POISSON STRUCTURES ON COMPLEX FLAG MANIFOLDS ASSOCIATED WITH REAL FORMS

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Abstract. For a complex semisimple Lie group G and a real form G_0 we define a Poisson structure on the variety of Borel subgroups of G with the property that all G_0-orbits in X as well as all Bruhat cells (for a suitable choice of a Borel subgroup of G) are Poisson submanifolds. In particular, we show that every non-empty intersection of a G_0-orbit and a Bruhat cell is a regular Poisson manifold and we compute the dimension of its symplectic leaves.

Dedicated to Alan Weinstein on
the occasion of his 60th Birthday.

1. Introduction.

Let G be a connected and simply connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$, and let X be the variety of Borel subalgebras of $\mathfrak{g}$. In this paper we use a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ to define a Poisson structure on X. This Poisson structure depends on a choice of a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ such that $\mathfrak{g}_0 \cap \mathfrak{b}$ is a maximally compact Cartan subalgebra of $\mathfrak{g}_0$. Instead of dealing with each real form individually, we fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. Then, as is shown in [6], a real from $\mathfrak{g}_v$ of $\mathfrak{g}$ can be constructed from each Vogan diagram v for $\mathfrak{g}$ such that $\mathfrak{g}_v \cap \mathfrak{b}$ is a maximally compact Cartan subalgebra of $\mathfrak{g}_v$. The corresponding Poisson structure on X is denoted by Π_v.

Let G_v be the real form of G corresponding to $\mathfrak{g}_v$, and let B be the Borel subgroup of G with Lie algebra $\mathfrak{b}$. The Poisson structure Π_v has the property that each G_v-orbit as well as each B-orbit in X is a Poisson submanifold. The B-orbits in X will be referred to as the Bruhat cells. We compute the rank of Π_v. In particular, if G_v-orbit $\mathcal{O}$ meets a Bruhat cell $\mathcal{C}$, they intersect transversally, and we find that all the symplectic leaves in $\mathcal{O} \cap \mathcal{C}$ have the same dimension, so $\mathcal{O} \cap \mathcal{C}$ is a regular Poisson manifold. Moreover, we show that

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all symplectic leaves in each connected component of $\mathcal{O} \cap \mathcal{C}$ are translates of each other by elements of a Cartan subgroup of G_v. We also show that the G_v-invariant Poisson cohomology for each open G_v-orbit in X is isomorphic to the de Rham cohomology of X.

Throughout this paper, if V is a set and σ is an involution on V, we will use V^σ to denote the fixed point set of σ in V.

2. Real forms of $\mathfrak{g}$ and Vogan diagrams

Let $\mathfrak{g}$ be a complex simple Lie algebra. In this section we recall the classification of real forms of $\mathfrak{g}$ by Vogan diagrams. Details can be found in [6, Chapter 6].

Suppose that $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$ and that τ_0 is the corresponding complex-conjugate linear involution on $\mathfrak{g}$. Let θ_0 be a Cartan involution of $\mathfrak{g}_0$, and let $\mathfrak{h}_0$ be a θ_0-stable maximally compact Cartan subalgebra of $\mathfrak{g}_0$. Set $\mathfrak{t}_0 = \mathfrak{h}_0^{\theta_0}$ and $\mathfrak{a}_0 = \mathfrak{h}_0^{-\theta_0}$ so that $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$. Let γ_0 be the complexification of θ_0. Then the Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_0 + i\mathfrak{h}_0$ of $\mathfrak{g}$ is γ_0-stable. Let Δ be the root system for $(\mathfrak{g}, \mathfrak{h})$. Since $\mathfrak{h}_0$ is a maximally compact Cartan subalgebra of $\mathfrak{g}_0$, there exists $x_0 \in i\mathfrak{t}_0$ that is regular for Δ. Define the subset Δ^+ of positive roots in Δ by $\alpha \in \Delta^+$ if and only if $\alpha(x_0) > 0$. Then $\gamma_0(\Delta^+) = \Delta^+$. Let $\Sigma \subset \Delta^+$ be the set of simple roots in Δ^+. Then $\gamma_0(\Sigma) = \Sigma$, so γ_0 gives rise to an involutive automorphism of the Dynkin diagram of $\mathfrak{g}$. Let $\mathcal{I}$ be the set of non-compact imaginary simple roots. The Vogan diagram of $\mathfrak{g}_0$ associated to the triple $(\theta_0, \mathfrak{h}_0, \Delta^+)$ is the Dynkin diagram $D(\mathfrak{g})$ of $\mathfrak{g}$ together with an involutive automorphism γ_0 on $D(\mathfrak{g})$ and the vertices corresponding to the simple roots in $\mathcal{I}$ painted black.

In general, a Vogan diagram for $\mathfrak{g}$ is defined to be a triple $(D(\mathfrak{g}), d, \mathcal{I})$, where $D(\mathfrak{g})$ is the Dynkin diagram of $\mathfrak{g}$, d is an involutive automorphism of $D(\mathfrak{g})$, and $\mathcal{I}$ is a subset of vertices of $D(\mathfrak{g})$ such that $d(\alpha) = \alpha$ for each $\alpha \in \mathcal{I}$. Every Vogan diagram for $\mathfrak{g}$ comes from a real form of $\mathfrak{g}$ (see below), although two different Vogan diagrams can come from isomorphic real forms. A non-redundant list of Vogan diagrams with the corresponding isomorphism class of real forms for all simple Lie algebras is given in [6]. Every Vogan diagram in the list in [6] is *normalized* in the sense that at most one vertex is painted black.

