On exponential sum over primes and application in Waring-Goldbach problem

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Abstract In this paper, we prove the following estimate on exponential sums over primes: Let $k \geq 1$, $\beta_k = 1/2 + \log k/\log 2$, $x \geq 1$ and $\alpha = a/q + \lambda$ subject to $(a,q) = 1$, $1 \leq a \leq q$, and $\lambda \in \mathbb{R}$. Then

$$\sum_{x<m\leq 2x} \Lambda(m)e(\alpha m^k) \ll (d(q))^{\beta_k}(\log x)^c \left(x^{1/2}\sqrt{q(1+|\lambda|x^k)} + x^{4/5} + \frac{x}{\sqrt{q(1+|\lambda|x^k)}} \right).$$

As an application, we prove that with at most $O(N^{7/8+\varepsilon})$ exceptions, all positive integers up to N satisfying some necessary congruence conditions are the sum of three squares of primes. This result is as strong as what has previously been established under the generalized Riemann hypothesis.

Keywords: exponential sums over primes, Waring-Goldbach problem, circle method.

1. Introduction

In this paper, we are concerned with estimates of the exponential sums

$$S_k(\alpha) = \sum_{x<m\leq 2x} \Lambda(m)e(m^k\alpha),$$

where k is a positive integer, x a big parameter, α a real number, $\Lambda(m)$ the von Mangoldt function, and $e(z) = e^{2\pi iz}$. These sums arise naturally and play important roles when solving the Waring-Goldbach type problems by the circle method.

Estimates for $S_k(\alpha)$ usually depends on the rational approximation of α. Let

$$\alpha = a/q + \lambda, \quad 1 \leq a \leq q, \quad (a,q) = 1. \tag{1.1}$$

The main result of this paper is the following

Theorem 1.1. *Fix $k \geq 1$, and let $\beta_k = 1/2 + \log k/\log 2$. We have*

$$S_k(\alpha) \ll (d(q))^{\beta_k}(\log x)^c \left(x^{1/2}\sqrt{q(1+|\lambda|x^k)} + x^{4/5} + \frac{x}{\sqrt{q(1+|\lambda|x^k)}} \right), \tag{1.2}$$

where $d(n)$ is the divisor function and c is an absolute positive constant.

When $k = 1$, Theorem 1.1 gives

Corollary 1.2. *Suppose α satisfies (1.1) with $|\lambda| \leq q^{-2}$. Then we have*

$$\sum_{x<m\leq 2x} \Lambda(m)e(m\alpha) \ll (d(q))^{1/2}(\log x)^c\{x^{1/2}q^{1/2} + x^{4/5} + xq^{-1/2}\}.$$

This is essentially the well-known result of Vinogradov in [1]. It is reproved by Vaughan [2] via an elementary identity now named after him. Actually Vaughan's estimate is slightly stronger, i.e. there is no $d(q)$ on the right, and the $c = 5$.

In the nonlinear cases $k \geq 2$, Theorem 1.1 partially improves the estimates of Ghosh [3] or Harman [4], and this improvement will be important in many applications. For example, Theorem 1.1 with $k = 2$ implies that, for $|\lambda| \leq 1/(qx^{7/5})$,

$$S_2(\alpha) \ll (d(q))^{3/2}(\log x)^c \left(x^{1/2}q^{1/2} + x^{4/5} + \frac{x}{q^{1/2}} \right). \tag{1.3}$$

This can be compared with Ghosh's estimate [3] that,

$$S_2(\alpha) \ll x^{1+\varepsilon} \left(\frac{1}{q} + \frac{1}{x^{1/2}} + \frac{q}{x^2} \right)^{1/4}. \tag{1.4}$$

It is easily seen that (1.3) is sharper than (1.4) when $q \leq x^{3/4}$. This will be crucial in our proof of Theorem 1.3 in §4.

Theorem 1.1 can be applied to a wide circle of problems of Waring-Goldbach type. Most of these problems can be studied by the circle method, and the quality of the arithmetic results obtained usually depends on the following key issues: (i) An asymptotic formula for the contribution from the major arcs $\mathfrak{M}$, with $\mathfrak{M}$ as large as possible; (ii) An upper bound, as small as possible, for the exponential sum $S_k(\alpha)$ on $\mathfrak{m}$. Several authors (see e.g. ref. [5], [6], [7], [8]) have successfully introduced a method to fulfill (i), which does not depend on the Deuring-Hilbronn phenomenon. Therefore, the quality of the arithmetic results obtained relies more heavily on (ii). Our Theorem 1.1 makes an effort in this direction. It can be applied, together with the estimates of Ghosh [3] and Harman [4] on $\mathfrak{m}$, to make a number of improvements. The idea is, roughly speaking, that we split $\mathfrak{m}$ into $\mathfrak{k} \cup \mathfrak{n}$, where $\mathfrak{k}$ is a subset such that our Theorem 1.1 gives a better bound than those of Ghosh or Harman. On $\mathfrak{n}$, Ghosh [3] or Harman [4] also gives a better bound since $\mathfrak{n}$ is much smaller than $\mathfrak{m}$. Therefore, a better bound of $S_k(\alpha)$ on $\mathfrak{m}$ follows, and hence a better arithmetic result.

