ON REPRESENTATION OF INTEGERS BY SUMS OF A CUBE AND THREE CUBES OF PRIMES

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Abstract. In this paper, we prove that all positive integers up to N but at most $O(N^{17/18+\varepsilon})$ exceptions, can be expressed by the sum of a cube and three cubes of primes.

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1. Introduction

We consider the expression of positive integers n as the sum of a cube and three cubes of primes, that is

$$n = m^3 + p_2^3 + p_3^3 + p_4^3, \tag{1.1}$$

where m is a positive integer and p_j are primes. In 1949, Roth [6] proved that almost all positive integers n can be written as (1.1). Precisely, let $E(N)$ denote the number of positive integers up N which cannot be written as (1.1), then Roth's theorem actually states that $E(N) \ll N \log^{-A} N$ for arbitrary $A > 0$. This result can be viewed as an approximation to the conjecture that all sufficiently large integers satisfying some necessary congruence conditions are the sum of four cubes of primes. As is well known that the quality of the approximation is indicated in the upper bound of $E(N)$. Recently, Roth's theorem has been improved by Ren [3] to $E(N) \ll N^{169/170}$, and by Ren and Tsang [4] to $E(N) \ll N^{1271/1296+\varepsilon}$. These improvements were obtained via new approaches to enlarge major arcs in the circle method used. For this, see for example [3], [4], [1]. In this paper, based on the major arcs estimate in [4], we use some new ideas to handle the minor arcs and prove the following.

Theorem 1. *For $E(N)$ defined above, we have*

$$E(N) \ll N^{17/18+\varepsilon}.$$

Notation. As usual, $\Lambda(n)$ stands for the von Mangoldt function. In our statement, N is a large positive integer, and $L = \log N$. The symbol $r \sim R$ means $R < r \le 2R$. The letters ε and A denote positive constants, which are arbitrarily small and arbitrarily large respectively.

2. proof of theorem 1

Following [4], we introduce notations

$$U = (N/9)^{1/3} \qquad \text{and} \qquad V = U^{5/6}. \tag{2.1}$$

In order to apply the circle method, for large positive integer N and positive real number θ, we let

$$P(\theta) = U^{\theta} \qquad \text{and} \qquad Q(\theta) = NP^{-1} = U^{3-\theta}. \tag{2.2}$$

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As usual, we define the major arcs $\mathfrak{M}(\theta)$ to be the union of all intervals $[a/q-1/(qQ(\theta)), a/q+1/(qQ(\theta))]$, where a, q are coprime integers and $1 \le a \le q \le P(\theta)$. Let the minor arcs $\mathfrak{m}(\theta)$ be the complement of $\mathfrak{M}(\theta)$ in the unit interval $I(\theta) = [1/Q(\theta), 1+1/Q(\theta)]$.

We define

$$T(\alpha) = \sum_{m \sim U} e(m^3\alpha),$$

and for $W > 0$

$$S(\alpha, W) = \sum_{m \sim W} \Lambda(m) e(m^3\alpha).$$

Let

$$R(n) = \sum_{\substack{n=m_1^3+\ldots+m_4^3 \\ m_1,m_4 \sim U, m_2,m_3 \sim V}} \Lambda(m_1)\Lambda(m_2)\Lambda(m_3).$$

Then

$$R(n) = \int_{I(\theta)} S(\alpha, U) S^2(\alpha, V) T(\alpha) e(-n\alpha) d\alpha = \int_{\mathfrak{M}(\theta)} + \int_{\mathfrak{m}(\theta)}. \tag{2.3}$$

Here for the major arcs estimate, we quote Theorem 2 in [4] and record it in the following lemma.

Lemma 2.1. *Let $\theta < 25/72$. For all integers n with $N/2 \le n \le N$, we have*

$$\int_{\mathfrak{M}(\theta)} S(\alpha, U) S^2(\alpha, V) T(\alpha) e(-n\alpha) d\alpha = \mathfrak{S}(n) J(n) + O(V^2 U^{-1} L^{-A}),$$

where $\mathfrak{S}(n)$ is the singular series in this problem which satisfies

$$(\log\log n)^{-c_0} \ll \mathfrak{S}(n) \ll \log n$$

for a certain positive constant c_0, and $J(n)$ is a multiple integral which satisfies

$$V^2 U^{-1} \ll J(n) \ll V^2 U^{-1}.$$

In this paper, we will concentrate on the minor arcs estimates. Our main result is the following.