For the purpose of defining Poisson structures on the variety of Borel subalgebras of $\mathfrak{g}$, we now recall the explicit construction of a real form of $\mathfrak{g}$ from a Vogan diagram [6, Theorem 6.88]. We need to fix the following data for $\mathfrak{g}$.

Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let Δ be the root system for $(\mathfrak{g}, \mathfrak{h})$. Fix a choice of positive roots Δ^+ and let Σ be the basis of simple roots. Let $\ll, \gg$ be the Killing form of $\mathfrak{g}$ and let root vectors $\{E_\alpha : \alpha \in \Delta\}$ be chosen such that $[E_\alpha, E_{-\alpha}] = H_\alpha$ for each $\alpha \in \Delta^+$, where H_α is the unique element of $\mathfrak{h}$ defined by $\ll H, H_\alpha \gg = \alpha(H)$ for all $H \in \mathfrak{h}$, and such that the numbers $m_{\alpha,\beta}$ given by $[E_\alpha, E_\beta] = m_{\alpha,\beta} E_{\alpha+\beta}$ when $\alpha + \beta \in \Delta$

are real. Define a compact real form $\mathfrak{k}$ of $\mathfrak{g}$ as

$$\mathfrak{k} = \mathrm{span}_{\mathbb{R}}\{\mathrm{i}H_\alpha,\ X_\alpha := E_\alpha - E_{-\alpha},\ Y_\alpha := \mathrm{i}(E_\alpha + E_{-\alpha})\}\ ,$$

and let θ be the complex conjugation of $\mathfrak{g}$ defining $\mathfrak{k}$. If d is an involutive automorphism of the Dynkin diagram of $\mathfrak{g}$, define γ_d to be the unique automorphism of $\mathfrak{g}$ satisfying $\gamma_d(H_\alpha) = H_{d(\alpha)}$ and $\gamma_d(E_\alpha) = E_{d(\alpha)}$ for each simple root α.

Given a Vogan diagram v for $\mathfrak{g}$, not necessarily normalized, with the involutive diagram automorphism d, let t_v be the unique element in the adjoint group of $\mathfrak{g}$ such that

$$\mathrm{Ad}_{t_v}(E_\alpha) = \begin{cases} E_\alpha \text{ if } \alpha \text{ is a blank vertex in } v \\ -E_\alpha \text{ if } \alpha \text{ is a painted vertex in } v \end{cases}$$

Define a complex conjugate linear involution

$$\tau_v := \mathrm{Ad}_{t_v} \circ \gamma_d \circ \theta.$$

Notation 2.1. We use $\mathfrak{g}_v = \mathfrak{g}^{\tau_v}$ to denote the real form of $\mathfrak{g}$ defined by τ_v. Set $\theta_v = \theta|_{\mathfrak{g}_v}$. Then θ_v is a Cartan involution of $\mathfrak{g}_v$, and $\mathfrak{h}^{\tau_v}$ is a θ_v-stable maximally compact Cartan subalgebra of $\mathfrak{g}_v$, with $\mathfrak{h} = \mathfrak{h}^{\tau_v} + \mathrm{i}\mathfrak{h}^{\tau_v}$. The complexification of τ_v is

$$\gamma_v := \tau_v\theta = \theta\tau_v = \mathrm{Ad}_{t_v}\gamma_d. \tag{2.1}$$

Since $\gamma_v(\Delta^+) = \Delta^+$, the Vogan diagram of $\mathfrak{g}_v$ associated to the triple $(\theta_v, \mathfrak{h}^{\tau_v}, \Delta^+)$ is v.

One of the advantages of introducing the real form $\mathfrak{g}_v$ is as follows. We say that a real subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ is *Lagrangian* if its real dimension is equal to the complex dimension of $\mathfrak{g}$ and if $\mathrm{Im} \ll x_1, x_2 \gg = 0$ for all $x_1, x_2 \in \mathfrak{l}$. A decomposition $\mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$ is called a *Lagrangian splitting* if both $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are Lagrangian. Let $\mathfrak{n}$ be the subalgebra of $\mathfrak{g}$ spanned by the set of all positive root vectors for Δ^+. The following fact is easy to prove.

Lemma 2.2. *Let $\mathfrak{l}_d := \mathfrak{h}^{-\tau_v} + \mathfrak{n}$. Then $\mathfrak{g} = \mathfrak{g}_v + \mathfrak{l}_d$ is a Lagrangian splitting of $\mathfrak{g}$.*

Let $\mathfrak{a} = \mathrm{span}_{\mathbb{R}}\{\mathrm{i}H_\alpha : \alpha \in \Sigma\}$, and let $\mathfrak{t} = \mathrm{i}\mathfrak{a}$. We note that since

$$\mathfrak{h}^{-\tau_v} = \mathfrak{h}^{-\gamma_d\circ\theta} = \mathfrak{t}^{-\gamma_d} + \mathfrak{a}^{\gamma_d},$$

the Lagrangian complement $\mathfrak{l}_d$ of $\mathfrak{g}_v$ depends only on d, and in the case when $d = 1$, we have $\mathfrak{l}_d = \mathfrak{a} + \mathfrak{n}$. Note that $\mathfrak{h}^{\tau_v} = \mathfrak{h}^{\gamma_d\circ\theta} = \mathfrak{t}^{\gamma_d} + \mathfrak{a}^{-\gamma_d}$ also depends only on d.

