On the Existence of Periodic Solutions for $p$-Laplacian Generalized Liénard Equation

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Abstract—By employing Mawhin’s continuation theorem, the existence of periodic solutions of the $p$-Laplacian generalized Liénard equation with deviating argument

$$(\varphi_p(x'(t)))' + f(t, x(t))x'(t) + \beta(t)g(x(t - \tau(t))) = e(t)$$

under various assumptions are obtained.

Keywords—periodic solution, Mawhin’s continuation theorem, deviating argument.

1. INTRODUCTION

Consider the $p$-Laplacian generalized Liénard equation with a deviating argument

$$(\varphi_p(x'(t)))' + f(t, x(t))x'(t) + \beta(t)g(x(t - \tau(t))) = e(t), \quad (1.1)$$

where $p > 1$ is a constant; $\varphi_p : \mathbb{R} \to \mathbb{R}$, $\varphi_p(u) = |u|^{p-2}u$ is a one-dimensional $p$-Laplacian; $f = f(t, u) \in C(\mathbb{R}^2, \mathbb{R})$ is a periodic function with regard to $t$ with period $T > 0$; and $\beta, g, e, \tau \in C(\mathbb{R}, \mathbb{R})$, where $\beta, \tau, e$ are periodic functions with period $T$, $e(t) \not\equiv 0$, $\int_0^T e(s)ds = 0$, $\beta(t) > 0$ and $\tau(t) \geq 0$ for $t \in [0, T]$.

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There has been a great deal of work in the literature on such an equation which is used to describe fluid mechanical and nonlinear elastic mechanical phenomena. For example, in [1-3, 6, 10], by using the time maps and the phase plane analysis, the existence of periodic solutions to Eq.(1.1) for $p \neq 2$ and $\tau(t) \equiv 0$ was studied. On the other hand, for $p = 2$, $\tau(t) \not\equiv 0$ and $f(t, x(t))$ being replaced by $f(x(t))$, the existence of $T$–periodic solutions to several second order scalar differential equations were also studied in [5, 7-9]. In [9], S. Ma, Z. Wang and J. Yu studied delay Duffing equations of the type

$$x''(t) + m^2 x(t) + g(x(t - \tau)) = p(t). \quad (1.2)$$

By assuming that

$$\sup_{x \in \mathbb{R}} |g(x)| < \infty, \quad (1.3)$$

several sufficient conditions for the existence of periodic solutions of Eq.(1.2) were established. Recently, S. Lu and W. Ge in [7] discussed the existence of periodic solutions for the second order differential equation with multiple deviating arguments

$$x''(t) + f(x(t))x'(t) + \sum_{j=1}^{n} \beta_j(t)g(x(t - \gamma_j(t))) = p(t). \quad (1.4)$$

In their work, some linear growth condition imposed on $g(x)$ such as

$$\lim_{|x| \to +\infty} \frac{|g(x)|}{|x|} = r \in [0, +\infty). \quad (1.5)$$

was needed.

The main technique of these works [5, 7-9] is to convert the problem into the abstract form $Lx = Nx$, with $L$ being a non-invertible linear operator. Thus the existence of solutions of the problem can be given by Mawhin’s continuation theorem [4]. But as far as we are aware of, the corresponding problem of Eq.(1.1) with $p \neq 2$ and $\tau(t) \not\equiv 0$ has never been studied. This is mainly due to the facts that in this situation, on the one hand Mawhin’s continuation theorem is not applicable directly since the $p$-Laplacian $\varphi_p(u) = |u|^{p-2}u$ is not linear with respect to $u$ except when $p = 2$, and on the other hand, the crucial step $\int_0^T f(x(t))x'(t)dt = 0$ which is needed to obtain an a priori bound of periodic solutions for Eq.(1.1) is no longer valid.

In this paper, we get around with these difficulties by using some new techniques and translating Eq.(1.1) into a two-dimensional system on which Mawhin’s continuation theorem applies. This method can also be used to solve problems for other equations with $p$-Laplacian.

