}}$ define extension classes in $H^1(S, \Omega_{S/\mathcal{M}})$. One has (by the preceding proof combined with proof of Proposition 3.10 for the constant $2r$)

Corollary 3.12. *The extension classes (*e.c.[$\bullet$] *for short) satisfy*

$$\text{e.c.}[R^1\pi_*(\mathcal{G}_E)] = \frac{1}{2r}\text{e.c.}[\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\lambda_{\mathbb{E}})]$$

in $H^1(S, \Omega_{S/\mathcal{M}})$, *where* $\mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\lambda_{\mathbb{E}}) = R^0\pi_*(\mathcal{E}nd(\mathbb{E})^{-1} \to \mathcal{A}_{\mathbb{E},\pi/\mathcal{M}})$.

For ii) of the theorem, our proof will need the following result.

Proposition 3.13. *There exisits* $\widetilde{\mathrm{res}} : S(\mathcal{G}_E) \to \mathcal{A}^0_{E/S}$ *such that*

$$\begin{tikzcd}
0 \arrow[r] & \omega_{X/S} \arrow[r] \arrow[d] & S(\mathcal{G}_E) \arrow[r, "\widetilde{\mathrm{res}}"] \arrow[d, equal] & \mathcal{A}^0_{E/S} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{G}_E \arrow[r, "\iota"] & S(\mathcal{G}_E) \arrow[r] & T_{X/S} \arrow[r] & 0
\end{tikzcd}$$

is commutative. If KS_S *in* (3.10) *is an isomorphism, then*

$$0 \to \mathcal{O}_S \to R^1\pi_* S(\mathcal{G}_E) \to T_S \to 0.$$

Proof. For any local section $s \in S(\mathcal{G}_E)$ in Definition 3.5, we define

$$\widetilde{\mathrm{res}}(s) = (\chi\partial_t, \frac{1}{2}\sum_a(\mu_a + \nu_a)J_a),$$

which is independent of the choice of $\{J_a\}$, thus well-defined locally. To show it well-defined globally, we need to check the invariance under gauge and coordinate changes. The invariance under local coordinate changes is straightforward. Under the gauge change $g \in \mathrm{GL}_r(\mathcal{O}_X)$, the section s becomes into $s_g \otimes \partial_t$, where s_g is

$$\begin{aligned}&\chi\sum_a(0, gJ_ag^{-1}, 0)^{\otimes 2} + \sum_a(\mu_a - \chi\mathrm{Tr}(J_ag^{-1}g'))(0, gJ_ag^{-1}, 0) \otimes (0, 0, 1)\\ &+\sum_a(\nu_a - \chi\mathrm{Tr}(J_ag^{-1}g'))(0, 0, 1) \otimes (0, gJ_ag^{-1}, 0)\\ &+\left(\chi\sum_a\mathrm{Tr}(J_ag^{-1}g')^2 - \sum_a(\mu_a + \nu_a)\mathrm{Tr}(J_ag^{-1}g') + w\right)(0, 0, 1)^{\otimes 2}\end{aligned}.$$

Then $\widetilde{\mathrm{res}}(s_g \otimes \partial_t) = (\chi\partial_t, \frac{1}{2}\sum_a(\mu_a + \nu_a - 2\chi\mathrm{Tr}(J_ag^{-1}g'))gJ_ag^{-1})$ coincides with $(\chi\partial_t, -\chi\partial_t(g)g^{-1} + \frac{1}{2}\sum_a(\mu_a+\nu_a)gJ_ag^{-1})$ in $\mathcal{A}^0_{E/S} = \mathcal{A}_{E/S}/\mathcal{O}_X$ since $\chi\partial_t(g)g^{-1} = \chi g'g^{-1} = \sum_a \chi\mathrm{Tr}(J_ag^{-1}g'))gJ_ag^{-1}$ modulo $\mathcal{O}_X$. Thus $\widetilde{\mathrm{res}}$ is gauge invariant and defined globally. Then the rest of this proposition is obvious. □

Remark 3.14. i) $\mathrm{Sym}^2(a\otimes b) = \frac{1}{2}(a\otimes b + b\otimes a)$ for $a\,, b \in \mathcal{G}_E$. ii) Using Proposition 3.13 and assuming $R^1f_*\mathcal{O}_S = 0$ one has a quick interpretation of ii) of Theorem 3.7. Note both $R^1\pi_*S(\mathcal{G}_E)$ and $R^0\pi_*\,{}^{tr}\mathcal{A}^{\bullet}_{\mathbb{E}} \cong \mathcal{D}^{\leq 1}_S(\lambda_{\mathbb{E}})$ contain a subsheaf $R^1\pi_*(\mathcal{G}_E) \cong \mathcal{D}^{\leq 1}_{S/\mathcal{M}}(\lambda_{\mathbb{E}})$. Given two extensions $\mathcal{F}$, $\mathcal{F}'$ of T_S by $\mathcal{O}_S$ suppose their subsheafs with symbolic part in $T_{S/\mathcal{M}}$ are isomorphic, then $\mathcal{F} \cong \mathcal{F}'$ (non-canonically) provided $R^1f_*\mathcal{O}_S = 0$ (since $H^1(S, \Omega_S) \to H^1(S, \Omega_{S/\mathcal{M}})$ is injective if $R^1f_*\mathcal{O}_S = 0$ (and $\mathcal{M}$ is affine)). The ii) of Theorem 3.7 just proves an isomorphism of this kind without reference to $R^1f_*\mathcal{O}_S = 0$.

In what follows, for simplicity, we will cover $\mathcal{C}$ by two affine open sets V and $\dot{\mathcal{C}}$ (shrink $\mathcal{M}$ if necessary). Then we choose and fix the local coordinates (along the fibre) on V and $\dot{\mathcal{C}}$. Let $U = p^{-1}(V)$, $\dot{X} = p^{-1}(\dot{\mathcal{C}})$ and $\{U^i\}_i$ be an affine covering of S. Let $U_i := U \cap \pi^{-1}(U^i)$ and $\dot{X}_i = \dot{X} \cap \pi^{-1}(U^i)$. Then it is important that on each U_i (resp. $\dot{X}_i$) we use the local coordinate pulling back from V (resp. $\dot{\mathcal{C}}$). Thus any construction on U (resp. $\dot{X}$) using the local description (3.7) only depends on the trivialization of E over U_i (resp. $\dot{X}_i$). We start with the construction of $\gamma_{\mathbb{E},U} : S(\mathcal{G}_E)|_U \to {}^{tr}\mathcal{A}_{\mathbb{E}}^{-1}|_U$ (resp. $\gamma_{\mathbb{E},\dot{X}} : S(\mathcal{G}_E)|_{\dot{X}} \to {}^{tr}\mathcal{A}_{\mathbb{E}}^{-1}|_{\dot{X}}$) such that the following diagram $(*)$ is commutative over U (resp. over $\dot{X}$)