Remark 2.3. Recall [2, Definition 6.10] that two real forms τ_1 and τ_2 are said to be in the same *inner class* if there exists $g \in \mathrm{Int}(\mathfrak{g})$, the adjoint group of $\mathfrak{g}$, such that $\tau_1 = \mathrm{Ad}_g\tau_2$. Inner classes of real forms are in one-to-one correspondence with involutive automorphisms of the Dynkin diagram of $\mathfrak{g}$ [2, Proposition 6.12]. Let d be an involutive automorphism of $D(\mathfrak{g})$. Then as v runs over the collection of all Vogan diagrams with d as the diagram automorphism, the real form $\mathfrak{g}_v$ runs over all $\mathrm{Int}(\mathfrak{g})$-conjugacy classes of real forms of $\mathfrak{g}$ in the inner class corresponding to d.

3. The Poisson structure Π_v on X.

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, and let X be the variety of all Borel subalgebras of $\mathfrak{g}$. We keep the notation from Section 2. Let v be a Vogan diagram for $\mathfrak{g}$ and $\mathfrak{g}_v = \mathfrak{g}^{\tau_v}$ be the real form of $\mathfrak{g}$ constructed in Section 2. Let G be the connected and simple connected Lie group with Lie algebra $\mathfrak{g}$. Without any risk of confusion, we shall also denote by τ_v the lift of τ_v from $\mathfrak{g}$ to G, and we set $G_v = G^{\tau_v}$. It follows from [5, Theorem 8.2, p. 320] that the group G_v is connected.

In this section, we will start with a Vogan diagram v for $\mathfrak{g}$ and define a Poisson structure Π_v on X such that every G_v-orbit in X is a Poisson submanifold. This Poisson structure comes from an identification of X with the G-orbit through $\mathfrak{t}+\mathfrak{n}$ inside the variety $\mathcal{L}$ of Lagrangian subalgebras of $\mathfrak{g}$, which was studied in [3]. We now recall the relevant details.

Set $n = \dim_{\mathbb{C}}\mathfrak{g}$ and let $\mathrm{Gr}_{\mathbb{R}}(n,\mathfrak{g})$ be the Grassmannian of real n-dimensional subspaces of $\mathfrak{g}$. The set $\mathcal{L}$ of all Lagrangian subalgebras of $\mathfrak{g}$ is naturally a real subvariety of $\mathrm{Gr}_{\mathbb{R}}(n,\mathfrak{g})$. The natural action of G on $\mathrm{Gr}_{\mathbb{R}}(n,\mathfrak{g})$ gives rise to a Lie algebra anti-homomorphism κ from $\mathfrak{g}$ to the Lie algebra of vector fields on $\mathrm{Gr}_{\mathbb{R}}(n,\mathfrak{g})$, whose extension from $\wedge^2\mathfrak{g}$ to the space of bi-vector fields on $\mathrm{Gr}_{\mathbb{R}}(n,\mathfrak{g})$ will also be denoted by κ. Given a Lagrangian splitting $\mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$, we define the element $R_{\mathfrak{l}_1,\mathfrak{l}_2} \in \wedge^2\mathfrak{g}$ by:

$$\langle R_{\mathfrak{l}_1,\mathfrak{l}_2},\, (x_1+\xi_1)\wedge(x_2+\xi_2)\rangle = \langle \xi_2, x_1\rangle - \langle \xi_1, x_2\rangle, \qquad x_1, x_2 \in \mathfrak{l}_1,\ \xi_1,\xi_2 \in \mathfrak{l}_2, \tag{3.1}$$

where $\langle\, ,\, \rangle = \mathrm{Im} \ll\, ,\, \gg$. Set $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2} = \frac{1}{2}\kappa(R_{\mathfrak{l}_1,\mathfrak{l}_2})$. Clearly, $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ is tangent to every G-orbit in $\mathrm{Gr}_{\mathbb{R}}(n,\mathfrak{g})$, so it is tangent to $\mathcal{L}$.

Theorem 3.1. [3, Theorems 2.14 and 2.18] *The bi-vector field $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ restricts to a Poisson structure on $\mathcal{L}$. If L_1 and L_2 are the connected subgroups of G with Lie algebras $\mathfrak{l}_1$ and $\mathfrak{l}_2$ respectively, then all the L_1- as well as L_2-orbits in $\mathcal{L}$ are Poisson submanifolds with respect to $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$.*

For $\mathfrak{l} \in \mathcal{L}$, let $\mathfrak{n}(\mathfrak{l})$ be the normalizer subalgebra of $\mathfrak{l}$ in $\mathfrak{l}_1$. Let $\mathfrak{m}(\mathfrak{l})$ be the annihilator of $\mathfrak{n}(\mathfrak{l})$ in $\mathfrak{l}$, i.e. $\mathfrak{m}(\mathfrak{l}) = \{x \in \mathfrak{l} : \langle x, y\rangle = 0\ \forall y \in \mathfrak{n}(\mathfrak{l})\} \subset \mathfrak{l}$, and let $\mathcal{V}(\mathfrak{l}) = \mathfrak{n}(\mathfrak{l}) + \mathfrak{m}(\mathfrak{l})$.

Proposition 3.2. [3, Theorem 2.21] [9, Corollary 7.3] *For each $\mathfrak{l} \in \mathcal{L}$, the space $\mathcal{V}(\mathfrak{l})$ is a Lagrangian subalgebra of $\mathfrak{g}$. The co-dimension of the symplectic leaf of $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ through $\mathfrak{l}$ in the orbit $L_1\cdot\mathfrak{l}$ is equal to* $\dim(\mathcal{V}(\mathfrak{l})\cap\mathfrak{l}_2)$.

