Prolongations of infinitesimal linear automorphisms of projective varieties and rigidity of rational homogeneous spaces of Picard number 1 under Kähler deformation

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In this paper we complete the proof of the following.

Main Theorem. *Let $\pi : \mathcal{X} \to \Delta$ be a smooth and projective morphism from a complex manifold $\mathcal{X}$ to the unit disc Δ. Suppose for any $t \in \Delta - \{0\}$, the fiber $X_t := \pi^{-1}(t)$ is biholomorphic to a rational homogeneous space S of Picard number 1. Then the central fiber X_0 is also biholomorphic to S.*

A rational homogeneous space S of Picard number 1 can be written as $S = G/P$ for a complex simple Lie group G and a maximal parabolic subgroup P. There are examples of homogeneous spaces G/P of a complex simple Lie group G with a non-maximal parabolic subgroup P such that the analogue of Main Theorem for G/P does not hold. For example, the tangent bundle $T_{\mathbb{P}_{2m+1}}$ of an odd-dimensional projective space is a non-trivial extension of a line bundle L by the null-correlation bundle D and the rational homogenous space $\mathbb{P}(T_{\mathbb{P}_{2m+1}})$ can be deformed to $\mathbb{P}(D \oplus L)$. Thus the condition on the Picard number in the statement of the Main Theorem is necessary.

For the background and the history of the Main Theorem, we refer the readers to the introduction in [HM2] and the references therein. In [HM2] the authors established the case of irreducible Hermitian symmetric spaces. Our method consisted of studying the deformation of minimal rational curves and their associated varieties of minimal rational tangents (VMRTs). We developed a method for proving the integrability of distributions spanned by minimal rational tangents using projective-geometric properties of the VMRTs, and used this to prove the linear non-degeneracy of VMRTs at general points of the central fiber, in order to recover on it the structure of the Hermitian symmetric space S on the central fiber. Along this line of approach [Hw1] established the case of contact homogeneous spaces of Picard number 1, and [HM5] established the same for non-symmetric and non-contact rational homogeneous spaces S of Picard number 1, provided that S is associated to a long simple root. A feature common to all the rational homogeneous spaces $S = G/P$ treated in [HM2,5] and [Hw1] is the fact VMRTs span the minimal G-invariant holomorphic distribution, a feature which allows us to study distributions in the central fiber from the full space of minimal rational curves on it. The analogue is not true for S associated to a short simple root. Discounting those that are isomorphic to one defined by a long root in a different way (e.g. the underlying complex structure of (G_2, α_2), in standard notations, is isomorphic to the 5-dimensional

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hyperquadric), S is one of the following (see (3.1), (4.1) and (7.1) below for the definitions of these homogeneous spaces).

(1) symplectic Grassmannians

(2) homogeneous space of type (F_4, α_1)

(3) homogeneous space of type (F_4, α_2).

Among these, (3) can still be treated by modifying the above method ([HM6]). The method fails drastically for the cases (1) and (2). In this article we develop a new method, by studying the structure of limiting holomorphic vector fields on the central fiber, and settles the remaining two cases, thereby completing the proof of Main Theorem.

Of independent interest are our results on the vanishing order of holomorphic vector fields at a given general point on some Fano manifolds of Picard number 1 (Theorem 1.3.1, Theorem 1.3.2). For Fano manifolds X of Picard number 1, the first author considers in [Hw2] the question of bounding at a general point $x \in X$ the vanishing order $ord_x(Z)$ of any nonzero holomorphic vector field Z. In this case he showed that $ord_x(Z) \leq n := \dim X$, yielding thus a bound on the dimension d of the automorphism group of the order n^n. On the other hand, for $p = 0$ he showed that $ord_x(Z) \leq 0$, thus yielding $d \leq n$, with equality if and only if X is almost homogeneous. For $p \geq 1$ we expect that the projective geometry of the variety of minimal rational tangents $\mathcal{C}_x$ imposes serious constraints on holomorphic vector fields vanishing at x. In general, we have the following conjecture.

Conjecture 1. *Let X be a Fano manifold of Picard number 1. Then, at a general point x on X, there does not exist any nonzero holomorphic vector field vanishing at x to the order ≥ 3.*

Conjecture 1 yields the bound on d of the order n^3 by counting the number of coefficients of Taylor expansions of holomorphic vector fields at x. There are many examples where $p > 0$ and the bound $ord_x(Z) \leq 2$ is realized. In fact, this is the case for any rational homogeneous space S of Picard number 1. On the other hand in those cases the dimensions of automorphism groups are usually much smaller than those given by a simple counting of Taylor coefficients. This can be explained in part on a common basis, as follows. Let $d_k(x)$ denote the dimension of holomorphic vector fields vanishing at x to the order $\geq k$. Then, $d_2(x) \leq n$ for any x on the n-dimensional rational homogeneous space S. Furthermore, this bound is sharp, and is realized if and only if S is Hermitian symmetric. Consideration of these examples leads to the following conjecture on dimensions of automorphism groups.

Conjecture 2. *Let X be an n-dimensional Fano manifold of Picard number 1. Then, $dim(Aut(X)) \leq n^2 + 2n$, with equality if and only if X is biholomorphic to the projective space $\mathbb{P}_n$.*

We prove Conjecture 1 in the present article under the assumption that the VMRT at a general point is nonsingular, irreducible and linearly non-degenerate, assumptions which are always satisfied whenever the positive generator of the Picard group of X is very ample, provided that $c_1(X) > \frac{n+2}{2}$. Furthermore we prove Conjecture 2 under the additional assumption that $\mathcal{C}_x$ is linearly normal. Under such assumptions, we establish a bound on the dimension $d_2(x)$ of holomorphic vector fields vanishing to the order ≥ 2 at a given general point x, which gives the sharp bound $d_2(x) \leq n$.

Our approach for bounds on vanishing orders consists of studying induced families of holomorphic vector fields on the VMRT $\mathcal{C}_x$ with extra symmetry properties. We consider then the orbits on $\mathcal{C}_x$ under the flow defined by some of these vector fields, and proved for instance that the existence of a single nonzero holomorphic vector field vanishing at x to the order ≥ 3 implies that $\mathcal{C}_x$ is uniruled by lines. This leads to a contradiction by an inductive argument consisting of passing to the VMRT $\mathcal{C}'_{[\alpha]}$ at a general point $[\alpha]$ on $\mathcal{C}_x$. A crucial ingredient for the inductive argument is a proof of the linear non-degeneracy of $\mathcal{C}'_{[\alpha]}$, which is obtained from a projective-geometric criterion on the integrability of distributions spanned by VMRTs developed in [HM2].

The result on the vanishing order of vector fields is interwoven with another result of ours on the prolongation of infinitesimal linear automorphisms of projective varieties. By the above argument, we will show that the Lie algebra of a smooth linearly non-degenerate subvariety of the projective space has no second order prolongation unless it is the whole projective space (Theorem 1.1.2). Moreover, to have first order prolongation, the projective subvariety must satisfy very special geometric conditions (Theorem 1.1.3). As a by-product, we get a new geometric proof of the classification of irreducible linear Lie algebras of infinite type over $\mathbb{C}$ (Corollary 1.1.4). This classification was first stated with a sketch of proof by E. Cartan ([Ca]). Complete proofs appeared half a century later by the theory of filtered Lie algebras ([SS], [KN2], [GQS], see also [De] for a survey). All these works are essentially algebraic. Our proof is completely different from these and more geometric.

In [HM2,5] and [Hw1], using results on the theory of geometric structures due to Ochiai [Oc] and Yamaguchi [Ya] the problem on deformation rigidity is essentially solved whenever the VMRT at a general point of the central fiber is shown to be isomorphic as a projective subvariety to that of the model space S. For the deformation of symplectic Grassmannians or the homogeneous space of type (F_4, α_1), the latter fact can likewise be established. Here the problem on deformation rigidity is however much deeper, since there is an additional property, *viz.* the non-degeneracy of the Frobenius form on the minimal distribution D, which determines the complex structure of the homogenous space. We overcome the difficulty by studying the structure of the Lie algebra of limiting holomorphic vector fields on the central fiber, by using our results on prolongations and vanishing orders of vector fields. Since the VMRT at a general point of the central fiber is isomorphic to that of the model, we know enough about the Lie algebra of limiting holomorphic vector fields to recover $\mathbb{C}^*$-actions on the central fiber. On the central fiber one can define a meromorphic distribution D_0 which is the limit of the minimal invariant holomorphic distributions of general fibers. For the deformation of symplectic Grassmannians, when the Frobenius form on D_0 fails to be nondegenerate at general points, using $\mathbb{C}^*$-actions we can recover on some smooth modification $\mu : \widetilde{X}_0 \to X_0$ the structure of a holomorphic fiber bundle whose fiber is a Grassmannian and whose base is a symplectic Grassmannian. For the deformation of the homogeneous space of type (F_4, α_1), we can recover the structure of a holomorphic fiber bundle whose fiber is 8-dimensional hyperquadric and whose base is a 7-dimensional hyperquadric. This description of the central fiber leads to extra symmetry on the central fiber which gives a contradiction when one considers the isotropy representation at a distinguished point on X_0 associated to the modification μ.

There are three chapters. In the first chapter we prove results on prolongations of infinitesimal linear automorphisms of projective varieties and holomorphic vector fields for a general uniruled projective manifold admitting nonsingular, irreducible and linearly nondegenerate VMRTs. The second chapter is the proof of the rigidity for symplectic Grassmannians. In the third section we examine the structure of VMRTs on symplectic Grassmannians and prove that the VMRT at a general point of the central fiber is isomorphic to that of the model space as projective subvarieties. In the fourth section, we recall several geometric facts about the Lie algebra of holomorphic vector fields on symplectic Grassmannians. In the fifth section, we apply our general results on holomorphic vector fields to the central fiber X_0. Using the explicit structure of VMRTs on X_0, we obtain sharp bounds on $d_2(x)$ at a general point x of the central fiber which goes beyond the general result in the first section. This allows us to prove the existence of standard vector fields on the central fiber whose integrals give $\mathbb{C}^*$-actions. In the sixth section, we consider the meromorphic distribution $\mathcal{D}$ on the central fiber which is the limit of minimal invariant distributions on general fibers. We prove that degeneracy of the Frobenius form of $\mathcal{D}$ at general points leads to a contradiction by studying the structure of X_0 arising from the $\mathbb{C}^*$-actions and completes the proof of the rigidity of symplectic Grassmannians under Kähler deformation. The third chapter, consisting of the seventh and the eighth sections, is the proof of the rigidity for the homogeneous space of type (F_4, α_1). The proof is quite parallel to that of chapter two and in many places the arguments in chapter 2 work verbatim.

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Chapter I. Prolongation of infinitesimal linear automorphisms of projective varieties

§1 Prolongation of infinitesimal linear automorphisms of projective varieties and vector fields on uniruled manifolds

(1.1) Let V be a complex vector space of dimension n. Given a Lie subalgebra $\mathfrak{g} \subset End(V)$, let $\mathfrak{g}^{(k)}$ be the space of symmetric multi-linear mappings $\sigma : S^{k+1}V \to V$ such that for any fixed $v_1, \ldots , v_k \in V$, the endomorphism

$$v \in V \ \mapsto \sigma(v, v_1, \ldots , v_k) \in V$$

belongs to $\mathfrak{g}$. The space $\mathfrak{g}^{(k)}$ is called the k-th **prolongation** of $\mathfrak{g}$. The following properties are immediate.

(i) $\mathfrak{g}^{(0)} = \mathfrak{g}$.
(ii) If $\mathfrak{g}^{(k)} = 0$ for some $k \geq 0$, then $\mathfrak{g}^{(k+1)} = 0$.
(iii) If $\mathfrak{h} \subset \mathfrak{g} \subset End(V)$ is a Lie subalgebra, then $\mathfrak{h}^{(k)} \subset \mathfrak{g}^{(k)}$ for each $k \geq 0$.

Let $Y \subset \mathbb{P}V$ be a projective subvariety. Denote by $\widetilde{Y} \subset V$ the affine cone of Y. By a slight abuse of terminology, the space of endomorphisms

$$aut(\widetilde{Y}) := \{g \in End(V) : \ \exp(tg)(\widetilde{Y}) \subset \widetilde{Y}, t \in \mathbb{C}\}$$

where $\exp(tg)$ denotes the 1-parameter group of linear automorphisms of V, will be called the **Lie algebra of infinitesimal linear automorphisms** of $\widetilde{Y}$. This is an algebraic Lie subalgebra of $End(V)$ in the sense that it is the tangent algebra of an algebraic subgroup of $GL(V)$ (cf. [OV] p.123). Elements of $aut(\widetilde{Y})$ induces vector fields on Y. In this regard, the following elementary fact will be used frequently.

Lemma 1.1.1. *For $A \in End(V)$ and a subvariety $Y \subset \mathbb{P}V$, $A \in aut(\widetilde{Y})$ if and only if at every smooth point $y \in Y$ the vector $A(y)$ is contained in the affine tangent space $\widetilde{T}_y(\widetilde{Y})$.*

The main results of Chapter I are the following two theorems. Recall that $Y \subset \mathbb{P}V$ is **non-degenerate** if it is not contained in any hyperplane and is linearly normal if $H^0(Y, \mathcal{O}(1)) = V^*$ where $\mathcal{O}(1)$ is the hyperplane line bundle.

Theorem 1.1.2. *Let $Y \subset \mathbb{P}V$ be an irreducible, smooth, non-degenerate subvariety. Then $aut(\widetilde{Y})^{(2)} = 0$, unless $Y = \mathbb{P}V$.*

Theorem 1.1.3. *Let $Y \subset \mathbb{P}V$ be an irreducible, smooth, non-degenerate, and linearly normal subvariety different from $\mathbb{P}V$. Then the following holds.*

(i) There exists a natural injection $aut(\widetilde{Y})^{(1)} \subset V^*$, implying $\dim aut(\widetilde{Y})^{(1)} \leq n = \dim V$.

(ii) If $aut(\widetilde{Y})^{(1)} \neq 0$, then there exists a point $y_o \in Y$ such that Y is covered by conics passing through y_o.

(iii) Suppose $\mathfrak{g} \subset aut(\widetilde{Y})$ is an algebraic subalgebra with $\mathfrak{g}^{(1)} \neq 0$. Then for a general point $y \in Y$, there exists an element $E_y \in \mathfrak{g}$ which generates a $\mathbb{C}^*$-action on Y with an isolated fixed point at y such that the isotropy action on $T_y(Y)$ is the scalar multiplication by $\mathbb{C}^*$.

The proofs of Theorem 1.1.2 and Theorem 1.1.3 will be given in Section 2. Let us state two immediate corollaries. The first one is the classification of irreducible linear Lie algebras of infinite type over $\mathbb{C}$ ([Ca], [De], [GQS], [KN2], [SS]).

Corollary 1.1.4. *Let $\mathfrak{g} \subset \mathfrak{gl}(n)$ be a Lie subalgebra which acts irreducibly on $\mathbb{C}^n$. Then $\mathfrak{g}^{(2)} = 0$ unless $\mathfrak{g}$ acts transitively on $\mathbb{P}_{n-1}$, i.e., unless $\mathfrak{g} = \mathfrak{gl}(n), \mathfrak{sl}(n), \mathfrak{csp}(m)$ or $\mathfrak{sp}(m)$, where in the last two cases $n = 2m$.*

Proof. In this case, $\mathfrak{g}$ is reductive. Let $Y \subset \mathbb{P}_{n-1}$ be the highest weight variety of the irreducible representation. Then $\mathfrak{g}$ acts transitively on Y and $\mathfrak{g} \subset aut(\widetilde{Y})$. If $\mathfrak{g}^{(2)} \neq 0$, then $aut(\widetilde{Y})^{(2)} \neq 0$. Thus by Theorem 1.1.2, $Y = \mathbb{P}_{n-1}$ and $\mathfrak{g}$ acts transitively on $\mathbb{P}_{n-1}$. It is well-known that only the four listed Lie algebras act transitively on $\mathbb{P}_{n-1}$. $\square$

The next corollary is a weak form of a result of S. Kobayashi and T. Nagano ([KN1]).

Corollary 1.1.5. *Let $\mathfrak{g} \subset \mathfrak{gl}(n)$ be a Lie subalgebra which acts irreducibly on $\mathbb{C}^n$. Suppose $\mathfrak{g}^{(2)} = 0$. Then $\mathfrak{g}^{(1)} = 0$ unless the image of $\mathfrak{g}$ in $\mathfrak{sl}(n)$ is isomorphic to the semi-simple part of the isotropy representation of an irreducible Hermitian symmetric space of compact type of rank ≥ 2.*

Proof. As in Corollary 1.1.4, $\mathfrak{g}$ is reductive and $\mathfrak{g} \subset aut(\widetilde{Y})$ for the highest weight variety Y. Moreover, $\mathfrak{g}$ is an algebraic Lie subalgebra. The highest weight variety $Y \subset \mathbb{P}^{n-1}$ is a homogeneous variety satisfying the assumptions in Theorem 1.1.3. By Theorem 1.1.3 (iii), Y is a symmetric complex manifold in the sense of A. Borel ([Bo1]). Thus Y is a Hermitian symmetric space because a homogeneous symmetric complex manifold is a Hermitian symmetric space ([Bo1] Theorem 2.4.). By Theorem 1.1.3 (ii), Y must be either a Hermitian symmetric space of rank 2 or a second Veronese embedding of the projective space. By the polydisc theorem as in Section 3 of [HM1], this implies that Y is the highest weight variety associated with the isotropy representation of an irreducible Hermitian symmetric space of rank ≥ 2. Since $\exp(\mathfrak{g})$ contains all the symmetric involutions of the Hermitian symmetric space Y by Theorem 1.1.3 (iii), the image of $\mathfrak{g}$ in $\mathfrak{sl}(n)$ agrees with the image of the full Lie algebra $aut(\widetilde{Y})$ in $\mathfrak{sl}(n)$, which is exactly the semi-simple part of the isotropy representation associated to an irreducible Hermitian symmetric space of rank ≥ 2. $\square$

(1.2) Now let us see how prolongations of infinitesimal linear automorphisms of projective varieties arise in geometric problems. Let X be an n-dimensional complex manifold and $\mathcal{C} \subset \mathbb{P}T(X)$ be a subvariety in the projectivized tangent space of X which is projective and flat over X. For a point $x \in X$, let $\mathcal{C}_x$ be the fiber of $\mathcal{C}$ over x. For simplicity, let us assume that $\mathcal{C}_x$ is irreducible and reduced. Let $\mathfrak{f}$ be the Lie algebra of all germs of holomorphic vector fields at x which preserves $\mathcal{C}$ in the sense that the germ of the 1-parameter subgroup of biholomorphisms at x generated by an element of $\mathfrak{f}$ preserves $\mathcal{C}$. The Lie algebra $\mathfrak{f}$ is naturally filtered by the vanishing orders of vector fields at x. More precisely, let $\mathfrak{f}^l$ be the subspace of $\mathfrak{f}$ consisting of vector fields which vanishes at x to the order $\geq l+1$ where l is an integer ≥ -1. Then $\mathfrak{f}^l$ is a Lie subalgebra of $\mathfrak{f}$ and $[\mathfrak{f}^l, \mathfrak{f}^m] \subset \mathfrak{f}^{l+m}$. By definition, $\mathfrak{f}^{-1} = \mathfrak{f}$.

Proposition 1.2.1. *For each $k \geq 0$, regard the quotient space $\mathfrak{f}^k/\mathfrak{f}^{k+1}$ as a subspace of $S^{k+1}T_x^*(X) \otimes T_x(X) = Hom(S^{k+1}T_x(X), T_x(X))$ by taking the leading terms of the Taylor expansion of the vector fields at x. Then*

$$\mathfrak{f}^k/\mathfrak{f}^{k+1} \subset aut(\widetilde{\mathcal{C}}_x)^{(k)},$$

the k-th prolongation of the Lie algebra of infinitesimal linear automorphisms of the affine cone of the projective variety $\mathcal{C}_x$.

Proof. For a vector field Z on X vanishing to order $\geq k+1$ at x, its $(k+1)$-jet defines an element $j_x^{k+1}(Z)$ of $S^{k+1}T_x^*(X) \otimes T_x(X)$. This defines the inclusion $\mathfrak{f}^k/\mathfrak{f}^{k+1} \subset S^{k+1}T_x^*(X) \otimes T_x(X)$. For $v_1, \dots, v_k \in T_x(X)$, the endomorphism

$$v \in T_x(X) \rightarrow j_x^{k+1}(Z)(v, v_1, \dots, v_k) \in T_x(X)$$

can be defined using the lift of Z to $\mathbb{P}T(X)$ as follows. The vector field $Z \in \mathfrak{f}^k$ induces a vector field Z' on $\mathbb{P}T(X)$ which vanishes along $\mathbb{P}T_x(X)$ to the order $\geq k$. The k-th order term of the Taylor expansion of Z' at a point α of $\mathbb{P}T_x(X)$ defines $j_\alpha^k(Z')$, the k-jet of Z' at α, which is an element of

$$S^kT_\alpha^*(\mathbb{P}T(X)) \otimes T_\alpha(\mathbb{P}T(X)).$$

Since Z' vanishes to the order $\geq k$ along $\mathbb{P}T_x(X)$, the k-jet $j_\alpha^k(Z')$ belongs to

$$S^kN_\alpha^*(\mathbb{P}T_x(X) \subset \mathbb{P}T(X)) \otimes T_\alpha(\mathbb{P}T(X))$$

where $N^*(\mathbb{P}T_x(X) \subset \mathbb{P}T(X))$ denotes the conormal bundle of $\mathbb{P}T_x(X)$ in $\mathbb{P}T(X)$. Under the projection $\pi : \mathbb{P}T(X) \rightarrow X$, any fiber of this conormal bundle can be canonically identified with $T_x^*(X)$. Note that $j_\alpha^k(Z')$ is sent to zero when composed with the natural projection $d\pi : T_\alpha(\mathbb{P}T(X)) \to T_x(X)$ because it comes from Z which vanishes to the order $\geq k+1$ at x. Thus the k-jet $j^k(Z')$ of Z' along $\mathbb{P}T_x(X)$ defines an element of

$$H^0(\mathbb{P}T_x(X), S^k\pi^*T_x^*(X) \otimes T(\mathbb{P}T_x(X))) = S^kT_x^*(X) \otimes End(T_x(X)).$$

By a direct calculation, one can check that for $v_1, \ldots, v_k \in T_x(X)$,

$$j_x^{k+1}(Z)(v, v_1, \ldots, v_k) = j^k(Z')(v_1, \ldots, v_k)(v) \ \in T_x(X).$$

Now by the assumption that Z preserves $\mathcal{C}$, the induced vector field Z' on $\mathbb{P}T(X)$ must be tangent to $\mathcal{C}$. Thus $j^k(Z')$ restricted to $\mathcal{C}_x$ defines an element of $H^0(\mathcal{C}_x, S^k\pi^*T_x^*(X)\otimes T(\mathcal{C}_x))$. It follows that

$$j^k(Z')(v_1, \ldots, v_k)(v) \in \widetilde{T}_v(\widetilde{\mathcal{C}}_x) \ \text{ for } v \in \widetilde{\mathcal{C}}_x$$

which implies $j_x^{k+1}(Z) \in aut(\widetilde{\mathcal{C}}_x)^{(k)}$. $\square$

(1.3) An important example of the subvariety $\mathcal{C} \subset \mathbb{P}T(X)$ in Proposition 1.2.1 arises in the study of rational curves on uniruled varieties in the following manner. Let X be an n-dimensional uniruled projective manifold. Let $\mathcal{K}$ be an irreducible component of the normalized Chow space of minimal rational curves with respect to a choice of polarization on X, in the sense of [HM3]. We call $\mathcal{K}$ a **minimal rational component**. A general element $[C] \in \mathcal{K}$ corresponds to an immersed standard minimal rational curve, i.e., for the normalization $f : \mathbb{P}^1 \to C$ we have $f^*T_X \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q, 1+p+q = n$ for some $0 \leq p \leq n-1$. For a general point $x \in X$, the normalized Chow space $\mathcal{K}_x$ of members of $\mathcal{K}$ passing through x is a smooth projective variety. Let $\tau_x : \mathcal{K}_x \dashrightarrow \mathbb{P}T_x(X)$ be the rational map sending a member of $\mathcal{K}_x$ to its tangent direction at x. This rational map is in fact a finite morphism ([Ke, Theorem 3.4]) over its image $\mathcal{C}_x$. We call τ_x the **tangent morphism** at x and $\mathcal{C}_x$ the **variety of minimal rational tangents** ([HM3]). Consider the variety $\mathcal{C} \subset \mathbb{P}T(X)$ obtained by taking the union of $\mathcal{C}_x$'s. Then any vector field on X must preserve $\mathcal{C}$. Thus by Proposition 1.2.1, Theorem 1.1.2 immediately gives the following result.

Theorem 1.3.1. *Let X be an n-dimensional uniruled projective manifold admitting a minimal rational component whose associated variety of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ at a general point x is p-dimensional; $0 < p < n-1$; irreducible, nonsingular and nondegenerate. Then, there does not exist any nonzero holomorphic vector field vanishing at the general point x to the order ≥ 3.*

This is a special case of Conjecture 1 in the introduction. The next Theorem is a special case of Conjecture 2.

Theorem 1.3.2. *If in Theorem 1.3.1 we impose the further hypothesis that $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is linearly normal, then* $\dim(Aut(X)) < n^2 + 2n$.

Proof. Since $\mathcal{C}_x \neq \mathbb{P}T_x(X)$ by $p < n-1$, $\dim(aut(\widetilde{\mathcal{C}_x})) < n^2$. Thus

$$\dim(Aut(X)) \leq \dim(X) + \dim(aut(\widetilde{\mathcal{C}_x})) + \dim(aut(\widetilde{\mathcal{C}_x})^{(1)}) < n^2 + 2n$$

where we used $\dim(aut(\widetilde{\mathcal{C}_x})^{(1)}) \leq n$ in Theorem 1.1.3. $\square$

Theorems 1.3.1 and 1.3.2 are meant to give an indication that the geometry of the variety of minimal rational tangents at a general point is relevant to the Conjectures 1 and 2. Their statements and proofs are formulated with the aim to have an immediate link to the question of deformation rigidity on symplectic Grassmannians and the homogeneous space of type (F_4, α_1) to be explained in Section 5 and Section 8. There are nonetheless interesting classes of Fano manifolds of Picard number 1 satisfying the hypothesis of the theorems, leading to

Corollary 1.3.3. *Let X be an n-dimensional Fano manifold of Picard number 1, and denote by $\mathcal{O}(1)$ the positive generator of the Picard group. Suppose $\mathcal{O}(1)$ is very ample and $c_1(X) > \frac{n+2}{2}$. Then, at a general point of X there does not exist any holomorphic vector field vanishing to the order ≥ 3. When the assumption on the first Chern class is strengthened to $c_1(X) > \frac{2(n+2)}{3}$, then we have $\dim(Aut(X)) \leq n^2 + 2n$, with equality if and only if X is biholomorphic to the projective space $\mathbb{P}_n$.*

Proof. If X is biholomorphic to the projective space $\mathbb{P}_n$, then any holomorphic vector field Z vanishing at some point $x \in X$ to the order ≥ 3 must vanish identically on any line passing through x, i.e., Z must vanish identically. Furthermore, we have $\dim(Aut(\mathbb{P}_n)) = \dim(\mathfrak{sl}(n+1)) = n^2 + 2n$. It suffices therefore to consider X with $c_1(X) < n+1$. There exists a minimal rational component $\mathcal{K}$ whose members $[C]$ satisfy $K_X^{-1} \cdot C \leq n+1$ ([HM3]). Identify X as a projective submanifold by means of the projective embedding defined by $\mathcal{O}(1)$. Whenever $c_1(X) > \frac{n+1}{2}$, C is of degree 1 and hence a line. At a general point $x \in X$, any line passing through x is standard, and since a line through x is completely determined by its tangent at x, the tangent morphism $\tau_x : \mathcal{K}_x \to \mathcal{C}_x \subset \mathbb{P}T_x$ at x is an embedding. We have $p := \dim(\mathcal{C}_x) = c_1(X) - 2 > \frac{n-3}{2}$. Any two irreducible components of $\mathcal{C}_x$ must intersect whenever $2p \geq dim \mathbb{P}T_x = n-1$. This is the case if we impose the slightly stronger hypothesis that $c_1(X) > \frac{n+2}{2}$, as we do in the first half of Corollary 1.3.3, so that $p > \frac{n-2}{2}$ and $2p \geq n-1$. From the smoothness of $\mathcal{C}_x$ it follows that $\mathcal{C}_x$ is irreducible. By [HM3, 1.3.2], $\mathcal{C}_x \subset \mathbb{P}T_x$ must be linearly non-degenerate, otherwise the meromorphic distribution W on X spanned at general points of X must be integrable, contradicting with the assumption that X is of Picard number 1. Thus, the hypothesis of Theorem 1.3.1 is satisfied and we conclude the first half of Corollary 1.3.3.