$$\begin{tikzcd}
0 \arrow[r] & \omega_{X/S} \arrow[r] & {}^{tr}\mathcal{A}_{\mathbb{E}}^{-1} \arrow[r, "\mathrm{res}"] & \mathcal{A}_{\mathbb{E}/S} \arrow[r] & 0 \\
0 \arrow[r] & \omega_{X/S} \arrow[r] \arrow[u, "r\cdot\mathrm{id}"] \arrow[d] & S(\mathcal{G}_E) \arrow[r, "\widetilde{\mathrm{res}}"] \arrow[u, "\gamma_{\mathbb{E},U}"] \arrow[d, equal] & \mathcal{A}^0_{E/S} \arrow[r] \arrow[u, "\mathrm{ad}"] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{G}_E \arrow[r, "\iota"] & S(\mathcal{G}_E) \arrow[r] & T_{X/S} \arrow[r] & 0
\end{tikzcd}$$

where $\mathrm{ad} : \mathcal{A}^0_{E/S} \to \mathcal{A}_{\mathbb{E}/S}$ is induced by the morphism in Proposition 3.10 i) that maps $\mathcal{O}_X$ to zero. Note that, except $\gamma_{\mathbb{E},U}$ (resp. $\gamma_{\mathbb{E},\dot{X}}$), other morphisms in the diagram are well defined over the global $\mathcal{M}$ (i.e. need not shrink $\mathcal{M}$).

On each U_i, fix a trivilization of E on U_i and use the pullback coordinate of V, we define the morphism by using the local description (3.7). For any local section

$$\alpha = \begin{pmatrix} \chi \sum_a (0, J_a, 0) \otimes (0, J_a, 0) + \sum_a \mu_a (0, J_a, 0) \otimes (0, 0, 1) \\ + \sum_a \nu_a (0, 0, 1) \otimes (0, J_a, 0) + w(0, 0, 1) \otimes (0, 0, 1) \end{pmatrix} \otimes \partial_t$$

one defines that (with r the rank of E)

$$\gamma_{\mathbb{E},U}(\alpha) = (\chi, \frac{1}{2}\sum_a (\mu_a + \nu_a)\mathrm{ad}J_a, rw) \in {}^{tr}\mathcal{A}_{\mathbb{E}}^{-1}(U_i). \tag{3.14}$$

Lemma 3.15. *The assignment $\alpha \to \gamma_{\mathbb{E},U}(\alpha)$ constructed above is gauge-invariant (it is not, however, independent of the choice of coordinate on V). Equivalently, under another choice of trivialization of E (on U_i), such that $J_a \to gJ_ag^{-1}$ and $\alpha \to \alpha_g$, the assignment remains unchanged, i.e. $\gamma_{\mathbb{E},U}(\alpha_g)$ is obtained as the e_g-transformation of $\gamma_{\mathbb{E},U}(\alpha)$, where e_g is the gauge of $\mathbb{E}$ induced by g (cf. Proposition 3.10).*

Proof. As in the proof of Proposition 3.13, $\alpha_g = s_g \otimes \partial_t$. Then

$$\begin{aligned} \gamma_{\mathbb{E},U}(\alpha_g) = (\chi, \frac{1}{2}\sum_a (\mu_a + \nu_a - 2\chi\mathrm{Tr}(J_a g^{-1} g'))\mathrm{ad}(gJ_ag^{-1}), \\ r\chi \sum_a \mathrm{Tr}(J_a g^{-1} g')^2 - r\sum_a (\mu_a + \nu_a)\mathrm{Tr}(J_a g^{-1} g') + rw). \end{aligned}$$

The e_g-transformation $\gamma_{\mathbb{E},U}(\alpha)^g$ of $\gamma_{\mathbb{E},U}(\alpha)$ is

$$\begin{pmatrix} \chi, -\chi e_g' e_g^{-1} + \dfrac{1}{2}\displaystyle\sum_a (\mu_a+\nu_a) e_g \mathrm{ad}(J_a) e_g^{-1}, \quad rw+ \\ \mathrm{Tr}(-\dfrac{1}{2}\chi e_g'' e_g^{-1} + \chi (e_g' e_g^{-1})^2 - \dfrac{1}{2}\displaystyle\sum_a (\mu_a+\nu_a)\mathrm{ad}(J_a) e_g^{-1} e_g') \end{pmatrix}$$

Recall that $e_g : \mathbb{M}_r(\mathcal{O}_X) \to \mathbb{M}_r(\mathcal{O}_X)$ means $e_g(B) = gBg^{-1}$, we have $e_g \mathrm{ad}(J_a) e_g^{-1} = \mathrm{ad}(gJ_a g^{-1})$. Thus the second components of $\gamma_{\mathbb{E},U}(\alpha_g)$ and $\gamma_{\mathbb{E},U}(\alpha)^g$ will coincide if $e_g' e_g^{-1} = \sum_a \mathrm{Tr}(J_a g^{-1} g')\mathrm{ad}(gJ_a g^{-1})$, which is true since $e_g' e_g^{-1} = \mathrm{ad}(g' g^{-1})$, $e_g^{-1} e_g' = \mathrm{ad}(g^{-1} g')$. To finish the proof, we will show that their third components coincide. Since

$$\frac{1}{2}\sum_a (\mu_a+\nu_a)\mathrm{Tr}(\mathrm{ad}(J_a) e_g^{-1} e_g') = r\sum_a (\mu_a+\nu_a)\mathrm{Tr}(J_a g^{-1} g'),$$

it will be done if one can show the following identity

$$r\sum_a \mathrm{Tr}(J_a g^{-1} g')^2 = \mathrm{Tr}((e_g' e_g^{-1})^2 - \frac{1}{2} e_g'' e_g^{-1}). \tag{3.15}$$

Write $e_g'' e_g^{-1} = (e_g' e_g^{-1})' - e_g' (e_g^{-1})' = (e_g' e_g^{-1})' + e_g' e_g^{-1} e_g' e_g^{-1}$, then

$$\mathrm{Tr}(e_g'' e_g^{-1}) = \mathrm{Tr}((e_g' e_g^{-1})^2) = \mathrm{Tr}(\mathrm{ad}(g' g^{-1})\mathrm{ad}(g' g^{-1}))$$