2. PRELIMINARIES

Let $X$ and $Y$ be real Banach Spaces and let $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, here $D(L)$ denotes the domain of $L$. This means that $\text{Im} L$ is closed in $Y$ and $\dim \ker L = \dim(Y/\text{Im} L) < +\infty$. Consider the supplementary subspaces $X_1$ and $Y_1$ such that $X = \ker L \oplus X_1$ and $Y = \text{Im} L \oplus Y_1$ and let $P : X \to \ker L$ and $Q : Y \to Y_1$ be the natural
projections. Clearly, \( \text{Ker } L \cap (D(L) \cap X_1) = \{0\} \), thus the restriction \( L_P := L|_{D(L) \cap X_1} \) is invertible. Denote by \( K \) the inverse of \( L_P \).

Let \( \Omega \) be an open bounded subset of \( X \) with \( D(L) \cap \Omega \neq \emptyset \). A map \( N : \overline{\Omega} \to Y \) is said to be \textit{L-compact} in \( \overline{\Omega} \) if \( QN(\overline{\Omega}) \) is bounded and the operator \( K(I - Q)N : \overline{\Omega} \to X \) is compact. We first recall the famous Mawhin’s continuity theorem.

\textbf{THEOREM 2.1}[4] Suppose that \( X \) and \( Y \) are Banach spaces, and \( L : D(L) \subset X \to Y \) is a Fredholm operator with index zero. Furthermore, \( \Omega \subset X \) is an open bounded set and \( N : \overline{\Omega} \to Y \) is L-compact on \( \overline{\Omega} \). If

\begin{enumerate}
\item \( Lx \neq \lambda Nx, \forall x \in \partial \Omega \cap D(L), \lambda \in (0, 1); \)
\item \( Nx \notin \text{Im } L, \forall x \in \partial \Omega \cap \text{Ker } L; \) and
\item \( \text{deg}\{JQN, \Omega \cap \text{Ker } L, 0\} \neq 0, \) where \( J : \text{Im } Q \to \text{Ker } L \) is an isomorphism,
\end{enumerate}

then the equation \( Lx = Nx \) has a solution in \( \overline{\Omega} \cap D(L) \).

The next result is useful in obtaining an \textit{a priori} bound of periodic solutions.

\textbf{THEOREM 2.2}[7] Let \( 0 \leq \alpha \leq T \) be a constant, \( s \in C(\mathbb{R}, \mathbb{R}) \) be periodic with period \( T \), and \( \max_{t \in [0, T]} |s(t)| \leq \alpha \). Then for any \( u \in C^1(\mathbb{R}, \mathbb{R}) \) which is periodic with period \( T \), we have

\[
\int_0^T |u(t) - u(t - s(t))|^2 dt \leq 2\alpha^2 \int_0^T |u'(t)|^2 dt.
\]

\section{3. MAIN RESULTS}

In order to use Mawhin’s continuation theorem to study the existence of \( T \)-periodic solutions for Eq. (1.1), we rewrite Eq. (1.1) in the following form

\[
\begin{aligned}
x_1'(t) &= \varphi_1(x_2(t)) = |x_2(t)|^{q-2}x_2(t) \\
x_2'(t) &= -f(t, x_1(t), x_2(t)) - \beta(t)g(x_1(t - \tau(t))) + e(t),
\end{aligned}
\tag{3.1}
\]

where \( q > 1 \) is a constant with \( \frac{1}{p} + \frac{1}{q} = 1 \). Clearly, if \( x(t) = (x_1(t), x_2(t))^T \) is a \( T \)-periodic solution to Eq. (3.1), then \( x(t) \) must be a \( T \)-periodic solution to Eq. (1.1). Thus, the problem of finding a \( T \)-periodic solution for Eq. (1.1) reduces to finding one for Eq. (3.1).

Now, we set \( C_T = \{ \phi \in C(\mathbb{R}, \mathbb{R}) : \phi(t + T) \equiv \phi(t) \} \) with norm \( \| \phi \|_0 = \max_{t \in [0, T]} |\phi(t)| \). It is obvious that \( \beta, \tau, e \in C_T \). Set \( X = Y = \{ x = (x_1(\cdot), x_2(\cdot)) \in C(\mathbb{R}, \mathbb{R}^2) : x(t) \equiv x(t + T) \} \) with norm \( \| x \| = \max\{|x_1|_0, |x_2|_0\} \). Clearly, \( X \) and \( Y \) are Banach spaces. Define

\[
L : D(L) = \{ x = (x_1(\cdot), x_2(\cdot)) \in C^1(\mathbb{R}, \mathbb{R}^2) : x(t) \equiv x(t + T) \} \subset X \to Y
\]

by

\[
Lx := x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}
\]

and

\[
N : X \to Y
\]
\[ N_x := \begin{pmatrix} \varphi_q(x_2) \\ -f(t, x_1(t))\varphi_q(x_2(t)) - \beta(t)g(x_1(t - \tau(t))) + e(t) \end{pmatrix}. \]