Notation 3.3. Let v be a Vogan diagram for $\mathfrak{g}$. We denote by Π_v the Poisson structure on $\mathcal{L}$ defined by the Lagrangian splitting $\mathfrak{g} = \mathfrak{g}_v + \mathfrak{l}_d$ in Lemma 2.2. Let H, N, and B be respectively the connected subgroups of G with Lie algebras $\mathfrak{h}$, $\mathfrak{n}$, and $\mathfrak{b} = \mathfrak{h}+\mathfrak{n}$, so $B = HN$. Identify the G-orbit through $\mathfrak{t}+\mathfrak{n} \in \mathcal{L}$ with $G/B \cong X$. The induced Poisson structure on X will also be denoted by Π_v. Let $H^{-\gamma_d\circ\theta} = \{h \in H : \gamma_d\circ\theta(h) = h^{-1}\}$ and let $L_d = H^{-\gamma_d\circ\theta}N$. By the Bruhat lemma, orbits of L_d in $X \cong G/B$, which are the same as the N-orbits in X, are labeled by the elements in the Weyl group W of Δ. We refer to these N-orbits as the Bruhat cells in X.

By [3, Theorem 2.18], we have

Proposition 3.4. *Each G_v-orbit in X as well as each Bruhat cell in X is a Poisson submanifold with respect to Π_v.*

When v is the Vogan diagram with $d = 1$ and no vertex painted, we have $\tau_v = \theta$, so $\mathfrak{g}_v = \mathfrak{k}$. The Poisson structure Π_v in this case was first introduced in [11] and [13], and it has the property that its symplectic leaves are precisely the Bruhat cells (hence the name "Bruhat Poisson structure" in [11]). In [3] and [10] this Poisson structure was related to some earlier work of Kostant [7] and of Kostant-Kumar [8] on the Schubert calculus on X.

The splitting $\mathfrak{g} = \mathfrak{g}_v + \mathfrak{l}_d$ naturally defines a Lie bialgebra structure on $\mathfrak{g}_v$ and therefore a Poisson Lie group structure on G_v [11]. All the G_v-orbits in $\mathcal{L}$ become G_v-Poisson homogeneous spaces [3, 9]. We remark that in [1], Andruskiewitsch and Jancsa classified non-triangular Lie bialgebra structures on $\mathfrak{g}_v$ using Belavin-Drinfeld triples. The one defined by the splitting $\mathfrak{g} = \mathfrak{g}_v + \mathfrak{l}_d$ comes from the standard Belavin-Drinfeld triple. We refer to [1] for details.

Example. Here we take $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and

$$\mathfrak{g}_v = \mathfrak{su}(1,1) = \left\{ \begin{pmatrix} \mathrm{i}x & y + \mathrm{i}z \\ y - \mathrm{i}z & -\mathrm{i}x \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

Then $d = 1$ and $\mathfrak{l}_d = \mathfrak{a} + \mathfrak{n}$ consists of upper triangular matrices in $\mathfrak{sl}(2, \mathbb{C})$ with real diagonal entries. Identify G/B with $\mathbb{P}^1$ via the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [w_0 : w_1] = [aw_0 + bw_1 : cw_0 + dw_1]$$

of G on $\mathbb{P}^1$ and by taking $[1 : 0] \in \mathbb{P}^1$ as the basepoint. There are two Bruhat cells: the zero-dimensional basepoint $[1 : 0]$, and the other being the rest:

$$U_1 = \mathbb{P}^1 \backslash \{[1 : 0]\} = \{[w_0 : w_1],\ w_1 \neq 0\}.$$

In terms of the holomorphic coordinate z on U_1 given by $z = w_0/w_1$ the Poisson structure Π_v, up to a scalar multiple, is given by:

$$\Pi_v = \mathrm{i}(1 - |z|^2) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \bar{z}}.$$

Setting $u = 1/z$, we see that in the u-coordinate on the open set

$$U_0 = \{[w_0 : w_1] \in \mathbb{P}^1, w_0 \neq 0\} = \{[1 : u], u \in \mathbb{C}\},$$

we have

$$\Pi_v = \mathrm{i}(|u|^2-1)|u|^2\frac{\partial}{\partial u}\wedge\frac{\partial}{\partial\bar{u}}.$$

Thus Π_v vanishes precisely at the basepoint $[1:0]$ and at every point of the form $[z:1]$ with $|z|=1$. If we identify $\mathbb{P}^1$ with the unit sphere S^2 in $\mathbb{R}^3$ via:

$$\mathbb{P}^1 \longrightarrow S^2:\ [w_0,w_1]\longmapsto\left(\frac{2\mathrm{Re}(w_0\overline{w_1})}{|w_0|^2+|w_1|^2},\ \frac{2\mathrm{Im}(w_0\overline{w_1})}{|w_0|^2+|w_1|^2},\ \frac{|w_0|^2-|w_1|^2}{|w_0|^2+|w_1|^2}\right),\tag{3.2}$$

then we see that Π_v vanishes at the "North pole" $(0,0,1)$ and at every point on the Equator $x_3=0$. Under this identification, there are exactly three orbits of $\mathrm{SU}(1,1)$ on S^2: the Northern hemisphere, the Equator, and the Southern hemisphere. Each one of these three orbits is clearly a Poisson submanifold.

4. Symplectic leaves of Π_v in X.

Suppose that $\mathcal{O}$ is a G_v-orbit in X and $\mathcal{C}$ is a Bruhat cell such that $\mathcal{O}\cap\mathcal{C}\neq\emptyset$. Since $\mathfrak{g}=\mathfrak{g}_v+\mathfrak{l}_d$, $\mathcal{O}$ and $\mathcal{C}$ intersect transversally. By Proposition 3.4, $\mathcal{O}\cap\mathcal{C}$ is a Poisson submanifold of Π_v. In this section we show that $(\mathcal{O}\cap\mathcal{C},\Pi_v)$ is a regular Poisson manifold and we compute the dimension of its symplectic leaves.