Lemma 2.2. *We have*

$$\int_{\mathfrak{m}(25/72-\varepsilon)} |S(\alpha, U)|^2 |S(\alpha, V)|^4 |T(\alpha)|^2 d\alpha \ll U^{5/2+\varepsilon} V^2.$$

We will prove this lemma in §3.

Proof of Theorem 1. We start from (2.3), where the major arcs estimate is taken care of by Lemma 2.1. As regards the minor arcs, by Bessel's inequality and Lemma 2.2, we have

$$\sum_{N/2 < n \le N} \left| \int_{\mathfrak{m}(25/72-\varepsilon)} S(\alpha, U) S^2(\alpha, V) T(\alpha) e(-n\alpha) d\alpha \right|^2$$
$$\ll \int_{\mathfrak{m}(25/72-\varepsilon)} |S(\alpha, U)|^2 |S(\alpha, V)|^4 |T(\alpha)|^2 d\alpha \ll U^{5/2+\varepsilon} V^2. \tag{2.4}$$

By a standard argument we derive that for all $N/2 < n \le N$ but at most $O(U^{9/2+3\varepsilon}V^{-2})$ exceptions,

$$\int_{\mathfrak{m}(25/72-\varepsilon)} S(\alpha,U)S^2(\alpha,V)T(\alpha)e(-n\alpha)d\alpha \ll V^2U^{-1-\varepsilon}.$$

This together with Lemma 2.1 proves that for these n, there holds

$$R(n) = \mathfrak{S}(n)J(n) + O(V^2U^{-1}L^{-A}),$$

and hence n can be written as (1.1). Let $F(N)$ be the number of the exceptional n above, then we have

$$F(N) \ll U^{9/2+3\varepsilon}V^{-2} = N^{17/18+\varepsilon}.$$

The assertion of Theorem 1 now follows from $E(N) = \sum_{j\ge 0} F(N/2^j)$. □

3. PROOOF OF LEMMA 2.2

To prove Lemma 2.2, we need the following lemmas. Lemma 3.1 is obtained by letting $k = 3$ in Theorem 1 of [5]; Lemma 3.2 is due to Vaughan [7]; and Lemma 3.3 is Lemma 2.4 in [2].

Lemma 3.1. *Suppose $\alpha = a/q + \lambda$ where $a,\ q \ge 1$, $(a,q) = 1$ and $\lambda \in \mathbb{R}$. Then we have*

$$S(\alpha, W) \ll q^{\varepsilon}(\log^c W)\left\{W^{1/2}q^{1/2}\sqrt{1+|\lambda|W^3} + W^{4/5} + \frac{Wq^{-1/2}}{\sqrt{1+|\lambda|W^3}}\right\},$$

where c is an absolute positive constant.

Lemma 3.2. *Let Z_0 denote the number of solutions of the equation $m_1^3 + n_1^3 + n_2^3 = m_2^3 + n_3^3 + n_4^3$ subject to $m_j \sim U$ and $n_j \sim V$. Then $Z_0 \ll U^{1+\varepsilon}V^2$.*

Lemma 3.3. *For $k \ge 3$, let $\omega_k(q)$ be the multiplicative function defined by*

$$\omega_k(p^{ku+v}) = \begin{cases} kp^{-u-1/2}, & \text{when} \quad u \ge 0 \quad \text{and} \quad v = 1, \\ p^{-u-1}, & \text{when} \quad u \ge 0 \quad \text{and} \quad 2 \le v \le k. \end{cases}$$

Suppose that η and ξ are real numbers satisfying $\eta > 0$, $\xi \ge 2\eta + 2$ and $\xi \ge k\eta + 1$. Then whenever $X \ge 2$, one has

$$\sum_{1\le q\le X} q^{\eta}\omega_k^{\xi}(q) \ll \begin{cases} 1, & \text{when} \quad \xi > k\eta + 1, \\ \log X, & \text{when} \quad \xi = k\eta + 1, \end{cases}$$

where the implied constant depends at most on k, η and ξ.