We illustrate the application of Theorem 1.1 by making the improvement in Theorem 1.3 on the exceptional set in the representation of certain positive integers n by the sum of three squares of primes,

$$n = p_1^2 + p_3^2 + p_3^2. \tag{1.5}$$

Necessary conditions for this representation are

$$n \equiv 3(\mathrm{mod}24), \qquad n \not\equiv 0(\mathrm{mod}5). \tag{1.6}$$

Now let $E(N)$ be the number of positive $n \leq N$ satisfying (1.6) but cannot be written as (1.5). Hua [9] was the first to prove that $E(N) \ll N \log^{-A} N$ for some positive A. Later Schwarz [10] showed that Hua's bound holds for any $A > 0$, and this was further improved by Leung and Liu [11] to $N^{1-\delta}$ for some computable but small δ depending on the constants in Deuring-Heilbronn phenomenon. An approach free of the Deuring-Heilbronn phenomenon was introduced in Bauer, Liu, and Zhan [12], where it was proved that $E(N) \ll N^{151/160+\varepsilon}$. The exponent 151/160 has subsequently reduced to 47/50 and then to 11/12 by Liu and Zhan [7][8] respectively. Under the Generalized Riemann Hypothesis, it was proved in ref. [12] that [1] the exponent can be further reduced to 7/8.

[1]Note that the displayed formula in Theorem 2 of ref. [12] should read $E_2(x) \ll x^{7/8+\varepsilon}$.

Using our Theorem 1.1, we are able to establish unconditionally the above conditional result.

Theorem 1.3. *Unconditionally, we have*

$$E(N) \ll N^{7/8+\varepsilon}.$$

In finishing this paper, we learn that Kumchev [13] proves new estimates on $S_k(\alpha)$. But we remark that his results and methods are different from ours.

Notation. As usual, $\varphi(n)$ stands for the Euler function. We use $\chi \bmod q$ and $\chi^0 \bmod q$ to denote a Dirichlet character and the principal character modulo q. Also $L(s,\chi)$ is the Dirichlet L-function. The symbol $r \sim R$ means $R < r \le 2R$. The letter ε denotes positive constant which is arbitrarily small. The letter c is written for an absolute positive constant which may not necessarily the same at each occurrence.

2. Mean-Value estimate of a Dirichlet polynomial

Let $M \ge 2$ be a real number. For $j = 1, ..., 10$, let M_j be positive integers such that

$$2^{-9}M < M_1 \cdots M_{10} \le M, \quad \text{and} \quad 2M_6, ..., 2M_{10} \le (2M)^{1/5}. \tag{2.1}$$

Let

$$a_j(m) = \begin{cases} \log m, & \text{if} \quad j = 1, \\ 1, & \text{if} \quad j = 2, ..., 5, \\ \mu(m), & \text{if} \quad j = 6, ..., 10. \end{cases} \tag{2.2}$$

We define the following functions of a complex variable s:

$$f_j(s,\chi) = \sum_{m \sim M_j} \frac{a_j(m)\chi(m)}{m^s}, \qquad F(s,\chi) = f_1(s,\chi) \cdots f_{10}(s,\chi). \tag{2.3}$$

Then we have the following mean-value estimate.

Lemma 2.1. *Let $\beta \ge 1$, $2 \le T \le M^{\beta}$ and $2 < q \le M^{2\beta}$. Then we have*

$$\sum_{\chi \bmod q} \int_{-T}^{T} \left| F\left(\frac{1}{2} + it, \chi\right) \right| dt \ll (d(q))^{1/2} (\log M)^c \{qT + (qT)^{1/2} M^{3/10} + M^{1/2}\}, \tag{2.4}$$

where c is an absolute positive constant.

To prove Lemma 2.1, we quote the following two well-known results (see for example ref. [14], Theorems 2.5, 3.10, 3.11 and 3.17).

Lemma 2.2. *Let $T, N_0, q \ge 1$ and $N \ge 0$. Let a_n, $n \in \mathbb{N}$ be any complex numbers. Then we have*

$$\sum_{\chi \bmod q} \int_{-T}^{T} \left| \sum_{n=N_0}^{N_0+N} \frac{a_n \chi(n)}{n^{it}} \right|^2 dt \ll \sum_{n=N_0}^{N_0+N} (qT + n)\, |a_n|^2.$$

Lemma 2.3. *Let $T \ge 1$, $b_0 = 4$ and $b_1 = 8$. Then for $k = 0$, 1, we have*

$$\int_{-T}^{T} \left| \zeta^{(k)}\left(\frac{1}{2} + it\right) \right|^4 dt \ll T \log^{b_k}(T+2);$$

and for $q > 2$,

$$\sum_{\chi \bmod q} \int_{-T}^{T} \left| L^{(k)}\left(\frac{1}{2} + it, \chi\right) \right|^4 dt \ll qT \log^{(b_k+1)} q(T+2).$$

Lemma 2.4. *Let a_n, $n \in \mathbb{N}$ be complex numbers and let the series $F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ be absolutely convergent for $\sigma = \mathrm{Re}(s) > \sigma_a$. Let*

$$A(x) = \max_{x/2 \le n \le 2x} |a_n|, \quad x \ge 1; \quad B(\sigma) = \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma}}, \qquad \sigma > \sigma_a.$$

Then for $T \ge 1$ and for any $s_0 = \sigma_0 + it_0$ and $b > 0$ with $\sigma_0 + b > \sigma_a$, one has

$$\sum_{n \le x} \frac{a_n}{n^{s_0}} = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} F(s_0 + s) \frac{x^s}{s} ds + R(x, T),$$

where, on writing $\|x\|$ for the distance from x to the nearest integer N,

$$\begin{aligned} R(x,T) \ \ll \ & \frac{x^b B(b+\sigma_0)}{T} + x^{1-\sigma_0} A(x) \min\left\{1, \frac{\log x}{T}\right\} \\ & + x^{-\sigma_0} |a_N| \min\left\{1, \frac{x}{T\|x\|}\right\}. \end{aligned} \tag{2.5}$$

This is Perron's formula, a proof can be found, for example, in ref. [15].