It is well-known [14] that there are only finitely many G_v-orbits in X. We first recall from [12, Section 6] some facts about these orbits.

Let $N_G(\mathfrak{h})$ be the normalizer subgroup of $\mathfrak{h}$ in G. Set

$$\mathcal{Z}=\{g\in G:\ g^{-1}\tau_v(g)\in N_G(\mathfrak{h})\}.$$

Then H acts on $\mathcal{Z}$ from the right by right multiplication, and G_v acts on $\mathcal{Z}$ from the left by left multiplication. Let Z be the double coset space

$$Z=G_v\backslash\mathcal{Z}/H.$$

For each $z\in Z$, choose any $g_z\in\mathcal{Z}$ in the double coset z and define $\mathcal{O}_z$ to be the G_v-orbit in X through $g_zB\in X\cong G/B$. Clearly, $\mathcal{O}_z$ is independent of the choice of g_z. According to [12, Theorem 6.1.4], the map $z\mapsto\mathcal{O}_z$ is a one-to-one correspondence between the set Z and the set of G_v-orbits in X. Let $W=N_G(\mathfrak{h})/H$ be the Weyl group. Thus we also have the map

$$\varphi:\ Z\longrightarrow W:\ z=G_vg_zH\longmapsto g_z^{-1}\tau_v(g_z)H\in W.$$

According to [12, Theorem 6.4.2], the codimension of the G_v-orbit $\mathcal{O}_z$ in X equals $l(\varphi(z))$, where l is the length function on the Weyl group W. We also introduce the map:

$$\sigma_z=\varphi(z)\tau_v:\ \mathfrak{h}\longrightarrow\mathfrak{h}.$$

For any g_z in the double coset z, we also have $\sigma_z=\mathrm{Ad}_{g_z}^{-1}\circ\tau_v\circ\mathrm{Ad}_{g_z}$, so σ_z is an involution.

Assume now that $z \in Z$ and $w \in W$ are such that $\mathcal{O}_z \cap \mathcal{C}_w \neq \emptyset$, where $\mathcal{C}_w$ is the Bruhat cell in X corresponding to w, i.e. the N-orbit through $w \in G/B$. Then $\dim_{\mathbb{R}} \mathcal{C}_w = 2l(w)$, and since $\mathcal{O}_z$ and $\mathcal{C}_w$ intersect transversally, we have

$$\dim(\mathcal{O}_z \cap \mathcal{C}_w) = 2l(w) - l(\varphi(z)).$$

Define now

$$\delta_{z,w} = \dim(\mathfrak{h}^{w\sigma_z w^{-1}} \cap \mathfrak{h}^{-\tau_v}).$$

Theorem 4.1. *Each symplectic leaf in the intersection $\mathcal{O}_z \cap \mathcal{C}_w$ has dimension equal to*

$$\dim(\mathcal{O}_z \cap \mathcal{C}_w) - \delta_{z,w} = 2l(w) - l(\varphi(z)) - \delta_{z,w}.$$

Proof. We use Proposition 3.2 to compute dimensions of the symplectic leaves in $\mathcal{O}_z \cap \mathcal{C}_w$. Let $x = g_z B \in X$ be a point in $\mathcal{O}_z \cap \mathcal{C}_w$, where $g_z \in \mathcal{Z}$ lies in the double coset z. Let $\mathfrak{l}_x = \mathrm{Ad}_{g_z}(\mathfrak{t} + \mathfrak{n}) \in \mathcal{L}$. Let $\mathfrak{n}(\mathfrak{l}_x) = \mathfrak{g}_v \cap \mathrm{Ad}_{g_z}(\mathfrak{h} + \mathfrak{n})$ be the normalizer subalgebra of $\mathfrak{l}_x$ in $\mathfrak{g}_v$, $\mathfrak{m}(\mathfrak{l}_x)$ the annihilator subspace of $\mathfrak{n}(\mathfrak{l}_x)$ in $\mathfrak{l}_x$, and $\mathcal{V}(\mathfrak{l}_x) = \mathfrak{n}(\mathfrak{l}_x) + \mathfrak{m}(\mathfrak{l}_x)$. We claim that $\mathcal{V}(\mathfrak{l}_x) = \mathrm{Ad}_{g_z}(\mathfrak{h}^{\sigma_z} + \mathfrak{n})$. Indeed, it follows from the definition of σ_z that

$$\mathrm{Ad}_{g_z}(\mathfrak{h}^{\sigma_z}) \subset \mathfrak{g}_v \cap \mathrm{Ad}_{g_z}(\mathfrak{h} + \mathfrak{n}) = \mathfrak{n}(\mathfrak{l}_x)\ .$$

It is also clear that $\mathrm{Ad}_{g_z}\mathfrak{n} \subset \mathfrak{m}(\mathfrak{l}_x)$, so

$$\mathrm{Ad}_{g_z}(\mathfrak{h}^{\sigma_z} + \mathfrak{n}) \subset \mathfrak{n}(\mathfrak{l}_x) + \mathfrak{m}(\mathfrak{l}_x) = \mathcal{V}(\mathfrak{l}_x)\ .$$

Since both $\mathrm{Ad}_{g_z}(\mathfrak{h}^{\sigma_z} + \mathfrak{n})$ and $\mathcal{V}(\mathfrak{l}_x)$ have the same dimension, they must coincide.