(using $(e_g e_g^{-1})' = 0$ and $\mathrm{Tr}((e_g' e_g^{-1})') = \mathrm{Tr}(e_g' e_g^{-1})' = 0$ here). Let $(g' g^{-1})_0$ be the traceless part of $g' g^{-1}$. Then R.H.S of (3.15) equals to

$$\frac{1}{2}\mathrm{Tr}(\mathrm{ad}(g' g^{-1})\mathrm{ad}(g' g^{-1})) = r\mathrm{Tr}((g' g^{-1})_0 (g' g^{-1})_0).$$

By the choice of $\{J_a\}_a$, we have $(g^{-1} g')_0 = \sum_a \mathrm{Tr}(J_a g^{-1} g') J_a$. Then L.H.S of (3.15) equals to $r\mathrm{Tr}((g^{-1} g')_0 g^{-1} g') = r\mathrm{Tr}((g^{-1} g')_0 (g^{-1} g')_0)$. Thus (3.15) is true since $g(g^{-1} g')_0 g^{-1} = (g' g^{-1})_0$. We are done. □

We have constructed $\gamma_{\mathbb{E},U}$ (the construction of $\gamma_{\mathbb{E},\dot{X}}$ is similar) such that the above diagram $(*)$ is commutative over U (resp. over $\dot{X}$). It is also easy to see that $\gamma_{\mathbb{E},U}$, $\gamma_{\mathbb{E},\dot{X}}$ induce (through ι) $\frac{1}{2}\cdot\widetilde{\mathrm{ad}}$ (where $\widetilde{\mathrm{ad}} : \mathcal{G}_E \to \mathcal{G}_{\mathbb{E}}$ is the map in (ii) of Proposition 3.10). However, $\gamma_{\dot{U}} := \gamma_{\mathbb{E},U} - \gamma_{\mathbb{E},\dot{X}}$ may not vanish on $S(\mathcal{G}_E)|_{\dot{U}}$ (but vanishes on $\mathcal{G}_E|_{\dot{U}}$), which defines a morphism $\gamma_{\dot{U}} : S(\mathcal{G}_E)|_{\dot{U}} \to \omega_{X/S}|_{\dot{U}}$ that induces $\widetilde{\gamma}_{\dot{U}} : T_{X/S}|_{\dot{U}} \to \omega_{X/S}|_{\dot{U}}$, i.e., a section

$$\widetilde{\gamma}_{\dot{U}} \in \mathcal{H}om(T_{X/S}, \omega_{X/S})(\dot{U}) = \omega^2_{X/S}(p^{-1}(\dot{V}))$$

where $\dot{V} := V \cap \dot{\mathcal{C}}$. $[\widetilde{\gamma}_{\dot{U}}]$ defines a class of $H^1(\mathcal{C}, p_*\omega^2_{X/S}) = H^1(\mathcal{C}, \omega^2_{\mathcal{C}/\mathcal{M}})$, which vanishes since we assume $\mathcal{M}$ affine. Thus there exist

$$\widetilde{\psi}_U \in \mathcal{H}om(T_{X/S}, \omega_{X/S})(U), \quad \widetilde{\psi}_{\dot{X}} \in \mathcal{H}om(T_{X/S}, \omega_{X/S})(\dot{X})$$

such that $\widetilde{\gamma}_{\dot{U}} = \widetilde{\psi}_U - \widetilde{\psi}_{\dot{X}}$. Let ψ_U, $\psi_{\dot{X}}$ denote the induced morphisms

$$\psi_U : S(\mathcal{G}_E)|_U \to S(\mathcal{G}_E)/\mathcal{G}_E|_U \cong T_{X/S}|_U \xrightarrow{\widetilde{\psi}_U} \omega_{X/S}|_U$$

$$\psi_{\dot{X}} : S(\mathcal{G}_E)|_{\dot{X}} \to S(\mathcal{G}_E)/\mathcal{G}_E|_{\dot{X}} \cong T_{X/S}|_{\dot{X}} \xrightarrow{\widetilde{\psi}_{\dot{X}}} \omega_{X/S}|_{\dot{X}}.$$

Then it is easy to see that $\gamma_{\dot{U}} = \gamma_{\mathbb{E},U} - \gamma_{\mathbb{E},\dot{X}} = \psi_U - \psi_{\dot{X}}$. Let

$$\beta_U := \gamma_{\mathbb{E},U} - \psi_U, \quad \beta_{\dot{X}} := \gamma_{\mathbb{E},\dot{X}} - \psi_{\dot{X}}.$$

Thus, by shrinking $\mathcal{M}$, we have proved the following

Proposition 3.16. *The β_U and $\beta_{\dot{X}}$ define a morphism*

$$\beta : S(\mathcal{G}_E) \to {}^{tr}\mathcal{A}^{-1}_{\mathbb{E}},$$

which induces (through ι) $\frac{1}{2}\widetilde{\mathrm{ad}}$ on $\mathcal{G}_E$, such that the following diagram is commutative

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \omega_{X/S} & \longrightarrow & {}^{tr}\mathcal{A}^{-1}_{\mathbb{E}} & \xrightarrow{\mathrm{res}} & \mathcal{A}_{\mathbb{E}/S} & \longrightarrow & 0 \\
 & & {\scriptstyle r\cdot\mathrm{id}}\big\uparrow & & {\scriptstyle\beta}\big\uparrow & & {\scriptstyle\mathrm{ad}}\big\uparrow & & \\
0 & \longrightarrow & \omega_{X/S} & \longrightarrow & S(\mathcal{G}_E) & \xrightarrow{\widetilde{\mathrm{res}}} & \mathcal{A}^0_{E/S} & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\| & & \big\downarrow & & \\
0 & \longrightarrow & \mathcal{G}_E & \xrightarrow{\iota} & S(\mathcal{G}_E) & \longrightarrow & T_{X/S} & \longrightarrow & 0
\end{array}$$

We shall now prove ii) of Theorem 3.7.