It is easy to see that \( \text{Ker } L = \mathbb{R}^2 \) and \( \text{Im } L = \{ y \in Y : \int_0^T y(s)ds = 0 \} \). So \( L \) is a Fredholm operator with index zero. Let \( P : X \to \text{Ker } L \) and \( Q : Y \to \text{Im } Q \subset \mathbb{R}^2 \) be defined by

\[ Px = \frac{1}{T} \int_0^T x(s)ds, \quad Qy = \frac{1}{T} \int_0^T y(s)ds, \]

and let \( K \) denote the inverse of \( L|_{\text{Ker } P} \). Obviously, \( \text{Ker } L = \text{Im } Q = \mathbb{R}^2 \) and

\[ [Kg](t) = \int_0^T G(t, s)g(s)ds, \quad (3.2) \]

where

\[ G(t, s) := \begin{cases} \frac{s}{T}, & 0 \leq s < t \leq T. \\ \frac{t}{T}, & 0 \leq t \leq s \leq T. \end{cases} \]

From (3.2), one can easily see that \( N \) is \( L \)-compact on \( \bar{\Omega} \), where \( \Omega \) is an open, bounded subset of \( X \).

For the sake of convenience, we denote by \( \beta_1 = \max_{t \in [0, T]} \beta(t), \beta_0 = \min_{t \in [0, T]} \beta(t) \). Obviously \( \beta_1 \geq \beta_0 > 0 \). Moreover, we list the following assumptions which will be used repeatedly in the sequel.

- \([H1]\) There is a constant \( r \geq 0 \) such that \( \lim_{|x| \to +\infty} \sup_{t \in [0, T]} |\frac{g(x)}{x}| \leq r \).
- \([H2]\) There is a constant \( A > 0 \) such that \( \text{sgn}(x)g(x) > \frac{|x|^p}{\delta_0} \) for \( |x| > A \).
- \([H3]\) There is a constant \( \sigma > 0 \) such that \( \inf_{(t, u) \in [0, T] \times \mathbb{R}} |f(t, u)| \geq \sigma > 0 \).
- \([H4]\) There exist an integer \( m \) and a constant \( \delta \geq 0 \) such that \( \max_{t \in [0, T]} |\tau(t) - mT| \leq \delta \).
- \([H5]\) There exists a constant \( l > 0 \) such that \( |g(u) - g(v)| \leq l|u - v| \).

**THEOREM 3.1** If \([H1]-[H3]\) hold, then Eq.(1.1) has at least a non-constant \( T \)-periodic solution if \( r < \frac{\sigma}{mT} \).

**PROOF.** Consider the operator equation

\[ Lx = \lambda Nx, \quad \lambda \in (0, 1). \]

Let \( \Omega_1 \in \{ x \in X : Lx = \lambda Nx, \lambda \in (0, 1) \} \). If \( x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in \Omega_1 \), then from (3.3), we have

\[ \begin{cases} x_1'(t) = \lambda \varphi_q(x_1(t)) = \lambda |x_1(t)|^q - x_1(t) \\ x_2'(t) = -\lambda f(t, x_1(t))\varphi_q(x_2(t)) - \beta(t)g(x_1(t - \tau_1(t))) + \lambda e(t). \end{cases} \]

We first claim that there is a constant \( \xi \in R \) such that

\[ |x(\xi)| \leq A. \]

In view of \( \int_0^T x_1'(t)dt = 0 \), we know that there exist two constants \( t_1, t_2 \in [0, T] \) such that

\[ x_1'(t_1) \geq 0, \quad x_1'(t_2) \leq 0. \]