Let now S_x be the symplectic leaf of Π_v in X through x. By Proposition 3.2, the codimension of S_x in $\mathcal{O}_z$ is equal to $\dim(\mathcal{V}(\mathfrak{l}_x) \cap \mathfrak{l}_d)$. Let $\dot{w} \in N_G(\mathfrak{h})$ be a representative of w in K. Since $x \in \mathcal{C}_w$, there exist $n \in N$ and $b \in B$ such that $g_z = n\dot{w}b$. Then we have

$$\begin{aligned}
\mathcal{V}(\mathfrak{l}_x) \cap \mathfrak{l}_d &= (\mathrm{Ad}_{n\dot{w}b}(\mathfrak{h}^{\sigma_z} + \mathfrak{n})) \cap (\mathfrak{h}^{-\tau_v} + \mathfrak{n}) \\
&= \mathrm{Ad}_n\left((\mathrm{Ad}_{\dot{w}}(\mathfrak{h}^{\sigma_z} + \mathfrak{n})) \cap (\mathfrak{h}^{-\tau_v} + \mathfrak{n})\right) \\
&= \mathrm{Ad}_n\left(\mathfrak{h}^{w\sigma_z w^{-1}} \cap \mathfrak{h}^{-\tau_v} + (\mathrm{Ad}_{\dot{w}}\mathfrak{n}) \cap \mathfrak{n}\right),
\end{aligned}$$

where in the last line we have the direct sum of vector spaces. Since

$$\dim(\mathrm{Ad}_{\dot{w}}\mathfrak{n}) \cap \mathfrak{n} = \dim_{\mathbb{R}} X - \dim_{\mathbb{R}} \mathcal{C}_w\ ,$$

we have

$$\dim(\mathcal{V}(\mathfrak{l}_x) \cap \mathfrak{l}_d) = \delta_{z,w} + \dim_{\mathbb{R}} X - \dim_{\mathbb{R}} \mathcal{C}_w\ ,$$

and thus

$$\dim S_x = \dim \mathcal{O}_z - \dim(\mathcal{V}(\mathfrak{l}_x) \cap \mathfrak{l}_d) = \dim(\mathcal{O}_z \cap \mathcal{C}_w) - \delta_{z,w}.$$

□

Note, that the number $\delta_{z,w}$ depends only on d and the two Weyl group elements $\varphi(z)$ and w. Define $d : W \to W$ by $d(w) = \gamma_d w \gamma_d$. Following [12], we say that $w \in W$ is a

d-twisted involution if $d(w) = w^{-1}$. Denote by $\mathcal{I}_d$ the set of all d-twisted involutions in W. Clearly, every $\varphi(z)$ is in $\mathcal{I}_d$. The Weyl group W acts on $\mathcal{I}_d$ by

$$w_1 * w = w_1 w d(w_1^{-1}) \quad \text{for } w_1 \in W, \quad \text{and } w \in \mathcal{I}_d \ ,$$

and the set $\varphi(Z) \subset \mathcal{I}_d$ is W-invariant. In fact, the W-action on G/H, given by $w \cdot gH = gw^{-1}H$, commutes with the left action of G_v by left multiplication, and thus induces a left action of W on Z, which we denote by $w \cdot z$ for $w \in W$ and $z \in Z$. It is also easy to see that $\varphi : Z \to W$ is W-equivariant, i.e. $\varphi(w \cdot z) = w * \varphi(z)$ for all $w \in W$ and $z \in Z$. Similarly, the involution $\tau_v : G \to G$ gives rise to an involution on Z which depends only on d. Denote this involution by $z \to d(z)$. Then we also have $\varphi(d(z)) = d\varphi(z) = \varphi(z)^{-1}$. As maps on $\mathfrak{h}$, we see that $w\sigma_z w^{-1} = (w * \varphi(z))\tau_v$. Thus we also have:

$$\delta_{z,w} = \dim(\mathfrak{h}^{(w*\varphi(z))\tau_v} \cap \mathfrak{h}^{-\tau_v}).$$

Corollary 4.2. *1) When $w * \varphi(z) = 1$, symplectic leaves of Π_v in $\mathcal{O}_z \cap \mathcal{C}_w$ are precisely its connected components.*

2) Every open orbit $\mathcal{O}_z$ has an open symplectic leaf $\mathcal{O}_z \cap \mathcal{C}_{w_0}$, where w_0 is the longest element in W;

3) If $d = 1$, symplectic leaves in an open orbit $\mathcal{O}_z$ are precisely the connected components of intersections of Bruhat cells with $\mathcal{O}_z$.

Proof. 1) When $w * \varphi(z) = 1$, we have $\delta_{z,w} = 0$, so every symplectic leaf in $\mathcal{O}_z \cap \mathcal{C}_w$ is open in $\mathcal{O}_z \cap \mathcal{C}_w$.

2) Since $\mathcal{C}_{w_0}$ is dense in X, it intersects with every open orbit $\mathcal{O}_z$. Since an orbit $\mathcal{O}_z$ is open if and only if $\varphi(z) = 1$, statement 2) follows from 1) and the fact that w_0 commutes with d. The fact that $\mathcal{C}_{w_0} \cap \mathcal{O}_z$ is connected follows from the observation that $\mathcal{O}_z$ is a connected open complex submanifold of X and thus $\mathcal{O}_z \cap (X \setminus \mathcal{C}_{w_0})$ is a divisor in $\mathcal{O}_z$.

3) follows directly from 1). □

Consider now the group $H^{\tau_v} = H \cap G_v$. The Poisson structure Π_v on X is H^{τ_v}-invariant. Indeed, let $R \in \wedge^2\mathfrak{g}$ be the element given in (3.1) for $\mathfrak{l}_1 = \mathfrak{g}_v$ and $\mathfrak{l}_2 = \mathfrak{l}_d$. We can also represent R as $R = \sum_i \xi_i \wedge y_i$, where $\{y_i\}$ is a basis of $\mathfrak{g}_v$, and $\{\xi_i\}$ is the dual basis of $\mathfrak{l}_d$ with respect to the pairing between $\mathfrak{g}_v$ and $\mathfrak{l}_d$ given by $\langle \, , \, \rangle$, the imaginary part of the Killing form on $\mathfrak{g}$. If $h \in H^{\tau_v}$, then $\{\mathrm{Ad}_h y_i\}$ is a basis of $\mathfrak{g}_v$, and $\{\mathrm{Ad}_h \xi_i\}$ is its dual basis. Thus $\mathrm{Ad}_h R = R$. It follows that the Poisson structure Π_v on X is H^{τ_v}-invariant.