Proof. The proof is similar to that of i) of the theorem. By

$$0 \to \mathcal{A}^0_{E/S} \to \mathcal{A}^0_{E,\pi} \to \pi^{-1}T_S \to 0 \tag{3.16}$$

and by identifying its connecting map $T_S \to R^1\pi_*\mathcal{A}^0_{E/S}$ with the Kodaira-Spencer map KS_S (which will be treated more generally for $\log D$ in Proposition 4.5), we see that $R^1\pi_*\mathcal{A}^0_{E/S}$ vanishes in $R^1\pi_*\mathcal{A}^0_{E,\pi}$ if KS_S is an isomorphism. Thus, for any $\alpha \in S(\mathcal{G}_E)(\dot{U}_i)$, there exist $\tau_{\dot{X}_i} \in \mathcal{A}^0_{E,\pi}(\dot{X}_i)$ and $\tau_{U_i} \in \mathcal{A}^0_{E,\pi}(U_i)$ such that $\tau_{\dot{X}_i} - \tau_{U_i} = \widetilde{\mathrm{res}}(\alpha)$. Then, by Proposition 3.16, we see that $\mathrm{res}(\beta(\alpha)) = \mathrm{ad}(\widetilde{\mathrm{res}}(\alpha))$. Thus

$$\widetilde{\phi}(\alpha) := \{\mathrm{ad}(\tau_{\dot{X}_i}), \mathrm{ad}(\tau_{U_i}); \beta(\alpha)\}$$

is a cocycle in $C^0({}^{tr}\mathcal{A}^{-1}_{\mathbb{E}} \to \mathcal{A}^0_{\mathbb{E},\pi})$. It is clear that $\widetilde{\phi}$ induces a morphism

$$\phi : R^1\pi_*S(\mathcal{G}_E) \to R^0\pi_*({}^{tr}\mathcal{A}^{-1}_{\mathbb{E}} \to \mathcal{A}^0_{\mathbb{E},\pi}) = R^0\pi_*\,{}^{tr}\mathcal{A}^{\bullet}_{\mathbb{E}}.$$

Similarly, we can construct

$$\vartheta : R^1\pi_*\mathcal{A}^0_{E/S} \to R^0\pi_*(\mathcal{A}^0_{E/S} \to \mathcal{A}^0_{E,\pi}) \to R^0\pi_*\,{}^{tr}\mathcal{A}^\bullet_{\mathbb{E}},$$

which is an isomorphism such that the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^1\pi_*\omega_{X/S} & \longrightarrow & R^1\pi_*S(\mathcal{G}_E) & \xrightarrow{\widetilde{\mathrm{res}}} & R^1\pi_*\mathcal{A}^0_{E/S} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle r\cdot\mathrm{id}} & & \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \vartheta} & & \\ 0 & \longrightarrow & R^0\pi_*\omega_{X/S}[1] & \longrightarrow & R^0\pi_*\,{}^{tr}\mathcal{A}^\bullet_{\mathbb{E}} & \xrightarrow{\mathrm{res}} & R^0\pi_*\mathcal{A}^\bullet_{\mathbb{E},\pi} & \longrightarrow & 0 \end{array}$$

is commutaive. Thus ϕ must be an isomorphism. □

Remark 3.17. The ϕ in Theorem 3.7 i) is defined globally over $\mathcal{M}$, but the one in Theorem 3.7 ii) is defined only over an open set of $\mathcal{M}$. More precisely, there is an affine covering $\{\mathcal{M}_i\}_{i\in I}$ of $\mathcal{M}$ such that on each $\mathcal{C}\times_{\mathcal{M}} f^{-1}(\mathcal{M}_i)$ we can choose a $\beta_i : S(\mathcal{G}_E) \to {}^{tr}\mathcal{A}^{-1}_{\mathbb{E}}$ as the β in Proposition 3.16. Then, by using β_i and Theorem 3.7 ii), we get the isomorphism $\phi_i : R^1\pi_*S(\mathcal{G}_E) \to R^0\pi_*\,{}^{tr}\mathcal{A}^\bullet_{\mathbb{E}} = \mathcal{D}^{\le 1}_S(\lambda_{\mathbb{E}})$ on $f^{-1}(\mathcal{M}_i)$. For another choice $\{\beta'_i\}_{i\in I}$, the map $\beta_i - \beta'_i : S(\mathcal{G}_E) \to \omega_{X/S}$ induces

$$\phi_i - \phi'_i : R^1\pi_*S(\mathcal{G}_E) \to R^0\pi_*\omega_{X/S}[1] = \mathcal{O}_S$$

on $f^{-1}(\mathcal{M}_i)$, which vanishes on $R^1\pi_*(\mathcal{G}_E)$, thus $(\phi_i - \phi'_i) \in \Omega^1_{\mathcal{M}}(\mathcal{M}_i)$. Similarly, on $f^{-1}(\mathcal{M}_i\cap\mathcal{M}_j)$, $\phi_{ij} := \phi_i - \phi_j$ induces $\widetilde{\phi}_{ij} \in \Omega^1_{\mathcal{M}}(\mathcal{M}_i\cap\mathcal{M}_j)$. Thus $\{\widetilde{\phi}_{ij}\}$ defines a class in $\mathrm{H}^1(\mathcal{M}, \Omega^1_{\mathcal{M}})$.

To conclude this section, we describe the connecting maps $\tilde{\delta}$ and prove Lemma 2.4 (cf. Lemma 3.18).

Lemma 3.18. *The map $\tilde{\delta}$ induces the identity map on $R^1\pi_*(\mathcal{G}_E)$. More precisely, $\tilde{\delta}\circ\iota(\bullet) = \mathrm{Sym}^2((\bullet)\otimes 1)$.*

Proof. It is known (see Proposition 4.2 of [10]) that the connecting map

$$\tilde{\delta} : R^1\pi_*(\mathcal{G}_E \boxtimes \mathcal{G}_E(\Delta)|_\Delta) \to R^2(\pi\times\pi)_*(\mathcal{G}_E\boxtimes\mathcal{G}_E)$$

is dual (under Serre duality) to the restriction map

$$r : (\pi\times\pi)_*(\mathcal{G}^*_E\boxtimes\mathcal{G}^*_E\otimes\omega_{X/S}\boxtimes\omega_{X/S}) \to \pi_*(\mathcal{G}^*_E\otimes\mathcal{G}^*_E\otimes\omega^2_{\Delta/S}).$$

Working locally on S, we can assume that X is covered by affine open sets U and $\dot{X}$. Let $w\in\omega_{X/S}(\dot{U})$ be a base of $\omega_{X/S}$ on $\dot{U} := U\cap\dot{X}$ and $w^*\in T_{X/S}(\dot{U})$ be its dual base. Then, for any $\alpha\in\mathcal{G}_E(\dot{U})$,

$$\iota([\alpha]) = [\mathrm{Sym}^2(\alpha\otimes w)\otimes w^*] = \frac{1}{2}[(\alpha\otimes w + w\otimes\alpha)\otimes w^*].$$