For the second half under the stronger hypothesis $c_1(X) > \frac{2(n+2)}{3}$, we have $p = \dim(\mathcal{C}_x) > \frac{2(n-1)}{3}$. By Zak's solution to the Hartshorne Conjecture on linear normality [Za, Corollary

2.17, p.48], $\mathcal{C}_x \subset \mathbb{P}T_x$ is linearly normal, so that $\dim(Aut(X)) < n^2 + 2n$, by Theorem 1.3.2, as desired. $\square$

Before stating next Corollary, let us recall a few facts. A rational curve C on a projective manifold Y is said to be **free** if the restriction of the tangent bundle $T(Y)$ to C is semi-positive. For any family of rational curves sweeping out an open subset of Y, a general member of the family is a free rational curve. Deformations of a free rational curve C cover an open neighborhood of C. Here and throughout the paper, an open set refers to the complex topology, unless mentioned otherwise.

Associated with a minimal rational component $\mathcal{K}$, we have the universal family morphisms $\rho : \mathcal{U} \to \mathcal{K}$ and $\mu : \mathcal{U} \to X$. A **ruling** on X by members of $\mathcal{K}$ is a subvariety $\mathcal{K}'$ of dimension $n-1$ in $\mathcal{K}$ with the associated universal family morphisms $\rho' : \mathcal{U}' \to \mathcal{K}'$ and $\mu' : \mathcal{U}' \to X$, such that μ' is birational.

Lemma 1.3.4. *If there exists a ruling on X by members of $\mathcal{K}$, then $\mathcal{C}_x$ at a general point x is irreducible.*

Proof. Let $\mathcal{U}' \to \mathcal{K}'$ be a ruling by members of $\mathcal{K}$. The image of $\mathcal{U}'$ in $\mathcal{C}$ gives a section of the projection $\mathcal{C} \to X$ over a Zariski open subset of X. Thus $\mathcal{C}_x$ is irreducible for a general point x. $\square$

The following Corollary will be used in Section 2.

Corollary 1.3.5. *Let $X \subset \mathbb{P}_N$ be a projective submanifold of dimension n. Assume that X has a minimal rational component $\mathcal{K}$ consisting of lines of $\mathbb{P}_N$ lying on X such that*

(i) X is rationally 2-connected by members of $\mathcal{K}$, in other words, any two points of X can be joined by a connected union of two lines belonging to $\mathcal{K}$.

(ii) X has a ruling by lines belonging to $\mathcal{K}$.

Then at a general point $x \in X$, the variety of minimal rational tangents $\mathcal{C}_x$ defined by $\mathcal{K}$ is smooth irreducible and non-degenerate. In particular, there does not exist any holomorphic vector field vanishing at x to the order ≥ 3.

Proof. On a submanifold of the projective space uniruled by lines, the variety of minimal rational tangents at a general point is always nonsingular because the tangent morphism τ_x is an embedding as already noticed in the proof of Corollary 1.3.3. By the assumption (ii) and Lemma 1.3.4., $\mathcal{C}_x$ at general x is irreducible. It suffices to prove the non-degeneracy of $\mathcal{C}_x \subset \mathbb{P}T_x(X)$.

We will first establish the estimate $\dim(\mathcal{C}_x) \geq \frac{n-2}{2}$. For $x \in X$ general, any point $y \in X$ can be joined to x by no more than two lines belonging to $\mathcal{K}$. We may assume that $\dim(\mathcal{C}_y) = \dim(\mathcal{C}_x) := p$ for any $y \in X$ sufficiently close to x. Let $\mathcal{V}_1 \subset X$ be the union of all lines belonging to $\mathcal{K}$ passing through x. By assumption on X, there exists a subvariety $\mathcal{S} \subset \mathcal{V}_1$ such that the union of lines issuing from points on $\mathcal{S}$ cover X. A general choice C_0 of a line issuing from $\mathcal{S}$ is then free and deformations of C_0 sweep out an open subset in X ([Kl, II.3]). Let $\mathcal{V}_1^o \subset \mathcal{V}_1$ be any Zariski-dense open subset and $\mathcal{V}_2^o$ be the union of lines issuing from $\mathcal{V}_1^o$. From the freeness of C_0 there are deformations of C_0 arbitrarily close to C_0, which intersect $\mathcal{V}_1^o$. It follows that C_0 is contained in the closure $\mathcal{V}_2$ of $\mathcal{V}_2^o$. Since C_0 is a general line issuing from $\mathcal{S}, \mathcal{V}_2 = X$. We have $\dim(\mathcal{V}_1) = p + 1$ and $\dim(\mathcal{V}_2) \leq 2(p+1)$. From $\mathcal{V}_2 = X$ it follows that $2(p+1) \geq \dim(X) = n$, i.e., $p \geq \frac{n-2}{2}$.

Suppose $\mathcal{C}_z$ is degenerate for z general. Let W be the proper meromorphic distribution of rank $< n$ on X defined by

$$W_z := \text{ the linear span of } \mathcal{C}_z \text{ in } T_z(X)$$

at general z. If W is integrable the varieties $\mathcal{V}_1$ and $\mathcal{V}_2$ constructed by adjoining members of $\mathcal{K}$ starting with $x \in X$ will have open subsets lying inside the leaf of the integrable distribution through x, provided that we start with x outside the singularity set of W. But this violates the fact that $\mathcal{V}_2 = X$. We have thus proven that W is not integrable.

On the other hand, since $\mathcal{C}_z \subset \mathbb{P}W_z$ is a nonsingular subvariety of dimension p in the projective space $\mathbb{P}W_z$ of dimension $\leq n-2$. By the estimate $p \geq \frac{n-2}{2}$, noting that $\mathcal{C}_z$ is nonsingular and irreducible, [HM3, 1.3.2] applies to show that W is integrable, a plain contradiction. In other words, W cannot be a proper distribution, i.e., $\mathcal{C}_z$ is linearly non-degenerate, as desired. $\square$

§2. Proofs of Theorem 1.1.2 and Theorem 1.1.3

(2.1) For the proof of Theorem 1.1.2. and Theorem 1.1.3, we need to relate the prolongation of $aut(\widetilde{Y})$ to the geometry of Y. The following observation is crucial. For $A \in S^2V^* \otimes V$, the evaluation of A at two vectors $\alpha, \beta \in V$ will be denoted by $A_{\alpha\beta} \in V$.

Proposition 2.1.1. *Let $Y \subset \mathbb{P}V, Y \neq \mathbb{P}V$, be an irreducible smooth non-degenerate proper subvariety. For any $A \in aut(\widetilde{Y})^{(1)}$ and any $\alpha \in \widetilde{Y}, A_{\alpha\alpha} \in \mathbb{C}\alpha$.*

For the proof of Proposition 2.1.1, we will make use of Zak's Tangency Theorem for nonsingular projective subvarieties, as follows.

Lemma 2.1.2 [Za]. *Let $Y \subset \mathbb{P}V$ be a nonsingular and nonlinear projective subvariety. Then the Gauss map on Y is a birational map onto its image. In particular, the kernel of the projective second fundamental form $\overline{\sigma}$ on Y is zero at a general point of Y and the intersection of all projective tangent spaces of Y is empty.*

Proof of Proposition 2.1.1. By Lemma 1.1.1, in the notations of Proposition 2.1.1 for the linear homomorphism $A : S^2V \to V$, for any $\beta \in V$ and any nonzero $\alpha \in \widetilde{Y}$ we have $A_{\alpha\beta} \in \widetilde{T}_\alpha(\widetilde{Y}) =: P_\alpha$. In particular if β is itself a nonzero vector in $\widetilde{Y}$, we have from the symmetry of A the property that $A_{\alpha\beta} \in P_\alpha \cap P_\beta$. Fixing α, consider the endomorphism $A_\alpha \in End(V)$ defined by $A_\alpha(\beta) = A_{\alpha\beta}$. A_α corresponds to a linear holomorphic vector field on V to be denoted by the same symbol. Denote by ∇ covariant differentiation on the Euclidean space V with respect to the flat connection. From the property that $A_{\alpha\eta} \in P_\alpha$ for any $\eta \in V$, the vector field A_α has value in P_α at every point of V. It follows that the covariant derivative $\nabla_\xi A_\alpha$ of the vector field A_α with respect to any tangent vector $\xi \in P_\alpha$ gives an element in P_α. Since the restriction of A_α to $\widetilde{Y} - \{0\}$ gives a holomorphic vector field on $\widetilde{Y} - \{0\}$ it follows that $\sigma(\xi; A_\alpha(\alpha)) = 0$ for the Euclidean second fundamental form σ, for any $\xi \in P_\alpha$. In particular, $A_{\alpha\alpha} = A_\alpha(\alpha)$ lies in the kernel of the second fundamental form. For a general point $[\alpha] \in Y$ from Lemma 2.1.2 we must have $A_{\alpha\alpha} \in \mathbb{C}\alpha$. The same is true for any $[\alpha] \in Y$ by taking limits, as desired. $\square$

Proposition 2.1.3. *Let $Y \subset \mathbb{P}V, Y \neq \mathbb{P}V$, be an irreducible smooth non-degenerate subvariety. Suppose for an $A \in aut(\widetilde{Y})^{(1)}$, $A_{\alpha\alpha} = 0$ for any $\alpha \in \widetilde{Y}$. Then $A \equiv 0$.*

The proofs of Theorem 1.1.2 and Proposition 2.1.3 will be given simultaneously by induction on the dimension n in the following two steps.

Step 1: Proposition 2.1.3 for $\dim V < n$ and Corollary 1.3.5 for $\dim X < n$ implies Proposition 2.1.3 for $\dim V = n$.

Step 2: Proposition 2.1.3. for $\dim V = n$ implies Theorem 1.1.2 for dimension $\dim V = n$

Note that both Theorem 1.1.2 and Proposition 2.1.3 are obvious when $\dim V \leq 2$. Since Corollary 1.3.5 for $\dim X = n$ was proved using Theorem 1.1.2 for $\dim V = n$, establishing these two steps completes the proof.

(2.2) To establish the two steps explained in (2.1), we need two lemmas.

Lemma 2.2.1. *Under the hypothesis of Proposition 2.1.3, suppose there exists a non-zero $A \in aut(\widetilde{Y})^{(1)}$ such that $A_{\alpha\alpha} = 0$ for any $\alpha \in \widetilde{Y}$. Then, Y has a minimal rational component $\mathcal{K}$ consisting of lines such that Y is rationally 2-connected by members of $\mathcal{K}$ and Y has a ruling by members of $\mathcal{K}$.*

Proof. For any $\eta \in V$ consider the linear endomorphism $A_\eta : V \to V$ given by $A_\eta(\zeta) = A_{\eta\zeta}$. We sometimes identify A_η as equivalently a linear holomorphic vector field on V. For any $\gamma \in \widetilde{Y}$ we have $A_{\gamma\gamma} = 0$. For a nonzero $\alpha \in \widetilde{Y}$ and any holomorphic arc $\{\alpha(t) \in \widetilde{Y} : \alpha(0) = \alpha\}$, we have by expansion and symmetry $A_{\alpha\xi} = 0$, where $\xi = \alpha'(0)$. It follows that $A_{\alpha\xi} = 0$ for any $\xi \in P_\alpha := \widetilde{T}_\alpha(\widetilde{Y})$; i.e., $A_\alpha|_{P_\alpha} \equiv 0$. On the other hand, since $A_\alpha(\eta) = A_\eta(\alpha) \in P_\alpha$, we have $Im(A_\alpha) \subset P_\alpha$, so that $Im(A_\alpha) \subset P_\alpha \subset Ker(A_\alpha)$. In particular, $A_\alpha^2 \equiv 0$. Consider the 1-parameter group of linear transformations $\{\Phi_{\alpha,t} := \exp(tA_\alpha), t \in \mathbb{C}\}$ on V. From $A_\alpha^2 \equiv 0$ we deduce that

$$\Phi_{\alpha,t}(\eta) = \eta + tA_\alpha(\eta) = \eta + tA_{\alpha\eta}.$$

Thus the closures of the orbits of the vector field on Y induced by A_α are lines. Let $\mathcal{K}^o$ be the lines on Y defined by general orbits of A_α as α varies over general points of Y. Clearly $\mathcal{K}^o$ is an irreducible family. Let $\mathcal{K}$ be a minimal rational component containing $\mathcal{K}^o$.

For a general α, the orbits of A_α defines a ruling on X by curves belonging to $\mathcal{K}$. Let us show that X is rationally 2-connected by members of $\mathcal{K}$. Since $A \not\equiv 0$ and $Y \subset \mathbb{P}V$ is non-degenerate we may now pick a pair of distinct points $[\alpha], [\beta] \in Y$ such that $\beta \notin P_\alpha, \alpha \notin P_\beta$ and $A_{\alpha\beta} \neq 0$. Then, $\Phi_{\alpha,t}(\beta) = \beta + tA_{\alpha\beta}$ and $\Phi_{\beta,t}(\alpha) = \alpha + tA_{\beta\alpha}$; $A_{\beta\alpha} = A_{\alpha\beta}$. Since $\alpha \notin P_\beta$, $\beta \notin P_\alpha$ and $A_{\alpha\beta} \in P_\alpha \cap P_\beta$, $A_{\alpha\beta}$ is neither proportional to α nor to β, and $\mathbb{P}(\mathbb{C}\beta + \mathbb{C}A_{\alpha\beta})$ is a line on Y joining $[\beta]$ to $[\gamma]$; $\gamma := A_{\alpha\beta}$. Likewise $\mathbb{P}(\mathbb{C}\alpha + \mathbb{C}A_{\beta\alpha})$ is a line on Y joining $[\alpha]$ to $[\gamma]$. Since the preceding procedure applies to any general pair of distinct points $[\alpha], [\beta] \in Y$, we have proven that Y is rationally 2-connected by lines belonging to $\mathcal{K}$. $\square$

In order to do induction we need to examine varieties of minimal rational tangents on Y for the Chow space $\mathcal{K}$ of lines lying on Y. This choice of $\mathcal{K}$ will be implicit in the sequel. Under the assumption of Lemma 2.2.1 for the induction argument we need to produce, at a general point $[\alpha] \in Y$, a holomorphic vector field on Y vanishing at $[\alpha]$ to the order ≥ 2 whose 2-jet enjoys similar properties as in Lemma 2.2.1. We have

Lemma 2.2.2. *Under the hypothesis of Lemma 2.2.1, let $\mathcal{K}$ be a minimal rational component with the properties mentioned in Lemma 2.2.1 and $\mathcal{C}_{[\alpha]}$ be the variety of minimal rational tangents with respect to $\mathcal{K}$ at general $[\alpha] \in Y$. Consider the linear endomorphism $A_\alpha : V \to V$ defined by $A_\alpha(\eta) = A_{\alpha\eta}$. Denote by $\mathcal{Z}$ the holomorphic vector field on $\mathbb{P}V$ induced by A_α and write $\mathcal{Z}_0$ for the restriction $\mathcal{Z}|_Y$, which is a holomorphic vector field on Y. Then, $\mathcal{Z}_0$ vanishes at $[\alpha]$ to the order ≥ 2. Moreover, if we denote by $B : S^2T_{[\alpha]}(Y) \to T_{[\alpha]}(Y)$ the homomorphism corresponding to the second order term in the Taylor expansion of $\mathcal{Z}_0$ at $[\alpha]$, then $B_{\mu\mu} = 0$ whenever $[\mu] \in \mathcal{C}_{[\alpha]}$.*

Proof. The holomorphic vector field $\mathcal{Z}$ vanishes at $[\alpha]$ since $A_\alpha(\alpha) = A_{\alpha\alpha} = 0$. $\mathcal{Z}_0 = \mathcal{Z}|_Y$ vanishes to the order ≥ 2 at $[\alpha]$ because $A_\alpha(\xi) = A_{\alpha\xi} = 0$ for each $\xi \in P_\alpha$ as in the proof of Lemma 2.2.1. Finally, for $B : S^2T_{[\alpha]}(Y) \to T_{[\alpha]}(Y)$ corresponding to the second order term of $\mathcal{Z}_0$ at $[\alpha]$, for $\mu \in T_{[\alpha]}(Y)$, $B_{\mu\mu} = 0$ if there exists a local holomorphic curve Γ on Y passing through $[\alpha]$, tangent to μ, such that $Z_0|_\Gamma$ vanishes at $[\alpha]$ to the order ≥ 3. If $[\mu] \in \mathcal{C}_{[\alpha]}$, then there is a line $L := \mathbb{P}(\mathbb{C}\alpha + \mathbb{C}\xi)$, $\mu \equiv \xi \bmod \alpha$ such that $L \subset Y$. In this case we have $A_{\alpha\beta} = 0$ for any $[\beta] \in L$, so that we can take Γ to be the germ of L at $[\alpha]$ to have even $Z_0|_\Gamma \equiv 0$. The proof of Lemma 2.2.2 is complete. $\square$

We are ready for the proof of Theorem 1.1.2 by establishing Step 1 and Step 2 stated in (2.1).

Proof of Step 1. Let us prove Proposition 2.1.3 for $\dim V = n$. Under the assumption of Lemma 2.2.1, $\mathcal{C}_{[\alpha]}$ for $\mathcal{K}$ at general point $\alpha \in \widetilde{Y}$ is smooth irreducible and non-degenerate by using Corollary 1.3.5 for dimension $\dim Y < \dim V = n$. On the other hand, A gives rise to a holomorphic vector field $\mathcal{Z}_0$ on Y such that, in the notation of the lemma, $B_{\mu\mu} = 0$ for any $\mu \in \mathcal{C}_{[\alpha]}$. Since B belongs to $aut(\widetilde{\mathcal{C}}_{[\alpha]})^{(1)}$, by Proposition 2.1.3 for dimension $< n$, $B \equiv 0$. This means that $\mathcal{Z}_0$ vanishes to the order ≥ 3 at $[\alpha]$. By Corollary 1.3.5 for dimension $< n$,

$\mathcal{Z}_0 \equiv 0$. Recall that $\mathcal{Z}_0 = \mathcal{Z}|_Y$ for a global vector field $\mathcal{Z} \in \Gamma(\mathbb{P}V, T_{\mathbb{P}V})$ defined by A. $\mathcal{Z}_0 \equiv 0$ if and only if $A_{\alpha\eta} = 0$ for η such that $[\eta] \in Y$. Since $Y \subset \mathbb{P}V$ is linearly non-degenerate, we have $A_{\alpha\eta} = 0$ for any $\eta \in V$. Varying α and using linear non-degeneracy once more we conclude that $A \equiv 0$, a contradiction to the assumption that A is non-zero. $\square$

Proof of Step 2. Let $A : S^3V \to V$ be an element of $aut(\widetilde{Y})^{(2)}$. Fixing $\gamma \in \widetilde{Y} - \{0\}$ the linear homomorphism $B : S^2V \to V$ defined by $B_{\mu\nu} := A_{\mu\nu\gamma}$, $\mu, \nu \in V$ belongs to $aut(\widetilde{Y})^{(1)}$. It follows from Proposition 2.1.1 that for each $\alpha \in \widetilde{Y}$, we have $B_{\alpha\alpha} \in \mathbb{C}\alpha$. Thus, for any nonzero $\gamma \in \widetilde{Y}$, we have $A_{\alpha\alpha\gamma} \in \mathbb{C}\alpha \cap P_\gamma$. Given a nonzero $\alpha \in \widetilde{Y}$, if $A_{\alpha\alpha\gamma} \neq 0$ for some γ, then $[A_{\alpha\alpha\gamma}] = [\alpha]$ must belong to P_γ for any nonzero $\gamma \in \widetilde{Y}$, violating Zak's Theorem applied to the nonsingular projective subvariety $Y \subset \mathbb{P}V$. We conclude therefore that for any $\alpha, \gamma \in \widetilde{Y}$, we have $A_{\alpha\alpha\gamma} = 0$. Fixing now again γ we conclude that $B_{\alpha\alpha} = 0$ for any $\alpha \in \widetilde{Y}$. By Proposition 2.1.3, $B \equiv 0$. As the choice of $\gamma \in \widetilde{Y}$ is arbitrary, $A \equiv 0$ too. $\square$

(2.3) We now turn to the proof of Theorem 1.1.3. A direct consequence of Proposition 2.1.1 and Proposition 2.1.3 is the following.

Proposition 2.3.1. *Suppose $Y \subset \mathbb{P}V$ is irreducible smooth non-degenerate and linearly normal. For any non-zero $A \in aut(\widetilde{Y})^{(1)}$, there exists a unique non-zero linear functional $\lambda_A \in V^*$ such that $A_{\alpha\alpha} = <\lambda_A, \alpha> \alpha$ for any $\alpha \in \widetilde{Y}$.*

We start the proof of Theorem 1.1.3 with two Lemmas.

Lemma 2.3.2. *Let $Y \subset \mathbb{P}V, Y \neq \mathbb{P}V$, be a Veronese embedding of a projective space. If $aut(\widetilde{Y})^{(1)} \neq 0$, then Y is the second Veronese embedding.*

Proof. If $A \in aut(\widetilde{Y})^{(1)}$, $A_{\alpha\beta} \in P_\alpha \cap P_\beta$ for $\alpha, \beta \in \widetilde{Y}$. Thus Lemma 2.3.2 follows from the fact that the tangent spaces at two distinct points on the Veronese embedding have nonempty intersection only for the second Veronese embedding. In fact, when $V = S^kW$ for a $(p+1)$-dimensional vector space W and $Y \subset \mathbb{P}V$ is the Veronese variety of pure symmetric tensors, a tangent line at $[w^k] \in Y$ for $w \in W$ is of the form $[\mathbb{C}w^k + \mathbb{C}w^{k-1}u]$ for some $u \in W$. If tangent lines at two different points $[w_1^k] \neq [w_2^k]$ have a common point,

$$\begin{aligned} w_1^k + w_1^{k-1}u_1 &= aw_2^k + w_2^{k-1}u_2 \\ w_1^{k-1}(w_1 + u_1) &= w_2^{k-1}(aw_2 + u_2) \end{aligned}$$

for some $a \in \mathbb{C}$ and $u_1, u_2 \in W$. This is possible only when $k = 2$. $\square$

Lemma 2.3.3. *Suppose that $Y \subset \mathbb{P}V, Y \neq \mathbb{P}V$, is a smooth irreducible non-degenerate linearly normal subvariety different from the second Veronese embedding of a projective space. Let $A \in aut(\widetilde{Y})^{(1)}$. Suppose for some $\alpha \in V$, there exists an ample hypersurface $H \subset Y$ such that the endomorphism $A_\alpha \in End(V)$ annihilates vectors contained in $\widetilde{H}$, i.e.,*

$$A_\alpha(e) = 0 \textit{ for any } e \in \widetilde{E}.$$

Then $A_\alpha = 0$.

Proof. Otherwise the vector field on Y generated by A_α vanishes on an ample divisor. This implies Y is a projective space by [MS] or [Wa], and must be the second Veronese embedding by Lemma 2.3.2. $\square$

Proof of Theorem 1.1.3. (i) This is immediate from Proposition 2.3.1.

(ii) For $A \in aut(\widetilde{Y})^{(1)}$, we have $\lambda \in V^*$ such that $A_{\alpha\alpha} =< \lambda, \alpha > \alpha$ for any $\alpha \in \widetilde{Y}$ by Proposition 2.3.1. Let $H \subset Y$ be the zero divisor of λ on Y so that $A_{\alpha\alpha} = 0$ for any $\alpha \in \widetilde{H}$. When α is a smooth point of $\widetilde{H}$, we have $A_{\alpha\gamma} = 0$ for any $\gamma \in \widetilde{T}_\alpha(\widetilde{H})$ because for any curve $\alpha + t\gamma + \cdots$ on $\widetilde{H}$ through α where $(\cdots)$ stands for terms involving t^2 or higher factor,

$$0 = A_{\alpha+t\gamma+\cdots,\alpha+t\gamma+\cdots} = 2tA_{\alpha\gamma} + \cdots .$$

We may assume that there exists a smooth point α of $\widetilde{H}$ such that $A_\alpha \neq 0$. In fact, if $A_{\alpha\eta} = 0$ for all $\alpha \in \widetilde{H}$ and all $\eta \in \widetilde{Y}$, then for a general $\eta \in \widetilde{Y}$ with $A_\eta \neq 0$, the endomorphism A_η annihilates the vectors contained in $\widetilde{H}$. Thus Y must be the second Veronese embedding of a projective space by Lemma 2.3.3. The second Veronese embedding of a projective space clearly satisfies (ii).

Regarding A_α as an endomorphism of V preserving Y, we know $A_\alpha(V) \subset P_\alpha$ and $A_\alpha(\widetilde{T}_\alpha(\widetilde{H})) = 0$. We claim that A_α is a nilpotent endomorphism of V. Assuming this claim for the time being, let us finish the proof of (ii). By the assumption A_α cannot have an eigenvector in the 1-dimensional space $P_\alpha/\widetilde{T}_\alpha(\widetilde{H})$ and so $A_\alpha(P_\alpha) \subset \widetilde{T}_\alpha(\widetilde{H})$. The subspace $I_\alpha := A^2_\alpha(V) = A_\alpha(P_\alpha)$ is contained in $\widetilde{T}_\alpha(\widetilde{H})$ and $\dim I_\alpha \leq 1$. Thus for any general $\eta \in Y$, $A^2_\alpha(\eta) \in I_\alpha$ and $A^3_\alpha(\eta) = 0$. It follows that

$$\exp(tA_\alpha)(\eta) = \eta + tA_{\alpha\eta} + \frac{t^2}{2}A^2_\alpha(\eta).$$

So the orbit of $\eta \in Y$ under the $\mathbb{C}$-action induced by A_α is a conic curve whose limit is the point $[I_\alpha] \in Y$.