Proof of Lemma 2.2. Let $\alpha \in \mathfrak{m}(25/72 - \varepsilon)$. Then by Dirichlet's lemma on rational approximations, there exist coprime integers a, q and real number λ satisfying

$$1 \le q \le 24U^2, \qquad |\lambda| \le 1/(24qU^2) \tag{3.1}$$

such that $\alpha = a/q + \lambda$. If $U < q \le 24U^2$, we apply Weyl's inequality to get

$$|T(\alpha)| \ll U^{3/4+\varepsilon}. \tag{3.2}$$

If $1 \le q \le U$, we combine conclusions of Lemmas 6.1 and 6.2 in [8] and (2.1)-(2.3) in [2] to obtain

$$|T(\alpha)| \ll \frac{\omega(q)U}{1+|\lambda|U^3} + q^{1/2+\varepsilon}, \tag{3.3}$$

where $\omega(q) = \omega_3(q)$ is as defined in Lemma 3.3 and satisfies

$$q^{-1/2} \ll \omega(q) \ll q^{-1/3}. \tag{3.4}$$

Let $0 < b < 1$. Then for q, λ satisfying either $U^b \le q \le U$, or

$$1 \le q \le U^b \qquad \text{and} \qquad \omega(q)U^{b/3-3} < |\lambda| \le 1/(24qU^2),$$

one has

$$|T(\alpha)| \ll U^{1-b/3}. \tag{3.5}$$

Let $\mathfrak{D}(b)$ be the set of all $\alpha = a/q + \lambda \in \mathfrak{m}(25/72-\varepsilon)$ with q, λ satisfying

$$1 \le q \le U^b, \qquad |\lambda| \le \omega(q)U^{b/3-3}.$$

Then one concludes from (3.2)-(3.5) that

$$\max_{\mathfrak{m}(25/72-\varepsilon)\setminus\mathfrak{D}(b)} |T(\alpha)| \ll \max\Big\{U^{1-b/3}, U^{3/4+\varepsilon}\Big\}.$$

Therefore, on choosing $b = 3/4$, we obtain

$$\begin{aligned} &\int_{\mathfrak{m}(25/72-\varepsilon)} |S(\alpha,U)|^2|S(\alpha,V)|^4|T(\alpha)|^2 d\alpha \\ &\ll \int_{\mathfrak{D}(3/4)} |S(\alpha,U)|^2|S(\alpha,V)|^4|T(\alpha)|^2 d\alpha \\ &\qquad +U^{3/2+\varepsilon}\int_0^1 |S(\alpha,U)|^2|S(\alpha,V)|^4 d\alpha. \end{aligned}$$

Here by Lemma 3.2, the last term is

$$\le U^{3/2+\varepsilon}Z_0 \ll U^{5/2+\varepsilon}V^2.$$

So it remains to prove

$$\int_{\mathfrak{D}(3/4)} |S(\alpha,U)|^2|S(\alpha,V)|^4|T(\alpha)|^2 d\alpha \ll U^{5/2+\varepsilon}V^2. \tag{3.6}$$

For $\alpha = a/q + \lambda \in \mathfrak{D}(3/4)$, there holds either

$$U^{25/72-\varepsilon} < q \le U^{3/4}, \qquad |\lambda| \le \omega(q)U^{1/4-3};$$

or

$$1 \le q \le U^{25/72-\varepsilon}, \qquad 1/(qU^{191/72+\varepsilon}) < |\lambda| \le \omega(q)U^{1/4-3}.$$

By (3.3) where the right hand side is dominated by $\omega(q)U(1+|\lambda|U^3)^{-1}$, one has

$$\begin{aligned}
&\int_{\mathfrak{D}(3/4)}|S(\alpha,U)|^2|S(\alpha,V)|^4|T(\alpha)|^2d\alpha\\
&\ll U^2\sum_{1\le q\le U^{25/72-\varepsilon}}\sum_{a=1}^{q}\omega^2(q)\int_{1/(qU^{191/72+\varepsilon})<|\lambda|\le\omega(q)U^{1/4-3}}\frac{|S(a/q+\lambda,U)|^2|S(a/q+\lambda,V)|^4}{(1+|\lambda|U^3)^2}d\lambda\\
&\quad+U^2\sum_{U^{25/72-\varepsilon}<q\le U^{3/4}}\sum_{a=1}^{q}\omega^2(q)\int_{|\lambda|\le\omega(q)U^{1/4-3}}\frac{|S(a/q+\lambda,U)|^2|S(a/q+\lambda,V)|^4}{(1+|\lambda|U^3)^2}d\lambda\\
&:=M_1+M_2,\quad \text{say.}
\end{aligned}\tag{3.7}$$

To estimate M_1 and M_2, we observe that for $|\lambda|\le\omega(q)U^{1/4-3}$, there holds $|\lambda|V^3\le 1$. Hence by Lemma 3.1 we have

$$S(a/q+\lambda,V)\ll V^{\varepsilon}\Big\{V^{1/2}q^{1/2}+V^{4/5}+Vq^{-1/2}\Big\}.\tag{3.8}$$