We use the following identification

$$(\pi\times\pi)_*(\mathcal{G}^*_E\boxtimes\mathcal{G}^*_E\otimes\omega_{X/S}\boxtimes\omega_{X/S}) \cong \pi_*(\mathcal{G}^*_E\otimes\omega_{X/S})\otimes\pi_*(\mathcal{G}^*_E\otimes\omega_{X/S}).$$

For any $\beta_i \in \pi_*(\mathcal{G}_E^* \otimes \omega_{X/S})$ $(i=1,2)$, let $\beta_i|_{\dot{U}} = s_i \otimes w$, where $s_i \in \mathcal{G}_E^*(\dot{U})$ $(i=1,2)$. Then $r(\beta_1 \otimes \beta_2)|_{\dot{U}} = s_1 \otimes s_2 \otimes w \otimes w$. Thus

$$< \tilde{\delta}(\iota([\alpha])), \beta_1 \otimes \beta_2 >= \frac{1}{2}[(s_1(\alpha)s_2(w) + s_1(w)s_2(\alpha)) \cdot w]. \tag{3.17}$$

By $R^1\pi_*\omega_{X/S} = \mathcal{O}_S$, let $[f \cdot w] = 1 \in \mathcal{O}_S$ $(f \in \mathcal{O}_X(\dot{U}))$, we have

$$\begin{aligned} &< \mathrm{Sym}^2([\alpha] \otimes [f \cdot w]), \beta_1 \otimes \beta_2 > \\ &= \frac{1}{2}([s_1(\alpha) \cdot w][fs_2(w) \cdot w] + [fs_1(w) \cdot w][s_2(\alpha) \cdot w]). \end{aligned} \tag{3.18}$$

Note that $s_i(w) = \beta_i|_{\dot{U}}(w \otimes w^*)$ $(i=1,2)$, we can see that $s_i(w) = \beta_i(\mathrm{id})|_{\dot{U}}$, where the (global) section id is the image of 1 under $\mathcal{O}_X \to \mathcal{G}_E \otimes \omega_{X/S}^*$. Thus $s_i(w) \in \mathcal{O}_S$ since β_i $(i=1,2)$ are global sections of $(\mathcal{G}_E \otimes \omega_{X/S}^*)^*$. Then $[s_1(\alpha)s_2(w) \cdot w] = [s_1(\alpha) \cdot w][fs_2(w) \cdot w]$ and $[s_2(\alpha)s_1(w) \cdot w] = [s_2(\alpha) \cdot w][fs_1(w) \cdot w]$, which means that

$$< \tilde{\delta}(\iota([\alpha])), \bullet >=< \mathrm{Sym}^2([\alpha] \otimes 1), \bullet > .$$

□

4. The Generalization to Singular Cases

Let $\pi : \widetilde{X} \to \widetilde{S}$ be a proper morphism of relative dimension 1 between smooth varieties in characteristic 0 such that each fiber has at most ordinary double points as singularities. Let $\widetilde{f} : \widetilde{S} \to \widetilde{\mathcal{M}}$ be a smooth morphism where $\widetilde{\mathcal{M}}$ is a smooth variety. Let $B = \widetilde{\mathcal{M}} \setminus \mathcal{M}$ and $W = \widetilde{f}^{-1}(B)$ such that $D = \pi^{-1}(W)$ consists precisely of singular fibres. As before let $\omega_{\widetilde{X}/\widetilde{S}}$ be the relative dualizing sheaf (which is locally free as is well known). Let $T_{\widetilde{\mathcal{M}}}(B) \subset T_{\widetilde{\mathcal{M}}}$ be the subalgebra of vector fields that preserve B (cf. [2], Section 6).

Notation 4.1. In the notation of Section 3, define the following

$$\begin{aligned} T_{\widetilde{f}}(\log D) &= d(\widetilde{f} \circ \pi)^{-1}((\widetilde{f} \circ \pi)^{-1} T_{\widetilde{\mathcal{M}}}(B)) \subset T_{\widetilde{X}}; \\ T_\pi(\log D) &= T_{\widetilde{f}}(\log D) \cap d\pi^{-1}(\pi^{-1}T_{\widetilde{S}}) \subset T_{\widetilde{X}}; \\ T_{\widetilde{S}}(\log W) &= d\pi(T_\pi(\log D)) \subset T_{\widetilde{S}}; \\ T_{\widetilde{X}/\widetilde{S}}(\log D) &= T_{\widetilde{X}/\widetilde{S}} \cap T_\pi(\log D). \end{aligned}$$

Notation 4.2. Let E be a vector bundle on $\widetilde{X}$. Define the following

$$\begin{aligned} \mathcal{A}_{E,\pi}(\log D) &= \epsilon^{-1} T_\pi(\log D) \subset \mathcal{A}_E; \\ \mathcal{A}_{E/\widetilde{S}}(\log D) &= \epsilon^{-1} T_{\widetilde{X}/\widetilde{S}}(\log D) \subset \mathcal{A}_E; \\ \mathcal{A}_{\lambda_E}(\log W) &= \epsilon^{-1} T_{\widetilde{S}}(\log W) \subset \mathcal{A}_{\lambda_E}, \end{aligned}$$

where "ϵ" denotes symbol maps.

The ${}^{tr}\mathcal{A}_E^{-1}$ in Section 3 admits a generalization ${}^{tr}\mathcal{A}_E^{-1}(\log D)$ (cf. [13], p. 593) such that

$$0 \to \omega_{\widetilde{X}/\widetilde{S}} \to {}^{tr}\mathcal{A}_E^{-1}(\log D) \to \mathcal{A}_{E/\widetilde{S}}(\log D) \to 0, \tag{4.1}$$

is exact. Furthermore with ${}^{tr}\mathcal{A}_E^{\bullet}$ in Section 3 replaced by ${}^{tr}\mathcal{A}_E^{\bullet}(\log D)$ (with $\mathcal{A}_{E,\pi}(\log D)$ in degree 0), one has

Proposition 4.3. (cf. [13]) *There is a canonical isomorphism*

$$R^0\pi_*^{tr}\mathcal{A}_E^{\bullet}(\log D) \cong \mathcal{A}_{\lambda_E}(\log W)$$

that extends Proposition 3.1.