Assume now that $z \in Z$ and $w \in W$ are such that $\mathcal{O}_z \cap \mathcal{C}_w \neq \emptyset$. Clearly, H^{τ_v} leaves $\mathcal{O}_z \cap \mathcal{C}_w$ invariant. Since the Poisson structure Π_v is H^{τ_v}-invariant, if S_x is the symplectic leaf of Π_v through x, then $hS_x := \{hx_1 : x_1 \in S_x\}$ is the symplectic leaf of Π_v through

hx. Let $H_0^{\tau_v}$ be the identity connected component of H^{τ_v}, and for $x \in X$ define:

$$F_x := \bigcup_{h \in H_0^{\tau_v}} hS_x \ .$$

Proposition 4.3. *For any $x \in X$, the set F_x is a connected component of $\mathcal{O}_z \cap \mathcal{C}_w$.*

Proof. It is easy to see that if $F_{x_1} \cap F_{x_2} \neq \emptyset$, then $F_{x_1} = F_{x_2}$. The statement would follow once we prove that F_x is an open subset of $\mathcal{O}_z \cap \mathcal{C}_w$ for each x.

Let $x = g_z B \in \mathcal{O}_z \cap \mathcal{C}_w$ with $g_z \in \mathcal{Z}$ in the double coset z. For $y \in \mathfrak{h}^{\tau_v}$, let X_y be the vector field on X generating the action of $\exp(ty) \in H_0^{\tau_v}$ on X. We claim that $X_y(x) \in T_x S_x$ if and only if $y \in p(\mathfrak{h}^{(w*\varphi(z))\tau_v})$, where $p : \mathfrak{h} \to \mathfrak{h}^{\tau_v}$ is the projection with respect to the decomposition $\mathfrak{h} = \mathfrak{h}^{\tau_v} + \mathfrak{h}^{-\tau_v}$. Assume the claim. Then since the kernel of the map $p : \mathfrak{h}^{(w*\varphi(z))\tau_v} \to \mathfrak{h}^{\tau_v}$ has dimension $\dim(\mathfrak{h}^{(w*\varphi(z))\tau_v} \cap \mathfrak{h}^{-\tau_v}) = \delta_{z,w}$, the image of the map

$$J_x : \ \mathfrak{h}^{\tau_v} \longrightarrow T_x\mathcal{O}_z / T_x S_x : \ y \longmapsto X_y(x) + T_x S_x$$

has dimension equal to $\dim(\mathfrak{h}^{\tau_v}) - \dim(\mathfrak{h}^{(w*\varphi(z))\tau_v}) + \delta_{z,w} = \delta_{z,w}$. Thus J_x is onto, and the $H_0^{\tau_v}$-orbit in $\mathcal{O}_z \cap \mathcal{C}_w$ through x is transversal to the symplectic leaf S_x. It follows that F_x is open in $\mathcal{O}_z \cap \mathcal{C}_w$.

It remains to prove the claim. Denote also by $p : \mathfrak{g} \to \mathfrak{g}_v$ the projection with respect to the decomposition $\mathfrak{g} = \mathfrak{g}_v + \mathfrak{l}_d$, and let q be the projection $q : \mathfrak{g}_v \to \mathfrak{g}_v / \mathfrak{g}_v \cap \mathrm{Ad}_{g_z}\mathfrak{b} \cong T_x\mathcal{O}_z$. Then by [9, Corollary 7.3], we have $T_x S_x = (q \circ p)(\mathcal{V}(\mathfrak{l}_x))$, where, as in the proof of Theorem 4.1, $\mathcal{V}(l_x) = \mathrm{Ad}_{g_z}(\mathfrak{h}^{\sigma_z} + \mathfrak{n})$. Let $y \in \mathfrak{h}^{\tau_v}$. If $X_y(x) \in T_x S_x$, then there exist $y_1 \in \mathfrak{l}_d$ and $y_2 \in \mathfrak{g}_v$ with $y_1 + y_2 \in \mathcal{V}(\mathfrak{l}_x)$ such that $y - y_2 \in \mathfrak{g}_v \cap \mathrm{Ad}_{g_z}\mathfrak{b} \subset \mathcal{V}(\mathfrak{l}_x)$. Thus $y + y_1 = y - y_2 + y_1 + y_2 \in \mathcal{V}(\mathfrak{l}_x)$. Write $y_1 = \xi_1 + u_1$, where $\xi_1 \in \mathfrak{h}^{-\tau_v}$ and $u_1 \in \mathfrak{n}$. Then there exist $\xi_2 \in \mathfrak{h}^{\sigma_z}$ and $u_2 \in \mathfrak{n}$ such that $y + \xi_1 + u_1 = \mathrm{Ad}_{g_z}(\xi_2 + u_2)$. Write $g_z = n\dot{w}b$, where $n \in N, b \in B$, and $\dot{w}$ is a representative of w in K. Write $\mathrm{Ad}_{n^{-1}}(y + \xi_1 + u_1) = y + \xi_1 + u_1'$ and $\mathrm{Ad}_b(\xi_2 + u_2) = \xi_2 + u_2'$, where $u_1', u_2' \in \mathfrak{n}$. Then we have