For $q\le U^{25/72-\varepsilon}$, this gives

$$|S(a/q+\lambda,V)|^4\ll V^{\varepsilon}\Big\{V^{16/5}+V^4q^{-2}\Big\}.$$

By Lemma 3.1, we also have

$$\frac{|S(a/q+\lambda,U)|^2}{1+|\lambda|U^3}\ll U^{\varepsilon}\left\{Uq+\frac{U^{8/5}}{1+|\lambda|U^3}+\frac{U^2q^{-1}}{(1+|\lambda|U^3)^2}\right\}.\tag{3.9}$$

For $|\lambda|>1/(qU^{191/72-\varepsilon})$, this gives

$$\frac{|S(a/q+\lambda,U)|^2}{1+|\lambda|U^3}\ll qU^{47/36+\varepsilon}.$$

Therefore

$$\begin{aligned}
M_1&\ll U^{2+47/36+\varepsilon}\sum_{1\le q\le U^{25/72-\varepsilon}}q^2\omega^2(q)\left\{V^{16/5}+V^4q^{-2}\right\}\int_0^1\frac{d\lambda}{1+|\lambda|U^3}\\
&\ll U^{11/36+\varepsilon}\sum_{1\le q\le U^{25/72-\varepsilon}}\omega^2(q)\left\{q^2V^{16/5}+V^4\right\}.
\end{aligned}$$

By Lemma 3.3, for any $X>Y\ge 1$,

$$\sum_{Y\le q\le X}\omega^2(q)\ll\sum_{Y\le q\le X}q^{1/2}\omega^3(q)\ll 1.\tag{3.10}$$

So we get

$$M_1\ll U^{11/36+\varepsilon}\left\{V^{16/5}U^{25/36}+V^4\right\}\ll U^{2+\varepsilon}V^2.\tag{3.11}$$

We now turn to M_2. Again by (3.8) and (3.9), we have

$$|S(a/q+\lambda,V)|^2\ll V^{\varepsilon}\Big\{Vq+V^{8/5}+V^2q^{-1}\Big\},$$

and

$$\frac{|S(a/q+\lambda,U)|^2}{1+|\lambda|U^3} \ll U^{\varepsilon}\Big\{Uq+U^{8/5}+U^2q^{-1}\Big\}.$$

Thus

$$\begin{aligned} M_2 &\ll U^{2+\varepsilon} \sum_{U^{25/72-\varepsilon}<q\le U^{3/4}} \omega^2(q)\Big\{Vq+V^{8/5}+V^2q^{-1}\Big\}\Big\{Uq+U^{8/5}+U^2q^{-1}\Big\} \\ &\quad\times \int_{|\lambda|\le\omega(q)U^{1/4-3}} \sum_{a=1}^{q} |S(a/q+\lambda,V)|^2 d\lambda. \end{aligned} \tag{3.12}$$

We have

$$\begin{aligned} &\sum_{a=1}^{q} |S(a/q+\lambda,V)|^2 \\ &= \sum_{m_i\sim V} \Lambda(m_1)\Lambda(m_2)\sum_{a=1}^{q} e\left((a/q+\lambda)(m_1^3-m_2^3)\right) \\ &\ll qL^2\sharp\Big\{m_1^3\equiv m_2^3 \pmod q,\ m_i\sim V,\ (m_i,q)=1\Big\} \\ &\ll V^2L^2. \end{aligned}$$

Putting this into (3.12) and then applying the second inequality in (3.10), we get

$$M_2 \ll U^{-3/4+\varepsilon}V^2 \max_{U^{25/72-\varepsilon}<q\le U^{3/4}} q^{-1/2}\Big\{Vq+V^{8/5}+V^2q^{-1}\Big\}\Big\{Uq+U^{8/5}+U^2q^{-1}\Big\}.$$

Since $U=V^{5/6}$, we have for $q>U^{25/72-\varepsilon}$,

$$\begin{aligned} &\Big\{Vq+V^{8/5}+V^2q^{-1}\Big\}\Big\{Uq+U^{8/5}+U^2q^{-1}\Big\} \\ &\ll UVq^2+U^{8/5}Vq+U^{8/5}V^{8/5}+U^2V^{8/5}q^{-1}. \end{aligned}$$

Hence

$$\begin{aligned} M_2 &\ll U^{-3/4+\varepsilon}V^2\Big\{U^{1+9/8}V+U^{8/5+3/8}V+U^{8/5-25/144}V^{8/5}+U^{2-75/144}V^{8/5}\Big\} \\ &\ll U^{53/24+\varepsilon}V^2. \end{aligned}$$

This together with (3.7) and (3.11) proves (3.6), and hence finishes the proof of Lemma 2.2. □

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