Now we come back to moduli situation. Recall Notation 2.10 and the Kodaira-Spencer map (cf. [14], Remark 3.2.7)

$$\mathrm{KS} : \widetilde{f}^*T_{\widetilde{\mathcal{M}}}(B) \to R^1\pi_*T_{\widetilde{X}/\widetilde{S}}(\log D). \tag{4.2}$$

As the same as the situation of smooth curves, we have

Lemma 4.4. *The morphism $\widetilde{f} : \widetilde{S} \to \widetilde{\mathcal{M}}$ is smooth and*

$$T_{\widetilde{S}/\widetilde{\mathcal{M}}} = R^1\pi_*\mathcal{E}nd^0(E).$$

Proof. When C is irreducible, its fibre $\widetilde{f}^{-1}([C])$ is the moduli space of stable bundles with fixed determinant $\mathcal{N}|_C$. When C is reducible, $\widetilde{f}^{-1}([C])$ have a few disjoint irreducible components and each component consists of bundles with a fixed determinant that coincides with $\mathcal{N}|_C$ outside the node of C (cf. [11]). □

Proposition 4.5.
i) Assume the Kodaira-Spencer map KS (4.2) *is injective. Then*

$$R^0\pi_*(\mathcal{E}nd(E)^{-1} \to \mathcal{A}_{E,\pi}(\log D)) \to R^0\pi_*(\mathcal{E}nd(E) \to \mathcal{A}_{E,\pi}(\log D)) \to 0$$

is canonically isomorphic to

$$\mathcal{D}^{\leq 1}_{\widetilde{S}/\widetilde{\mathcal{M}}}(\lambda_E) \to T_{\widetilde{S}/\widetilde{\mathcal{M}}} \to 0.$$

ii) Assume that the KS (4.2) *is an isomorphism. Then*

$$R^1\pi_*\mathcal{A}^0_{E/\widetilde{S}}(\log D) \cong T_{\widetilde{S}}(\log W)$$

canonically, where $\mathcal{A}^0_{E/\widetilde{S}}(\log D) = \mathcal{A}_{E/\widetilde{S}}(\log D)/\mathcal{O}_{\widetilde{X}}$.

The following is left to the reader.

Lemma 4.6.

$$(\mathcal{A}_{E/\widetilde{S}}(\log D) \to \mathcal{A}_{E,\pi}(\log D)) \cong_{\text{q.i.}} (T_{\widetilde{X}/\widetilde{S}}(\log D) \to T_\pi(\log D)) \\ \cong_{\text{q.i.}} \pi^{-1} T_{\widetilde{S}}(\log W)$$

as quasi-isomorphisms.

We prove now Proposition 4.5.

Proof. i) From the exact sequence

$$0 \to (\mathcal{E}nd(E)^{-1} \to \mathcal{A}_{E,\pi}(\log D)) \to {}^{tr}\mathcal{A}^\bullet_E(\log D) \to T_{\widetilde{X}/\widetilde{S}}(\log D)[1] \to 0,$$

and passage to cohomology

$$0 \to R^0\pi_*(\mathcal{E}nd(E)^{-1} \to \mathcal{A}_{E,\pi}(\log D)) \to \mathcal{A}_{\lambda_E}(\log W) \xrightarrow{\epsilon} \\ R^0\pi_* T_{\widetilde{X}/\widetilde{S}}(\log D)[1] \to \cdots,$$

one has, via the injectivity of KS, that $\epsilon^{-1}(0)$ has symbolic part in $T_{\widetilde{S}/\widetilde{\mathcal{M}}}$. This gives one of the isomorphisms in i) (the one with $D^{\leq 1}_{\widetilde{S}/\widetilde{\mathcal{M}}}(\lambda_E)$). Further, by

$$0 \to \omega_{\widetilde{X}/\widetilde{S}}[1] \to (\mathcal{E}nd(E)^{-1} \to \mathcal{A}_{E,\pi}(\log D)) \to (\mathcal{E}nd(E) \to \mathcal{A}_{E,\pi}(\log D)) \to 0$$

and $R^0\pi_*\omega_{\widetilde{X}/\widetilde{S}}[1] \cong \mathcal{O}_{\widetilde{S}}$ it follows

$$R^0\pi_*(\mathcal{E}nd(E)^{-1} \to \mathcal{A}_{E,\pi}(\log D)) \to R^0\pi_*(\mathcal{E}nd(E) \to \mathcal{A}_{E,\pi}(\log D))$$

is nothing but the symbol map, completing the asserted isomorphisms.
ii) Write $B^\bullet(\log D)$ for $(\mathcal{E}nd(E) \to \mathcal{A}_{E,\pi}(\log D))$ and $\mathcal{A}^\bullet_{E,\pi}(\log D)$ for L.H.S. of Lemma 4.6. The following exact sequence

$$0 \to B^\bullet(\log D) \to \mathcal{A}^\bullet_{E,\pi}(\log D) \to T_{\widetilde{X}/\widetilde{S}}(\log D)[1] \to 0 \tag{4.3}$$

projects to

$$0 \to \mathcal{E}nd^0(E)[1] \to \mathcal{A}^0_{E/\widetilde{S}}(\log D)[1] \to T_{\widetilde{X}/\widetilde{S}}(\log D)[1] \to 0 \tag{4.4}$$

Computing direct images of (4.3) and (4.4), by i) just proved and Lemma 4.6, yields that $R^0\pi_*$ of (4.4) should be isomorphic to

$$0 \to T_{\widetilde{S}/\widetilde{\mathcal{M}}} \to T_{\widetilde{S}}(\log W) \to \widetilde{f}^* T_{\widetilde{\mathcal{M}}}(B) \to 0,$$

implying the assertion. □

Remark 4.7. i) $R^1\pi_*(\mathcal{G}_E) \cong R^0\pi_*(\mathcal{E}nd(\mathbb{E})^{-1} \to \mathcal{A}_{\mathbb{E},\pi}(\log D)) \cong \mathcal{D}^{\leq 1}_{\widetilde{S}/\widetilde{\mathcal{M}}}(\lambda_{\mathbb{E}})$ holds, cf. Theorem 3.7 and Remark 3.8. ii) For the family $\widetilde{\mathcal{C}} \to \widetilde{\mathcal{M}}$ there is a Kodaira-Spencer map (cf. [14], 3.1-2)

$$\rho_b : T_{\widetilde{\mathcal{M}},b} \to \operatorname{Ext}^1_{\mathcal{O}_{\widetilde{\mathcal{C}}_b}}(\Omega_{\widetilde{\mathcal{C}}_b}, \mathcal{O}_{\widetilde{\mathcal{C}}_b}),\ b \in \widetilde{\mathcal{M}} \tag{4.5}$$

The family $\widetilde{\mathcal{C}} \to \widetilde{\mathcal{M}}$ is a *local universal family* if ρ_b is an isomorphism at each $b \in \widetilde{\mathcal{M}}$. If $\widetilde{\mathcal{C}} \to \widetilde{\mathcal{M}}$ is a local universal family (cf. [14], Theorem 3.1.5 for the existence of such a family), then KS (4.2) is an isomorphism (cf. [14], Theorem 3.2.6).