It remains to prove that A_α is nilpotent. Assuming the contrary, let $s \neq 0$ be the semi-simple part of the Jordan decomposition of A_α. Then $s \in aut(\widetilde{Y})$ because $aut(\widetilde{Y})$ is the Lie algebra of an algebraic subgroup of $GL(V)$ (e.g. [OV] p.127). Moreover $s(V) \subset P_\alpha$ and $s(\widetilde{T}_\alpha(\widetilde{H})) = 0$. Thus there must be an $(n-1)$-dimensional eigenspace of s with eigenvalue 0 and a 1-dimensional eigenspace B of s with non-zero eigenvalue. Then the orbits of the 1-parameter subgroup $\{\exp(\lambda s), \lambda \in \mathbb{C}\}$ on $\mathbb{P}V$ must be lines passing through the point $[B] \in \mathbb{P}V$ and an invariant subvariety must be a cone with vertex at $[B]$. But Y is an invariant subvariety which is smooth and non-degenerate, a contradiction.

(iii) Given $A \in \mathfrak{g}^{(1)}$, let $\lambda \in V^*$ be as in Proposition 2.3.1. For $\alpha' \in P_\alpha$,

$$< \lambda, \alpha + t\alpha' + \cdots > (\alpha + t\alpha' + \cdots) = A_{\alpha+t\alpha'+\cdots,\alpha+t\alpha'+\cdots} = A_{\alpha\alpha} + 2tA_{\alpha\alpha'} + \cdots$$

where $(\cdots)$ stands for terms containing t^2-factor. It follows that

$$<\lambda, \alpha> \alpha' + <\lambda, \alpha'> \alpha = 2A_{\alpha\alpha'}.$$

This equation, together with the fact that α is an eigenvector of A_α with the eigenvalue $<\lambda, \alpha>$, implies that if we choose α outside the zero locus of λ, A_α acts on the tangent space

$$T_{[\alpha]}(Y) = Hom(\mathbb{C}\alpha, P_\alpha/\mathbb{C}\alpha)$$

as the scalar multiplication by $\frac{1}{2}$. In particular, the semi-simple part s of A_α under the Jordan decomposition is non-zero and s acts on $T_\alpha(Y)$ as the scalar multiplication by $\frac{1}{2}$. Since $\mathfrak{g}$ is an algebraic Lie subalgebra, $s \in \mathfrak{g}$. The $\mathbb{C}^*$-action $\{\exp(2ts), t \in \mathbb{C}^*\}$ on Y has an isolated fixed point $[\alpha]$ and the isotropy representation on $T_{[\alpha]}(Y)$ is the scalar multiplication by t. Putting $y = [\alpha]$ and $E_y = 2s$, we have (iii) $\square$

Chapter II. Rigidity of symplectic Grassmannians

§3. Varieties of minimal rational tangents on symplectic Grassmannians and the central fiber

(3.1) Let V be a 2ℓ-dimensional complex vector space with a symplectic form ω. Fix an integer $k, 1 < k < \ell$ and write $S = S_{k,\ell}$ for the variety of all k-dimensional isotropic subspaces of V. The aim of Chapter II is the proof of the following rigidity theorem.

Theorem 3.1.1. *Let $\pi : \mathcal{X} \to \Delta$ be a smooth and projective morphism from a complex manifold $\mathcal{X}$ to the unit disc Δ. Suppose for any $t \in \Delta - \{0\}$, the fiber $X_t := \pi^{-1}(t)$ is biholomorphic to $S_{k,\ell}$. Then the central fiber X_0 is also biholomorphic to $S_{k,\ell}$.*

In Section 3, we will study the variety of minimal rational tangents of S and X_0 at a general point.

(3.2) We can view S as a subvariety of the Grassmannian $\mathbb{G}(k, V)$ of k-dimensional subspaces of V. Fix a k-dimensional isotropic subspace $W \subset V$ and denote by $[W] \in S$ the corresponding point of S. The tangent space of $\mathbb{G}(k, V)$ at $[W]$ is naturally isomorphic to $Hom(W, V/W)$. By the inclusion $S \subset \mathbb{G}(k, V)$, we can see that the tangent space of S at $[W]$ is

$$T_{[W]}(S) = \{h \in Hom(W, V/W) : \ \forall w_1, w_2 \in W, \omega(h(w_1), w_2) + \omega(w_1, h(w_2)) = 0\}.$$

Let

$$W^\perp := \{v \in V : \ \omega(v, w) = 0 \text{ for all } w \in W\}$$

which is a subspace of dimension $2\ell - k$ containing W. Let

$$\psi : W^* \otimes (V/W) \to W^* \otimes W^*$$

be the projection defined by the composition of $V/W \to V/W^{\perp}$ with the isomorphism $V/W^{\perp} \cong W^*$ induced by ω. Then under the identification $Hom(W, V/W) = W^* \otimes (V/W)$,

$$T_{[W]}(S) = \psi^{-1}(S^2W^*) \subset W^* \otimes (V/W).$$

There is a natural subspace $D_{[W]}$ of the tangent space $\psi^{-1}(S^2W^*)$ defined by

$$D_{[W]} := W^* \otimes (W^{\perp}/W) = Ker(\psi).$$

This defines a natural distribution D on S of rank $k(2\ell-2k)$. The quotient space $T_{[W]}(S)/D_{[W]}$ can be naturally identified with S^2W^*. It follows that the dimension of S is

$$\dim D_{[W]} + \dim S^2W^* = \frac{1}{2}k(4\ell - 3k + 1).$$

Minimal rational curves of S are precisely lines of $\mathbb{G}(k, V)$ lying on S. Recall that the variety of minimal rational tangents of $\mathbb{G}(k, V)$ at $[W]$ consists of decomposable tensors in $T_{[W]}(\mathbb{G}(k, V)) = W^* \otimes (V/W)$. Thus the variety of minimal rational tangents of $S_{k,\ell} \subset \mathbb{G}(k, V)$ corresponds to the set of decomposable tensors in $T_{[W]}(S) \subset W^* \otimes (V/W)$. From $T_{[W]}(S) = \psi^{-1}(S^2W^*)$, a simple calculation shows that the affine cone of the variety of minimal rational tangents of S is

$$\widetilde{\mathcal{C}}_{[W]} = \{\lambda \otimes \mu \in W^* \otimes (V/W) : \ \mu^{\flat} \in \mathbb{C}\lambda\}$$

where $v^{\flat} \in W^*$ for $v \in V$ is defined by $v^{\flat}(w) := \omega(v, w)$ for all $w \in V$. The intersection $\mathcal{C}_{[W]} \cap \mathbb{P}D_{[W]}$ will be denoted by $\mathcal{E}_{[W]}$. Its affine cone is

$$\widetilde{\mathcal{E}}_{[W]} = \{\lambda \otimes \mu \in W^* \otimes (W^{\perp}/W)\}.$$

Thus $\mathcal{E}_{[W]}$ is isomorphic to $\mathbb{P}W^* \times \mathbb{P}(W^{\perp}/W)$.

Proposition 3.2.1. *The variety of minimal rational tangents $\mathcal{C}_{[W]}$ at $[W] \in S = S_{k,l}$ is isomorphic to the projectivization of the vector bundle $\mathcal{O}(-1)^{2l-2k} \oplus \mathcal{O}(-2)$ on $\mathbb{P}W^*$ embedded by the complete linear system associated to the dual tautological bundle of the projectivization. In particular, $\mathcal{C}_{[W]} \subset \mathbb{P}T_{[W]}(S)$ is non-degenerate and linearly normal.*

Proof. Under the projection of $\lambda \otimes \mu$ to λ, $\mathcal{E}_{[W]}$ is a trivial $\mathbb{P}_{2l-2k-1}$-bundle over $\mathbb{P}_{k-1} \cong \mathbb{P}W^*$ and $\mathcal{C}_{[W]}$ is a $\mathbb{P}_{2l-2k}$-bundle over $\mathbb{P}_{k-1} \cong \mathbb{P}W^*$. Let F be the vector bundle on $\mathbb{P}W^*$ such that $\mathbb{P}F = \mathcal{C}_{[W]}$ and F has a trivial subbundle isomorphic to $(W^{\perp}/W) \times \mathbb{P}W^*$ corresponding to $\mathcal{E}_{[W]}$. From the above description of $\mathcal{C}_{[W]}$, the vector bundle modulo the trivial subbundle is isomorphic to the tautological line bundle of $\mathbb{P}W^*$. Hence $F \cong \mathcal{O}^{2l-2k} \oplus \mathcal{O}(-1)$. The embedding $\mathcal{C}_{[W]} \subset \mathbb{P}T_{[W]}(S)$ restricts to the Segre embedding on $\mathcal{E}_{[W]}$. In other words the line bundle $\mathcal{O}(1)$ on $\mathbb{P}T_{[W]}(S)$ restricted to $\mathcal{E}_{[W]}$ is the dual tautological line bundle when we view $\mathcal{E}_{[W]}$ as the projectivization of the bundle $\mathcal{O}(-1)^{2l-2k}$ on $\mathbb{P}W^*$. Thus the line bundle $\mathcal{O}(1)$ of $\mathbb{P}T_{[W]}(S)$ restricted to $\mathcal{C}_{[W]}$ corresponds to the dual tautological line bundle when we view $\mathcal{C}_{[W]}$ as the projectivization of $\mathcal{O}(-1)^{2l-2k} \oplus \mathcal{O}(-2)$ on $\mathbb{P}W^*$. This finishes the proof because $h^0(\mathbb{P}W^*, \mathcal{O}(2) \oplus \mathcal{O}(1)^{2l-2k}) = \dim T_{[W]}(S)$. $\square$

(3.3) In this and the next subsection, we will study some geometric features of the projective variety $\mathcal{C}_{[W]} \subset \mathbb{P}T_{[W]}(S)$, forgetting that it is the variety of minimal rational tangents for S. To emphasize this, we will use the letter Z in place of $\mathcal{C}_{[W]}$.

Let Z be the projectivization of the vector bundle $\mathcal{O}(-1)^{2m} \oplus \mathcal{O}(-2)$ on the projective space $\mathbb{P}_{k-1}$. Here we will study some properties of Z as a complex manifold. Let $\vartheta : Z \to \mathbb{P}_{k-1}$ be the natural projection and ξ be the dual tautological line bundle of the projectivization so that $\vartheta_* \xi = \mathcal{O}(1)^{2m} \oplus \mathcal{O}(2)$. Then ξ is very ample and the complete linear system $|\xi|$ gives an embedding $\kappa : Z \subset \mathbb{P}H^0(Z, \xi)^*$. Let $R \subset Z$ be the hypersurface corresponding to $\mathbb{P}(\mathcal{O}(-1)^{2m})$. As a complex manifold, R is the product $\mathbb{P}_{2m-1} \times \mathbb{P}_{k-1}$ and the line bundle ξ restricted to R is the Segre line bundle. A choice of the $\mathcal{O}(-2)$-factor gives a section $\Sigma \subset Z$ of ϑ disjoint from R. The restriction of ξ to $\Sigma \cong \mathbb{P}_{k-1}$ is $\mathcal{O}(2)$.

Lemma 3.3.1. *In $Pic(Z)$, $\xi = [R] + \vartheta^* \mathcal{O}(2)$ where $[R]$ is the line bundle corresponding to the divisor R.*

Proof. Clearly, $\xi = [R] + \vartheta^* \mathcal{O}(b)$ for some integers b. Restriction to Σ, which is disjoint from $[R]$, shows that $b = 2$. $\square$

The next lemma is obvious.

Lemma 3.3.2. *Let $\mathcal{W}$ be a vector bundle on a complex manifold. The affine bundle $\mathbb{P}(\mathcal{W} \oplus \mathcal{O}) - \mathbb{P}\mathcal{W}$ has a vector bundle structure by choosing $\mathbb{P}\mathcal{O}$ as the zero section. This vector bundle is isomorphic to $\mathcal{W}$ itself.*

Lemma 3.3.3. *Let $\mathbb{P}_{n-1} \subset \mathbb{P}_n$ be a hyperplane. Consider a vector bundle $\mathcal{V}$ on $\mathbb{P}_n$ isomorphic to $\mathcal{O}(1)^{2m} \oplus \mathcal{O}$. Let $\mathcal{V}'$ be the subbundle corresponding to $\mathcal{O}(1)^{2m}$. A choice of $\mathcal{O}$-complement to $\mathcal{V}'$ in $\mathcal{V}$ over $\mathbb{P}_{n-1}$ can be extended to a choice of $\mathcal{O}$-complement on $\mathbb{P}_n$.*

Proof. Fix a complement of $\mathcal{V}'$ in $\mathcal{V}$ over $\mathbb{P}_n$ and identify $\mathbb{P}\mathcal{V} - \mathbb{P}\mathcal{V}'$ with $\mathcal{V}'$ by Lemma 3.3.2. Under this identification, a choice of $\mathcal{O}$-complement to $\mathcal{V}'$ in $\mathcal{V}$ corresponds to a section of the vector bundle $\mathcal{V}'$. Thus Lemma 3.3.3 follows from the fact that any section of $\mathcal{V}'$ on $\mathbb{P}_{n-1}$ can be extended to a section on $\mathbb{P}_n$. $\square$

Using these lemmas, we have the following result about curves of ξ-degree 2 on Z.

Proposition 3.3.4. *Let C be a rational curve on Z disjoint from R and of degree 2 with respect to ξ. Then C lies on some section Σ of ϑ disjoint from R and deformations of C fixing a point span an open set in Z.*

Proof. By Lemma 3.3.1, $\vartheta(C)$ is a line on $\mathbb{P}_{k-1}$. Regarding Z as the projectivization of $\mathcal{O}(1)^{2m} \oplus \mathcal{O}$ and R as the projectivization of $\mathcal{O}(1)^{2m}$ on $\mathbb{P}_{k-1}$, C gives a complement to R over the line $\vartheta(C)$. By successive use of Lemma 3.3.3, we can extend this to a complement of R over $\mathbb{P}_{k-1}$, which is Σ. Different choices of Σ passing through a fixed point correspond to sections of $\mathcal{O}(1)^{2m}$ on $\mathbb{P}_{k-1}$ vanishing at a given point. Since such sections generate the bundle at a general point, deformations of Σ fixing a point span an open set in Z. On the other hand, it is clear that deformations of the conic C inside the Veronese variety Σ fixing a point cover Σ. $\square$

As a consequence, we have the following rigidity result.

Proposition 3.3.5. *Let $\mu : \mathcal{M} \to \Delta$ be a smooth projective morphism with a line bundle ζ on $\mathcal{M}$ such that $M_t := \pi^{-1}(t)$ is biholomorphic to Z for each $t \in \Delta - \{0\}$ with $\zeta|_{M_t}$ isomorphic to ξ. Suppose for any flat family of curves $C_t \subset M_t$ the following holds:*

(i) if C_t is of degree 1 with respect to ζ then C_0 is irreducible;

(ii) if C_t is of degree 2 with respect to ζ then either C_0 is irreducible or C_0 has exactly two irreducible components of degree 1 with respect to ζ.

Then M_0 is also biholomorphic to Z.

Before going into the proof of Proposition 3.3.5, we need two lemmas. Let $C \subset Y$ be a free rational curve on a projective manifold Y. By Kodaira's stability ([Kd]), for any deformation $\{Y_t, t \in \Delta\}$ of Y, $C \subset Y$ can be deformed to free rational curves $C_t \subset Y_t$ for sufficiently small t.

Lemma 3.3.6. *In the situation of Proposition 3.3.5, let $\mathcal{R} \subset \mathcal{M}$ be the irreducible hypersurface corresponding to $R \subset Z$ on $M_t, t \neq 0$. For a general point $z \in M_0$ and any section $\nu : \Delta \to \mathcal{M}$ of μ with $\nu(0) = z$, there exists a family of irreducible curves $\{C_t, t \in \Delta\}$ such that $\nu(t) \in C_t$, $C_t \cap \mathcal{R} = \emptyset$ and C_t is of degree 2 for each $t \in \Delta$.*

Proof. Let $\{C_t \subset M_t, t \in \Delta\}$ be a family of degree-2 curves through $\nu(\Delta)$ such that they are disjoint from R and C_t is irreducible for $t \neq 0$. If C_0 is reducible, it has two components of degree 1 with respect to ζ by (ii). Let C_1, C_2 be two irreducible components of C_0 and assume $\nu(0) \in C_1$. Since ν_0 is general, we can assume that C_1 is free. Suppose that deformations of C_1 fixing $\nu(0)$ span an open set in M_0. By Kodaira's stability, we can deform C_1 out of M_0 to get a degree-1 curve in M_t, whose deformations fixing a point span an open set, a contradiction from Lemma 3.3.1. It follows that deformations of C_1 fixing $\nu(0)$ cannot sweep out an open subset in M_0. By Proposition 3.3.4, deformations of C_t fixing $\nu(t)$ sweep out an open subset in M_t. Thus deformations of $C_1 \cup C_2$ fixing one point sweep out an open subset in M_0. This means that we can choose C_t such that both C_1 and C_2 are free. But then $C_1 \cup C_2$ can be deformed to an irreducible free rational curve of degree 2 whose deformations fixing one point sweep out an open subset of X_0 by the smoothing argument of [Kl, II.6]. This irreducible degree 2 curve must be a limit of degree-2 curves disjoint from $\mathcal{R}$ in $M_t, t \neq 0$, by Kodaira's stability again. Thus we can choose C_t so that C_0 is irreducible. $\square$

A holomorphic vector field v on a projective manifold M is called a **$\mathbb{C}^*$-vector field** if the 1-parameter subgroup $\{\exp(\lambda v), \lambda \in \mathbb{C}\}$ has a period so that its image in $Aut(M)$ is isomorphic to the multiplicative group $\mathbb{C}^*$. Orbits of $\mathbb{C}^*$-vector fields can be compactified to rational curves by adding two limit points. These curves will be called **orbital curves**. Conversely, if the orbits of a vector field can be compactified to rational curves by adding two distinct points, the vector field is a $\mathbb{C}^*$-vector field. We need the following result of Bialynicki-Birula on the structure of the zero set and the orbital curves of a $\mathbb{C}^*$-vector field.

Lemma 3.3.7 [BB]. *The fixed point set of a $\mathbb{C}^*$-action on a projective manifold M is smooth. There is a unique component M^+ (resp. M^-) of the fixed point set where all the weights of the isotropy action on the tangent space are non-negative (resp. non-positive). The codimension of M^+ (resp. M^-) is equal to the number of strictly positive weights (resp. strictly negative weights) of the isotropy action. Generic orbital curves join M^+ and M^-. There exists an invariant Zariski open subset $\hat{M}^+$ (resp. $\hat{M}^-$) of M containing M^+ (resp. M^-) such that $\hat{M}^+$ (resp. $\hat{M}^-$) has the structure of a vector bundle over M^+ (resp. M^-) and the $\mathbb{C}^*$-action corresponds to the scalar multiplication on the vector bundle.*

Proof of Proposition 3.3.5 Choose a general section $\nu : \Delta \to \mathcal{M}$ disjoint from $\mathcal{R}$. For each $t \neq 0$, we can find a section Y_t of the projection $\vartheta_t : M_t \to \mathbb{P}_{k-1}$ corresponding to $\vartheta : Z \to \mathbb{P}_{k-1}$ such that $Y_t \cap \mathcal{R} = \emptyset$ and $\nu(t) \in Y_t$. We can choose the varieties Y_t so that the limit Y_0 exists. We know that conics on Y_t through $\nu(t)$ cover Y_t for all $t \neq 0$. By Lemma 3.3.6, the limit of a general choice of these conics will remain irreducible and cover the limit Y_0. This implies that Y_0 is irreducible.

Regarding M_t as the projectivization of $\mathcal{O}(1)^{2m} \oplus \mathcal{O}$ on $\mathbb{P}_{k-1}$ for $t \neq 0$, the hypersurface $R_t := \mathcal{R} \cap M_t$ and the section Y_t define a splitting of the vector bundle $\mathcal{O}(1)^{2m} \oplus \mathcal{O}$. Consider the $\mathbb{C}^*$-action on the vector bundle given by the action of $\lambda \in \mathbb{C}^*$ as

$$v \mapsto v \text{ for } v \in \mathcal{O}(1)^{2m}$$
$$v \mapsto \lambda v \text{ for } v \in \mathcal{O}, \text{ the factor determined by } Y_t.$$

This action induces a vector field E_t on $M_t, t \neq 0$, such that E_t is a $\mathbb{C}^*$-vector field whose orbital curves are lines joining Y_t to R_t. After multiplying by a suitable power of t, we can assume that the limit E_0 is a non-trivial vector field on M_0 vanishing on R_0 and Y_0 whose orbital curves are irreducible degree-1 curves joining R_0 and Y_0. Since Y_0 is irreducible and contains $\nu(0) \notin R_0$, Y_0 is not contained in R_0. Note that through each point of $Y_t, t \neq 0$, there exists an orbital curve of E_t which is not contained in Y_t. By the condition (i), a family of orbital curves of E_t converges to an irreducible orbital curve of E_0. We see that through each general point of Y_0, an orbital curve exists which is not contained in Y_0. These are general orbital curves and they intersect R_0 because they are limit of orbital curves of $E_t, t \neq 0$, which intersect Y_t. Since $Y_0 \not\subset R_0$, general orbital curves of E_0 have two distinct limit points. This means E_0 is a $\mathbb{C}^*$-vector field on X. It follows that $\mathcal{R}$ and $\mathcal{Y}$ are smooth families. By the same argument as in [HM2, Section 3], $\mathcal{R}$ and $\mathcal{Y}$ are trivial families of $\mathbb{P}_{2m-1} \times \mathbb{P}_{k-1}$ and $\mathbb{P}_{k-1}$. By changing the sign, we can say $Y_0 = M_0^+$ and $R_0 = M_0^-$ in the notation of Lemma 3.3.7. Since all the orbital curves intersect both M_0^+ and M_0^-, $\hat{M}_0^+ = M_0 - M_0^-$ and $\hat{M}_0^- = M_0 - M_0^+$. Since R_0 is a hypersurface in M_0, the $\mathbb{C}^*$-action on $M_0 - Y_0$ is the scalar multiplication on the line bundle $\hat{M}_0^-$ over R_0 and there exists a unique orbital curve through each point of R_0. Thus the collection of all orbital curves passing through two distinct points on Y_0 define two disjoint projective subvarieties in M_0. This induces a morphism $M_0 \to Y_0 \cong \mathbb{P}_{k-1}$, which is the limit of the $\mathbb{P}_{2m}$-bundle structure on $M_t, t \neq 0$. It follows that M_0 is a $\mathbb{P}_{2m}$-bundle over Y_0, and it is easy to check that this bundle is biholomorphic to Z. $\square$

(3.4) Let us continue to use the notation of (3.3). Fix a k-dimensional vector space $\mathbf{U}$ and a $2m$-dimensional vector space $\mathbf{Q}$. Let $\mathbf{t}$ be the tautological line bundle on $\mathbb{P}\mathbf{U}$. The vector bundle $(\mathbf{Q} \otimes \mathbf{t}) \oplus \mathbf{t}^{\otimes 2}$ on $\mathbb{P}\mathbf{U}$ is isomorphic to $\mathcal{O}(-1)^{2m} \oplus \mathcal{O}(-2)$. Let us make an identification of Z with the projective bundle $\vartheta : \mathbb{P}((\mathbf{Q} \otimes \mathbf{t}) \oplus \mathbf{t}^{\otimes 2}) \to \mathbb{P}\mathbf{U}$ and then ξ, the dual tautological line bundle on Z, satisfies $\vartheta_* \xi = (\mathbf{Q}^* \otimes \mathbf{h}) \oplus \mathbf{h}^{\otimes 2}$ where $\mathbf{h}$ is the hyperplane line bundle on $\mathbb{P}\mathbf{U}$, namely, the dual bundle of $\mathbf{t}$. So

$$H^0(Z, \xi) = H^0(\mathbb{P}\mathbf{U}, \vartheta_* \xi) = H^0(\mathbb{P}\mathbf{U}, (\mathbf{Q}^* \otimes \mathbf{h}) \oplus \mathbf{h}^{\otimes 2}) = (\mathbf{U}^* \otimes \mathbf{Q}^*) \oplus S^2 \mathbf{U}^*.$$

Let $\mathbf{T} := H^0(Z, \xi)^*$. The line bundle ξ is very ample defining an embedding $\kappa : Z \subset \mathbb{P}\mathbf{T}$. The hypersurface $R \subset Z$ is naturally identified with $\mathbb{P}(\mathbf{Q} \otimes \mathbf{t})$. Let $\mathbf{D} \subset \mathbf{T}$ be the subspace spanned by $\kappa(R)$. Then $\mathbf{D} = \mathbf{U} \otimes \mathbf{Q}$ and $\mathbf{T}/\mathbf{D} = S^2\mathbf{U}$. In terms of the decomposition $\mathbf{T} = (\mathbf{U} \otimes \mathbf{Q}) \oplus S^2\mathbf{U}$, the affine cone over Z is

$$\widetilde{Z} = \{\lambda \otimes \mu + \mathbb{C}\lambda^2 : \ \lambda \in \mathbf{U}, \mu \in \mathbf{Q}\}.$$

The next lemma follows directly from this expression.