Lemma 2.5. *If there exist M_i, M_j with $1 \le i < j \le 5$ such that $M_i M_j > M^{2/5}$, then Lemma 2.1 is true.*

Proof. Without loss of generality, we may assume that $i = 1$ and $j = 2$. By applying Lemma 2.4 with $T = M^{\beta}$, $s_0 = 1/2 + it$ and $b = 1/2 + 1/\log M$, we get

$$\begin{aligned} & f_1\left(\frac{1}{2} + it, \chi\right) \\ & = -\frac{1}{2\pi i} \int_{1/2+1/L-iM^{\beta}}^{1/2+1/L+iM^{\beta}} L'\left(\frac{1}{2} + it + w, \chi\right) \frac{(2M_1)^w - M_1^w}{w} dw + O(1), \end{aligned} \tag{2.6}$$

where we have written $L = \log M$. Now we move the integral leftward along the rectangular contour with vertices $\pm iM^{\beta}$, $1/2 + 1/L \pm iM^{\beta}$ to the line $\mathrm{Re}(w) = 0$. Note that the integrand is regular inside the contour except for the simple pole at $w = 1/2 - it$ when $\chi = \chi^0$. We have

$$L(s, \chi^0) = \zeta(s) \prod_{p|q} (1 - p^{-s}), \tag{2.7}$$

and

$$L'(s, \chi^0) = \zeta'(s) \prod_{p|q} (1 - p^{-s}) + \zeta(s) \sum_{p_1|q} \frac{\log p_1}{{p_1}^s} \prod_{\substack{p|q \\ p \ne p_1}} \left(1 - p^{-s}\right). \tag{2.8}$$

So the residue at the simple pole is

$$\left\{ \sum_{p_1|q} \frac{\log p_1}{p_1} \prod_{\substack{p|q \\ p \ne p_1}} \left(1 - p^{-1}\right) \right\} \frac{(2M_1)^{1/2-it} - M_1^{1/2-it}}{1/2 - it},$$

which can be estimated as

$$\ll \left\{ \sum_{p|q} \frac{\log p}{p} \right\} \left\{ \prod_{p|q} (1-p^{-1}) \right\} \frac{M_1^{1/2}}{1+|t|} \ll \frac{M_1^{1/2}(\log\log q)^2}{1+|t|},$$

by elementary estimates

$$\sum_{p|q} \frac{\log p}{p} \ll \log\log q, \qquad \prod_{p|q}(1-p^{-1}) \ll \log\log q.$$

Thus (2.6) becomes

$$\begin{aligned} f_1\left(\frac{1}{2}+it,\chi\right) &= -\frac{1}{2\pi i}\left\{ \int_{1/2+1/L-iM^\beta}^{-iM^\beta} + \int_{-iM^\beta}^{iM^\beta} + \int_{iM^\beta}^{1/2+1/L+iM^\beta} \right\} \\ &\quad +O\left(\frac{\delta_\chi M_1^{1/2} L}{1+|t|}\right) + O(1), \end{aligned} \tag{2.9}$$

where $\delta_\chi = 1$ or 0 according as $\chi = \chi^0$ or not.

Now we recall the following bound: For $\sigma \geq 1/2$ and $|t| \geq 2$,

$$L^{(k)}(\sigma+it,\chi) \ll (\log^{k+2} q(|t|+2)) \max\{1, q^{(1-\sigma)/2}(|t|+2)^{1-\sigma}\}. \tag{2.10}$$

When $\chi \neq \chi^0$, this can be found, for example, in ref. [15], p.269, (13) and p.271, Exercise 6. When $\chi = \chi^0$, the above inequality can be derived from (2.7) and (2.8), by applying the well-known bound (see for example ref. [15], p.140, Theorem 2):

$$\zeta^{(k)}(\sigma+it) \ll (\log^{k+1}|t|)\max\{1,|t|^{1-\sigma}\}, \quad \text{for} \quad \sigma \geq 1/2 \quad \text{and} \quad |t| \geq 2,$$

and the elementary estimates: For $0 < \sigma < 1$,

$$\prod_{p|q}(1-p^{-s}) \ll \sum_{d|q} d^{-\sigma} \leq \sum_{r \leq d(q)} r^{-\sigma} \ll (d(q))^{1-\sigma}, \tag{2.11}$$

and

$$\begin{aligned} &\sum_{p_1|q} \frac{\log p_1}{{p_1}^s} \prod_{\substack{p|q \\ p\neq p_1}} (1-p^{-s}) \\ &\ll \left| \left(\prod_{p|q}(1-p^{-s}) \right) \left(\sum_{p|q} \frac{\log p}{p^\sigma} \right) \right| \ll (d(q))^{1-\sigma}(\log q) \sum_{r \leq \log q} r^{-\sigma} \\ &\ll (d(q))^{1-\sigma}(\log q)^{2-\sigma}. \end{aligned} \tag{2.12}$$