Combining the above with the 2nd half of Section 2, we are now ready to generalize Theorem 3.7 in the context of *log geometry*.

Theorem 4.8. *Suppose* KS *in* (4.2) *is an isomorphism* (cf. *ii) of Remark 4.7). Then, over any affine open set U_i of $\widetilde{\mathcal{M}}$, there is an isomorphism*

$$\phi_i : R^1\pi_* S(\mathcal{G}_E) \cong \mathcal{D}^{\leq 1}_{\widetilde{S}}(\lambda_{\mathbb{E}})(\log W).$$

Proof. Note the above assumption for KS is, via ii) of Proposition 4.5, in correspondence to KS_S in Theorem 3.7. It follows that all key ingredients in proof of Theorem 3.7 admit corresponding counterparts for log geometry, such as Proposition 4.3 and Proposition 4.5. Thus the generalization of Theorem 3.7 in *log* context is immediate. $\square$

To complete this paper, we prove the coherence of $f_{T*}\Theta^k$ (for definitions of f_T and f_Z, see Notation 2.10), for which our proof need a result on the density of locally free sheaves.

Proposition 4.9. *The fibre of $f_Z : Z \to \widetilde{\mathcal{M}}$ at any point of $B = \widetilde{\mathcal{M}}\backslash\mathcal{M}$ has a dense open set of locally free sheaves.*

Proof. Let X_0 be a fibre of $\mathcal{C} \to \widetilde{\mathcal{M}}$ at $0 \in B$. If X_0 is reducible, the lemma is known (see Theorem 1.6 and Lemma 2.2 of [11]). Thus we assume that X_0 is irreducible. Let $\mathcal{U}_0$ be the moduli space of semistable sheaves of rank r and degree d (without fixed determinant) on X_0. We need to show that $Z_0 := f_Z^{-1}(0)$ $(\subset \mathcal{U})$ contains a dense set of locally free sheaves with the fixed determinant $\mathcal{N}_0$. Let $J(X_0)$ be the Jacobian of X_0, which consists of line bundles of degree 0 (thus non-compact for the singular curve X_0). Then we have a morphism

$$\phi_0 : Z_0 \times J(X_0) \to \mathcal{U}_0, \quad \text{where } \phi_0(E, L) = E \otimes L.$$

Now we prove that ϕ_0 has fibre dimension at most 1, namely, for any $[F_0] \in \mathcal{U}_0$, the fibre

$$\phi_0^{-1}([F_0]) = \{(F, L)|F \otimes L = F_0\} \subset Z_0 \times J(X_0)$$

has at most dimension 1 (for simplicity, we assume that X_0 has only one node x_0). One can check that for any $[F] \in Z_0$ it satisfies

$$\frac{\wedge^r F}{\text{torsion}} \subset \mathcal{N}_0 \quad \text{and} \quad m_{x_0}^{r(F)}\mathcal{N}_0 \subset \frac{\wedge^r F}{\text{torsion}}$$

where m_{x_0} is the ideal sheaf of the node $x_0 \in X_0$ and $r(F) = d - \deg(\frac{\wedge^r F}{\text{torsion}})$. Let $\rho : \widetilde{X_0} \to X_0$ be the normalization and $\rho^{-1}(x_0) = \{x_1, x_2\}$. Then the above condition implies that

$$\frac{\rho^*(\wedge^r F)}{\text{torsion}} = \rho^*\mathcal{N}_0(-n_1x_1 - n_2x_2), \quad n_1 + n_2 \leq 2r(F),\ n_1 \geq 0, n_2 \geq 0,$$

where $r(F) = d - \deg(\frac{\wedge^r F_0}{\text{torsion}}) = r(F_0)$ since $F \otimes L = F_0$ and thus

$$\frac{\wedge^r F_0}{\text{torsion}} = \frac{\wedge^r F}{\text{torsion}} \otimes L^r.$$

Thus, for any $(F, L) \in \phi_0^{-1}(F_0)$, L has to satisfy

$$(\rho^* L)^r = (\rho^*\mathcal{N}_0)^{-1}(n_1x_1 + n_2x_2) \otimes \frac{\rho^* \wedge^r F_0}{\text{torsion}}$$

which is a finite set since there are only finitely many choices of (n_1, n_2). The pullback map $\rho^* : J(X_0) \to J(\widetilde{X_0})$ has 1-dimensional kernel. Thus $\dim(\phi_0^{-1}([F_0])) \leq 1$. We shall now prove the density of locally free sheaves.

Let Z_0^i be an irreducible component of Z_0, then

$$\dim(Z_0^i \times J(X_0)) \geq \dim(Z_\eta \times J(X_0)) = \dim(\mathcal{U}_0). \tag{4.6}$$

If Z_0^i contains no locally free sheaf, then $\phi(Z_0^i \times J(X_0))$ falls into the subvariety $\mathcal{U}_0^n \subset \mathcal{U}_0$ of non-locally free sheaves. $\mathcal{U}_0^n$ has a dense open set $\mathcal{U}_0^n(1)$ consisting of torsion free sheaves F of the following type, said to be type 1. Namely,

$$F \otimes \hat{\mathcal{O}}_{X_0,x_0} = \hat{\mathcal{O}}_{X_0,x_0}^{(r-1)} \oplus \hat{m}_{x_0}.$$

If $\phi((F, L)) = F_0 \in \mathcal{U}_0^n(1)$, then F is also of type 1 (tensoring a line bundle does not change its type). By Remark 8.1 of [9],

$$L_0 = \frac{\wedge^r F}{\text{torsion}}$$

is a torsion free (but non-locally free) sheaf of degree d. But $L_0 \subset \mathcal{N}_0$, thus $L_0 = \mathcal{N}_0$ since they have the same degree, a contradiction with that L_0 is not locally free. Thus $\phi(Z_0^i \times J(X_0))$ falls into a subvariety of codimension at least two, which contradicates the dimension of fibres and (4.6). □

Theorem 4.10. *i) $f_{T*}\Theta^k$ is coherent; ii) $f_{T*}\Theta^k = \widetilde{f}_*\Theta^k$ if either $g \geq 3$ or $r \geq 3$.*