Lemma 3.4.1. *For the point $\alpha = \lambda^2 \in S^2\mathbf{U}$, the affine tangent space to $\widetilde{Z}$ at α is*

$$P_\alpha = Span\{\lambda \otimes \mu, \lambda \odot \zeta : \ \mu \in \mathbf{Q}, \zeta \in \mathbf{U}\}.$$

For the point $\beta = \lambda \otimes \mu + \lambda^2$, the tangent space to $\widetilde{Z}$ at β is

$$P_\beta = Span\{\lambda \otimes \theta, \zeta \otimes \mu + 2\lambda \odot \zeta : \ \theta \in \mathbf{Q}, \zeta \in \mathbf{U}\}.$$

For the point $\gamma = \lambda \otimes \mu$ of the affine cone $\widetilde{R}$ over $R \subset Z$, the tangent space to $\widetilde{R}$ at γ is

$$P'_\gamma = Span\{\lambda \otimes \theta, \zeta \otimes \mu : \ \theta \in \mathbf{Q}, \zeta \in \mathbf{U}\}.$$

Proposition 3.4.2. *Let $\mathcal{T} \subset \mathbb{P}\Lambda^2\mathbf{T}$ be the variety of tangential lines to $Z \subset \mathbb{P}\mathbf{T}$ under the Plücker embedding $\mathbb{G}(2, \mathbf{T}) \subset \mathbb{P}\Lambda^2\mathbf{T}$. Then $\mathcal{T}$ is non-degenerate in $\mathbb{P}\Lambda^2\mathbf{T}$.*

Proof. Let $\Upsilon \subset \Lambda^2\mathbf{T}$ be the linear span of the tangential lines to Z. Note that Z is invariant under the natural action of $GL(\mathbf{U}) \times GL(\mathbf{Q})$ on $\mathbf{T} = (\mathbf{U} \otimes \mathbf{Q}) \oplus S^2\mathbf{U}$. So Υ is also $GL(\mathbf{U}) \times GL(\mathbf{Q})$-invariant. As $GL(\mathbf{U}) \times GL(\mathbf{Q})$-modules

$$\Lambda^2\mathbf{T} = \Lambda^2(\mathbf{U} \otimes \mathbf{Q}) \oplus \Lambda^2(S^2\mathbf{U}) \oplus (\mathbf{U} \otimes \mathbf{Q} \otimes S^2\mathbf{U}).$$

From the expression of P'_γ in Lemma 3.4.1, we see that vectors of the form

$$(\lambda \otimes \mu) \wedge (\lambda \otimes \theta), \ (\lambda \otimes \mu) \wedge (\zeta \otimes \mu)$$

are in Υ. Thus $\Lambda^2(\mathbf{U} \otimes \mathbf{Q}) \subset \Upsilon$. From the expression of P_α in Lemma 3.4.1, we see that Υ contains non-zero vectors in $\Lambda^2(S^2\mathbf{U})$, which is an irreducible $GL(\mathbf{U})$-module. Thus $\Lambda^2(S^2\mathbf{U}) \subset \Upsilon$. Now recall the irreducible $GL(\mathbf{U}) \times GL(\mathbf{Q})$-module decomposition (e.g. [OV], p.300 table 5)

$$\mathbf{U} \otimes \mathbf{Q} \otimes S^2\mathbf{U} = (S^3\mathbf{U} \oplus \Gamma) \otimes \mathbf{Q}$$

where Γ is the irreducible $GL(\mathbf{U})$-module defined as the kernel of the product map $\mathbf{U} \otimes S^2\mathbf{U} \to S^3\mathbf{U}$. Again from the expression of P_α, we see that $S^3\mathbf{U} \subset \Upsilon$. Finally, the expression of P_β in Lemma 3.4.1 gives the element of Υ of the form

$$(\lambda \otimes \mu + \lambda^2) \wedge (\zeta \otimes \mu + 2\lambda \odot \zeta) \equiv \lambda^2 \wedge (\zeta \otimes \mu) - 2(\lambda \odot \zeta) \wedge (\lambda \otimes \mu) \mod \Upsilon.$$

The left hand side cannot be contained in $\Lambda^2(\mathbf{U} \otimes \mathbf{Q}), \Lambda^2(S^2\mathbf{U})$ or $S^3\mathbf{U} \otimes \mathbf{Q}$. Thus Υ contains $\Gamma \otimes \mathbf{Q}$, too. $\square$

(3.5) Let us consider the situation of Theorem 3.1.1. Using the results from (3.3) and (3.4), we will show that the variety of minimal rational tangents at a general point $x \in X_0$ is isomorphic to $\mathcal{C}_o$ at a base point $o \in S$.

Proposition 3.5.1. *In the notation of Theorem 3.1.1, choose a section $\sigma : \Delta \to \mathcal{X}$ of π such that $x_0 := \sigma(0)$ is a general point of X_0. Let $\mu : \mathcal{M} \to \Delta$ be the family where $M_t := \mu^{-1}(t)$ is the normalized Chow space of minimal rational curves through $x_t := \sigma(t)$. Then $M_t \cong \mathcal{C}_o$ for each $t \in \Delta$.*

Proof. We know that $M_t \cong \mathcal{C}_o$ for each $t \in \Delta - \{0\}$. We want to show that $M_0 \cong \mathcal{C}_o$. By the same proof as in [HM2, Proposition 4], μ is smooth and projective. We can find a line bundle on $\mathcal{M}$ whose restriction to $M_t, t \neq 0$, corresponds to the line bundle ξ on $\mathcal{C}_o$. Let us denote this line bundle by the same symbol ξ. By the same argument as in [HM5, Lemma 1 and Lemma 2], the limits of lines on M_t remain to be irreducible curves of ξ-degree 1 on M_0. Similarly, limits of degree 2 curves are either irreducible, or have two components. Thus Proposition 3.5.1 follows from Proposition 3.3.5. $\square$

Proposition 3.5.2. *The tangent morphism $\tau_{x_0} : M_0 \to \mathcal{C}_{x_0} \subset \mathbb{P}T_{x_0}(X_0)$ is the embedding defined by the complete linear system of the line bundle ξ. In particular, the variety of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X_0)$ at a general point $x \in X_0$ is isomorphic to $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ at a base point $o \in S$.*

Proof. By Proposition 3.5.1, $M_0 \cong C_0$. The tangent morphism $\tau_t : M_t \cong \mathcal{C}_o \to \mathbb{P}T_{x_t}(X_t)$ is the one defined by the complete linear system associated to ξ for $t \neq 0$. Thus τ_{x_0} must be defined by a subsystem of ξ and it suffices to show that the image of the tangent map is linearly non-degenerate. But this follows from the same argument as in the Hermitian symmetric case as given in [HM2, (5.1), Proposition 16 and proof of Theorem 1 in (5.2)], together with the linear non-degeneracy of $\mathcal{T}_o \subset \mathbb{P}\Lambda^2 T_o(S)$ as given by Proposition 3.4.2. □

§4. Lie algebra of vector fields on symplectic Grassmannians

(4.1) In this section, we study the Lie algebra of holomorphic vector fields on the symplectic Grassmannian. It will be useful to start the discussion with general rational homogenous spaces and their associated graded Lie algebras, cf. Yamaguchi [Ya]. Let $\mathfrak{g}$ be a simple Lie algebra. Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and denote by Φ the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Any $\rho \in \Phi$ can be expressed uniquely as an integral linear combination of the simple roots $\alpha_i, 1 \le i \le \ell$. Fix a simple root α_k. For $m \in \mathbb{Z}$, denote by Φ_m the set of roots whose coefficient in α_k is equal to m. Let μ be the maximal integer such that $\Phi_m \neq \emptyset$. We call μ the depth of $(\mathfrak{g}, \alpha_k)$. We have a decomposition of $\mathfrak{g}$ as a complex graded Lie algebra

$$\mathfrak{g} = \mathfrak{g}_{-\mu} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{\mu},$$

where for $m \neq 0$, $\mathfrak{g}_m$ is the direct sum of root spaces $\mathfrak{g}_\rho$, $\rho \in \Phi_m$, and $\mathfrak{g}_0$ is the direct sum of $\mathfrak{h}$ and root spaces $\mathfrak{g}_\rho, \rho \in \Phi_0$. This decomposition of $\mathfrak{g}$ is said to be associated to the simple root α_k. The subalgebra

$$\mathfrak{p} = \mathfrak{g}_{-\mu} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$$

is called the parabolic subalgebra associated to α_k. The nilradical of $\mathfrak{p}$ is $\mathfrak{g}_{-\mu} \oplus \cdots \oplus \mathfrak{g}_{-1}$. Let G be a simple Lie group associated to $\mathfrak{g}$ and $P \subset G$ be the subgroup associated to $\mathfrak{p}$. We say that the homogeneous space $S := G/P$ is of type $(\mathfrak{g}, \alpha_k)$.

Consider the case of depth 2, $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. The decomposition of $\mathfrak{g}$ has a geometric interpretation in terms of holomorphic vector fields on $S = G/P$. Let $o \in S$ be the base point fixed by P. When $\mathfrak{g}$ is regarded as the Lie algebra of holomorphic vector fields on S, the parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is the Lie subalgebra of holomorphic vector fields vanishing at o. For any element $v \in \mathfrak{p}$, let $\rho_o(v)$ be the linear transformation of the tangent space $T_o(S)$ given by the isotropy representation. The tangent space $T_o(S)$ can be naturally identified with $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $\mathfrak{g}_1$ defines a G-invariant subbundle D of the tangent bundle $T(S)$. By the G-invariance of the distribution D on S, $\rho_o(v)$ preserves D_o.

Proposition 4.1.1. *Let $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a decomposition of a simple Lie algebra $\mathfrak{g}$ as a graded Lie algebra of depth 2 as described above, $\mathfrak{p} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \subset \mathfrak{g}$ the parabolic subalgebra, and $o \in S$ be the unique fixed point of the parabolic subgroup $P = \exp(\mathfrak{p})$. Define*

$$J := \{v \in \mathfrak{p} : \ \rho_o(v)|_{D_o} \equiv \lambda \cdot id \quad \text{for some} \quad \lambda \in \mathbb{C}\},$$

$$I := \{v \in \mathfrak{p} : \ \rho_o(v)|_{D_o} \equiv 0\},$$

$$H := \{v \in \mathfrak{p} : \ \rho_o(v) \equiv 0\}.$$

Then, H agrees with $\mathfrak{g}_{-2}$, I agrees with $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, and J agrees with $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{z}(\mathfrak{g}_0)$, where $\mathfrak{z}(\mathfrak{g}_0)$ stands for the 1-dimensional center of $\mathfrak{g}_0$.

Proof. As is well-known (cf. Yamaguchi [Ya]), $\mathfrak{g}_i, i \neq 0$, is irreducible as a $\mathfrak{g}_0$-module. $\mathfrak{g}_0$ is reductive with 1-dimensional center $\mathfrak{z}(\mathfrak{g}_0)$. Furthermore, $[\mathfrak{g}_{-1}, \mathfrak{g}_2] = \mathfrak{g}_1$. Under the identification $T_o(S) = \mathfrak{g}/\mathfrak{p}$, for $v \in \mathfrak{p}$, we have

$$\rho_o(v) = ad(v)|_{\mathfrak{g}_1 \oplus \mathfrak{g}_2} \mod \mathfrak{p} \ \in End(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \cong End(T_o(S)).$$

Thus $v = g_{-2} + g_{-1} + g_0 \in J$ if and only if $g_0 \in J$, which is the case if and only if $[g_0, h_1] = \lambda h_1$ for some $\lambda \in \mathbb{C}$ and for any $h_1 \in \mathfrak{g}_1$. By the irreducibility of $\mathfrak{g}_1$ as a $\mathfrak{g}_0$-module and Schur's Lemma, $v \in J$ if and only if $g_0 \in \mathfrak{z}(\mathfrak{g}_0)$, so that $J = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{z}(\mathfrak{g}_0)$. Likewise $v \in I$ if and only if $g_0 = 0$, i.e., $I = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$. Finally, $v \in H$ if and only if $[g_{-1}, \mathfrak{g}_2] = 0$. This forces $g_{-1} = 0$ by the irreducibility of $\mathfrak{g}_{-1}$ and the fact that $[\mathfrak{g}_{-1}, \mathfrak{g}_2] = \mathfrak{g}_1$, so that $H = \mathfrak{g}_{-2}$. □

For $v \in H$, let $\varrho_o(v) \in S^2 T_o^*(S) \otimes T_o(S)$ be the element defined by the 2-jet of v at o as in Proposition 2.2.1. For $v \in \mathfrak{g}$, let $\epsilon_o(v) \in T_o(S)$ be the value of v at o.

Proposition 4.1.2. *In the situation of Proposition 4.1.1, the adjoint representation gives injections*

$$\mathfrak{g}_{-1} \subset Hom(\mathfrak{g}_2, \mathfrak{g}_1)$$

$$\mathfrak{g}_{-2} \subset Hom(\mathfrak{g}_2, \mathfrak{g}_0).$$

For a vector field $v \in \mathfrak{g}_{-1}$ and a vector field $w \in \mathfrak{g}_2$, $\epsilon_o([v, w])$ is determined by $\rho_o(v)$ and $\epsilon_o(w)$. For a vector field $v \in \mathfrak{g}_{-2}$ and a vector field $w \in \mathfrak{g}_2$, $\rho_o([v, w])$ is determined by $\varrho_o(v)$ and $\epsilon_o(w)$.

Proof. The injections follow from the irreducibility of $\mathfrak{g}_{-1}$ and $\mathfrak{g}_{-2}$ as $\mathfrak{g}_0$-modules as in the proof of Proposition 4.1.1.

The second and the third statements are essentially local. For local vector fields v, w in a neighborhood of o with $\epsilon_o(v) = 0$, $\rho_o(v)(w)$ is defined by the derivative of v with respect to the vector field w at o and agrees with $\epsilon_o([w, v])$. Similarly, for local vector fields u, v, w with $\epsilon_o(v) = \rho_o(v)$, $\varrho_o(v)(w, u)$ is defined by the derivative of $[w, v]$ with respect to u at o, which agrees with $\rho_o([w, v])(u)$ because $\rho_o(v) = 0$. □

Proposition 4.1.3. *Let $\mathfrak{g}$ be as in Proposition 4.1.1 and assume that* $\dim \mathfrak{g}_2 > 1$. *A graded vector space automorphism of $\mathfrak{g}$ preserving the graded Lie algebra structure of $\oplus_{i=0}^{i=2}\mathfrak{g}_i$ and the adjoint actions of $\mathfrak{g}_{-1} \subset Hom(\mathfrak{g}_2, \mathfrak{g}_1)$ and $\mathfrak{g}_{-2} \subset Hom(\mathfrak{g}_2, \mathfrak{g}_0)$ is a Lie algebra automorphism.*

Proof. By [Ya, Theorem 5.2] the graded Lie algebra $\mathfrak{g}$ with $\dim \mathfrak{g}_2 > 1$, is the prolongation of the graded Lie algebra $\oplus_{i=0}^{i=2}\mathfrak{g}_i$ in the sense of [Ya, p. 429-430], i.e., it is the maximal graded Lie algebra extending $\oplus_{i=0}^{i=2}\mathfrak{g}_i$ by adding terms with negative degrees so that the action of $\mathfrak{g}_j$ on $\oplus_{i=j+1}^{i=2}\mathfrak{g}_i$ is faithful ([Ya, p.433]). Thus any graded Lie algebra extending $\oplus_{i=0}^{i=2}\mathfrak{g}$ by adding terms with negative degrees has a natural graded Lie algebra injection into $\mathfrak{g}$. This implies Proposition 4.1.3. $\square$

We will also need corresponding results for Hermitian symmetric space. Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the gradation with G/P isomorphic to an irreducible Hermitian symmetric space of rank ≥ 2. In this case, $\mathfrak{g}_{-1}$ is the set of vector fields v with $\rho_o(v) = 0$. The following two propositions can be proved in a similar way as the above two propositions.

Proposition 4.1.4. *When $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is the gradation associated to an irreducible Hermitian symmetric space of rank ≥ 2, the adjoint representation gives injections $\mathfrak{g}_{-1} \subset Hom(\mathfrak{g}_1, \mathfrak{g}_0)$. For a vector field $v \in \mathfrak{g}_{-1}$ and a vector field $w \in \mathfrak{g}_1$, $\rho_o([v, w])$ is determined by $\varrho_o(v)$ and $\epsilon_o(w)$.*

Proposition 4.1.5. *For $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ as in Proposition 4.1.4, a graded vector space automorphism of $\mathfrak{g}$ preserving the graded Lie algebra structure of $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ and the adjoint actions of $\mathfrak{g}_{-1} \subset Hom(\mathfrak{g}_1, \mathfrak{g}_0)$ is a Lie algebra automorphism.*

(4.2) The symplectic Grassmannian $S = S_{k,\ell}$ is the rational homogeneous space of type $(\mathfrak{g}, \alpha_k) = (\mathfrak{sp}_\ell, \alpha_k)$ where $1 < k < \ell$ under the usual numeration of simple roots of the symplectic Lie algebra $\mathfrak{sp}_\ell$ (cf. [Ya] p. 454). The depth is $\mu = 2$ and $\dim \mathfrak{g}_2 > 1$. So Propositions 4.1.1, 4.1.2 and 4.1.3 can be applied. The distribution D defined using $\mathfrak{g}_1$ is a unique G-invariant distribution on $S_{k,\ell}$. So it is exactly the distribution D defined in (3.2) by the uniqueness. In particular, $\dim \mathfrak{g}_1 = k(2\ell - 2k)$ and $\dim \mathfrak{g}_2 = \frac{1}{2}k(k+1)$.

Now let us see what the variety of minimal rational tangents tells us about the vector fields on S. We start with examining its infinitesimal linear automorphisms $aut(\widetilde{\mathcal{C}}_o)$. For simplicity, let us use the notation of (3.4), identifying $\mathcal{C}_o$ with $Z = \mathbb{P}((\mathbf{Q} \otimes \mathbf{t}) \oplus \mathbf{t}^{\otimes 2})$. Since any endomorphism of $\mathbf{T}$ preserving $\widetilde{Z}$ must preserve $\widetilde{R}$, we have the restriction homomorphism $\chi : aut(\widetilde{Z}) \to aut(\widetilde{R})$ where $aut(\widetilde{R})$ denotes the endomorphisms of $\mathbf{D}$ preserving $\widetilde{R}$.

Proposition 4.2.1. *The homomorphism $\chi : aut(\widetilde{Z}) \to aut(\widetilde{R})$ is surjective. Let $\mathfrak{n}$ be the nil-radical of $Ker(\chi)$. Then $Ker(\chi)/\mathfrak{n}$ is 1-dimensional and represented by homotheties on* $\mathbf{T}/\mathbf{D} = S^2\mathbf{U}$. *Furthermore*

$$aut(\widetilde{R}) = \mathfrak{z} \oplus \mathfrak{sl}(\mathbf{U}) \oplus \mathfrak{sl}(\mathbf{Q})$$

$$\mathfrak{n} = \mathbf{U}^* \otimes \mathbf{Q}$$

where $\oplus$ is a Lie algebra direct sum and $\mathfrak{z}$ denotes the 1-dimensional center consisting of homotheties on $\mathbf{D}$.

Proof. Recalling $\mathbf{T} = H^0(\mathbb{P}\mathbf{U}, (\mathbf{Q}^* \otimes \mathbf{h}) \oplus \mathbf{h}^{\otimes 2})^*$, there is a natural injective homomorphism

$$H^0(\mathbb{P}\mathbf{U}, End((\mathbf{Q} \otimes \mathbf{t}) \oplus \mathbf{t}^{\otimes 2})) \longrightarrow End(\mathbf{T}).$$

The image of this injection is precisely $aut(\widetilde{Z})$. The surjectivity of χ follows from the fact that any endomorphism of $\mathbf{Q} \otimes \mathbf{t}$ can be extended to an endomorphism of $(\mathbf{Q} \otimes \mathbf{t}) \oplus \mathbf{t}^{\otimes 2}$. $Ker(\chi)$ consists of elements of $H^0(\mathbb{P}\mathbf{U}, End((\mathbf{Q} \otimes \mathbf{t}) \oplus \mathbf{t}^{\otimes 2}))$ which annihilates $\mathbf{Q} \otimes \mathbf{t}$ and $\mathfrak{n} \subset Ker(\chi)$ corresponds to those elements with zero trace. Thus $\mathfrak{n}$ is the abelian Lie algebra

$$H^0(\mathbb{P}\mathbf{U}, Hom(\mathbf{t}^{\otimes 2}, \mathbf{Q} \otimes \mathbf{t})) = H^0(\mathbb{P}\mathbf{U}, \mathbf{Q} \otimes \mathbf{h}) = \mathbf{U}^* \otimes \mathbf{Q}$$

and the statement about $Ker(\chi)/\mathfrak{n}$ is obvious. Finally, for the Segre variety $R \subset \mathbb{P}(\mathbf{U} \otimes \mathbf{Q})$, the isomorphism $aut(\widetilde{R}) = \mathfrak{z} \oplus \mathfrak{sl}(\mathbf{U}) \oplus \mathfrak{sl}(\mathbf{Q})$ is well-known. $\square$

Proposition 4.2.2. *For the linear Lie algebra $aut(\widetilde{Z}) \subset End(\mathbf{T})$, $aut(\widetilde{Z})^{(2)} = 0$ and there is a natural inclusion $aut(\widetilde{Z})^{(1)} \subset S^2\mathbf{U}^*$.*

Proof. The vanishing of the second prolongation is a direct consequence of Theorem 1.1.2. Let $A \in aut(\widetilde{Z})^{(1)} \subset S^2\mathbf{T}^* \otimes \mathbf{T}$. We will denote by $A_{\alpha\beta}$ the value of A on $\alpha, \beta \in \mathbf{T} = (\mathbf{U} \otimes \mathbf{Q}) \oplus S^2\mathbf{U}$. Recall that when $\alpha, \beta \in \widetilde{Z}$, the vector $A_{\alpha\beta}$ is in the intersection $P_\alpha \cap P_\beta$ where $P_\alpha \subset \mathbf{T}$ (resp. P_β) denotes the affine tangent space to $\widetilde{Z}$ at α (resp. β). Fix $\alpha \in \widetilde{Z} \cap S^2\mathbf{U}$, say $\alpha = \lambda^2, \lambda \in \mathbf{U}$. Let $\beta = \zeta \otimes \mu + \zeta^2$ for some $\zeta \in \mathbf{U}, \mu \in \mathbf{Q}$, be a general point of $\widetilde{Z}$. Using Lemma 3.4.1, we see

$$P_\alpha \cap P_\beta = \mathbb{C}(\lambda \otimes \mu + 2\lambda \odot \zeta).$$

Replacing μ by $t\mu$ with $t \to \infty$, we get $\beta_t = t\zeta \otimes \mu + \zeta^2$ with $[\beta_t] \to [\zeta \otimes \mu]$. Write $\delta = \zeta \otimes \mu$. We have $\lim\limits_{t\to\infty} \frac{1}{t}\beta_t = \delta$. Then $A_{\alpha\beta} \in P_\alpha \cap P_\beta$ implies

$$A_{\alpha\beta_t} \in \mathbb{C}(2\lambda \odot \zeta + \zeta \otimes t\mu) = \mathbb{C}\Big(\frac{2}{t}\lambda \odot \zeta + \zeta \otimes \mu\Big) \ , \quad \text{so that}$$

$$A_{\alpha\delta} = \lim_{t\to\infty} \frac{1}{t} A_{\alpha\beta_t} \in \mathbb{C}(\zeta \otimes \mu) \ .$$

We conclude that $A_{\alpha\delta} \in \mathbb{C}\delta$ for any $\delta \in \widetilde{Z} \cap \mathbf{D} = \widetilde{R}$ and $\alpha \in \widetilde{Z} \cap S^2\mathbf{U}$. In particular, if $A_{\alpha\delta} \neq 0$ then $\delta \in P_\alpha$. Since we can choose the factor $S^2\mathbf{U}$ so that it contains any point of $\widetilde{Z} - \widetilde{R}$, this means that δ is contained in the tangent space P_α for any choice of $\alpha \in \widetilde{Z}$. This is impossible by Lemma 2.1.2. Thus $A_{\alpha\delta} = 0$ for all $\alpha \in \widetilde{Z}$ and $\delta \in \widetilde{R}$. By linearity, $A_{\alpha\delta} = 0$ if $\delta \in \mathbf{D}$. Thus the value of A is determined by its values on $S^2\mathbf{U}$. Since for $\alpha, \beta \in \widetilde{Z} \cap S^2\mathbf{U}$, $P_\alpha \cap P_\beta \subset S^2\mathbf{U}$, we see that A induces an element A' of $S^2(S^2\mathbf{U}^*) \otimes S^2\mathbf{U}$

and A is determined by A'. Moreover A' is an element of $aut(\widetilde{Z} \cap S^2\mathbf{U})^{(1)}$. In other words we have a natural injection

$$aut(\widetilde{Z})^{(1)} \subset aut(\widetilde{Z} \cap S^2\mathbf{U})^{(1)}.$$

But $Z \cap \mathbb{P}S^2\mathbf{U}$ is the second Veronese embedding of the projective space $\mathbb{P}\mathbf{U}$. Thus

$$aut(\widetilde{Z} \cap S^2\mathbf{U})^{(1)} \subset \dim S^2\mathbf{U}^*$$

by Theorem 1.1.3 (i). This completes the proof of Proposition 4.2.2. $\square$

Recall that for any vector field v vanishing at o, $\rho_o(v) \in End(T_o(S))$ is the endomorphism induced by the first jet of v at o. If $\rho_o(v) = 0$, $\varrho_o(v) \in S^2T^*_o(S) \otimes T_o(S)$ is the element induced by the 2-jet of v at o. Denote by χ_o the surjective homomorphism $aut(\widetilde{\mathcal{C}}_o) \to aut(\widetilde{\mathcal{E}}_o)$ and by $\mathfrak{n}_o$ the nil-radical of $Ker(\chi_o)$.

Proposition 4.2.3. *Let us choose identification of $\mathcal{C}_o$ with $Z = \mathbb{P}((\mathbf{Q} \otimes \mathbf{t}) \oplus \mathbf{t}^{\otimes 2})$, inducing identifications $D_o = \mathbf{U} \otimes \mathbf{Q}$ and $T_o(S)/D_o = S^2\mathbf{U}$ in the notation of (3.4). Denoting by $\epsilon_o(v)$ the value of a vector field v on S at the base point $o \in S$ and by ς_o the projection $T_o(S) \to T_o(S)/D_o$, we have isomorphisms*

$$\varsigma_o \circ \epsilon_o : \mathfrak{g}_2 \to T_o(S)/D_o = S^2\mathbf{U}$$

$$\epsilon_o : \mathfrak{g}_1 \to D_o = \mathbf{U} \otimes \mathbf{Q}.$$

By taking 1-jets at o of vector fields vanishing at o, we have homomorphism $\rho_o : \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \to aut(\widetilde{\mathcal{C}}_o)$. It gives an injection

$$\chi_o \circ \rho_o : \mathfrak{g}_0 \to aut(\widetilde{\mathcal{E}}_o) = aut(\widetilde{R}) = \mathfrak{z} \oplus \mathfrak{sl}(\mathbf{U}) \oplus \mathfrak{sl}(\mathbf{Q})$$

and an isomorphism

$$\rho_o : \mathfrak{g}_{-1} \to \mathfrak{n}_o = \mathbf{U}^* \otimes \mathbf{Q}.$$

By taking 2-jets at o of vector fields vanishing to order ≥ 2 at o, we get an isomorphism

$$\varrho_o : \mathfrak{g}_{-2} \to aut(\widetilde{\mathcal{C}}_o)^{(1)} = S^2\mathbf{U}^*.$$

Proof. From $aut(\widetilde{\mathcal{C}}_o)^{(2)} = 0$ in Proposition 4.2.2, ϱ_o must be injective. Since $\dim \mathfrak{g}_{-2} = \dim \mathfrak{g}_2$ by [Ya] and $\dim \mathfrak{g}_2 = \dim \mathbf{T}/\mathbf{D}$, $\varrho_o : \mathfrak{g}_{-2} \to aut(\widetilde{\mathcal{C}}_o)^{(1)} \cong aut(\widetilde{Z})^{(1)}$ and the inclusion $aut(\widetilde{Z})^{(1)} \subset S^2\mathbf{U}^*$ in Proposition 4.2.2 must be isomorphisms. This implies that $\rho_o : \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \to aut(\widetilde{\mathcal{C}}_o)$ is injective and its image is isomorphic to the Lie algebra $\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ mod $\mathfrak{g}_{-2}$, which has nil-radical $\mathfrak{g}_{-1}$.