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From the first equation of (3.4), we have $x_2(t) = \varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)$. So

$$x_2(t_1) = \frac{1}{\lambda^{p-1}}|x_1'(t_1)|^{p-2}x_1'(t_1) \geq 0,$$

$$x_2(t_2) = \frac{1}{\lambda^{p-1}}|x_1'(t_2)|^{p-2}x_1'(t_2) \leq 0.$$

Let $t_3, t_4 \in [0, T]$ be, respectively, the maximum point and minimum point of $x_2(t)$. Clearly, we have

$$x_2(t_3) \geq 0, \quad x_2'(t_3) = 0, \quad (3.7)$$

$$x_2(t_4) \leq 0, \quad x_2'(t_4) = 0. \quad (3.8)$$

From [H3] and by continuity, $f$ will not change sign for $(t, u) \in [0, T] \times \mathbb{R}$. Without loss of generality, suppose $f(t, u) > 0$ for $(t, u) \in [0, T] \times \mathbb{R}$ and upon substitution of (3.7) into the second equation of (3.4), we have

$$-\lambda \beta(t_3)g(x_1(t_3 - \tau(t_3)) + \lambda e(t) = \lambda f(t, x_1(t_3))\varphi_q(x_2(t_3)) \geq 0,$$

i.e.,

$$g(x_1(t_3, \tau(t_3))) \leq \frac{e(t_3)}{\beta(t_3)} \leq \frac{|e|_0}{\beta_0}. \quad (3.9)$$

From (H2) we see that

$$x_1(t_3 - \tau(t_3)) < A. \quad (3.10)$$

Similarly, from (3.8) we have

$$g(x_1(t_4 - \tau(t_4))) \geq \frac{e(t_4)}{\beta(t_4)} \geq \frac{|e|_0}{\beta_0}, \quad (3.11)$$

and again by (H2),

$$x_1(t_4 - \tau(t_4)) > -A. \quad (3.12)$$

Case (1) If $x_1(t_3 - \tau(t_3)) \in (-A, A)$, define $\xi = t_3 - \tau(t_3)$. Obviously $|x(\xi)| \leq A$.

Case (2) If $x_1(t_3 - \tau(t_3)) < -A$, from (3.12) and the fact that $x(t)$ is a continuous function in $\mathbb{R}$, there exists a constant $\xi$ between $x_1(t_3 - \tau(t_3))$ and $x_1(t_4 - \tau(t_4))$ such that $|x_1(\xi)| = A$.

This proves (3.5).

Next, in view of $\xi \in \mathbb{R}$, there is an integer $k$ and a constant $t_5 \in [0, T]$ such that $\xi = kT + t_5$, hence $|x_1(\xi)| = |x_1(t_5)| \leq A$. So

$$|x_1|_0 \leq A + \int_0^T |x_1(s)| ds. \quad (3.13)$$

Substituting $x_2(t) = \varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)$ into the second equation of (3.4),

$$[\varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)]' + \lambda f(t, x_1(t))[\varphi_q\left(\frac{1}{\lambda}x_1'(t)\right)] + \lambda \beta(t)g(x_1(t - \tau_1(t))) = \lambda e(t),$$

i.e.,

$$[\varphi_p(x_1'(t))]' + \lambda^{p-1}f(t, x_1(t))x_1'(t) + \lambda^p \beta(t)g(x_1(t - \tau_1(t))) = \lambda^p e(t). \quad (3.14)$$
Multiplying both sides of Eq. (3.14) by \( x'_1(t) \) and integrating over \([0, T]\), we have

\[
\int_0^T f(t, x_1(t))[x'_1(t)]^2 dt = -\lambda \int_0^T \beta(t)g(x_1(t - \tau_1(t)))x'_1(t)dt + \lambda \int_0^T e(t)x'_1(t)dt . \tag{3.15}
\]

It follows from [H3] that

\[
\begin{align*}
\sigma \int_0^T |x'_1(t)|^2 dt \\
&\leq \int_0^T |f(t, x_1(t))||x'_1(t)|^2 dt \\
&= \int_0^T \beta(t)g(x_1(t - \tau_1(t)))|x'_1(t)|^2 dt + \int_0^T e(t)|x'_1(t)|^2 dt \\
&\leq \int_0^T \beta(t)\|g(x_1(t - \tau_1(t)))\| |x'_1(t)|^2 dt + \int_0^T |e(t)|x'_1(t)^2 dt \\
&\leq \beta_1 \int_0^T |g(x_1(t - \tau_1(t)))||x'_1(t)|^2 dt + \int_0^T e(t)|x'_1(t)|^2 dt . \tag{3.16}
\end{align*}
\]