Proof. Let $\iota : \widetilde{Z} \to Z$ be the normalisation of Z and $\widetilde{\Theta} = \iota^*(\Theta)$. Write $f_{\widetilde{Z}} : \widetilde{Z} \to \widetilde{\mathcal{M}}$ and $f_{\iota^{-1}(T)} : \iota^{-1}(T) \to \widetilde{\mathcal{M}}$. Then $\iota^{-1}(T) \cong T$ since T is normal, and $\mathcal{F} := f_{T*}\Theta^k \cong f_{\iota^{-1}(T)*}(\widetilde{\Theta}^k)$. On the other hand, since each fibre of $f_Z : Z \to \widetilde{\mathcal{M}}$ contains a dense open set of locally free sheaves, we have $\mathrm{codim}(Z \setminus T) \geq 2$. Thus $\widetilde{Z} \setminus \iota^{-1}(T) = \iota^{-1}(Z \setminus T)$ has codimension at least 2 since $\iota : \widetilde{Z} \to Z$ is a finite map. By Hartogs type extension theorem,

$$\mathcal{F} \cong f_{\iota^{-1}(T)*}(\widetilde{\Theta}^k) \cong f_{\widetilde{Z}*}(\widetilde{\Theta}^k),$$

which is coherent, hence i). The claim ii) that $f_{T*}\Theta^k = \widetilde{f}_*\Theta^k$ follows also from the Hartogs type theorem because T is normal and $T \setminus \widetilde{S}$ is of codimension at least 2 when $g > 2$ or $r > 2$ (cf. [7]). □

Finally, we prove the following lemma, though not strictly needed for this paper, which gives the relationships of $S(\mathcal{G}_E)$, $\mathcal{G}_{E^*}$ and $\mathcal{A}_{E^*/S}$. We remark that the morphism $\widetilde{\mathrm{res}}$ in Proposition 3.13 can be induced by pairings in this lemma..

Lemma 4.11. *There are canonical isomorphisms:*
i) $S(\mathcal{G}_E)/\omega_{X/S} \cong (\mathcal{G}_{E^*})^*$,
ii) $(\mathcal{G}_E)^* \cong \mathcal{A}^0_{E^*/S}$ *where* $\mathcal{A}^0_{E^*/S} = \mathcal{A}_{E^*/S}/\mathcal{O}_X$.

Proof. i) One constructs a non-degenerate pairing

$$\mathcal{G}_{E^*} \times S(\mathcal{G}_E)/\omega_{X/S} \to \mathcal{O}_X$$

using local description (3.7) (also cf. [2]). We define

$$< s_1, s_2 > := \chi\nu + \frac{1}{2}\sum_a (\nu_a + \mu_a)\mathrm{Trace}(J_a \cdot {}^tB) \tag{4.7}$$

for $s_1 = (0, B, \nu)$ and

$$s_2 = \begin{pmatrix} \chi \sum_a (0, J_a, 0) \otimes (0, J_a, 0) + \sum_a \mu_a (0, J_a, 0) \otimes (0, 0, 1) \\ + \sum_a \nu_a (0, 0, 1) \otimes (0, J_a, 0) + w(0, 0, 1) \otimes (0, 0, 1) \end{pmatrix} \otimes \partial_t.$$

One sees that $\omega_{X/S}$ is contained in the kernel of the pairing (4.7). The pairing is obviously invariant under coordinates change $y(t)$. If the trivialization of E is changed by a gauge g, then E^* is changed by a gauge $({}^tg)^{-1}$. Thus the verification of (4.7) being g-invariant is easily reduced to an identity

$$\sum_a \mathrm{Tr}(-J_a g^{-1} g') \cdot \mathrm{Tr}(J_a \cdot {}^tB) = \mathrm{Tr}(B({}^tg)(({}^tg)^{-1})'). \tag{4.8}$$

The L.H.S. of (4.8) equals $\mathrm{Tr}(-g^{-1}g' \cdot {}^tB)$ (B being traceless). The R.H.S. of (4.8), after transposition, is

$$\mathrm{Tr}((g^{-1})'g \cdot {}^tB) = \mathrm{Tr}\big((-g^{-1}g'g^{-1})g \cdot^t B\big).$$

ii) Define a non-degenerate pairing

$$\mathcal{G}_E \times \mathcal{A}^0_{E^*/S} \to \mathcal{O}_X \tag{4.9}$$

by $< (0, B_1, \nu), (\chi, B_2) >= \nu\chi + \mathrm{Trace}(B_1 \cdot^t B_2)$. We check that it is independent of choices of coordinates and gauges.

$$\begin{aligned}(0, B_1, \nu) := s_1 &\xrightarrow{g} \big(0, gB_1g^{-1}, \mathrm{Tr}(-B_1g^{-1}g') + \nu\big);\\ (\chi, B_2) := s_2 &\xrightarrow{{}^tg^{-1}} \Big(\chi, -\chi\big(({}^tg^{-1})' \cdot {}^tg\big) + ({}^tg^{-1})B_2 \cdot^t g\Big);\\ s_1 \xrightarrow{y(t)} (0, B_1, \nu y'); &\qquad s_2 \xrightarrow{y(t)} \big(\chi y'^{-1}, B_2\big).\end{aligned} \tag{4.10}$$

The $y(t)$-change is obvious. For g-change, we have

$$\begin{aligned}&\chi\nu + \mathrm{Tr}(B_1 \cdot^t B_2) \longrightarrow \chi\nu + \chi\mathrm{Tr}(-B_1g^{-1}g') +\\ &\mathrm{Tr}\Big(gB_1g^{-1} \cdot^t \big(-\chi(({}^tg^{-1})' \cdot {}^tg)\Big) + \mathrm{Tr}\Big(gB_1g^{-1} \cdot^t \big({}^tg^{-1}B_2 \cdot^t g\big)\Big).\end{aligned} \tag{4.11}$$

The 1st (resp. 2nd) term in the last line equals

$$\begin{aligned}-\chi\mathrm{Tr}\big(gB_1g^{-1} \cdot g(g^{-1})'\big) &= -\chi\mathrm{Tr}\big(gB_1g^{-1} \cdot g(-g^{-1}g'g^{-1})\big)\\ &= \chi\mathrm{Tr}\big(g(B_1g^{-1}g')g^{-1}\big) = \chi\mathrm{Tr}(B_1g^{-1}g')\\ &\big(\text{resp.} \quad \mathrm{Tr}(gB_1 \cdot^t B_2g^{-1}) = \mathrm{Tr}(B_1 \cdot^t B_2)\big).\end{aligned} \tag{4.12}$$

It follows from (4.11) and (4.12) that the pairing $\chi\nu + \mathrm{Tr}(B_1\, {}^tB_2)$ in (4.9) is globally defined. □

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