Applying (2.10) with $\sigma = 1/2+u$, we see that the contribution from the two horizontal segments in (2.9) is

$$\ll L^3 \max_{0\leq u \leq 1/2+1/L} q^{(1-(1/2+u))/2} M^{\beta(1-(1/2+u))} \frac{M_1^u}{M^\beta} \ll M^{u(1-2\beta)} L^3 \ll 1,$$

since $q \leq M^{2\beta}$ and $\beta \geq 1$. Moreover, on the vertical segment from $-iM^\beta$ to iM^β, one has

$$\frac{(2M_1)^{iv} - M_1^{iv}}{iv} \ll \frac{1}{1+|v|}.$$

Therefore on writing

$$g_1(t,\chi)=\int_{-M^\beta}^{M^\beta}\left|L'\left(\frac{1}{2}+it+iv,\chi\right)\right|\frac{dv}{1+|v|}+1, \tag{2.13}$$

we have

$$f_1\left(\frac{1}{2}+it,\chi\right)\ll g_1(t,\chi)+\frac{\delta_\chi M_1^{1/2}L}{1+|t|}. \tag{2.14}$$

Similarly we have

$$f_2\left(\frac{1}{2}+it,\chi\right)\ll g_2(t,\chi)+\frac{\delta_\chi M_2^{1/2}L}{1+|t|}, \tag{2.15}$$

where

$$g_2(t,\chi)=\int_{-M^\beta}^{M^\beta}\left|L\left(\frac{1}{2}+it+iv,\chi\right)\right|\frac{dv}{1+|v|}+1.$$

Write

$$g_3(t,\chi)=\prod_{j=3}^{10}f_j\left(\frac{1}{2}+it,\chi\right)=\sum_{M_3\cdots M_{10}<m\le 2^8M_3\cdots M_{10}}\frac{b(m)\chi(m)}{m^{1/2+it}}, \tag{2.16}$$

where $|b(m)|\le d_8(m)$ and $d_k(m)$ denotes the number of ways of expressing m as a product of k factors. Then we have

$$F\left(\frac{1}{2}+it\right)=f_1\left(\frac{1}{2}+it,\chi\right)f_2\left(\frac{1}{2}+it,\chi\right)g_3(t,\chi).$$

Therefore by (2.14) and (2.15), we get

$$\begin{aligned}&\sum_{\chi \bmod q}\int_{-T}^{T}\left|F\left(\frac{1}{2}+it,\chi\right)\right|dt\\ &\ll\sum_{\chi \bmod q}\int_{-T}^{T}|g_1(t,\chi)g_2(t,\chi)g_3(t,\chi)|\,dt+M_1^{1/2}L\int_{-T}^{T}\frac{|g_2(t,\chi^0)g_3(t,\chi^0)|}{1+|t|}dt\\ &+M_2^{1/2}L\int_{-T}^{T}\frac{|g_1(t,\chi^0)g_3(t,\chi^0)|}{1+|t|}dt+M_1^{1/2}M_2^{1/2}L^2\int_{-T}^{T}\frac{|g_3(t,\chi^0)|}{(1+|t|)^2}dt\\ &:=H_1+H_2+H_3+H_4,\end{aligned} \tag{2.17}$$

say. By (2.16), one has

$$|g_3(t,\chi^0)|\ll\sum_{M_3\cdots M_{10}<m\le 2^8M_3\cdots M_{10}}\frac{d_8(m)}{m^{1/2}}\ll(M_3...M_{10})^{1/2}L^7, \tag{2.18}$$

by making use of (12.1.3) and (12.1.4) in ref. [16]. So we get

$$H_4\ll M^{1/2}L^9. \tag{2.19}$$

Again by (2.18), we have

$$H_3\ll(M_2M_3...M_{10})^{1/2}L^8\int_{-T}^{T}\frac{|g_1(t,\chi^0)|}{1+|t|}dt.$$

By (2.14), we have

$$\begin{aligned}\int_{-T}^{T}\frac{\left|g_1\left(t,\chi^0\right)\right|}{1+|t|}dt &\ll \int_{-T}^{T}\int_{-M^\beta}^{M^\beta}\frac{\left|L'\left(\frac{1}{2}+i(t+v),\chi^0\right)\right|}{(1+|t|)(1+|v|)}dvdt+L\\ &\ll \int_{-2M^\beta}^{2M^\beta}\left|L'\left(\frac{1}{2}+iw,\chi^0\right)\right|\int_{-T}^{T}\frac{dtdw}{(1+|t|)(1+|w-t|)}+L.\end{aligned}$$

Let $I=[w/2,3w/2]\cap[-T,T]$. Then for $t\notin I$, one has $|w-t|\geq|w|/2$; while for $t\in I$, one has $|t|\geq|w|/2$. Therefore

$$\begin{aligned}&\int_{-T}^{T}\frac{dt}{(1+|t|)(1+|w-t|)}\\ &\ll \frac{1}{1+|w|}\int_I\frac{dt}{1+|w-t|}+\frac{1}{1+|w|}\int_{[-T,T]\setminus I}\frac{dt}{1+|t|}\ll\frac{L}{1+|w|}.\end{aligned}$$