By Proposition 4.1.1, $\rho_o(\mathfrak{g}_{-1})$ lies in the kernel of $\chi_o : aut(\widetilde{\mathcal{C}}_o) \to aut(\widetilde{\mathcal{E}}_o)$. Since $\dim \mathfrak{g}_{-1} = \dim \mathfrak{g}_1$ by [Ya] and $\dim \mathfrak{g}_1 = \dim \mathfrak{n}_o$ by Proposition 4.2.1, $\rho_o|_{\mathfrak{g}_{-1}}$ gives the isomorphism to $\mathfrak{n}_o$. This implies that $\rho_o(\mathfrak{g}_0)$ intersects $\mathfrak{n}_o$ at 0. By Proposition 4.1.1 and the description of $Ker(\chi_o)$ in Proposition 4.2.1, $\rho(\mathfrak{g}_0)$ intersects $Ker(\chi_o)$ at 0. So the homomorphism $\chi_o \circ \rho_o : \mathfrak{g}_0 \to aut(\widetilde{\mathcal{E}}_o)$ is injective. The statements about ϵ_o and $\varsigma_o \circ \epsilon_o$ are obvious. □

(4.3) Let us consider the situation of Proposition 4.1.1, i.e., a gradation of a simple Lie algebra $\mathfrak{g}$ with depth 2. Under the decomposition $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, there is a unique element $E \in \mathfrak{z}(\mathfrak{g}_0)$, called the characteristic element, such that $[E, g] = ig$ if and only if $g \in \mathfrak{g}_i$. A holomorphic vector field on $S = G/P$ will be called a **characteristic vector field** if it corresponds to the characteristic element under a choice of the Cartan subalgebra and simple roots when we regard $\mathfrak{g}$ as a Lie algebra of holomorphic vector fields on S. From Proposition 4.1.1, E belongs to the subspace J. We observe that a general element of J gives a characteristic vector field on S. More precisely, we have

Proposition 4.3.1. *Let $I \subset J \subset \mathfrak{p} \subset \mathfrak{g}$ be the subalgebras defined by a choice of the base point $o \in S$ as in Proposition 4.1.1. Let E' be any element of $J - I$ with $\rho_o(E')|_{D_o} = id$. Then E' is a characteristic element of a decomposition $\mathfrak{g} = \mathfrak{g}'_{-2} \oplus \mathfrak{g}'_{-1} \oplus \mathfrak{g}'_0 \oplus \mathfrak{g}'_1 \oplus \mathfrak{g}'_2$ with respect to some choice of a Cartan subalgebra $\mathfrak{h}' \subset \mathfrak{g}$ and a system of roots.*

Proof. We have the characteristic vector field $E \in J - I$ associated to a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{p} \subset \mathfrak{g}$. It suffices to show that any element E' as in the statement of Proposition 4.3.1 is conjugate to E by the adjoint action of P on $\mathfrak{p}$. This is equivalent to showing that the P-orbit of $[E]$ in the projective space $\mathbb{P}J$ is the affine space $\mathbb{P}J - \mathbb{P}I$. In fact, we can show that the orbit of $[E]$ under the unipotent subgroup $\exp(I)$ is $\mathbb{P}J - \mathbb{P}I$ as follows. Since

$$\dim[\mathfrak{g}_{-1} + \mathfrak{g}_{-2}, E] = \dim(\mathfrak{g}_{-1} + \mathfrak{g}_{-2}) = \dim(\mathbb{P}J - \mathbb{P}I),$$

the $\exp(I)$-orbit of $[E]$ in $\mathbb{P}J$ is open. But the orbit of a unipotent group acting on an affine space must be closed ([Bo2] Proposition 4.10, p.88). Thus the orbit must be the whole affine space. □

Corollary 4.3.2. *A holomorphic vector field v on S is a characteristic vector field if it vanishes at a point $o \in S$ and its first jet at o satisfies*

$$\rho_o(v)|_{D_o} \equiv id.$$

Now consider the case of the symplectic Grassmannian $S = S_{k,\ell}$. From Corollary 4.3.2, we can describe the $\mathbb{C}^*$-action on $S = S_{k,\ell}$ generated by a characteristic vector field as follows. Let $[W] \in S_{k,\ell}$ be as in (3.2). Pick a complement $W^\sharp$ of W in $W^\perp$. The symplectic form ω is non-degenerate on $W^\sharp$. Thus $(W^\sharp)^\perp$ is a complementary subspace of $W^\sharp$ in V, on which

ω is non-degenerate. Choose an isotropic complement $W^\diamond$ to W in $(W^\sharp)^\perp$. Consider the symplectic action of $\lambda \in \mathbb{C}^*$ on V defined by

$$w \mapsto \lambda^{-1}w \text{ for } w \in W$$
$$w \mapsto w \text{ for } w \in W^\sharp$$
$$w \mapsto \lambda w \text{ for } w \in W^\diamond.$$

The induced action on $D_{[W]} = W^* \otimes (W^\perp/W) = W^* \otimes W^\sharp$ is multiplication by λ. Thus the vector field generating this action must be a characteristic vector field by Corollary 4.3.2. Conversely, any $\mathbb{C}^*$-action defined by a characteristic vector field must be the above form associated to some choice of $[W] \in S_{k,\ell}$ and the complements $W^\sharp$ and $W^\diamond$.

Given a characteristic vector field E at $[W]$ determined by a choice of complement $W^\sharp$ and $W^\diamond$. Let $W = W' \oplus W''$ be any direct sum decomposition with $\dim W' = 2$ and $\dim W'' = k-2$. Define a new symplectic $\mathbb{C}^*$-action on V where $\lambda \in \mathbb{C}^*$ acts as

$$v \mapsto \lambda v \text{ for } v \in W''$$

$$v \mapsto v \text{ for } v \in W' \oplus W^\sharp \oplus (W^\diamond \cap (W'')^\perp)$$

$$v \mapsto \lambda^{-1} v \text{ for } v \in W^\diamond \cap (W')^\perp.$$

This induces a $\mathbb{C}^*$-action on S. A fixed point component of this action on $S_{k,\ell}$ consists of k-dimensional isotropic subspaces contained in $(W'')^\perp$ which contains W''. This is naturally isomorphic to the set of isotropic 2-planes in $W' \oplus W^\sharp \oplus (W^\diamond \cap (W'')^\perp)$. The vector field F generating this $\mathbb{C}^*$-action will be called **the slicing vector field** determined by E and $W = W' \oplus W''$. Thus the component of the zero set of the slicing vector field containing $[W]$ is biholomorphic to $S_{2,\ell-k+2}$.

Proposition 4.3.3. *The slicing vector field F determined by a characteristic vector field E at $o = [W] \in S$ and a decomposition $W = W' \oplus W''$ is the unique element of $\mathfrak{g}_0$ satisfying $\rho(F)|_{W' \otimes (W^\perp/W)} \equiv id$ and $\rho_o(F)|_{W'' \otimes (W^\perp/W)} \equiv 0$.*

Proof. It is clear that F satisfies the stated conditions. Thus Proposition 4.3.3 is a direct consequence of the injectivity of $\chi_o \circ \rho_o : \mathfrak{g}_0 \to End(W \otimes (W^\perp/W))$ in Proposition 4.2.3. $\square$

§5. Limit vector fields on the central fiber

(5.1) Now let us go to the situation of Theorem 3.1.1. From Proposition 3.5.2, there exists a subvariety $B \subset X_0$ such that $\mathcal{C}_x \subset \mathbb{P}T_x(X_0)$ is isomorphic to $Z \subset \mathbb{P}\mathbf{T}$ for any $x \in X_0 - B$. Let $\mathcal{E}_x \subset \mathcal{C}_x$ be the subvariety corresponding to $R \subset Z$. Denote by $\mathcal{D}$ the distribution on $\mathcal{X} - B$ defined by the linear span of $\mathcal{E}_x$.

Fix a holomorphic section $\sigma : \Delta :\to \mathcal{X} - B$, $x_t := \sigma(t)$, $x := x_0$. Fix a vector bundle U of rank k and another vector bundle Q of rank $2m$ on $\sigma(\Delta)$. We can choose an isomorphism

$$(\ddagger) \ \{\mathcal{C}_{x_t} : \ t \in \Delta\} \cong \{\mathbb{P}((Q_{x_t} \otimes \mathbf{t}) \oplus \mathbf{t}^{\otimes 2}) : \ t \in \Delta\}$$

where on the right hand side Q is regarded as a vector bundle on $\mathbb{P}U$ pulled back from $\sigma(\Delta)$ by the projection $\mathbb{P}U \to \sigma(\Delta)$ and $\mathbf{t}$ denotes the tautological line bundle on $\mathbb{P}U_{x_t}$. We will keep the choice of σ and the isomorphism ($\ddagger$) throughout Section 5. This induces isomorphisms

$$aut(\widetilde{\mathcal{E}}_{x_t}) \cong \mathfrak{z}_{x_t} \oplus \mathfrak{sl}(U_{x_t}) \oplus \mathfrak{sl}(Q_{x_t})$$

$$\mathfrak{n}_{x_t} \cong U^*_{x_t} \otimes Q_{x_t}$$

$$aut(\widetilde{\mathcal{C}}_{x_t})^{(1)} \cong S^2 U^*_{x_t}$$

from Proposition 4.2.1 and Proposition 4.2.3. Here $\mathfrak{z}_{x_t}$ denotes the 1-dimensional center consisting of homotheties on $\mathcal{D}_{x_t}$ and $\mathfrak{n}_{x_t}$ is the nil-radical of $Ker(\chi_{x_t})$ where $\chi_{x_t} : aut(\widetilde{\mathcal{C}}_{x_t}) \to aut(\widetilde{\mathcal{E}}_{x_t})$ is the natural homomorphism.

For the relative tangent bundle T^π of $\pi : \mathcal{X} \to \Delta$ write $\mathcal{T}^\pi$ for the associated relative tangent sheaf and $\mathcal{L}$ for the direct image $\pi_*(\mathcal{T}^\pi)$. $\mathcal{L}$ is a locally free sheaf with fiber at $t \neq 0$,

$$L^t \cong \text{ the Lie algebra of holomorphic vector fields on } X_t \cong \mathfrak{g}.$$

The Lie algebra L^0 with $\dim L^0 = \dim \mathfrak{g}$ is called the **Lie algebra of limit vector fields** on X_0. For $v \in L^t$, let $\epsilon_{x_t}(v) \in T_{x_t}(X_t)$ be the evaluation at x_t. When $\epsilon_{x_t}(v) = 0$, let $\rho_{x_t}(v)$ be the endomorphism of $T_{x_t}(X_t)$ given by the first jet of v. Following Proposition 4.1.1, define, for each $t \in \Delta$,

$$J^t := \{v \in L^t : \ \epsilon_{x_t}(v) = 0, \rho_{x_t}(v)|_{\mathcal{D}_{x_t}} \equiv \mu \cdot id \text{ for some } \mu \in \mathbb{C}\}$$

$$I^t := \{v \in L^t : \ \epsilon_{x_t}(v) = 0, \rho_{x_t}(v)|_{\mathcal{D}_{x_t}} \equiv 0\}$$

$$H^t := \{v \in L^t : \ \epsilon_{x_t}(v) = 0, \rho_{x_t}(v) = 0\}.$$

These are subalgebras of the Lie algebra L^t for each $t \in \Delta$.

Proposition 5.1.1. *The homomorphism defined by 2-jets of vector fields at x_t*

$$\varrho_{x_t} : H^t \to aut(\widetilde{\mathcal{C}}_{x_t})^{(1)} \cong S^2 U^*_{x_t}$$

is an isomorphism for each $t \in \Delta$. The homomorphism defined by 1-jets of vector fields at x_t

$$\rho_{x_t} : I^t \to \mathfrak{n}_{x_t} \cong U^*_{x_t} \otimes Q_{x_t}$$

is an isomorphism for each $t \in \Delta$. The homomorphism defined by 1-jets of vector fields at x_t

$$\chi_{x_t} \circ \rho_{x_t} : J^t/I^t \to aut(\widetilde{\mathcal{E}}_{x_t})$$

is injective and has image $\mathfrak{z}_{x_t}$, the 1-dimensional center of $aut(\widetilde{\mathcal{E}}_{x_t})$, for each $t \in \Delta$. In particular,

$$\{H^t \subset I^t \subset J^t : \ t \in \Delta\}$$

form a subbundle of the locally free sheaf $\mathcal{L}$ on Δ.

Proof. For $t \neq 0$, this follows from Proposition 4.1.1 and Proposition 4.2.3. Let us show it for $t = 0$. By Theorem 1.3.1, X_0 cannot have a vector field vanishing to the order ≥ 3 at $x = x_0$, which implies the injectivity of $\varrho_x : H^0 \to S^2U^*_{x_0}$. Since $\dim H^0 \geq \dim H^t = \dim S^2U^*_{x_t}$ by Proposition 4.1.1 and Proposition 4.2.3, we conclude that ϱ_x is an isomorphism. This implies that

$$Ker(\rho_x : J^0 \to End(T_x(X_0))) = H^0$$

and the quotient map $\rho_x : J^0/H^0 \to End(T_x(X_0))$ is injective. Since $\rho_x(I^0/H^0) \subset \mathfrak{n}_x$ while

$$\dim(I^0/H^0) \geq \dim(I^t/H^t) = \dim \mathfrak{n}_{x_t} = \dim \mathfrak{n}_x \text{ for } t \neq 0,$$

we conclude that $\rho_x : I^0/H^0 \to \mathfrak{n}_x$ is an isomorphism. This implies the injectivity of $\chi_{x_t} \circ \rho_{x_t} : J^t/H^t \to aut(\widetilde{E}_{x_t})$ for all $t \in \Delta$. Certainly the image of $\chi_{x_t} \circ \rho_{x_t}$ should be the 1-dimensional $\mathfrak{z}_{x_t}$. □

(5.2) From Proposition 5.1.1 we deduce the extension of characteristic vector fields to X_0. More precisely, we have

Proposition 5.2.1. *Let $x_t = \sigma(t)$ be as in (5.1). There exists a family of $\mathbb{C}^*$-vector fields E_t on X_t with an isolated zero at x_t for all $t \in \Delta$ such that E_t is a characteristic vector field at x_t for each $t \neq 0$.*

Proof. By choosing a section of the invertible sheaf $\{J^t/I^t, t \in \Delta\}$ in Proposition 5.1.1 over Δ, we get a holomorphic family of holomorphic vector fields $\{E_t : t \in \Delta\}$ such that for all $t \in \Delta$, E_t vanishes at x_t, and the 1-jet $\rho_{x_t}(E_t) \in End(T_{x_t}(X_t))$ at x_t restricts to the identity map on $\mathcal{D}_{x_t}$. By Corollary 4.3.2, E_t is a characteristic vector field on X_t for $t \neq 0$. The 1-parameter subgroup on X_t, given by $\{\exp(\lambda E_t), \lambda \in \mathbb{C}\}$ is of minimum period 2π for each $t \neq 0$, and the same must hold true for $t = 0$ by continuity. Thus E_0 is also a $\mathbb{C}^*$-vector field with an isolated zero at x_0 and $\rho_{x_0}(E_0)$ restricts to the identity map on $\mathcal{D}_{x_0}$. □

A $\mathbb{C}^*$-vector field on X_0 with an isolated zero x will be called a **characteristic vector field at** x if it is the limit of a family of characteristic vector fields at x_t for a section of π as in Proposition 5.2.1. For any point $x \in X_0 - B$, there exists a characteristic vector field at x by Proposition 5.2.1.

Let us fix a family E_t of characteristic vector fields. For each $t \neq 0$, we have the eigenspace decomposition of the adjoint action of $E_t, t \neq 0$,

$$L^t = L^t_{-2} \oplus L^t_{-1} \oplus L^t_0 \oplus L^t_1 \oplus L^t_2$$

which is conjugate to $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. For each i, $-2 \leq i \leq 2$, the i-eigenspace L^0_i of L^0 under the adjoint action of E_0 must have dimension at least $\dim L^t_i$. Since $\dim L^0 = \dim L^t$, we have the eigenspace decomposition

$$L^0 = L^0_{-2} \oplus L^0_{-1} \oplus L^0_0 \oplus L^0_1 \oplus L^0_2.$$

In analogy with Proposition 4.2.3, we have

Proposition 5.2.2. *Let $L^0 = L^0_{-2} \oplus L^0_{-1} \oplus L^0_0 \oplus L^0_1 \oplus L^0_2$ be the eigenspace decomposition associated to a choice of a family of characteristic vector fields E_t along x_t as above. Denoting by ϵ_x the evaluation of vector fields on X_0 at $x = x_0 \in X_0$ and denoting by ς_x the projection $T_x(X_0) \to T_x(X_0)/\mathcal{D}_x$, we have two isomorphisms*

$$\varsigma_x \circ \epsilon_x : L^0_2 \to T_x(X_0)/\mathcal{D}_x \cong S^2 U_x$$

$$\epsilon_x : L^0_1 \to \mathcal{D}_x \cong U_x \otimes Q_x.$$

By taking 1-jets at $x = x_0$ of vector fields vanishing at x, we have the homomorphism $\rho_x : L^0_0 \oplus L^0_{-1} \to aut(\widetilde{\mathcal{C}_x})$ which induces an injection

$$\chi_x \circ \rho_x : L^0_0 \to aut(\widetilde{\mathcal{E}_x}) \cong \mathfrak{z}_x \oplus \mathfrak{sl}(U_x) \oplus \mathfrak{sl}(Q_x)$$

and an isomorphism

$$\rho_x : L^0_{-1} \to \mathfrak{n}_x \cong U^*_x \otimes Q_x.$$

By taking 2-jets at x of vector fields vanishing to order ≥ 2 at x, we get an isomorphism

$$\varrho_x : L^0_{-2} \to aut(\widetilde{\mathcal{C}_x})^{(1)} \cong S^2 U^*_x.$$

Proof. The statements about $\chi_x \circ \rho_x$, ρ_x and ϱ_x can be proved in exactly the same way as in Proposition 4.2.3. It remains to prove the statements about ϵ_x and $\varsigma_x \circ \epsilon_x$. By considering dimensions, it suffices to show that non-zero elements of L^0_2 and L^0_1 do not vanish at x. We will check this for L^0_2. The proof for L^0_1 will be exactly the same. Suppose there exists a non-zero element $v \in L^0_2$ vanishing at x. Note that

$$\rho_x : \{\text{vector fields vanishing at } x\} \to End(T_x(X_0))$$

is a Lie algebra homomorphism. Since $[E_0, v] = 2v$, we see that $\rho_x(v) \in End(T_x(X_0))$ satisfies $[\rho_x(E_0), \rho_x(v)] = 2\rho_x(v)$ inside the Lie algebra $End(T_x(X_0))$. $\rho_x(E_0) \in End(T_x(X_0))$ is a semi-simple endomorphism with eigenvalues 1 and 2. $\rho_x(v)$ must preserve $\mathcal{D}_x$ which is the eigenspace of $\rho_x(E_0)$ with eigenvalue 1. This implies that $\rho_x(v)$ is an eigenvector of the adjoint action of $\rho_x(E_0)$ only if it annihilates $\mathcal{D}_x$, in which case $[\rho_x(E_0), \rho_x(v)] = -\rho_x(v)$. It follows that $\rho_x(v) = 0$ and v should vanish to the order ≥ 2 at x. This implies that $v \in H^0$. But by Proposition 5.1.1 and the statement about ϱ_x in Proposition 5.2.2, we have $H^0 = L^0_{-2}$. This gives $v \in L^0_{-2}$, a contradiction. $\square$

(5.3) Now we study the distribution $\mathcal{D}$ on $X_0 - B$. The identification (‡) in (5.1) induces identifications $\mathcal{D}_{x_t} \cong U_{x_t} \otimes Q_{x_t}$ and $T^\pi_{x_t}/\mathcal{D}_{x_t} \cong S^2 U_{x_t}$.

Proposition 5.3.1. *In terms of the identifications $\mathcal{D}_{x_t} \cong U_{x_t} \otimes Q_{x_t}$ and $T^\pi_{x_t}/\mathcal{D}_{x_t} \cong S^2 U_{x_t}$, the Frobenius bracket tensor $[,] : \Lambda^2 \mathcal{D}_{x_t} \to T^\pi_{x_t}/\mathcal{D}_{x_t}$ for the distribution $\mathcal{D}$ at x_t is given by*

$$[u \otimes w, v \otimes z] = \nu_{x_t}(w, z)\, u \odot v$$

for $u, v \in U_{x_t}$ and $w, z \in Q_{x_t}$ where $\nu : \Lambda^2 Q \to \mathcal{O}_{\sigma(\Delta)}$ is a bundle homomorphism such that ν_{x_t} is non-degenerate for $t \neq 0$.

Proof. It suffices to show that $[u \otimes w, v \otimes z]$ is proportional to $u \odot v$. Moreover by continuity, it is enough to check it for $t \neq 0$. Thus the problem is about the distribution D on the symplectic Grassmannian S. At the point $[W] \in S$, the Frobenius bracket of D,

$$[,] : \Lambda^2 D_{[W]} = \Lambda^2(W^* \otimes (W^\perp/W)) \to T_{[W]}(S)/D_{[W]} = S^2 W^*$$

must be invariant under the natural action of $GL(W) \times Sp(W^\perp/W)$ arising from the isotropy action of the stabilizer of $[W]$ from which it is easy to see that $[u \otimes w, v \otimes z] = c\omega(w, z) u \odot v$ for some constant $c \neq 0$ and the symplectic form ω on $W^\perp/W$. It is clear that ν_{x_t} is non-degenerate for $t \neq 0$. $\square$

Let us recall the following lemma, which can be proved easily by modifying the standard proof of the existence of symplectic basis on a symplectic vector space.

Lemma 5.3.2. *Let $Q = \{Q_t, t \in \Delta\}$ be a vector bundle of rank $2q$ on the unit disc with an anti-symmetric form $\nu_t : \Lambda^2 Q_t \to \mathbb{C}$ on the fiber depending holomorphically on t such that ν_t is non-degenerate for $t \neq 0$ and ν_0 has rank $2r, r \leq q$. Then there exist sections $f_1, \ldots, f_{2q}$ of the dual bundle Q^* which gives a basis of Q^*_t for each t with respect to which*

$$\nu_t = f_1 \wedge f_2 + \cdots + f_{2r-1} \wedge f_{2r} + a_1(t) f_{2r+1} \wedge f_{2r+2} + \cdots + a_{q-r}(t) f_{2q-1} \wedge f_{2q}$$

for some holomorphic functions $a_1(t), \ldots, a_{q-r}(t)$ on Δ vanishing at $t = 0$.

Let $2r$ be the rank of ν_x, x_0. By Lemma 5.3.2, we can choose a subbundle $Q' \subset Q$ of rank $2r$ on $\sigma(\Delta)$ such that

(†) ν is non-degenerate on Q'_{x_t} for all $t \in \Delta$ and the complementary subspaces $(Q'_{x_t})^{\perp}, t \neq 0$, in Q_{x_t} with respect to the symplectic from $\nu_{x_t}, t \neq 0$, converges to the kernel of ν_{x_0} on Q_{x_0}.

Choose a family of characteristic vector fields $E_t, x_t = \sigma(t)$ as in Proposition 5.2.1, inducing the eigenspace decomposition $L^t = L^t_{-2} \oplus L^t_{-1} \oplus L^t_0 \oplus L^t_1 \oplus L^t_2$. For $t \neq 0$, let $\mathfrak{h}^t = \mathfrak{h}^t_{-2} \oplus \mathfrak{h}^t_{-1} \oplus \mathfrak{h}^t_0 \oplus \mathfrak{h}^t_1 \oplus \mathfrak{h}^t_2$ be the graded subalgebra of L^t defined as follows using the notation of Proposition 5.2.2.

$$\begin{aligned}
&\mathfrak{h}^t_2 = L^t_2 \\
&\mathfrak{h}^t_1 = \{v \in L^t_1 :\ \epsilon_{x_t}(v) \in U_{x_t} \otimes Q'_{x_t}\} \\
&\mathfrak{h}^t_0 = \{v \in L^t_0 :\ \chi_{x_t} \circ \rho_{x_t}(v) \in \mathfrak{z}_{x_t} \oplus \mathfrak{sl}(U_{x_t}) \oplus \mathfrak{sp}(Q'_{x_t})\} \\
&\mathfrak{h}^t_{-1} = \{v \in L^t_{-1} :\ \rho_{x_t}(v) \in U^*_{x_t} \otimes Q'_{x_t}\} \\
&\mathfrak{h}^t_{-2} = L^t_{-2}
\end{aligned}$$

where in the third line $\mathfrak{sp}(Q'_{x_t})$ is regarded as a subalgebra of $\mathfrak{sl}(Q_{x_t})$ by

$$\mathfrak{sp}(Q'_{x_t}) = \{v \in \mathfrak{sl}(Q_{x_t}) :\ v((Q'_{x_t})^{\perp}) = 0, v(Q'_{x_t}) \subset Q'_{x_t} \text{ and } v \text{ preserves } \nu_{x_t}\}.$$

When $r = 0$, $\mathfrak{h}^t_1 = \mathfrak{h}^t_{-1} = 0$. From the description of $\mathfrak{h}^t_i$, it is clear that the limit $\mathfrak{h}^0_i$ as $t \to 0$ is defined and we have also a graded Lie subalgebra $\mathfrak{h}^0 = \mathfrak{h}^0_{-2} + \mathfrak{h}^0_{-1} + \mathfrak{h}^0_0 + \mathfrak{h}^0_1 + \mathfrak{h}^0_2$ of L^0. Note that by our choice (†) of Q',

$$\begin{aligned}
&\mathfrak{h}^0_0 = \{v \in L^0_0 :\ \chi_{x_0} \circ \rho_{x_0}(v) \in \mathfrak{z}_{x_0} \oplus \mathfrak{sl}(U_{x_0}) \oplus \mathfrak{sp}(Q'_{x_0})\} \\
&\mathfrak{h}^0_{-1} = \{v \in L^0_0 :\ \rho_{x_0}(v) \in U^*_{x_0} \otimes Q'_{x_0}\}
\end{aligned}$$

where $\mathfrak{sp}(Q'_{x_0})$ is regarded as a subalgebra of $\mathfrak{sl}(Q_{x_0})$ by

$$\mathfrak{sp}(Q'_{x_0}) = \{v \in \mathfrak{sl}(Q_{x_0}) :\ v(Ker(\nu_{x_0})) = 0, v(Q'_{x_0}) \subset Q'_{x_0} \text{ and } v \text{ preserves } \nu_{x_0}\}.$$

To determine the Lie algebra structure of $\mathfrak{h}^t$, let $\mathbf{g} = \mathfrak{sp}(k+r)$ and consider the gradation with respect to the simple root α_k. When $r > 0$, the gradation on $\mathbf{g}$ has depth 2, $\mathbf{g} = \mathbf{g}_{-2} \oplus \mathbf{g}_{-1} \oplus \mathbf{g}_0 \oplus \mathbf{g}_1 \oplus \mathbf{g}_2$, and Proposition 4.1.2 and Proposition 4.1.3 hold for $\mathbf{g}$. When $r = 0$, the gradation has depth 1, $\mathbf{g} = \mathbf{g}_{-1} \oplus \mathbf{g}_0 \oplus \mathbf{g}_1$, and is associated to the Lagrangian Grassmannian, which is a Hermitian symmetric space of rank ≥ 2. Proposition 4.1.4 and Proposition 4.1.5 hold for $\mathbf{g}$. It is easy to check that when $r > 0$, the graded Lie algebra $\mathfrak{h}^t$ for each $t \in \Delta - \{0\}$ is isomorphic to $\mathbf{g}$. When $r = 0$, $\mathfrak{h}^t = \mathfrak{h}^t_{-2} \oplus \mathfrak{h}^t_0 \oplus \mathfrak{h}^t_2$ for each $t \in \Delta - \{0\}$ is isomorphic to $\mathbf{g} = \mathbf{g}_{-1} \oplus \mathbf{g}_0 \oplus \mathbf{g}_1$ as graded Lie algebras, if we view elements of $\mathfrak{h}^t_{-2}$ as elements of degree -1 and elements of $\mathfrak{h}^t_2$ as elements of degree 1. We claim the isomorphism $\mathfrak{h}^t \cong \mathbf{g}$ holds for $t = 0$, too:

Proposition 5.3.3. *The graded Lie subalgebra* $\mathfrak{h}^0$ *defined above is isomorphic to* $\mathfrak{h}^t \cong \mathbf{g}$.