For \( \varepsilon = \frac{1}{2}(\frac{\sigma}{\beta_1} - r) \), by [H1] there is a constant \( A_1 > 0 \) such that

\[
g(x_1(t - \tau_1(t))) \leq (r + \varepsilon)|x_1(t - \tau(t))| \quad \text{for} \quad |x_1(t - \tau(t))| \geq A_1 . \tag{3.17}
\]

Define

\[
E_1 = \{ t \in [0, T]||x_1(t - \tau(t))|| < A_1 \}, \quad E_2 = \{ t \in [0, T]|x_1(t - \tau(t))| \geq A_1 \} .
\]

Then (3.16) can be transformed into

\[
\begin{align*}
\sigma \int_0^T |x'_1(t)|^2 dt \\
&\leq \beta_1 \int_{E_1} |g(x_1(t - \tau_1(t)))||x'_1(t)||x'_1(t)|dt + \beta_1 \int_{E_2} |g(x_1(t - \tau_1(t)))||x'_1(t)||x'_1(t)|dt + \int_0^T \varepsilon |x'_1(t)|^2 dt \\
&\leq \beta_1 g_A + \varepsilon |x_0| \int_0^T |x'_1(t)|dt + \beta_1 (r + \varepsilon)|x_0| \int_0^T |x'_1(t)|dt \\
&= \beta_1 g_A + \varepsilon |x_0| \int_0^T |x'_1(t)|dt + \beta_1 (r + \varepsilon)A + \int_0^T |x'_1(t)|dt \int_0^T |x'_1(t)|dt \\
&\leq T \frac{2}{3}\beta_1 g_A + \varepsilon |x_0 + \beta_1 (r + \varepsilon)A| \left( \int_0^T |x'_1(t)|^2 dt \right)^{\frac{1}{2}} + \beta_1 (r + \varepsilon)T \int_0^T |x'_1(t)|^2 dt ,
\end{align*}
\]

i.e.,

\[
\left( \sigma - \beta_1 (r + \varepsilon)T \right) \int_0^T |x'_1(t)|^2 dt \leq c_3 \left( \int_0^T |x'_1(t)|^2 dt \right)^{\frac{1}{2}} , \tag{3.18}
\]

where \( g_A := \max_{|u| \leq A_1} |g(u)| \) and \( c_3 := T \frac{2}{3}\beta_1 g_A + |x_0| + \beta_1 (r + \varepsilon)A \). In view of \( r < \frac{\sigma}{\beta_1 T} \) and \( \varepsilon = \frac{1}{2}(\frac{\sigma}{\beta_1} - r) \), it is easy to see that \( \sigma - \beta_1 (r + \varepsilon)T = \frac{1}{2}(\sigma - \beta_1 T r) > 0 \). So from (3.18) we have

\[
\int_0^T |x'_1(t)|^2 dt \leq \left( \frac{2c_3}{\sigma - \beta_1 T r} \right)^2
\]

and so

\[
\int_0^T |x'_1(t)|dt \leq T \frac{2}{3} \left( \int_0^T |x'_1(t)|^2 dt \right)^{\frac{1}{2}} = \frac{2T \frac{2}{3} c_3}{\sigma - \beta_1 T r} := A_2 . \tag{3.19}
\]

Hence

\[
|x_1|_0 \leq A + \int_0^T |x'_1(t)|dt \leq A + A_2 := M_1 . \tag{3.20}
\]

By the first equation of (3.4), we have

\[
\int_0^T |x_2(s)|^{q-2} x_2(s) ds = 0 , \tag{3.21}
\]
which implies that there is a constant $t_2 \in [0,T]$ such that $x_2(t_2) = 0$. So

$$|x_2|_0 \leq \int_0^T |x_2'(s)|ds .$$

(3.22)