Thus we get

$$\begin{aligned}\int_{-T}^{T}\frac{\left|g_1\left(t,\chi^0\right)\right|}{1+|t|}dt &\ll L\int_{-2M^\beta}^{2M^\beta}\left|L'\left(\frac{1}{2}+iw,\chi^0\right)\right|\frac{dw}{1+|w|}+L\\ &\ll L^2\max_{1\leq X\leq M^\beta}\frac{1}{X}\int_{-2X}^{2X}\left|L'\left(\frac{1}{2}+iw,\chi^0\right)\right|dw+L.\end{aligned}$$

By (2.8), (2.11), (2.12) and Hölder's inequality, one easily obtains

$$\begin{aligned}&\left\{\int_{-2X}^{2X}\left|L'\left(\frac{1}{2}+iw,\chi^0\right)\right|dw\right\}^4\\ &\ll d^2(q)X^3\int_{-2X}^{2X}\left|\zeta'\left(\frac{1}{2}+iw\right)\right|^4dw+d^2(q)L^6X^3\int_{-2X}^{2X}\left|\zeta\left(\frac{1}{2}+iw\right)\right|^4dw\\ &\ll d^2(q)X^4L^{10},\end{aligned}$$

by Lemma 2.3. Hence we get

$$\int_{-T}^{T}\frac{\left|g_1\left(t,\chi^0\right)\right|}{1+|t|}dt\ll d^{1/2}(q)L^5.$$

This establishes

$$H_3\ll(M_2...M_{10})^{1/2}d^{1/2}(q)L^{13}\ll M^{1/2}d^{1/2}(q)L^{13}. \tag{2.20}$$

Similarly one can prove that

$$H_2\ll M^{1/2}d^{1/2}(q)L^{12}. \tag{2.21}$$

To bound H_1, we write

$$G_k=\sum_{\chi \bmod q}\int_{-T}^{T}|g_k(t,\chi)|^4dt,\quad k=1,\ 2;\quad G_3=\sum_{\chi \bmod q}\int_{-T}^{T}\left|g_3\left(t,\chi\right)\right|^2dt.$$

Then by Hölder's inequality, we have

$$H_1\ll G_1^{1/4}G_2^{1/4}G_3^{1/2}. \tag{2.22}$$

By Lemma 2.2 and (2.16), we have

$$\begin{aligned} G_3 &\ll \sum_{M_3\cdots M_{10}<m\le 2^8 M_3\cdots M_{10}} (qT+m)\,\frac{d_8^2(m)}{m} \\ &\ll (qT+M_3\cdots M_{10})L^c \ll (qT+M^{3/5})L^c. \end{aligned} \tag{2.23}$$

since $M_3\cdots M_{10} \ll M/(M_1M_2) < M^{3/5}$.

Now it remains to bound G_1 and G_2. By (2.13) and Hölder's inequality, we have

$$G_1 \ll L^3 \sum_{\chi \bmod q} \int_{-T}^{T}\int_{-M^\beta}^{M^\beta} \left|L'\left(\frac{1}{2}+it+iv,\chi\right)\right|^4 \frac{dvdt}{1+|v|} + qT.$$

Wtite $\int_{-M^\beta}^{M^\beta} = \int_{-2T}^{2T} + \int_{2T<|v|\le M^\beta}$. Then the first term on the right splits accordingly into two quantities which we denote by G_{11} and G_{12}, respectively. We have

$$\begin{aligned} G_{11} &= L^3 \sum_{\chi \bmod q} \int_{-2T}^{2T} \frac{dv}{1+|v|} \int_{-T+v}^{T+v} \left|L'\left(\frac{1}{2}+iw,\chi\right)\right|^4 dw \\ &\ll L^4 \sum_{\chi \bmod q} \int_{-3T}^{3T} \left|L'\left(\frac{1}{2}+iw,\chi\right)\right|^4 dw \ll qTL^{13}, \end{aligned}$$

by Lemma 2.3. To bound G_{12}, we let $w=t+v$. We observe that $2T<|v|\le M^\beta$ and $|t|\le T$ imply $|v|\ge |w|/2$ and $T<|w|\le 2M^\beta$. So it follows that

$$\begin{aligned} G_{12} &\ll TL^3 \sum_{\chi \bmod q} \int_{T}^{2M^\beta} \left|L'\left(\frac{1}{2}+iw,\chi\right)\right|^4 \frac{dw}{1+|w|} \\ &\ll TL^4 \max_{T\le X\le M^\beta} \frac{1}{X} \sum_{\chi \bmod q} \int_{-2X}^{2X} \left|L'\left(\frac{1}{2}+iw,\chi\right)\right|^4 dw \ll qTL^{13}. \end{aligned}$$

This proves

$$G_1 \ll qTL^{16}. \tag{2.24}$$

Similar argument also leads to

$$G_2 \ll qTL^{12}. \tag{2.25}$$

Putting (2.23)-(2.25) into (2.22), we get

$$H_1 \ll (qT)^{1/2}\,(qT+M^{3/5})^{1/2}L^c \ll \left(qT+(qT)^{1/2}M^{3/10}\right)L^c.$$

This together with (2.17) and (2.19)-(2.21) prove that

$$\sum_{\chi \bmod q} \int_{T}^{2T} \left|F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll \left(qT+(qT)^{1/2}M^{3/10}+M^{1/2}\right) d^{1/2}(q)L^c.$$

This proves Lemma 2.5. □

Lemma 2.6. *If there is a partition $\{J_1, J_2\}$ of the set $\{1, ..., 10\}$ such that*

$$\prod_{j\in J_1} M_j + \prod_{j\in J_2} M_j \ll M^{3/5},$$

then Lemma 2.1 is true.