Proof. First, we will give the proof when $r > 0$. Let $\mathbf{U}$ be a vector space of dimension k and $\mathbf{Q}'$ be a vector space of dimension $2r$ with a fixed symplectic form ϖ. Define

$$\begin{aligned}
\mathbf{h}_2 &= S^2\mathbf{U} \\
\mathbf{h}_1 &= \mathbf{U} \otimes \mathbf{Q}' \\
\mathbf{h}_0 &= \mathbb{C} \oplus \mathfrak{sl}(\mathbf{U}) \oplus \mathfrak{sp}(\mathbf{Q}') \\
\mathbf{h}_{-1} &= \mathbf{U}^* \otimes \mathbf{Q}' \\
\mathbf{h}_{-2} &= S^2\mathbf{U}^*.
\end{aligned}$$

Now fix a trivialization of the vector bundle $U|_{\sigma(\Delta)} \cong \mathbf{U} \times \sigma(\Delta)$ and a trivialization of the symplectic vector bundle $Q'|_{\sigma(\Delta)} \cong \mathbf{Q}' \times \sigma(\Delta)$. Then we get natural isomorphisms

$$\mathfrak{h}_2^t \cong S^2 U_{x_t} \cong S^2\mathbf{U} \cong \mathbf{h}_2$$

$$\mathfrak{h}_1^t \cong U_{x_t} \otimes Q'_{x_t} \cong \mathbf{U} \otimes \mathbf{Q}' \cong \mathbf{h}_1$$

$$\mathfrak{h}_0^t \cong \mathfrak{z}_{x_t} \oplus \mathfrak{sl}(U_{x_t}) \oplus \mathfrak{sp}(Q'_{x_t}) \cong \mathbb{C} \oplus \mathfrak{sl}(\mathbf{U}) \oplus \mathfrak{sp}(\mathbf{Q}') = \mathbf{h}_0$$

$$\mathfrak{h}_{-1}^t \cong U^*_{x_t} \otimes Q'_{x_t} \cong \mathbf{U}^* \otimes \mathbf{Q}' = \mathbf{h}_{-1}$$

$$\mathfrak{h}_{-2}^t \cong S^2 U^*_{x_t} \cong S^2\mathbf{U}^* = \mathbf{h}_{-2}$$

for each $t \in \Delta$. This gives a natural vector space isomorphism $\varphi_t : \mathfrak{h}^t \to \mathbf{h} := \mathbf{h}_{-2} \oplus \mathbf{h}_{-1} \oplus \mathbf{h}_0 \oplus \mathbf{h}_1 \oplus \mathbf{h}_2$.

Fix $t_1 \neq 0$ and consider the vector space isomorphism $\psi_t : \mathfrak{h}^{t_1} \to \mathfrak{h}^t$ defined by $\psi_t := \varphi_t^{-1} \circ \varphi_{t_1}$. $\mathbf{h}_0 \oplus \mathbf{h}_1 \oplus \mathbf{h}_2$ has a natural graded Lie algebra structure given by $[u_1 \otimes q_1, u_2 \otimes q_2] = \varpi(q_1, q_2)\ u_1 \odot u_2$ for $u_1, u_2 \in \mathbf{h}_1$ and the natural action of $\mathbf{h}_0$ on $\mathbf{h}_1$ and $\mathbf{h}_2$. By the construction, φ_t preserves the natural graded Lie algebra structures on $\mathfrak{h}_0^t \oplus \mathfrak{h}_1^t \oplus \mathfrak{h}_2^t$ and $\mathbf{h}_0 \oplus \mathbf{h}_1 \oplus \mathbf{h}_2$. Thus ψ_t preserves the graded Lie algebra structure on $\mathfrak{h}_0^t \oplus \mathfrak{h}_1^t \oplus \mathfrak{h}_2^t$. For $t \neq 0$, the adjoint action gives injections $\mathfrak{h}_{-1}^t \subset Hom(\mathfrak{h}_2^t, \mathfrak{h}_1^t), \mathfrak{h}_{-2}^t \subset Hom(\mathfrak{h}_2^t, \mathfrak{h}_0^t)$ from Proposition 4.1.2. By Proposition 4.1.2, these injections are determined by the ρ_{x_t}-values of vector fields in $\mathfrak{h}_{-1}^t$ and the ϱ_{x_t}-values of vector fields in $\mathfrak{h}_{-2}^t$. Thus these injections are preserved by ψ_t because the isomorphism φ_t is defined by ρ_{x_t} on $\mathfrak{h}_{-1}^t$ and ϱ_{x_t} on $\mathfrak{h}_{-2}^t$. This implies that ψ_t is a Lie algebra homomorphism for $t \neq 0$ by Proposition 4.1.3. By continuity, the vector space isomorphism $\psi_0 : \mathfrak{h}^{t_1} \to \mathfrak{h}_0^t$ is also a Lie algebra homomorphism.

The proof for the case of $r = 0$ can be done in the same way using Proposition 4.1.4 and Proposition 4.1.5. □

Corollary 5.3.4. *The orbit of x under* $\exp(\mathfrak{h}^0) \subset Aut(X_0)$ *is isomorphic to* $S_{k,k+r}$*. In particular, if* $r = \ell - k$*,* $X_0 \cong G/P$*.*

Proof. $\mathfrak{h}^0 \cong \mathbf{g} \cong \mathfrak{sp}(k+r)$ and the isotropy subalgebra at x is $\mathfrak{h}^0_{-2} \oplus \mathfrak{h}^0_{-1} \oplus \mathfrak{h}^0_0$ which is conjugate to the parabolic subalgebra of $\mathfrak{sp}(k+r)$ associated to $S_{k,k+r}$. □

(5.4) We want to show that L^0 contains also a $\mathbb{C}^*$-vector field with a component of the zero set isomorphic to a deformation of $S_{2,\ell-k+2}$, by taking the limit of a family of slicing vector fields. Under a choice of the identification (‡) in (5.1), choose a subbundle U' of rank 2 in U and a complementary subbundle U'' of rank $k-2$. Consider the decomposition $L^0 = L^0_{-2} \oplus L^0_{-1} \oplus L^0_0 \oplus L^0_1 \oplus L^0_2$ with respect to a choice of a family of characteristic vector fields E_t along σ. Let $\mathcal{F}^t \subset L^t_0$ be the subspace defined by

$$\mathcal{F}^t = \{v \in L^t_0 : \ \rho_{x_t}(v)|_{U'_{x_t} \otimes Q_{x_t}} \equiv \lambda \cdot id \text{ for some } \lambda \in \mathbb{C}, \rho_{x_t}(v)|_{U''_{x_t} \otimes Q_{x_t}} \equiv 0\}.$$

For $t \neq 0$, $\mathcal{F}^t$ is 1-dimensional subspace generated by the slicing vector field determined by the choice of E_t and $U_{x_t} = U'_{x_t} \oplus U''_{x_t}$ as explained in Proposition 4.3.3. Since $\chi_{x_t} \circ \rho_{x_t} : L^t_0 \to End(\mathcal{D}_{x_t})$ is injective for all t, the limit $\mathcal{F}^0 \subset \mathcal{L}^0_0$ is 1-dimensional and $\{\mathcal{F}^t\}$ form a line bundle over Δ. By choosing a non-vanishing section of this line bundle, we get a vector filed F_t which is the slicing vector field for $t \neq 0$. The 1-parameter subgroup $\{\exp(\lambda F_t), \lambda \in \mathbb{C}\}$ has minimal period 2π for $t \neq 0$ and the same holds for $\{\exp(\lambda F_0), \lambda \in \mathbb{C}\}$ by continuity. Thus F_0 is a $\mathbb{C}^*$-vector field.

Let $\mathcal{X}' \subset \mathcal{X}$ be the submanifold such that the fiber $X'_t = \mathcal{X}' \cap X_t$ is the component of the zero set of F_t containing x_t. Then $\pi' = \pi|_{\mathcal{X}'} : \mathcal{X}' \to \Delta$ is a smooth projective morphism with $X'_t \cong S_{2,\ell-k+2}$ for all $t \neq 0$.

At x_0, the variety of minimal rational tangents $\mathcal{C}'_{x_0} \subset \mathbb{P}T_{x_0}(X'_0)$ is precisely the tangent vectors to minimal rational curves on X_0 through x_0 which are fixed by the isotropy action of $\{\exp(\lambda F_0), \lambda \in \mathbb{C}\}$. Under the identification $T_{x_0}(X_0) = (U_{x_0} \otimes Q_{x_0}) \oplus S^2 U_{x_0}$ induced by (‡),

$$\widetilde{\mathcal{C}'}_{x_0} = \widetilde{\mathcal{C}}_{x_0} \cap [(U'_{x_0} \otimes Q_{x_0}) \oplus S^2 U'_{x_0}].$$

As a submanifold of $\mathcal{C}_{x_0} \cong Z$, $\mathcal{C}'_{x_0}$ is isomorphic to the restriction of the $\mathbb{P}_{2\ell-2k-1}$-bundle Z over $\mathbb{P}_{k-1}$ to a line in $\mathbb{P}_{k-1}$. It is clear that any deformation of such submaifold of Z comes from a deformation of the line in $\mathbb{P}_{k-1}$. It follows that there exists some $B' \subset X'_0$ such that $\mathcal{C}'_y \subset \mathcal{C}_y$ is isomorphic to $\mathcal{C}'_{x_0} \subset \mathcal{C}_{x_0}$ for any $y \in X'_0 - B'$. The natural subbundle $\mathcal{D}' \subset \mathcal{D}|_{\mathcal{X}'-B}$ arising from the invariant distributions on $S_{2,\ell-k+2}$ satisfies

$$\mathcal{D}'_y = \mathcal{D}_y \cap T_y(X'_t) \text{ for } y \in X'_t - B', t \in \Delta.$$

The Frobenius bracket for $\mathcal{D}'$ must be the restriction of the Frobenius bracket for $\mathcal{D}$. Thus under the identifications $\mathcal{D}'_{x_t} \cong U'_{x_t} \otimes Q_{x_t}$ and $T_{x_t}(X'_t)/\mathcal{D}'_{x_t} \cong S^2 U'_{x_t}$, the Frobenius bracket for $\mathcal{D}'$ is $[u \otimes a, v \otimes b] = \nu_{x_t}(a,b)\ u \odot v$ for $u, v \in U'_{x_t}$ and $a, b \in Q_{x_t}$ where ν_{x_t} is the same antisymmetric form on Q_{x_t} as before.

Proposition 5.4.1. *Suppose Theorem 3.1.1 is true for $S = S_{k,\ell}$ with $k = 2$. Then it is true for all $k, 2 \leq k < \ell$.*

Proof. By Corollary 5.3.4, Theorem 3.1.1 is true if and only if the rank $2r$ of ν_{x_0} is $2(\ell-k)$. Taking the subfamily X'_t as above, and applying Theorem 3.1.1 for $k = 2$ to this subfamily, we see that $X'_0 \cong S_{2,\ell-k+2}$ which implies the rank $2r$ of ν_{x_0} is $2[(\ell-k+2)-2] = 2(\ell-k)$. $\square$

§6. Structure of the foliation on the central fiber

(6.1) In Section 6, we will assume $k = 2$. We will extend the isomorphism (‡) in (5.1) to an open neighborhood. There exist a vector bundle U of rank 2 and a vector bundle Q of rank $2\ell - 4$ on $X_0 - B$ where B is as in (5.1) such that $\mathcal{D} \cong U \otimes Q$ and $T^\pi/\mathcal{D} \cong S^2U$ on $X_0 - B$. We will fix such an identification. The Frobenius bracket for $\mathcal{D}$ at $y \in X_0 - B$ is given by $[u \otimes w, v \otimes z] = \nu_y(w, z) u \odot v$ for some anti-symmetric form ν_y on Q_y. We will assume that the rank of ν_y is $2r, r < \ell - 2$ for a general $y \in X_0 - B$. We will get a contradiction from this assumption, which will prove Theorem 3.1.1 by Corollary 5.3.4 and Proposition 5.4.1.

Lemma 6.1.1. *Let $Ker(\nu_x) \subset Q_x$ be the the kernel of the anti-symmetric form ν_x on Q_x at $x \in \Omega$. Then, the meromorphic distribution $U \otimes Ker(\nu)$ is integrable on X_0.*

Proof. For $u \neq 0$, $[u \otimes w, \eta] = 0$ for every $\eta \in U_y \otimes Q_y$ if and only if $\nu_y(w, z) = 0$ for every $z \in Q_y$. In other words, $U \otimes Ker(\nu)$ is the Cauchy characteristic (cf. [HM3, 3.1]) of $\mathcal{D}$, hence integrable. $\square$

Let $\mathcal{P}$ be the meromorphic foliation on X_0 defined by $U \otimes Ker(\nu)$. From now on, assume that $B \subset X_0$ is the smallest subvariety such that the distribution $\mathcal{P}$ is holomorphic on $X_0 - B$, and that the variety of minimal rational tangents $\mathcal{C}_y \subset \mathbb{P}T_y(X_0)$ is isomorphic to the model $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ for each $y \in X_0 - B$. To start with, we will show that the leaves of $\mathcal{P}$ are quasi-projective. More precisely, we have

Lemma 6.1.2. *For any $y \in X_0 - B$ the leaf of $\mathcal{P}|_{X_0 - B}$ passing through y is closed in $X_0 - B$, and its topological closure in X_0 is a projective subvariety of X_0.*

Proof. On X_0 a minimal rational curve C not lying on B and tangent to $\mathcal{P}$ on $C - B$ will be called a $\mathcal{P}$-minimal rational curve. By Proposition 5.2.1, at each point $y \in X_0 - B$, there exists a characteristic vector field E_y which is the limit of characteristic vector fields on $X_t, t \neq 0$. From (4.3) one can see that the orbital curves of a characteristic vector field at $o \in S$ are rational curves and those tangent to $\mathcal{C}_o$ are minimal rational curves. It follows that the orbital curves of E_y through y are rational curves and by the invariance of $\mathcal{P}$ under $\{\exp(\lambda E_y), \lambda \in \mathbb{C}\}$, those tangent to $\mathcal{C}_y \cap \mathbb{P}\mathcal{P}_y$ are $\mathcal{P}$-minimal rational curves. Thus the tangent vectors to $\mathcal{P}$-minimal rational curves through y are precisely the decomposable tensors in $\mathcal{P}_y = U_y \otimes Ker(\nu_y)$, which is exactly $\mathcal{C}_y \cap \mathbb{P}\mathcal{P}_y$.

Pick $y \in X_0 - B$ and denote by $\Lambda(y)$ the local leaf of $\mathcal{P}$ passing through y. Starting with y we construct projective varieties $\{y\} = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots$ inductively, as follows. Let $\mathcal{V}_1$ be the union of $\mathcal{P}$-minimal rational curves passing thorough y. Having constructed $\mathcal{V}_k$, we define

$$\mathcal{V}_{k+1} := \text{ closure of } \bigcup_{C:\mathcal{P}\text{-minimal rational curve, } C\cap(\mathcal{V}_k - B)\neq\emptyset} C.$$

$\mathcal{C}_y \cap \mathbb{P}\mathcal{P}_y$ is irreducible and linearly non-degenerate in $\mathbb{P}\mathcal{P}_y$. It follows by induction that each $\mathcal{V}_k$ is irreducible. Since $\mathcal{P}$ is integrable $\mathcal{V}_k - B$ is contained in $\Lambda(y)$ for all k. There is some positive integer N such that $\mathcal{V}_{N+1} = \mathcal{V}_N$. We claim that $\mathcal{V}_N$ contains $\Lambda(y)$ as an open set, from which Lemma 6.1.2 follows. Otherwise $dim(\mathcal{V}_N) < dim(\Lambda(y))$. For a smooth point $w \in \mathcal{V}_N - B$, the union of vectors tangent to $\mathcal{P}$-minimal rational curves span $\mathcal{P}_w$ linearly. Hence, there exists a $\mathcal{P}$-minimal rational curve C through w lying outside $\mathcal{V}_N$, and $\mathcal{V}_N \subsetneq \mathcal{V}_{N+1}$, a contradiction. $\square$

For simplicity, set $a = \ell - 2 - r$ so that the rank of $\mathcal{P}$ is $4(\ell - 2 - r) = 4a$. For a given $x \in X_0 - B$, let $N = \exp(\mathfrak{h}^0)\cdot x \cong S_{2,r+2}$ be the submanifold through x as defined in Corollary 5.3.4. Note $rank(\mathcal{P}) + \dim S_{2,r+2} = \dim S_{2,\ell}$. By the action of the group $\exp(\mathfrak{h}^0) \subset Aut(X_0)$, we see that the foliation $\mathcal{P}$ is defined everywhere in a neighborhood of N and is transversal to N. From Lemma 6.1.2 we can produce a holomorphic fiber bundle, as follows.

Lemma 6.1.3. *For each $w \in N$ denote by V_w the leaf of $\mathcal{P}|_{X_0-B}$ passing through w, and by $P_w \in X_0$ the projective variety which is the topological closure of V_w in X_0. Then, there is a locally trivial holomorphic fiber bundle $\mu : Y \to N$, together with a canonical birational morphism $f : Y \to X_0$, such that f maps any fiber $\mu^{-1}(w)$ biholomorphically onto $P_w \subset X_0$.*

Proof. Write $\Theta = \exp(\mathfrak{h}^0)$ and $\Phi = \exp(\mathfrak{h}^0_{-2} \oplus \mathfrak{h}^0_{-1} \oplus \mathfrak{h}^0_0)$ in the notation of (5.3). Then Θ is isogenous to $Sp(r+2)$, $\Phi \subset \Theta$ is maximal parabolic, and $N = \Theta(x) \cong \Theta/\Phi$. Θ acts transitively on the space $\{[P_w] : w \in N\}$ of such projective subvarieties, with isotropy at the point $[P_x]$ given by $\Phi \subset \Theta$. Define $\mathcal{R} = \{(y,\theta) \in X_0 \times \Theta : y \in \theta(P_x)\}$. Θ acts on $\mathcal{R}$ by $\theta(y,\psi) = (\theta(y), \theta \circ \psi)$. The quotient $Y := \mathcal{R}/\Phi$ admits a canonical projection onto $N = \Theta/\Phi$, giving a locally trivial holomorphic fiber bundle $\mu : Y \to N$ whose fiber over $w \in N$ corresponds to P_w. Furthermore, there is a canonical map $f : Y \to X_0$ defined by the projection $X_0 \times \Theta \to X_0$ which is a birational morphism. $\square$

Note that blowing-down in $f : Y \to X_0$ occurs precisely when points of X_0 lies on a positive-dimensional algebraic family of leaves P_w. Since X_0 is of Picard number 1, this shows

Lemma 6.1.4. *For any $w \in N$, the leaf V_w cannot contain a complete curve.*

Since X_0 is of Picard number 1 and the projective submanifold $N \subset X_0$, $N \cong S_{2,r+2}$ is disjoint from B, we have proven

Lemma 6.1.5. *The set B is of codimension ≥ 2 in X_0.*

(6.2) Let us study the leaf V_x more carefully. It is a quasi-projective complex manifold whose tangent bundle can be written as the tensor product of a vector bundle U of rank 2 and a vector bundle $Ker(\nu)$ of rank $2a$. This means that it has the G-structure modelled on the Grassmannian. Let us recall some basic facts about the geometry of the Grassmannian.

Let $\mathbb{V}$ be a vector space and $\mathbb{G}(2, \mathbb{V})$ be the Grassmannian of 2-planes in $\mathbb{V}$. For a point $[\mathbb{W}] \in \mathbb{G}(2, \mathbb{V})$ corresponding to a 2-plane $\mathbb{W} \subset \mathbb{V}$, an **Euler vector field** at $[\mathbb{W}]$ is a vector field inducing a $\mathbb{C}^*$-action on $\mathbb{G}(2, \mathbb{V})$ with an isolated fixed point at $[\mathbb{W}]$ such that the isotropy action on $T_{[\mathbb{W}]}(\mathbb{G}(2, \mathbb{V}))$ is the identity. This is an analogue of the characteristic vector field in (4.3). A choice of an Euler vector field at $[\mathbb{W}]$ corresponds to a choice of a complementary subspace $\mathbb{U} \subset \mathbb{V}, \mathbb{W} \oplus \mathbb{U} = \mathbb{V}$, which is the eigenspace decomposition of the induced $\mathbb{C}^*$-action on $\mathbb{V}$. Recall that a choice of $\mathbb{U}$ determines an affine open set $\mathcal{U} \subset \mathbb{G}(2, \mathbb{V})$ consisting of 2-planes transversal to $\mathbb{U}$. This $\mathcal{U}$ has a natural vector space structure with the origin at $[\mathbb{W}]$, canonically isomorphic to $\mathbb{W}^* \otimes \mathbb{U}$. An open subset of this type will be called a **standard open set**. This way, a choice of an Euler vector field on the Grassmannian corresponds to a choice of a standard open set and a point on the standard open set.

On a complex manifold M, a vector bundle $\mathcal{W}$ of rank 2, another vector bundle $\mathcal{Q}$ of rank $2a$ and a fixed isomorphism of $\mathcal{W} \otimes \mathcal{Q}$ with the tangent bundle $T(M)$ is called a **G-structure on M modelled on $\mathbb{G}(2, \mathbb{C}^{2+2a})$**. Clearly, the Grassmannian itself has such a G-structure where $\mathcal{W}$ and $\mathcal{Q}$ correspond to the dual universal bundle and the dual universal quotient bundle. The G-structure on M is said to be **flat** if there exists an unramified holomorphic map $\delta : M \to \mathbb{G}(2, \mathbb{C}^{2+2a})$, called a **developing map** such that the vector bundles $\mathcal{W}$, $\mathcal{Q}$ and the isomorphism $\mathcal{W} \otimes \mathcal{Q} \cong T(M)$ are pull-backs of those on the Grassmannian. Given such a G-structure on M, there exist canonically defined sections of $S^j T^*(M) \otimes T(M) \otimes \Lambda^2 T^*(M), j \geq 0$, called curvature tensors ([HM1, Proposition 2]) such that the G-structure is flat if and only if the curvature tensors vanish. For the Grassmannian, the following lemma is well-known.

Lemma 6.2.1. *For a standard open set $\mathcal{U} \subset \mathbb{G}(2, \mathbb{C}^{2+2a})$, any biholomorphic automorphism of $\mathcal{U}$ preserving the G-structure is an affine transformation of $\mathcal{U}$.*

We have the following result for the G-structure on the leaves of $\mathcal{P}$.

Lemma 6.2.2. *The G-structure modelled on $\mathbb{G}(2, \mathbb{C}^{2+2a})$ on V_x is flat. In other words, there exists a developing map $\delta : V_x \to \mathbb{G}(2, \mathbb{C}^{2+2a})$, which is an unramified holomorphic map such that the G-structure on V_x is the pull-back of that of $\mathbb{G}(2, \mathbb{C}^{2+2a})$ by δ.*

Proof. By Proposition 5.2.1, for any point $y \in V_x$, there exists a $\mathbb{C}^*$-action on V_x preserving the G-structure with an isolated fixed point at y, which acts on the tangent space $T_x(V_x)$ by homothety. The curvature tensors of the G-structure are holomorphic sections of $S^j T^*(V_x) \otimes T(V_x) \otimes \Lambda^2 T^*(V_x), j \geq 0$. Thus under the $\mathbb{C}^*$-action the curvature tensors must be multiplied by nontrivial scalars at y. Since the curvature tensors are invariants of the G-structure, this implies that the curvature tensors vanish. $\square$

(6.3) Let us study the compactified leaves P_w of $\mathcal{P}$ more precisely. Our first aim is to prove that it is smooth by realizing it as a fixed point set of a $\mathbb{C}^*$-action in $Aut(X_0)$. Since $\mathcal{P}$ is invariant under the group $\exp(L^0)$ of automorphisms, germs of leaves $\Lambda(y)$ at $y \in N$ are mapped to each other under elements of $\exp(L^0)$. In particular, the $\mathbb{C}^*$-action $\exp(\mathbb{C}E_w)$ defined by a characteristic vector field E_w at $w \in N$ must preserve the germ of the leaf through w. We claim

Lemma 6.3.1. *Denote by $\mathcal{V}$ the open subset $\mu^{-1}(X_0 - B)$ in Y. Then $\mu|_{\mathcal{V}} : \mathcal{V} \to N$ carries the structure of a holomorphic vector bundle of rank $4a$, with fibers again denoted by V_w. Furthermore, the fibers R_w of $\mu : Y \to N$ over $w \in N$ are smooth compactifications of the vector spaces V_w.*

Proof. Under the developing map $\delta : V_w \to \mathbb{G}(2, \mathbb{C}^{2+2a})$, a characteristic vector field E_y at $y \in V_w$ in the sense of (5.2) is sent to one of the Euler vector fields of $\mathbb{G}(2, \mathbb{C}^{2+2a})$. Since orbits of the $\mathbb{C}^*$-action $\exp(\mathbb{C}E_y)$ are sent to orbits of the Euler vector field, the image of V_w must cover a standard open set of $\mathbb{G}(2, \mathbb{C}^{2+2a})$. Moreover δ sends open pieces of $\mathcal{P}$-minimal rational curves on V_w to open pieces of minimal rational curves on $\mathbb{G}(2, \mathbb{C}^{2+2a})$. This implies that δ defines a rational map $P_w \dashrightarrow \mathbb{G}(2, \mathbb{C}^{2+2a})$ by [HM4, Proposition 4.3]. Thus $\delta(V_w)$ is a constructible set containing a standard open set $\mathcal{U}$. Suppose $\delta(V_w)$ contains a point outside $\mathcal{U}$. Then the complement of $\delta(V_w)$ is of codimension ≥ 2 in $\mathbb{G}(2, \mathbb{C}^{2+2a})$ which implies that V_w contains a complete curve, a contradiction to Lemma 6.1.4. Thus the image of V_w is exactly one standard open set. Since a standard open set is simply connected, the developing map $\delta : V_w \to \mathcal{U}$ is biholomorphic. By δ, V_w has a natural vector space structure inherited from $\mathcal{U}$ with center $\delta(w)$. This vector space structure is uniquely determined independent of the choice of the developing map because of Lemma 6.2.1.

To prove that $P_w \subset X_0$ is smooth it suffices to identify it as an irreducible component of the zero set of some $\mathbb{C}^*$-vector field on X_0. From the structure of $\mu|_{\mathcal{V}} : \mathcal{V} \to N$ as a holomorphic vector bundle there is a $\mathbb{C}^*$-action on $\mathcal{V}$ corresponding to multiplication by $\mathbb{C}^*$ on the fibers. Write E' for an infinitesimal generator of this $\mathbb{C}^*$-action so that for a constant vector field u on $V_x, x \in N$, we have $[E'|_{V_x}, u] = u$. The holomorphic vector field E' is defined on $\mathcal{V} \cong X_0 - B$, and must therefore descend to a vector field on X_0 by Hartogs' extension, since $B \subset X_0$ is of codimension ≥ 2, by Lemma 6.1.5. Define now $E^\sharp = E_w - E'$ where E_w is a characteristic vector field at w in the sense of (5.2). Then, $E^\sharp$ vanishes at w, and the first-order term $\rho_w(E^\sharp)$ is semi-simple with 0-eigenspace $\mathcal{P}_w = U_w \otimes Ker(\nu_w)$. Thus $P_w \subset X_0$ is an irreducible component of the zero set of $E^\sharp$.