On the other hand, taking absolute value and integrating over $[0,T]$ on both sides of the second equation of (3.4), we obtain

$$\int_0^T |x_2'(s)|ds \leq \int_0^T |f(t, x_1(t))||x_1'(t)|dt + \lambda \int_0^T |\beta(t)g(x_1(t) - \tau_1(t))|dt + \lambda \int_0^T |e(t)|dt$$

$$\leq f_{M_1}A_2T + \beta_1(g_{M_1}T) + |e|_1 ,$$

where $f_{M_1} := \max_{t \in [0,T], |u| \leq M_1} f(t, u)$, $g_{M_1} := \max_{|u| \leq M_1} |g(u)|$ and $|e|_1 := \int_0^T |e(t)|dt$. So from (3.22), we have

$$|x_2|_0 \leq f_{M_1}A_2T + \beta_1(g_{M_1}T) + |e|_1 := M_2 .$$

(3.23)

Let $\Omega_2 := \{x \in Ker L : Nx \in Im L\}$. If $x \in \Omega_2$, then $x \in Ker L$ and $QN x = 0$. From assumption $\int_0^T e(t)dt = 0$ we see that

$$\begin{cases}
|x_2|^{q - 2}x_2 = 0 \\
g(x_1) = 0 .
\end{cases}$$

(3.24)

So

$$|x_1| \leq A \leq M_1 , \quad x_2 = 0 \leq M_2 .$$

(3.25)

Let $\Omega = \{x = (x_1, x_2) \in X : |x_1|_0 < N_1, |x_2|_0 < N_2\}$, where $N_1$ and $N_2$ are constants with $N_1 > M_1$, $N_2 > M_2$ and $(N_2)^q > A\overline{\beta}g_A$, where $g_A := \max_{|u| \leq A} |g(u)|$ and $\overline{\beta} := \frac{1}{T} \int_0^T \beta(t)dt$. Then $\overline{\Omega}_1 \subset \Omega, \overline{\Omega}_2 \subset \Omega$. From (3.20), (3.22) and (3.25), it is obvious that conditions (1) and (2) of Theorem 2.1 are satisfied.

Next, we claim that condition (3) of Theorem A is also satisfied. For this, define the isomorphism $J : Im Q \rightarrow Ker L$ by $J(x_1, x_2) := (-x_2, x_1)$ and let $H(v, \mu) := \mu v + (1 - \mu)JQNa, \quad (v, \mu) \in \Omega \times [0,1]$. By simple calculation, we obtain, for $(x, \mu) \in \partial(\Omega \cap Ker L) \times [0,1],

$$x^T H(x, \mu) = \mu(x_1^2 + x_2^2) + (1 - \mu)(\overline{\beta}x_1 g(x_1) + |x_2|^q) > 0 .$$

Hence

$$deg\{JQNa, \Omega \cap Ker L, 0\} = deg\{H(x, 0), \Omega \cap Ker L, 0\}$$

$$= deg\{H(x, 1), \Omega \cap Ker L, 0\} = deg\{J, \Omega \cap Ker L, 0\}$$

$$\neq 0 ,$$

and so condition (3) of Theorem 2.1 is satisfied.

Therefore, by Theorem 2.1, we conclude that equation

$$Lx = Nx$$

has a solution $x(t) = (x_1(t), x_2(t))^T$ on $\overline{\Omega}$, i.e., Eq.(1.1) has a $T$–periodic solution $x_1(t)$ with $|x_1|_0 \leq M_1$. 

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Finally, observe that \( x_1(t) \) is not a constant. For if not, it follows from (3.14) that \( e(t) = c\beta(t) \geq c \) which will contradict to \( e(t) \not\equiv 0 \) and \( \int_0^T e(s)ds = 0 \). This completes the proof of Theorem 3.1.

**THEOREM 3.2** If (H2)-(H5) hold, then equation (1.1) has a non-constant \( T \)-periodic solution if \( \sqrt{2}\beta_1 \delta < \sigma \).

**PROOF.** Let \( \Omega_1 \) be defined as in Theorem 3.1. If \( x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in \Omega_1 \), then from the proof of Theorem 3.1 we see that

\[
[\varphi_p(x_1'(t))]' + \lambda^{p-1} f(t, x_1(t))x_1'(t) + \lambda^p \beta(t)g(x_1(t - \tau_1(t))) = \lambda^p e(t),
\]

(3.26)

and

\[
|x_1|_0 \leq A + \int_0^T |x_1'(s)|ds.
\]

(3.27)

We claim that \( |x_1|_0 \) is bounded.