Proof. For $\nu = 1, 2$, define

$$F_\nu(s,\chi) := \prod_{j\in J_\nu} f_j(s,\chi) = \sum_{n\le N_\nu} \frac{b_\nu(n)\chi(n)}{n^s},$$

where $N_\nu = \prod_{j\in J_\nu}(2M_j)$ and $b_\nu(n) \ll Ld_{10}(n)$. By Lemma 2.2, we have

$$\sum_{\chi \bmod q} \int_{-T}^{T} \left| F_1\left(\frac{1}{2}+it,\chi\right)\right|^2 dt \ll L^2 \sum_{n\le N_1} (qT+n)\frac{d_{10}^2(n)}{n} \ll (qT+N_1)L^c,$$

Similarly

$$\sum_{\chi \bmod q} \int_{-T}^{T} \left| F_2\left(\frac{1}{2}+it,\chi\right)\right|^2 dt \ll (qT+N_2)L^c,$$

Therefore by Cauchy's inequality we get

$$\begin{aligned}\sum_{\chi \bmod q} \int_{-T}^{T} \left| F\left(\frac{1}{2}+it,\chi\right)\right| dt &\ll (qT+N_1)^{1/2}(qT+N_2)^{1/2}L^c \\ &\ll (qT+(qT)^{1/2}M^{3/10}+M^{1/2})L^c,\end{aligned}$$

since $N_1 + N_2 \ll M^{3/5}$ and $N_1N_2 \ll M$. This proves Lemma 2.6. □

Proof of Lemma 2.1. In view of Lemma 2.5, we may assume that $M_iM_j \le M^{2/5}$ holds for all i, j with $1 \le i < j \le 5$. Then it follows that there is at most one of M_j with $j \le 5$ such that $M_j > M^{1/5}$. Without loss of generality, we may assume that this possible exceptional M_j is M_1. Then we have $M_j \le M^{1/5}$ for $j = 2, ..., 5$, and also for $j = 6, ..., 10$, by assumption. Let $l \ge 2$ be the integer such that

$$M_1 \cdots M_l \le M^{2/5}, \quad \text{but} \quad M_1 \cdots M_{l+1} > M^{2/5}.$$

Let $J_1 = \{1, 2, ..., l+1\}$, and $J_2 = \{l+2, ..., 10\}$. Then it is easy to check that

$$M^{2/5} \ll \prod_{j\in J_1} M_j \ll M^{3/5}.$$

This means that the assumption of Lemma 2.6 is satisfied. And the assertion of Lemma 2.1 thus follows. □

3. The proof of Theorem 1.1

Lemma 3.1. *Under conditions of Theorem 1, we assume further that*

$$2 < q \le x, \qquad q|\lambda| < x^{1-k}. \tag{3.1}$$

Then we have

$$\sum_{\chi \bmod q} \left| \sum_{x<m\le 2x} \Lambda(m)\chi(m)e(\lambda m^k) \right|$$
$$\ll d^{1/2}(q)\log^c x \left\{ qx^{1/2}\sqrt{1+|\lambda|x^k} + q^{1/2}x^{4/5} + \frac{x}{\sqrt{1+|\lambda|x^k}} \right\}.$$

Proof. By integration by parts, we have

$$\sum_{x<m\le 2x} \Lambda(m)\chi(m)e(\lambda m^k) = \int_x^{2x} e(\lambda u^k)d\sum_{x<m\le u}\Lambda(m)\chi(m). \tag{3.2}$$

Now we apply Heath-Brown's identity (see ref. [17], Lemma 1) with $k=5$ which reveals that for $m \le 2x$,

$$\Lambda(m) = \sum_{j=1}^{5}(-1)^{j-1}\binom{5}{j} \sum_{\substack{m_1\cdots m_{2j}=m \\ m_{j+1},\dots,m_{2j}\le (2x)^{1/5}}} (\log m_1)\mu(m_{j+1})\cdots\mu(m_{2j}).$$

On putting this in (3.2), we find that the sum over m becomes a linear combination of $O(L^{10})$ terms of the form

$$\mathit{\Sigma}(u;\mathbf{M}) = \sum_{\substack{m_1\sim M_1 \\ x<m_1\cdots m_{10}\le u}} \cdots \sum_{m_{10}\sim M_{10}} a_1(m_1)\chi(m_1)\cdots a_{10}(m_{10})\chi(m_{10}),$$

where $a_j(m)$, $j=1,\ 2,\dots,10$ are defined by (2.2), and M_j are positive integers such that (2.1) holds with $M=x$. Here $\mathbf{M}$ is written for the vector $(M_1, M_2, \dots, M_{10})$.