It remains therefore to show that $E^\sharp = E_w - E'$ generates a $\mathbb{C}^*$-action. Since both $\{\exp(\lambda E_w), \lambda \in \mathbb{C}\}$ and $\{\exp(\lambda E'), \lambda \in \mathbb{C}\}$ have minimal period 2π, it is sufficient to show that the two vector fields commute. But $\exp(\mathbb{C}E_w)$ respects the vector bundle structure and $\exp(\mathbb{C}E')$ is the scalar multiplication. So they must commute.

Since the above holds for any $w \in N$, we have proven that $\mu : Y \to N$ is a holomorphic fiber bundle with isomorphic smooth fibers R_w which are smooth compactifications of $V_w \cong \mathbb{C}^{4a}$. $\square$

Lemma 6.3.2. *For each $w \in N$, the compactified leaf $P_w \subset X_0$ of $\mathcal{P}$ passing through w is biholomorphic to the Grassmannian $\mathbb{G}(2, \mathbb{C}^{2+2a})$ of 2-planes in $\mathbb{C}^{2+2a}$, such that $V_w \subset P_w$ agrees with a standard open subset of $\mathbb{G}(2, \mathbb{C}^{2+2a})$.*

Proof. We know that the developing map defines a birational map $\delta : P_w \dashrightarrow \mathbb{G}(2, \mathbb{C}^{2+2a})$ sending V_w to a standard open set $\mathcal{U}$ of the Grassmannain and open pieces of minimal rational curves on V_w to open pieces of minimal rational curves of the Grassmannian. Write $\mathcal{J} = \mathbb{G}(2, \mathbb{C}^{2+2a}) - \mathcal{U}$ resp. $\mathcal{H} = P_w - V_w$ for the divisors at infinity. $\mathcal{J}$ is irreducible but $\mathcal{H}$ may be reducible. Let $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_q$ be the decomposition of $\mathcal{H}$ into its irreducible components.

We claim that the strict image $\delta^{-1}(\mathcal{J})$ must be an irreducible component of $\mathcal{H}$, say $\mathcal{H}_0$. Suppose not. Then $\delta^{-1}(\mathcal{H})$ is of codimension ≥ 2 in P_w. All minimal rational curves of the Grassmannian intersect $\mathcal{J}$. Thus a general minimal rational curve of P_w must intersect $\delta^{-1}(\mathcal{J})$. But a general minimal rational curve on the smooth variety P_w can be deformed to be disjoint from any given set of codimension ≥ 2 (c.f. [HM2, proof of Proposition 12]). This is a contradiction.

By the above claim, there exists a subvariety $A \subset \mathcal{J}$, A of codimension ≥ 2 in $\mathbb{G}(2, \mathbb{C}^{2+2a})$, such that δ^{-1} is a holomorphic map outside A and sends $\mathcal{J} - A$ dominantly over $\mathcal{H}_0$. Thus, δ^{-1} gives a biholomorphism $\mathbb{G}(2, \mathbb{C}^{2+2a}) - A \cong P_w - (\mathcal{H}_1 \cup \cdots \cup \mathcal{H}_q) - \mathcal{I}$ for some proper subvariety $\mathcal{I} \subset \mathcal{H}_0, \mathcal{I} \not\subset (\mathcal{H}_1 \cup \cdots \cup \mathcal{H}_q)$. From Hartogs' extension for the G-structure modelled on the Grassmannian ([HM2, Proposition 1]), δ can be extended across general points of $\mathcal{I}$ as an unramified holomorphic map. Thus we may assume that $\mathcal{I} = \emptyset$.

Suppose $A \neq \emptyset$. Choose a family of minimal rational curves $\{l_s : s \in \Delta\}$ on $\mathbb{G}(2, \mathbb{C}^{2+2a})$ such that l_0 intersects A but is not contained in A; all l_s with $s \neq 0$ are disjoint from A and are the strict images of a family of minimal rational curves C_s on P_w. Then the limit C_0 is an irreducible curve because C_0 has degree 1 with respect to the ample line bundle on X_0. Thus the strict image of C_0 must be l_0, implying $C_0 \cap (\mathcal{H}_1 \cup \cdots \cup \mathcal{H}_q) \neq \emptyset$. So $C_0 \cdot (\mathcal{H}_1 \cup \cdots \cup \mathcal{H}_q) > 0$. But $C_s, s \neq 0$, is disjoint from $\mathcal{H}_1 \cup \cdots \cup \mathcal{H}_q$, a contradiction. We conclude that $A = \emptyset$ and $P_w \cong \mathbb{G}(2, \mathbb{C}^{2+2a})$. $\square$

Summarizing, we have proved

Proposition 6.3.3. *Let $\pi : \mathcal{X} \to \Delta$ be a regular family of projective manifolds, $X_t := \pi^{-1}(t)$, such that $X_t \cong S = S_{2,\ell}$ for $t \neq 0$. Suppose the antisymmetric form ν on X_0 as in (5.3) has rank $2r, 1 \leq r < \ell - 2$ at general points of X_0. Then,*

1. *there exists a subvariety $B \subset X_0$ of codimension ≥ 2 and a projective submanifold $N \subset X_0 - B$ biholomorphic to the symplectic Grassmannian $S_{2,2+r}$;*

2. *$X_0 - B$ can be realized as the total space of a holomorphic vector bundle $\mu^o : \mathcal{V} \to N$ of rank $4a, a = \ell - r - 2$, with $N \subset X_0 - B$ identified as the zero section;*
3. *$\mu^o : \mathcal{V} \to N$ can be compactified to a holomorphic fiber bundle $\mu : Y \to N \cong S_{2,r+2}$ of Grassmannians $\mathbb{G}(2, \mathbb{C}^{2+2a})$ equipped with a birational morphism $f : Y \to X_0$, which sends each fiber Y_w of μ biholomorphically to P_w.*

(6.4) Using the vector bundle structure on $X_0 - B$ in Proposition 6.3.3, we will find a special submanifold in X_0, which will lead to the contradiction we want.

We need to recall the geometry of $\mathbb{G}(2, \mathbb{V})$ when $\dim \mathbb{V} = 4$. Let $\mathcal{U} \subset \mathbb{G}(2, \mathbb{V})$ be a standard open set determined by a choice of a subspace $\mathbb{U}$ of codimension 2. Since $\dim \mathbb{V} = 4$, $\mathbb{U}$ itself is a point of $\mathbb{G}(2, \mathbb{V})$. It is easy to check that the point $[\mathbb{U}]$ is the unique singular point of the hypersurface $\mathbb{G}(2, \mathbb{V}) - \mathcal{U}$. Now suppose we are given a point $[\mathbb{W}] \in \mathcal{U}$. The isotropy subgroup $P_{[\mathbb{W}]}$ of $PGL(\mathbb{V})$ has a Levi factor isogenous to $P(GL(\mathbb{W}) \times GL(\mathbb{U}))$. In fact, a choice of the standard open set containing $[\mathbb{W}]$ determines a Levi factor of $P_{[\mathbb{W}]}$. From this description, the following is obvious.

Lemma 6.4.1. *For a standard open set $\mathcal{U}$ and a point $[\mathbb{W}] \in \mathcal{U}$ of the 4-dimensional Grassmannian $\mathbb{G}(2, \mathbb{V}), \dim V = 4$, a Levi factor of the isotropy subgroup $P_{[\mathbb{W}]}$ preserving $\mathcal{U}$ is also a Levi factor of the isotropy subgroup at the unique singular point of the hypersurface $\mathbb{G}(2, \mathbb{V}) - \mathcal{U}$.*

Proposition 6.4.2. *In the notation of Proposition 6.3.3, there is a complex submanifold $X' \subset X_0$, with the following properties:*

(1) *$X' - B \subset X_0 - B \cong \mathcal{V}$ is canonically identified with the total space of a holomorphic vector subbundle $\mathcal{V}' \subset \mathcal{V}$,*

(2) *the compactification Y' of $f^{-1}(X' - B) \cong f^{-1}(\mathcal{V}')$ is the total space $\mu' : Y' \to N$ of a bundle of Grassmannians $\mathbb{G}(2, \mathbb{C}^4)$ for $\mu' = \mu|_{Y'}$; where each fiber $V'_w \subset Y'_w$ is a standard neighborhood of $\mathbb{G}(2, \mathbb{C}^4)$.*

(3) *the holomorphic map $f' := f|_{Y'} : Y' \to X'$ is a modification of the projective manifold X', such that, writing $\Gamma \subset Y' - \mathcal{V}'$ for the section of μ' consisting of the unique isolated singularity Γ_w of $Y'_w - V'_w$, $f'(\Gamma)$ is a single point $\iota \in X'$.*

Proof. Note $\mathcal{V} \cong U \otimes \Xi$ as a homogeneous vector bundle on N under the action of $\exp(\mathfrak{h}^0)$ where Ξ is the vector subbundle of $Q|_N$ whose fiber $\Xi_w \subset Q_w$ is the kernel of ν_w at $w \in N$. Since the isotropy subgroup $\exp(\mathfrak{h}^0_{-2} \oplus \mathfrak{h}^0_{-1} \oplus \mathfrak{h}^0_0)$ at w acts trivially on Ξ_w from the definition of $\mathfrak{h}^0_0$ in (5.3), Ξ is a trivial vector bundle on N. Let now $\Xi'_w \subset \Xi_w$ be any choice of a 2-dimensional vector subspace, and $\Xi' \subset \Xi$ be the corresponding trivial holomorphic vector subbundle. Define $\mathcal{V}' := U \otimes \Xi'$ as a subbundle of $\mathcal{V}$. For any fiber V'_w over a point $w \in N$, $\mathbb{P}(V'_w) \cong \mathbb{P}(U_w \otimes \Xi'_w) \cong \mathbb{P}^1 \times \mathbb{P}^1$, so that the closure of V'_w in Y_w is biholomorphic to $\mathbb{G}(2, \mathbb{C}^4)$, giving a holomorphic fiber bundle $\mu|_{Y'} : Y' \to N$ of Grassmannians $\mathbb{G}(2, \mathbb{C}^4)$. Define now $X' = f(Y')$.

To show that $X' \subset X_0$ is smooth, we will find a $\mathbb{C}^*$-action on X_0 with X' as a component of the fixed point set. Since Ξ is a trivial holomorphic vector bundle, we can identify Ξ with $\Xi_w \times N$. Choose a trivial subbundle $\Xi'' \subset \Xi$ complementary to Ξ', i.e., $\Xi_w = \Xi'_w \oplus \Xi''_w$. For $\lambda \in \mathbb{C}^*$ we have automorphisms ψ_λ of the vector bundle $\mu^o : \mathcal{V} \to N$ corresponding to assigning $(q', q'') \in \Xi' \oplus \Xi''$ to $(q', \lambda q'')$. This gives a $\mathbb{C}^*$-action on $\mathcal{V}$ which is the homothety on $U_w \otimes \Xi''_w$ and fixes $V'_w = U_w \otimes \Xi'_w$ at each $w \in N$. The corresponding vector field descends to $X_0 - B$ to give a holomorphic vector field $E^\natural$ extendible to X_0 by Hartogs. Then $E^\natural$ generates a $\mathbb{C}^*$-action whose fixed point set contains X' as an irreducible component.

Properties (1) and (2) are immediate from the definition of X'. For (3), note first of all that $f' : Y' \to X'$ must contract the hypersurface $Y' - \mathcal{V}'$ because $X' \cap B$ is of codimension ≥ 2 in X' as in Lemma 6.1.5. Moreover the scalar multiplication on the vector bundle $\mathcal{V}'$ induces $\mathbb{C}^*$-actions on Y' and X' with respect to which f' is equivariant. The restriction of this $\mathbb{C}^*$-action on each fiber Y'_w of μ' is equivalent to the one induced by an Euler vector field on $\mathbb{G}(2, \mathbb{C}^4)$ and has an isolated fixed point at Γ'_w. As noted in (6.1), blowing-down in $f' : Y' \to X'$ occurs precisely when points of X' lie on a family of cycles $\{f'(Y'_{w_s})\}$ for some holomorphic arc $\{w_s \in N : s \in \Delta\}$. Thus $f'(\Gamma_{w_0}) \in P'_{w_s} := f'(Y'_{w_s})$ for all $s \in \Delta$, implying that $f'(\Gamma_{w_0})$ must be a fixed point of P'_{w_s} for all $s \in \Delta$. Since $\Gamma_w \in Y'_w - V'_w$ is an isolated fixed point of the $\mathbb{C}^*$-action, $f'(\Gamma_{w_0}) = f'(\Gamma_{w_s})$ for small s. Thus $f'|_\Gamma$ has positive dimensional fibers. But $\Gamma \cong S_{2,r+2}$ has Picard number 1, so the image must be one point. $\square$

End of Proof of Theorem 3.1.1. We are ready to get a contradiction. First note that $dim(N) = 4r + 3$, $dim(Y') = dim(X') = dim(N) + dim(\mathbb{G}(2, \mathbb{C}^4)) = 4r + 7$.

We proceed to consider isotropy representations at $\iota \in X'$. $Aut(N)$ embeds as a Lie subgroup G' of $Aut(X')$. G' is isogenous to $Sp(r+2)$ and acts canonically and holomorphically on Y'. G' preserves the vector bundle $\mu|_{\mathcal{V}'} : \mathcal{V}' \to N$ as a group of bundle maps. Hence it preserves the total space of $\mu' : Y' \to N$. As a consequence, G' preserves the section Γ of singular points of $Y' - \mathcal{V}'$. By Proposition 6.4.2, $f(\Gamma) = \iota$, so that the point $\iota \in X'$ is fixed under the action of G' on X'. As a result, we obtain an isotropy action of G' on $T_\iota(X') \cong \mathbb{C}^{4r+7}$.

Next we will augment the group of the isotropy action at $T_\iota(X')$. Since $\mathcal{V}' \cong U \otimes \Xi'$ as vector bundles on N and Ξ' is a trivial vector bundle of rank 2 on N, $SL(2) \cong SL(\Xi'_z)$ for a base point $z \in N$ acts on $\mathcal{V}'$ as vector bundle automorphisms. This action can be extended to Y' and hence descends to an action on X' fixing ι. The latter action clearly commutes with the action of G', giving an action of $SL(\Xi'_z) \times G'$ on X'.

Both $SL(\Xi'_z)$ and G' act non-trivially on $T_\iota(X')$ by [BB, Lemma 2.4]. Thus we have a representation of $SL(2) \times Sp(r+2)$, which is isogenous to $SL(\Xi'_z) \times G'$, on $T_\iota(X')$ where both factors act non-trivially. But any irreducible $SL(2) \times Sp(r+2)$-representation space where both factors act non-trivially is a tensor product $R_1 \otimes R_2$ where R_1 is a nontrivial irreducible $SL(2)$-representation space and R_2 is a nontrivial irreducible $Sp(r+2)$-representation space. We have $\dim R_1 \geq 2$ and $\dim R_2 \geq 2r+4$, so that $\dim(R_1 \otimes R_2) \geq 4r+8$. But this contradicts with the fact that $T_\iota(X')$ is of dimension $4r+7$. The proof of Theorem 3.1.1 is complete. $\square$

Chapter III. Rigidity of the homogeneous space of type (F_4, α_1)

§7. Geometry of the homogeneous space of type (F_4, α_1)

(7.1) In Chapter III, S denotes the 15-dimensional F_4-homogeneous space associated to the short simple root α_1 where the numbering of the short roots $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ of F_4 is chosen such that α_1 and α_2 are short and the highest long root is of the form $2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4$.

Let Υ be the 16-dimensional E_6-Hermitian symmetric space of compact type. Regard Υ as a submanifold of $\mathbb{P}_{26}$ by the minimal E_6-equivariant embedding. The semi-simple part of the isotropy subalgebra of E_6 at a base point $o \in \Upsilon$ is $\mathfrak{so}(10)$ and the isotropy representation is the 16-dimensional spin representation. Υ is covered by lines and the variety of minimal rational tangents of Υ at a base point $o \in \Upsilon$ corresponds to the 10-dimensional spinor variety $\mathbb{S}_5 \subset \mathbb{P}_{15}$, which is the highest weight orbit of the spin representation of $\mathfrak{so}(10)$. Note that all smooth (resp. singular) hyperplane sections of $\mathbb{S}_5 \subset \mathbb{P}T_o(\Upsilon)$ are conjugate under the action of $Spin(10)$ because the number of orbits of the isotropy representation of $Spin(10)$ on $\mathbb{P}T_o^*(\Upsilon)$ is 2, the rank of the symmetric space Υ ([Ts, Lemma 2.4]).

It is well-known that S is a smooth hyperplane section of Υ (e.g. [Za, p.59]). Minimal rational curves on S are lines of Υ that lie on the hyperplane. Thus the variety of minimal rational tangents $\mathcal{C}_o$ at a base point $o \in S$ is a smooth hyperplane section of $\mathbb{S}_5$. Note that $c_1(\mathbb{S}_5) = 8$ and $\mathbb{S}_5$ is a 10-dimensional Fano manifold of coindex 3 in the sense of [Mu]. Thus $\mathcal{C}_o$ is a 9-dimensional Fano manifold of coindex 3.

Proposition 7.1.1. *Given any projective and smooth morphism $\mathcal{M} \to \Delta$ from a complex manifold $\mathcal{M}$ to the unit disc, if one of the fiber is biholomorphic to $\mathcal{C}_o$, then any other fiber is also biholomorphic to $\mathcal{C}_o$. In particular, $H^1(\mathcal{C}_o, T(\mathcal{C}_o)) = 0$.*

Proof. By [Mu], any projective and smooth deformation of $\mathcal{C}_o$ must be a hyperplane section of $\mathbb{S}_5$ and so biholomorphic to $\mathcal{C}_o$ because all smooth hyperplane sections of $\mathbb{S}_5$ are conjugate to each other. The vanishing of $H^1(\mathcal{C}_o, T(\mathcal{C}_o))$ follows from the fact that deformations of Fano manifolds are unobstructed because $H^2(\mathcal{C}_o, T(\mathcal{C}_o)) = H^{n-2}(\mathcal{C}_o, \Omega^1 \otimes K_{\mathcal{C}_o}) = 0$ by Kodaira-Nakano vanishing. $\square$

Proposition 7.1.2. *$\mathcal{C}_o \subset \mathbb{P}T_o(S)$ is linearly normal and the variety of tangential lines to $\mathcal{C}_o$ is non-degenerate in $\mathbb{P}\Lambda^2 T_o(S)$.*

Proof. The linear normality is a consequence of Zak's linear normality theorem [Za] because $\mathcal{C}_o \subset \mathbb{P}T_o(S) \cong \mathbb{P}_{14}$ and $\dim \mathcal{C}_o = 9$. The second statement follows from [HM3, Proof of Proposition 1.3.2], which is a consequence of Zak's theorem on tangencies. $\square$

Proposition 7.1.3. *The Lie algebra of infinitesimal linear automorphisms $aut(\widetilde{\mathcal{C}_o})$ has dimension 31.*

Proof. Since all infinitesimal linear automorphisms of $\mathcal{C}_o$ induce linear transformations of $T_o(S)$, it suffices to show $h^0(\mathcal{C}_o, T(\mathcal{C}_o)) = 30$. Consider the exact sequence associated to the realization of $\mathcal{C}_o$ as a hyperplane section of $\mathbb{S}_5$

$$0 \longrightarrow H^0(\mathcal{C}_o, T(\mathcal{C}_o)) \longrightarrow H^0(\mathcal{C}_o, T(\mathbb{S}_5)|_{\mathcal{C}_o}) \longrightarrow H^0(\mathcal{C}_o, \mathcal{O}(1)) \longrightarrow H^1(\mathcal{C}_o, T(\mathcal{C}_o)).$$

Note $H^1(\mathcal{C}_o, T(\mathcal{C}_o)) = 0$ from Proposition 7.1.1 and $h^0(\mathcal{C}_o, \mathcal{O}(1)) = 15$ because $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ is linearly normal by Proposition 7.1.2. Thus it suffices to show $h^0(\mathcal{C}_o, T(\mathbb{S}_5)|_{\mathcal{C}_o}) = 45$. Using the long exact sequence associated to the short exact sequence on $\mathbb{S}_5$

$$0 \longrightarrow T(\mathbb{S}_5) \otimes \mathcal{O}(-1) \longrightarrow T(\mathbb{S}_5) \longrightarrow T(\mathbb{S}_5)|_{\mathcal{C}_o} \longrightarrow 0$$

and the vanishing

$$H^0(\mathbb{S}_5, T(\mathbb{S}_5) \otimes \mathcal{O}(-1)) = H^0(\mathbb{S}_5, \Omega^9(7)) = 0$$

$$H^1(\mathbb{S}_5, T(\mathbb{S}_5) \otimes \mathcal{O}(-1)) = H^1(\mathbb{S}_5, \Omega^9(7)) = 0$$

from [Sn, Theorem 3.4], we get $h^0(\mathcal{C}_o, T(\mathbb{S}_5)) = h^0(\mathbb{S}_5, T(\mathbb{S}_5)) = \dim \mathfrak{so}(10) = 45$. $\square$

(7.2) The grading on the Lie algebra $\mathfrak{g}$ of type F_4 induced by the parabolic subalgebra $\mathfrak{p}$ associated to the simple root α_1 is

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

with

$$\dim \mathfrak{g}_2 = \dim \mathfrak{g}_{-2} = 7$$

$$\dim \mathfrak{g}_1 = \dim \mathfrak{g}_{-1} = 8$$

$$\dim \mathfrak{p} = \dim(\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0) = 37.$$

By the identification $T_o(S) = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, the subspace $\mathfrak{g}_1$ defines a distribution D on $S = G/P, G = \exp \mathfrak{g}, P = \exp(\mathfrak{p})$. Proposition 4.1.1, Proposition 4.3.1 and Corollary 4.3.2 hold for S.

$\mathfrak{g}_0$ is reductive with semi-simple part isomorphic to $\mathfrak{so}(7)$. As $\mathfrak{g}_0$-modules, $\mathfrak{g}_1$ is isomorphic to the spin representation of $\mathfrak{so}(7)$ and $\mathfrak{g}_2$ is isomorphic to the basic representation of $\mathfrak{so}(7)$. Thus there are exactly two orbits on $\mathbb{P}\mathfrak{g}_1$ (resp. $\mathbb{P}\mathfrak{g}_2$) consisting of a smooth hyperquadric Q_o (resp. R_o) of dimension 6 (resp. 5) and its complement. The variety of minimal rational tangents $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ cannot contain $\mathfrak{g}_1$ or $\mathfrak{g}_2$. Thus

$$\mathcal{C}_o \cap \mathbb{P}\mathfrak{g}_1 = Q_o$$

$$\mathcal{C}_o \cap \mathbb{P}\mathfrak{g}_2 = R_o.$$

A **conformal structure** on a vector space V means a choice of a smooth hyperquadric in $\mathbb{P}V$. Denote by $\mathfrak{co}(V)$ the Lie algebra of infinitesimal linear automorphisms of the affine cone of the smooth hyperquadric. $\mathfrak{g}_2$ has a conformal structure given by $R_o \subset \mathbb{P}\mathfrak{g}_2$. It is easy to check that $\dim \mathfrak{co}(\mathfrak{g}_2) = 22$. $aut(\widetilde{\mathcal{C}}_o)$ preserves $D_o \subset T_o(S)$ inducing a homomorphism

$$\chi_o : aut(\widetilde{\mathcal{C}}_o) \to End(T_o(S)/D_o).$$

Proposition 7.2.1. *Let $\mathfrak{n}_o$ be the nil-radical of $Ker(\chi_o)$. Then $Ker(\chi_o)/\mathfrak{n}_o$ is 1-dimensional and represented by homotheties on D_o. By the homomorphism $\rho_o : \mathfrak{g}_{-1} \to aut(\widetilde{\mathcal{C}}_o)$, arising from Proposition 4.1.1, $\mathfrak{g}_{-1} \cong \mathfrak{n}_o$ and by the homomorphism $\chi_o \circ \rho_o : \mathfrak{g}_0 \to End(T_o(S)/D_o)$, $\mathfrak{g}_0 \cong \mathfrak{co}(T_o(S)/D_o)$. In particular, the image of χ_o is $\mathfrak{co}(T_o(S)/D_o)$.*

Proof. The action of $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ on $T_o(S) \cong \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is effective by [Ya, Lemma 3.2]. Thus it is effective on $\mathcal{C}_o$ by the non-degeneracy of $\mathcal{C}_o$. It follows that

$$\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathbb{C}id_{T_o(S)} \subset aut(\widetilde{\mathcal{C}}_o).$$

The inclusion is in fact an equality because both sides have the same dimension by Proposition 7.1.3. Since $\mathfrak{g}_{-1}$ is the nil-radical of the graded Lie algebra $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \mod \mathfrak{g}_{-2}$, it acts trivially on the irreducible $\mathfrak{p}$-module $T_o(S)/D_o(S)$. Thus we have $\mathfrak{n}_o \cong \mathfrak{g}_{-1}$ and the statement about $Ker(\chi_o)/\mathfrak{n}_o$ is obvious. By the definition of the conformal structure on $T_o(S)/D_o$, the image of $\chi_o \circ \rho_o : \mathfrak{g}_0 \to End(T_o(S)/D_o)$ is in $\mathfrak{co}(T_o(S)/D_o)$. Since this is injective by the above description of $Ker(\chi_o)$, it is an isomorphism because $\dim \mathfrak{g}_0 = \dim \mathfrak{co}(\mathfrak{g}_2)$. $\square$

Proposition 7.2.2. *$aut(\widetilde{\mathcal{C}}_o)^{(2)} = 0$ and there is a natural inclusion $aut(\widetilde{\mathcal{C}}_o)^{(1)} \subset (T_o(S)/D_o)^*$.*

Proof. The vanishing of $aut(\widetilde{\mathcal{C}}_o)^{(2)}$ is a direct consequence of Theorem 1.1.2. By the linear normality of $\mathcal{C}_o$ in Proposition 7.1.2, Theorem 1.1.3 induces an injection $aut(\widetilde{\mathcal{C}}_o)^{(1)} \subset T^*_o(S)$. By Proposition 2.3.1, for each $A \in aut(\widetilde{\mathcal{C}}_o)^{(1)}$, there exists $\lambda \in T^*_o(S)$ such that $A_{\alpha\alpha} =< \lambda, \alpha > \alpha$ for each $\alpha \in \widetilde{\mathcal{C}}_o$. The restrictions of elements of $aut(\widetilde{\mathcal{C}}_o)^{(1)}$ to $D_o \cong \mathfrak{g}_1$ define elements of the first prolongation of the spin representation of $\mathfrak{so}(7)$, which vanishes by Corollary 1.1.5 because the spin representation of $\mathfrak{so}(7)$ is not the semi-simple part of the isotropy of a Hermitian symmetric space. Thus λ annihilates D_o. It follows that the value of A is determined by its values on $T_o(S)/D_o$, inducing an injection of $aut(\widetilde{\mathcal{C}}_o)^{(1)}$ into the first prolongation of the standard representation of $\mathfrak{so}(7)$ on $T_o(S)/D_o$. By Theorem 1.1.3, this can be regarded as a subspace of $(T_o(S)/D_o)^*$. $\square$

Recall that for any vector field v vanishing at o, $\rho_o(v) \in End(T_o(S))$ is the endomorphism induced by the first jet of v at o. If $\rho_o(v) = 0$, let $\varrho_o(v) \in S^2T^*_o(S) \otimes T_o(S)$ be the element induced by the 2-jet of v at o. Denote by χ_o the projection $aut(\widetilde{\mathcal{C}}_o) \to \mathfrak{co}(T_o(S)/D_o)$. The following Proposition can be proved using Propositions 7.2.1 and 7.2.2 in the same way as the proof of Proposition 4.2.3. In particular, the inclusion $aut(\widetilde{\mathcal{C}}_o)^{(1)} \subset (T_o(S)/D_o)^*$ in Proposition 7.2.2 is an equality.