$$y + \xi_1 + u_1' = \mathrm{Ad}_{\dot{w}}(\xi_2 + u_2').$$

Since $y + \xi_1, \mathrm{Ad}_{\dot{w}}\xi_2 \in \mathfrak{h}$ and $u_1', \mathrm{Ad}_{\dot{w}}u_2' \in \mathfrak{n} + \mathfrak{n}_-$, where $\mathfrak{n}_- = \theta(\mathfrak{n})$, we have $y + \xi_1 = \mathrm{Ad}_{\dot{w}}\xi_2 \in \mathfrak{h}^{(w*\varphi(z))\tau_v}$. Thus $y \in p(\mathfrak{h}^{(w*\varphi(z))\tau_v})$. Conversely, if $y \in \mathfrak{h}^{\tau_v}$ is such that $y + \xi_1 \in \mathfrak{h}^{(w*\varphi(z))\tau_v} = \mathrm{Ad}_{\dot{w}}\mathfrak{h}^{\sigma_z}$ for some $\xi_1 \in \mathfrak{h}^{-\tau_v}$, write $y + \xi_1 = \mathrm{Ad}_{\dot{w}}\xi_2$ for $\xi_2 \in \mathfrak{h}^{\sigma_z}$. Let $\mathrm{Ad}_{b^{-1}}\xi_2 = \xi_2 + u_2$ for some $u_2 \in \mathfrak{n}$. We then have

$$\mathrm{Ad}_n(y + \xi_1) = \mathrm{Ad}_{n\dot{w}b}(\xi_2 + u_2) \in \mathcal{V}(\mathfrak{l}_x).$$

On the other hand, let $\mathrm{Ad}_n(y + \xi_1) = y + \xi_1 + u_1$ with $u_1 \in \mathfrak{n}$. We see that $y = p(\mathrm{Ad}_n(y + \xi_1))$ so $X_y(x) \in T_x S_x$. □

5. Invariant Poisson cohomology of open orbits.

Let $\mathcal{O}_z$ be a G_v-orbit in X equipped with the Poisson structure Π_v. Then $(\mathcal{O}_z, \Pi_v)$ is a Poisson homogeneous space for the Poisson Lie group G_v. The G_v-invariant Poisson cohomology of $(\mathcal{O}_z, \Pi_v)$, denoted by $H^\bullet_{\Pi_v, G_v}(\mathcal{O}_z)$, is defined as the cohomology of the cochain complex $(\chi^\bullet_{G_v}(\mathcal{O}_z), \partial_{\Pi_v})$, where $\chi^\bullet(\mathcal{O}_z)^{G_v}$ is the space of all G_v-invariant complex multi-vector fields on $\mathcal{O}_z$, $d_{\Pi_v}(V) = [\Pi_v, V]$, and $[\cdot, \cdot]$ is the Schouten bracket of the multi-vector fields.

Proposition 5.1. *When $\mathcal{O}_z$ is an open G_v-orbit in X, the G_v-invariant Poisson cohomology $H^\bullet_{\Pi_v, G_v}(\mathcal{O}_z)$ is isomorphic to the de Rham cohomology of X.*

Proof. As in the proof of Theorem 4.1, let $x = g_z B \in X$ be an arbitrary point in $\mathcal{O}_z$, where $g_z \in \mathcal{Z}$ is in the coset z, and let $\mathcal{V}(\mathfrak{l}_x) = \mathrm{Ad}_{g_z}(\mathfrak{h}^{\sigma_z} + \mathfrak{n})$. Since $\mathcal{O}_z$ is open, the stabilizer subalgebra of $\mathfrak{g}_v$ at x is $\mathfrak{g}_v \cap \mathcal{V}(\mathfrak{l}_x) = \mathrm{Ad}_{g_z}(\mathfrak{h}^{\sigma_z})$. By [9, Theorem 7.5], the G_v-invariant Poisson cohomology $H^\bullet_{\Pi_v, G_v}(\mathcal{O}_z)$ is isomorphic to the relative Lie algebra cohomology of the Lie algebra $\mathcal{V}(\mathfrak{l}_x) \otimes \mathbb{C}$ relative to the subalgebra $(\mathrm{Ad}_{g_z}(\mathfrak{h}^{\sigma_z})) \otimes \mathbb{C}$. Thus the G_v-invariant Poisson cohomology is isomorphic to the $\mathfrak{h}$-invariant part of the Lie algebra cohomology of the direct sum Lie algebra $\mathfrak{n} \oplus \mathfrak{n}$ with coefficients in $\mathbb{C}$, which by Kostant's theorem [7], is isomorphic to the de Rham cohomology of X. □

6. Remarks.

We have constructed a Poisson structure Π_v on X for each Vogan diagram v for $\mathfrak{g}$ (which is not necessarily normalized). In particular, each Bruhat cell $\mathcal{C}_w$ in X carries the Poisson structure Π_v. It would be interesting to study connections between the Poisson structures for different v. Especially interesting are the properties of Π_v that depend only on the inner class d of the real form $\mathfrak{g}_v$. We also remark that the Poisson structure Π_v is defined on the whole variety $\mathcal{L}$ of Lagrangian subalgebras of $\mathfrak{g}$. We have only been looking at the restriction of Π_v to a particular G-orbit, namely the G-orbit through the Lagrangian subalgebra $\mathfrak{t} + \mathfrak{n}$. There are many other interesting G-orbits in $\mathcal{L}$, such as the G-orbit through a given real form of $\mathfrak{g}$. It would be interesting to study the properties of the Poisson structure Π_v on these orbits as well as on their closures with respect to both the classical topology and the Zariski topology.

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