Let $f_j(s,\chi)$ and $F(s,\chi)$ be defined by (2.3). Then by Lemma 2.4, we have

$$\mathit{\Sigma}(u;\mathbf{M}) = \frac{1}{2\pi i}\int_{1+1/L-iT}^{1+1/L+iT} F(s,\chi)\frac{u^s - x^s}{s}ds + O\left(\frac{uL^2}{T} + L\min\{1, \frac{u}{T\|u\|}\}\right),$$

where $L=\log x$ and $T\ge 2$ is a parameter. We now move the integral along the rectangular contour with vertices $1/2\pm iT$, $1+1/L\pm iT$ to the line Re $s=1/2$. Then the integral on the two horizontal segments is

$$\ll |F(\sigma \pm iT,\chi)|\frac{x^\sigma}{T} \ll \left(\prod_{j=1}^{10} M_j^{1-\sigma}\right)\frac{x^\sigma L}{T} \ll \frac{xL}{T}.$$

Therefore we get

$$\mathit{\Sigma}(u;\mathbf{M}) = \frac{1}{2\pi}\int_{-T}^{T} F\left(\frac{1}{2}+it,\chi\right)\frac{u^{1/2+it}-x^{1/2+it}}{1/2+it}dt + R(u),$$

where

$$R(u) \ll \frac{xL^2}{T} + L\min\left\{1, \frac{u}{T\|u\|}\right\}. \tag{3.3}$$

Hence the right-hand side of (3.2) becomes a linear combination of $O(L^{10})$ terms of the form

$$\begin{aligned}&\frac{1}{2\pi i}\int_x^{2x} e(\lambda u^k)d\Sigma(u;\mathbf{M})\\&=\frac{1}{2\pi}\int_{-T}^{T} F\left(\frac{1}{2}+it,\chi\right)\int_x^{2x} u^{-1/2+it}e(\lambda u^k)dudt+O\left(\left|\int_x^{2x} e(\lambda u^k)dR(u)\right|\right).\end{aligned}\tag{3.4}$$

Without loss of generality, we assume that $\|x\|=1/4$. Then by (3.3) we have

$$\int_x^{2x} e(\lambda u^k)dR(u) \ll (1+|\lambda|x^k)\frac{xL^2}{T}+|\lambda|x^{k-1}L\int_x^{2x}\min\left\{1,\frac{u}{T\|u\|}\right\}du.$$

But

$$\int_x^{2x}\min\left\{1,\frac{u}{T\|u\|}\right\}du \ll \sum_{x<m\le 2x}\int_0^{1/2}\min\left\{1,\frac{m+t}{Tt}\right\}dt \ll \frac{x^2L}{T}.$$

Thus on taking $T=T_0=(1+|\lambda|x^k)^{3/2}qL^2$, we see that the error term in (3.4) is $O\left(xq^{-1}/\sqrt{1+|\lambda|x^k}\right)$. On the other hand, we have

$$\int_x^{2x} u^{-1/2+it}e(\lambda u^k)du=\frac{1}{k}\int_{x^k}^{(2x)^k} v^{-1+1/(2k)}e\left(\frac{t}{2k\pi}\log v+\lambda v\right)dv.\tag{3.5}$$

Here by Lemmas 4.3 and 4.5 in ref. [16], the right integral is

$$\ll x^{1/2}\min\left\{1,\frac{1}{\min\limits_{x^k<v\le(2x)^k}|t+2k\pi\lambda v|},\frac{1}{\sqrt{|t|}}\right\}.\tag{3.6}$$

Note that

$$\min_{x^k<v\le(2x)^k}|t+2k\pi\lambda v| \gg \begin{cases}|\lambda|x^k, & \text{if} \quad |t|<k\pi|\lambda|x^k,\\ |t|, & \text{if} \quad |t|\ge 4k\pi|\lambda|(2x)^k.\end{cases}\tag{3.7}$$

Hence we deduce from (3.4)-(3.7) that

$$\begin{aligned}&\int_x^{2x} e(\lambda u^k)d\Sigma(u;\mathbf{M})\\&\ll \frac{x^{1/2}}{\sqrt{1+|\lambda|x^k}}\int_{|t|\le 4k\pi|\lambda|(2x)^k}\left|F\left(\frac{1}{2}+it,\chi\right)\right|dt\\&\quad +x^{1/2}\int_{4k\pi|\lambda|(2x)^k<|t|\le T_0}\left|F\left(\frac{1}{2}+it,\chi\right)\right|\frac{dt}{1+|t|}+\frac{xq^{-1}}{\sqrt{1+|\lambda|x^k}}.\end{aligned}$$

Accordingly we have

$$
\begin{aligned}
&\sum_{\chi \bmod q}\left|\sum_{x<m\le 2x}\Lambda(m)\chi(m)e(\lambda m^k)\right|\\
&\ll \frac{x^{1/2}}{\sqrt{1+|\lambda|x^k}}\sum_{\mathbf{M}}\sum_{\chi \bmod q}\int_{|t|\le 4k\pi|\lambda|(2x)^k}\left|F\left(\frac{1}{2}+it,\chi\right)\right|dt\\
&\quad +x^{1/2}L\max_{4k\pi|\lambda|(2x)^k<T\le T_0}\frac{1}{1+T}\sum_{\mathbf{M}}\sum_{\chi \bmod q}\int_{T<|t|\le 2T}\left|F\left(\frac{1}{2}+it,\chi\right)\right|dt+\frac{xL^{10}}{\sqrt{1+|\lambda|x^k}}.
\end{aligned}
$$