Multiplying both sides of Eq.(3.26) by \( \int f(t, x_1(t))(x_1'(t))^2 dt \) and integrating over \([0, T]\), we have

\[
\int_0^T f(t, x_1(t))(x_1'(t))^2 dt + \lambda \int_0^T \beta(t)g(x_1(t - \tau_1(t)))x_1'(t)dt = \lambda \int_0^T e(t)x_1'(t)dt.
\]

(3.28)

By (3.28) and (H3),

\[
\sigma \int_0^T |x_1(t)|^2 dt \\
\leq \int_0^T |f(t, x_1(t))|(x_1'(t))^2 dt \\
= |\int_0^T f(t, x_1(t))(x_1'(t))^2 dt| \\
\leq \int_0^T \beta(t)|g(x_1(t - \tau_1(t)))|x_1(t)|dt + |\int_0^T e(t)x_1'(t)dt| \\
\leq \beta_1 |\int_0^T [g(x_1(t - \tau_1(t))) - g(x_1(t))]|x_1'(t)|dt + |\int_0^T g(x_1(t))x_1'(t)dt| + |\int_0^T e(t)x_1'(t)dt|.
\]

(3.29)

Considering \( \int_0^T g(x_1(t))x_1'(t)dt = 0 \) and by assumption (H5), we have from (3.29) that

\[
\sigma \int_0^T |x_1(t)|^2 dt \\
\leq \beta_1 |\int_0^T [g(x_1(t - \tau_1(t))) - g(x_1(t))]|x_1'(t)|dt| + |\int_0^T e(t)x_1'(t)dt| \\
\leq \beta_1 |\int_0^T |x_1(t - \tau_1(t)) - x_1(t)||x_1'(t)|dt| + |\int_0^T e(t)x_1'(t)dt| \\
\leq \beta_1 |\int_0^T |x_1(t - \tau_1(t)) - x_1(t)|^2 dt|^{\frac{1}{2}} \left(\int_0^T |x_1(t)|^2 dt\right)^{\frac{1}{2}} + \frac{1}{2} \left(\int_0^T |e(t)|^2 dt\right)^{\frac{3}{2}} \left(\int_0^T |x_1(t)|^2 dt\right)^{\frac{1}{2}}.
\]

(3.30)

By (H4), and applying Theorem 2.2, we obtain

\[
\left(\int_0^T |x_1(t - \tau_1(t)) - x_1(t)|^2 dt\right)^{\frac{1}{2}} = \left(\int_0^T |x_1(t - \tau_1(t) + nT) - x_1(t)|^2 dt\right)^{\frac{1}{2}} \leq \sqrt{2}\delta \left(\int_0^T |x_1(t)|^2 dt\right)^{\frac{1}{2}}.
\]

(3.31)

Substituting (3.31) into (3.29) yields

\[
(\sigma - \sqrt{2}\beta_1 \delta)\left(\int_0^T |x_1(t)|^2 dt\right) \leq \left(\int_0^T |x_1(t)|^2 dt\right)^{\frac{1}{2}} \left(\int_0^T |e(t)|^2 dt\right)^{\frac{1}{2}}.
\]

(3.32)

As \( \sqrt{2}\beta_1 \delta < \sigma \), we obtain

\[
\left(\int_0^T |x_1(t)|^2 dt\right)^{\frac{1}{2}} \leq \frac{\left(\int_0^T |e(t)|^2 dt\right)^{\frac{1}{2}}}{\sigma - \sqrt{2}\beta_1 \delta}.
\]

(3.33)
Hence (3.27) can be transformed into

\[ |x_1|_0 \leq A + \int_0^T |x_1'(t)| dt \leq A + T^{\frac{1}{2}} \left( \int_0^T |x_1(t)|^2 dt \right)^{\frac{1}{2}} \leq A + \frac{T^{\frac{1}{2}} \left( \int_0^T |x(t)|^2 dt \right)^{\frac{1}{2}}}{\sigma - \sqrt{2}\beta_1 l^2} \cdot \]

This proves the claim and the rest of the proof of the theorem is identical to that of Theorem 3.1.

REFERENCES


[5] X. Huang and Z. Xiang, On the existence of 2\( \pi \)-periodic solutions of Duffing type equation \( x''(t) + g(x(t - \tau)) = p(t) \), *Chinese Science Bulletin.*, **39**(1994), 201-203 [in Chinese].