Proposition 7.2.3. *Denoting by $\epsilon_o(v)$ the value of a vector field v on S at the base point $o \in S$ and by ς_o the projection $T_o(S) \to T_o(S)/D_o$, we have isomorphisms*

$$\varsigma_o \circ \epsilon_o : \mathfrak{g}_2 \to T_o(S)/D_o$$

$$\epsilon_o : \mathfrak{g}_1 \to D_o.$$

By taking 1-jets at o of vector fields vanishing at o, we have the homomorphism $\rho_o : \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \to aut(\widetilde{\mathcal{C}}_o)$. Then ρ_0 is injective and induces isomorphisms

$$\chi_o \circ \rho_o : \mathfrak{g}_0 \to \mathfrak{co}(T_o(S)/D_o)$$

and

$$\rho_o : \mathfrak{g}_{-1} \to \mathfrak{n}_o.$$

By taking 2-jets at o of vector fields vanishing to order ≥ 2 at o, we get an isomorphism

$$\varrho_o : \mathfrak{g}_{-2} \to aut(\widetilde{\mathcal{C}}_o)^{(1)} = (T_o(S)/D_o)^*.$$

§8. Proof of the rigidity

(8.1) The aim of this section is to prove the following rigidity.

Theorem 8.1.1. *Let $\pi : \mathcal{X} \to \Delta$ be a smooth and projective morphism from a complex manifold $\mathcal{X}$ to the unit disc Δ. Suppose for any $t \in \Delta - \{0\}$, the fiber $X_t := \pi^{-1}(t)$ is biholomorphic to S, the homogeneous space of type (F_4, α_1). Then the central fiber X_0 is also biholomorphic to S.*

From Proposition 7.1.1, the following counterpart of Proposition 3.5.1 is immediate.

Proposition 8.1.2. *In the notation of Theorem 8.1.1, choose a section $\sigma : \Delta \to \mathcal{X}$ of π such that $x_0 := \sigma(0)$ is a general point of X_0. Let $\nu : \mathcal{M} \to \Delta$ be the family where $M_t := \nu^{-1}(t)$ is the normalized Chow space of minimal rational curves through $x_t := \sigma(t)$. Then $M_t \cong \mathcal{C}_o$ for each $t \in \Delta$.*

Using Proposition 7.1.2 in place of Proposition 3.4.2, the following Proposition can be proved in exactly the same way as Proposition 3.5.2.

Proposition 8.1.3. *In the situation of Theorem 8.1.1, the variety of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X_0)$ at a general point $x \in X_0$ is isomorphic to $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ at a base point $o \in S$.*

(8.2) By Proposition 8.1.3, there exists a subvariety $B \subset X_0$ such that $\mathcal{C}_x \subset \mathbb{P}T_x(X_0)$ is isomorphic to $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ for each $x \in \mathcal{X} - B$. We assume that B is the smallest subvariety with this property. Denote by $\mathcal{D}$ the distribution on $\mathcal{X} - B$ defined by the linear span of the subvariety of $\mathcal{C}_x$ corresponding to $Q_o = \mathcal{C}_o \cap \mathbb{P}\mathfrak{g}_1$ for each $x \in \mathcal{X} - B$.

For the relative tangent bundle T^π of $\pi : \mathcal{X} \to \Delta$ write $\mathcal{T}^\pi$ for the associated relative tangent sheaf and $\mathcal{L}$ for the direct image $\pi_*(\mathcal{T}^\pi)$. $\mathcal{L}$ is a locally free sheaf with fiber at $t \neq 0$,

$$L^t \cong \text{ the Lie algebra of holomorphic vector fields on } X_t \cong \mathfrak{g}.$$

As in (5.1), L^0 is the Lie algebra of limit vector fields on X_0. Choose a section $\{x_t \in X_t, t \in \Delta\}$ with $x_0 \notin B$. For $v \in L^t$, let $\epsilon_{x_t}(v) \in T_{x_t}(X_t)$ be the evaluation at x_t. When $\epsilon_{x_t}(v) = 0$, let $\rho_{x_t}(v)$ be the endomorphism of $T_{x_t}(X_t)$ given by the first jet of v. As in (5.1), define, for each $t \in \Delta$,

$$J^t := \{v \in L^t : \ \epsilon_{x_t}(v) = 0, \rho_{x_t}(v)|_{\mathcal{D}_{x_t}} \equiv \mu \cdot id \text{ for some } \mu \in \mathbb{C}\}$$

$$I^t := \{v \in L^t : \ \epsilon_{x_t}(v) = 0, \rho_{x_t}(v)|_{\mathcal{D}_{x_t}} \equiv 0\}$$

$$H^t := \{v \in L^t : \ \epsilon_{x_t}(v) = 0, \rho_{x_t}(v) = 0\}.$$

These are subalgebras of the Lie algebra L^t for each $t \in \Delta$.

For $x \in \mathcal{X} - B$, $T^\pi_x/\mathcal{D}_x$ has the conformal structure conjugate to that of $T_o(S)/D_o$. Let $\chi_x : aut(\widetilde{\mathcal{C}}_x) \to \mathfrak{co}(T^\pi_x/\mathcal{D}_x)$ be the natural projection and $\mathfrak{n}_x$ be the nil-radical of $Ker(\chi_x)$. Using Proposition 7.2.3, the next Proposition can be proved in a similar manner as Proposition 5.1.1.

Proposition 8.2.1. *The homomorphism defined by 2-jets of vector fields at x_t*

$$\varrho_{x_t} : H^t \to aut(\widetilde{\mathcal{C}}_{x_t})^{(1)} \cong (T^\pi_{x_t}/\mathcal{D}_{x_t})^*$$

is an isomorphism for each $t \in \Delta$. The homomorphism defined by 1-jets of vector fields at x_t

$$\rho_{x_t} : I^t \to \mathfrak{n}_{x_t}$$

is an isomorphism for each $t \in \Delta$. The homomorphism defined by 1-jets of vector fields at x_t

$$\chi_{x_t} \circ \rho_{x_t} : J^t/I^t \to \mathfrak{co}(T^\pi_{x_t}/\mathcal{D}_{x_t})$$

is injective and has image $\mathfrak{z}_{x_t}$, the 1-dimensional center of $\mathfrak{co}(T^\pi_{x_t}/\mathcal{D}_{x_t})$ for each $t \in \Delta$. In particular,

$$\{H^t \subset I^t \subset J^t, t \in \Delta\}$$

form a subbundle of the locally free sheaf $\mathcal{L}$ on Δ.

From Corollary 4.3.2 and Proposition 8.2.1 we deduce the extension of characteristic vector fields to X_0 as in Proposition 5.2.1:

Proposition 8.2.2. *Let $x_t = \sigma(t)$ be a section of $\pi : \mathcal{X} \to \Delta$ with $x_0 \notin B$. There exists a family of $\mathbb{C}^*$-vector fields E_t on X_t with an isolated zero at x_t for all $t \in \Delta$ such that E_t is a characteristic vector field at x_t for each $t \neq 0$.*

Let us fix a family E_t of characteristic vector fields as in Proposition 8.2.2. For each $t \neq 0$, we have the eigenspace decomposition of the adjoint action of E_t on L^t,

$$L^t = L^t_{-2} \oplus L^t_{-1} \oplus L^t_0 \oplus L^t_1 \oplus L^t_2$$

which is conjugate to $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. For each i, $-2 \leq i \leq 2$, the i-eigenspace L^0_i of L^0 under the adjoint action of E_0 must have dimension at least $\dim L^t_i$. Since $\dim L^0 = \dim L^t$, we have the eigenspace decomposition

$$L^0 = L^0_{-2} \oplus L^0_{-1} \oplus L^0_0 \oplus L^0_1 \oplus L^0_2.$$

The next Proposition can be proved as Proposition 5.2.2 and Proposition 7.2.3.

Proposition 8.2.3. *Let $L^0 = L^0_{-2} \oplus L^0_{-1} \oplus L^0_0 \oplus L^0_1 \oplus L^0_2$ be the eigenspace decomposition associated to a choice of a family of characteristic vector fields E_t along x_t as above. Denoting by ϵ_x the evaluation of vector fields on X_0 at $x = x_0 \in X_0$ and denoting by ς_x the projection $T_x(X_0) \to T_x(X_0)/\mathcal{D}_x$, we have two isomorphisms*

$$\varsigma_x \circ \epsilon_x : L^0_2 \to T^\pi_x(X_0)/\mathcal{D}_x$$

$$\epsilon_x : L^0_1 \to \mathcal{D}_x.$$

By taking 1-jets at $x = x_0$ of vector fields vanishing at x, we have homomorphisms ρ_x : $L^0_0 \oplus L^0_{-1} \to aut(\widetilde{\mathcal{C}}_x)$ which is injective and induces isomorphisms

$$\chi_x \circ \rho_x : L^0_0 \to \mathfrak{co}(T_x(X_0)/\mathcal{D}_x)$$

$$\rho_x : L^0_{-1} \to \mathfrak{n}_x.$$

By taking 2-jets at x of vector fields vanishing to order ≥ 2 at x, we get an isomorphism

$$\varrho_x : L^0_{-2} \to aut(\widetilde{\mathcal{C}}_x)^{(1)} \cong (T^\pi_x/\mathcal{D}_x)^*.$$

Choose a family of characteristic vector fields $E_t, x_t = \sigma(t)$ as in Proposition 8.2.3, inducing the eigenspace decomposition $L^t = L^t_{-2} \oplus L^t_{-1} \oplus L^t_0 \oplus L^t_1 \oplus L^t_2$. For $t \neq 0$, let $\mathfrak{h}^t_1 = L^t_2, \mathfrak{h}^t_0 = L^t_0$ and $\mathfrak{h}^t_{-1} = L^t_{-2}$. Then $\mathfrak{h}^t = \mathfrak{h}^t_{-1} \oplus \mathfrak{h}^t_0 \oplus \mathfrak{h}^t_1$ is a subalgebra of L^t.

Let $\mathbf{g} = \mathfrak{so}(9)$ and $\mathbf{g} = \mathbf{g}_{-1} \oplus \mathbf{g}_0 \oplus \mathbf{g}_1$ be the gradation of $\mathbf{g}$ regarded as the Lie algebra of holomorphic vector fields on the 7-dimensional hyperquadric $\mathbb{Q}_7$. Then it is easy to check that the Lie algebra $\mathfrak{h}^t = \mathfrak{h}^t_{-1} \oplus \mathfrak{h}^t_0 \oplus \mathfrak{h}^t_1$ is isomorphic to $\mathbf{g}$ for each $t \in \Delta - \{0\}$. Applying Proposition 4.1.4 and Proposition 4.1.5 to $\mathbf{g}$, the following counterpart of Proposition 5.3.3 holds.

Proposition 8.2.4. *The graded Lie subalgebra* $\mathfrak{h}^0$ *defined above is isomorphic to* $\mathfrak{h}^t \cong \mathbf{g}$.

Proof. Let $\mathbf{h}_1$ be a 7-dimensional vector space with a conformal structure, $\mathbf{h}_0 = \mathfrak{co}(\mathbf{h}^1)$ and $\mathbf{h}_{-1} = \mathbf{h}_1^*$. By fixing a trivialization of the family

$$\{\mathcal{C}_{x_t}, t \in \Delta\} \cong \mathcal{C}_o \times \Delta,$$

we get natural identifications of $(T^\pi/\mathcal{D})_{x_t}$ with $\mathbf{h}_1$ for all t preserving the conformal structure. We have a family of vector space isomorphisms $\varphi_t : \mathfrak{h}^t \to \mathbf{h} = \mathbf{h}_{-1} \oplus \mathbf{h}_0 \oplus \mathbf{h}_1$ arising from Proposition 8.2.3. For a fixed $t_1 \neq 0$, the vector space isomorphism $\psi_t : \mathfrak{h}^{t_1} \to \mathfrak{h}^t$ defined by $\psi_t := \varphi_t^{-1} \circ \varphi_{t_1}$ preserves the Lie algebra structure, by the same argument as in the proof of Proposition 5.3.3. By continuity, we get a Lie algebra isomorphism $\psi_0 : \mathfrak{h}^{t_1} \cong \mathfrak{h}^0$. $\square$

Corollary 8.2.5. *The orbit of* x *under* $\exp(\mathfrak{h}^0) \subset Aut(X_0)$ *is isomorphic to* $\mathbb{Q}_7$.

(8.3) To prove Theorem 8.1.1, we assume that $X_0 \not\cong S$ and get a contradiction from this assumption.

Proposition 8.3.1. *In the notation of Proposition 8.2.3, if all vector fields on* X_0 *are limit vector fields, i.e.,* $L^0 = H^0(X_0, T(X_0))$, *then* $X_0 \cong S$. *If* $L^0 \neq H^0(X_0, T(X_0))$, *there exists a* $\mathbb{C}^*$*-vector field* $\mathbf{E} \in H^0(X_0, T(X_0)) - L^0$ *with* $\epsilon_{x_t}(\mathbf{E}) = 0$ *and* $\rho_{x_t}(\mathbf{E}) = id_{T_{x_0}(X_0)}$.

Proof. Since X_0 is Fano, $H^i(X_0, T(X_0)) = 0$ for all $i \geq 2$. Thus if $h^0(X_0, T(X_0)) = h^0(X_t, T(X_t))$ for $t \neq 0$, then $h^1(X_0, T(X_0)) = h^1(X_t, T(X_t)) = 0$ by the local rigidity of S and we get $X_0 \cong S$. Since

$$52 = \dim \mathfrak{g} = \dim L^0 \leq h^0(X_0, T(X_0)) \leq \dim S + \dim aut(\widetilde{\mathcal{C}_o}) + \dim aut(\widetilde{\mathcal{C}_o})^{(1)} = 53,$$

$L^0 \neq H^0(X_0, T(X_0))$ implies the existence of a vector field $\mathbf{E}$ vanishing at x_0 with $\rho_{x_0}(\mathbf{E}) = id_{T_{x_0}(X_0)} \in aut(\widetilde{\mathcal{C}}_{x_0})$. By taking Zariski closure of the analytic subgroup $exp(\mathbb{C}\mathbf{E})$, we can choose $\mathbf{E}$ to be a $\mathbb{C}^*$-vector field not contained in L^0. $\square$

By Proposition 8.3.1, a $\mathbb{C}^*$-vector field $\mathbf{E}$ with isolated zero at x_0 and $\rho_{x_0}(\mathbf{E}) = id$ exists for each choice of the section $\{x_t \in X_t : x_0 \notin B\}$.

Lemma 8.3.2. *The distribution* $\mathcal{D}$ *is integrable on* X_0.

Proof. The distribution $\mathcal{D}$ is invariant under the vector field $\mathbf{E}$ because the fiber subbundle $\mathcal{C} \subset \mathbb{P}T(X_0 - B)$ is invariant under $Aut(X_0)$. Let $F : \Lambda^2\mathcal{D} \to T(X_0 - B)/\mathcal{D}$ be the Frobenius bracket tensor of the distribution. On the one hand, F_{x_o} is invariant under the isotropy action of $\{\exp(\lambda\mathbf{E}) : \lambda \in \mathbb{C}\}$ by the invariance of the distribution $\mathcal{D}$. On the other hand, F_{x_0} is multiplied by λ^{-1} by the action of $\exp(\lambda\mathbf{E})$ because it acts on $T_{x_0}(X_0)$ as the multiplication by λ. Thus $F_{x_0} = 0$. Since x_0 can be chosen to be any point of $X_0 - B$, we get $F \equiv 0$. $\square$

As in Lemma 6.1.2, we have

Lemma 8.3.3. *For any $y \in X_0 - B$ the leaf of $\mathcal{D}|_{X_0 - B}$ passing through y is closed in $X_0 - B$, and its topological closure is a projective subvariety of X_0.*

Proof. On X_0 a minimal rational curve C not lying on B and tangent to $\mathcal{D}$ on $C - B$ will be called a $\mathcal{D}$-minimal rational curve. By Proposition 8.2.2, at each point $y \in X_0 - B$, there exists a characteristic vector field E_y which is the limit of characteristic vector fields on $X_t, t \neq 0$. The orbital curves of E_y through y are rational curves and by the invariance of $\mathcal{D}$ under $\{\exp(\lambda E_y), \lambda \in \mathbb{C}\}$, those tangent to $\mathcal{D}_y \subset \mathcal{C}_y$ at y are $\mathcal{D}$-minimal rational curves. Thus the tangent vectors to $\mathcal{D}$-minimal rational curves through y are precisely those corresponding to $Q_o \subset \mathcal{C}_o$, which is exactly $\mathcal{C}_y \cap \mathbb{P}\mathcal{D}_y$. The rest of the proof is identical with Lemma 6.1.2. $\square$

A **conformal structure** on a complex manifold M means a fiber subbundle $R \subset \mathbb{P}T(M)$ whose fibers are smooth hyperquadrics. This is the G-structure modelled on the quadric $\mathbb{Q}_n, n = \dim M$. It has similar properties as the G-structure modelled on the Grassmannian in (6.2). For example, there exists an unramified holomorphic map $\delta : M \to \mathbb{Q}_n$ such that the conformal structure on M coincides with the pull-back of the conformal structure on $\mathbb{Q}_n$ under δ if and only if a naturally defined tensor (the Weyl tensor) of type $T^*(M) \otimes T(M) \otimes \Lambda^2 T^*(M)$ vanishes.

The foliation $\mathcal{D}$ is transversal to $N \cong \mathbb{Q}_7$, the orbit of x_0 under $\exp \mathfrak{h}^0$. The leaves of $\mathcal{D}$ have conformal structures given by $\mathcal{C}_y \cap \mathbb{P}\mathcal{D}_y$ at $y \in X_0 - B$. This conformal structure is invariant under $\{\exp(\lambda \mathbf{E}), \lambda \in \mathbb{C}\}$. Using this conformal structure in place of the G-structure modelled on $\mathbb{G}(2, \mathbb{C}^{2+2a})$ the arguments in (6.2) and (6.3) work verbatim to give the following. Let us just mention that a standard open set on the quadric $\mathbb{Q}_n$ means the complement of a singular hyperplane section. Such an open set is biregular to the affine space $\mathbb{C}^n$ and an analogue of Lemma 6.2.1 holds.

Proposition 8.3.4. *Let $\pi : \mathcal{X} \to \Delta$ be a regular family of projective manifolds, $X_t := \pi^{-1}(t)$, such that $X_t \cong S$ for $t \neq 0$. Suppose $X_0 \ncong S$. Then,*

1. *there exists a subvariety $B \subset X_0$ of codimension ≥ 2 and a projective submanifold $N \subset X_0 - B$ biholomorphic to the 7-dimensional hyperquadric $\mathbb{Q}_7$;*
2. *$X_0 - B$ can be realized as the total space of a holomorphic vector bundle $\mu^o : \mathcal{V} \to N$ of rank 8 with $N \subset X_0 - B$ identified as the zero section;*
3. *$\mu^o : \mathcal{V} \to N$ can be compactified to a holomorphic fiber bundle $\mu : Y \to N \cong \mathbb{Q}_7$ of 8-dimensional hyperquadrics $\mathbb{Q}_8$ equipped with a birational morphism $f : Y \to X_0$, which sends each fiber Y_w of μ biholomorphically to a submanifold P_w of X_0 compactifying the fiber $V_w \subset X_0 - B$ of μ^o such that $V_w \subset Y_w$ corresponds to a standard open set in $\mathbb{Q}_8$.*

End of Proof of Theorem 8.1.1. From the geometry of $\mathbb{Q}_8$, the hypersurface $Y_w - V_w$ has a unique singular point Γ_w defining a section $\Gamma \subset Y - \mathcal{V}$ of μ. By the same reasoning as in Proposition 6.4.2, $f(\Gamma)$ should be a single point $\iota \in X_0$.

We proceed to consider the isotropy representations at $\iota \in X'$. $Aut(N)$ embeds as a Lie subgroup G' of $Aut(X_0)$. G' is isogenous to $SO(9)$ and arguing as in (6.3), the point $\iota \in X'$ is fixed under the action of G' on X'. As a result, we obtain an isotropy action of G' on $T_\iota(X_0)$. By [BB, Lemma 2.4], this is a non-trivial representation of G'. Since an irreducible representation of $\mathfrak{so}(9)$ with dimension $\leq 15 = \dim X_0$ is isomorphic to the standard representation of dimension 9, there must be a unique G'-submodule $U \subset T_\iota(X_0)$ of dimension 9 equivalent to the standard representation and a 6-dimensional trivial representation complementary to U. For a point $w \in N$, the semisimple factor of the isotropy subgroup G'_w is isogenous to $SO(7)$ and its action on $T_w(R_w)$ is the spin representation. The isotropy action of G'_w on $T_\iota(P_w)$ is also the spin representation in analogy with Lemma 6.4.1. In particular, $T_\iota(P_w)$ is contained in the non-trivial G'-module $U \subset T_\iota(X_0)$ for any choice of $w \in N$.

Now consider the $\mathbb{C}^*$-subgroup $G^\flat \subset Aut(X_0)$ corresponding to the scalar multiplication of the vector bundle $\mathcal{V}$. $G^\flat$ commutes with G' because elements of G' induce vector bundle homomorphisms of $\mathcal{V}$. We claim that the action of $G^\flat$ on $T_\iota(X_0)$ has at least two distinct weights. Suppose not. Then orbital curves of $G^\flat$ on X_0 through ι have distinct tangent vectors at ι by Lemma 3.3.7. Since general orbital curves lie in P_w for general $w \in N$, this implies that $\{T_\iota(P_w) : \text{ general } w \in N\}$ span $T_\iota(X_0)$, a contradiction to $T_\iota(P_w) \subset U$.

Thus we have the isotropy representation of $G^\flat \times G'$ on $T_\iota(X_0)$ such that $G^\flat$ has two distinct weights and the action of G' has 9-dimensional irreducible module U. This forces the dimension of $T_\iota(X_0)$ to be at least $2 \times 9 = 18$, a contradiction to $\dim X_0 = 15$. $\square$

References

[BB] Bialynicki-Birula, A.: Some theorems on actions of algebraic groups. Ann. Math. **98** (1973) 480-497

[Bo1] Borel, A.: Symmetric compact complex spaces. Arch. Math. **33** (1979) 49-56

[Bo2] Borel, A.: *Linear algebraic groups.* Graduate texts in Math. **126** Springer-Verlag, 1991

[Ca] Cartan, E.: Les groupes de transformations continus, infinis, simples. Ann. scient. Ec. Norm. Sup. **26** (1909) 93-161

[De] Demazure, M.: Classification des Algèbres de Lie filtrées. Séminare Bourbaki, Exposé 326 (1967)

[GQS] Guillemin, V., Quillen, D. and Sternberg, S.: The classification of the irreducible complex algebras of infinite type. J. Analyse Math. **18** (1967) 107-112

[HM1] Hwang, J.-M. and Mok, N.: Uniruled projective manifolds with irreducible reductive G-structures. J. reine angew. Math. **490** (1997) 55-64

[HM2] Hwang, J.-M. and Mok, N.: Rigidity of irreducible Hermitian symmetric spaces of the compact type under Kähler deformation. Invent. math. **131** (1998) 393-418

[HM3] Hwang, J.-M. and Mok, N.: Varieties of minimal rational tangents on uniruled manifolds. in Several Complex Variables, ed. by M. Schneider and Y.-T. Siu, MSRI Publications

37, Cambridge University Press (1999) 351-389

[HM4] Hwang, J.-M. and Mok, N.: Cartan-Fubini type extension of holomorphic maps for Fano manifolds of Picard number 1. Journal Math. Pures Appl. **80** (2001) 563-575

[HM5] Hwang, J.-M. and Mok, N.: Deformation rigidity of the rational homogeneous space associated to a long simple root. to appear in Ann. scient. Ec. Norm. Sup.

[HM6] Hwang, J.-M. and Mok, N.: Deformation rigidity of the 20-dimensional F_4-homogeneous space associated to a short root. Preprint.

[Hw1] Hwang, J.-M.: Rigidity of homogeneous contact manifolds under Fano deformation. J. reine angew. Math. **486** (1997) 153-163

[Hw2] Hwang, J.-M.: On the vanishing orders of vector fields on Fano varieties of Picard number 1. Compositio Math. **125** (2001) 255-262

[Kd] Kodaira, K.: On stability of compact submanifolds of complex manifolds. Amer. J. Math. **85** (1963) 79-94

[Ke] Kebekus, S.: Families of singular rational curves. J. Alg. Geom. **11** (2002) 245-256

[Kl] Kollár, J.: *Rational curves on algebraic varieties.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 3 Folge, Band 32, Springer Verlag, 1996

[KN1] Kobayashi, S. and Nagano, T.: On filtered Lie algebras and geometric structures I. Jour. Math. Mech. **13** (1964) 875-907

[KN2] Kobayashi, S. and Nagano, T.: On filtered Lie algebras and geometric structures III. Jour. Math. Mech. **14** (1965) 679-706

[MS] Mori, S. and Sumihiro, H.: On Hartshorne's conjecture. J. Math. Kyoto Univ. **18** (1978) 523-533

[Mu] Mukai, S.: Biregular classification of Fano 3-folds and Fano manifolds of coindex 3. Proc. Natl. Acad. Sci. USA **86** (1989) 3000-3002

[Oc] Ochiai, T.: Geometry associated with semisimple flat homogeneous spaces. Trans. A.M.S. **152** (1970) 159-193.

[OV] Onishchik, A. L. and Vinberg, E. B.: *Lie groups and algebraic groups.* Springer Verlag, 1990

[Sn] Snow, D.: Vanishing theorems on compact Hermitian symmetric spaces. Math. Zeit. **198** (1988) 1-20

[SS] Singer, I. and Sternberg, S.: The infinite groups of Lie and Cartan. J. anal. Math. **15** (1965) 1-114

[Ts] Tsai, I.-H.: Rigidity of holomorphic maps from compact Hermitian symmetric spaces to smooth projective varieties. J. Alg. Geom. **2** (1993) 603-634.

[Wa] Wahl, J.: A cohomological characterization of $\mathbb{P}^n$. Invent. math. **72** (1983) 315-322

[Ya] Yamaguchi, K.: Differential systems associated with simple graded Lie algebras. Adv. Study Pure Math. **22** *Progress in differential geometry* (1993) 413-494

[Za] Zak, F.L.: *Tangents and secants of algebraic varieties*, Translations of Mathematical Monographs, Vol.127, Amer. Math. Soc., Providence 1993.

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