Now Lemma 3.1 follows by appealing to Lemma 2.1. □

Proof of Theorem 1.1. Since (1.2) is trivial for $q>x$ or $q|\lambda|\ge x^{1-k}$, we may assume (3.1). For Dirichlet character $\chi \bmod q$, we define

$$C_k(\chi,a)=\sum_{h=1}^{q}\overline{\chi}(h)e\left(\frac{ah^k}{q}\right).$$

Then Vinogradov's bound (see e.g. ref. [18]) gives

$$|C_k(\chi,a)|\le 2(d(q))^{\alpha_k}q^{1/2},\quad \text{where}\quad \alpha_k=\log k/\log 2. \tag{3.8}$$

Let $\alpha=a/q+\lambda$ with $(a,q)=1$ and $\lambda\in\mathbb{R}$. By orthognolity of Dirichlet characters, we have

$$
\begin{aligned}
S_k(\alpha) &= \sum_{\chi \bmod q}\frac{C_k(\chi,a)}{\varphi(q)}\sum_{x<m\le 2x}\Lambda(m)\chi(m)e(\lambda m^k)+O(L^2)\\
&\ll (d(q))^{\alpha_k}q^{-1/2}L\sum_{\chi \bmod q}\left|\sum_{x<m\le 2x}\chi(m)\Lambda(m)e(\lambda m^k)\right|+O(L^2),
\end{aligned}
$$

by (3.8). Now the assertion of Theorem 1.1 follows immediately by Lemma 3.1. □

4. Proof of Theorem 1.3

We will consider n with $N/2<n\le N$. Let

$$R(n)=\sum_{n=n_1^2+n_2^2+n_3^2}\Lambda(n_1)\Lambda(n_2)\Lambda(n_3).$$

Then $R(n)$ is the number of weighted representations of n by (1.5). To apply the circle method, we let

$$P=N^{1/6-\varepsilon},\qquad P^*=N^{1/4-\varepsilon},\qquad Q=N/(PL^{14}),\qquad Q^*=N/(P^*L^{14}).$$

Now define the major arcs $\mathfrak{M}$ be the union of intervals $[a/q-1/qQ,a/q+1/qQ]$ with $1\le a\le q\le P$, $(a,q)=1$, and the minor arcs $\mathfrak{m}$ by $\mathfrak{m}=[1/Q,1+1/Q]\backslash\mathfrak{M}$. Then

$$R(n)=\int_{1/Q}^{1+1/Q}S_2^3(\alpha)e(-n\alpha)d\alpha=\int_{\mathfrak{M}}+\int_{\mathfrak{m}}. \tag{4.1}$$

The integral on the major arcs can be treated by Theorem 2 of ref. [8], which states that, for $N/2 < n \leq N$,

$$\int_{\mathfrak{M}} = \mathfrak{S}(n, P) n^{1/2} (1 + O(\log^{-A} N)) \tag{4.2}$$

where

$$\mathfrak{S}(n, P) \gg \log^{-B} N,$$

for all n satisfying (1.6) with at most $O(N^{3/8+\varepsilon})$ exceptions.

Now we bound the contribution from the minor arcs. By Dirirchlet's lemma on rational approximations, every $\alpha \in \mathfrak{m}$ can be written as (1.1) with

$$1 \leq q \leq Q^*, \qquad |\lambda| \leq 1/(qQ^*).$$

We let $\mathfrak{n}$ be the set of $\alpha \in \mathfrak{m}$ satisfying (1.1) such that

$$P^* < q \leq Q^*, \qquad |\lambda| \leq 1/(qQ^*).$$

On $\mathfrak{n}$, Ghosh's bound (1.4) with $x = N^{1/2}$ gives

$$\max_{\alpha \in \mathfrak{n}} |S_2(\alpha)| \ll N^{7/16+\varepsilon}. \tag{4.3}$$

Let $\mathfrak{k}$ be the complement of $\mathfrak{n}$ in $\mathfrak{m}$, so that $\mathfrak{m} = \mathfrak{k} \cup \mathfrak{n}$. For $\alpha \in \mathfrak{k}$, we have either

$$P < q \leq P^*, \qquad |\lambda| \leq 1/(qQ^*),$$

or

$$q \leq P, \qquad 1/(qQ) < |\lambda| \leq 1/(qQ^*).$$

In either case, we have, with $x = N^{1/2}$,

$$N^{1/12-\varepsilon} \ll \sqrt{\min\left(P, \frac{x^2}{Q}\right)} \ll \sqrt{q(1+|\lambda|x^2)} \ll \sqrt{P^* + \frac{x^2}{Q^*}} \ll N^{1/8+\varepsilon}.$$

Therefore, Theorem 1.1 gives

$$\max_{\alpha \in \mathfrak{k}} |S_2(\alpha)| \ll N^{5/12+\varepsilon}. \tag{4.4}$$

Collecting (4.3) and (4.4), we have

$$\max_{\alpha \in \mathfrak{m}} |S_2(\alpha)| \ll N^{7/16+\varepsilon}. \tag{4.5}$$

Now we consider the mean square of the integral over $\mathfrak{m}$ in (4.1), and get by Bessel's inequality

$$\sum_{N/2<n\leq N} \left| \int_{\mathfrak{m}} \right|^2 \ll \int_{\mathfrak{m}} |S_2(\alpha)|^6 d\alpha \ll \max_{\alpha \in \mathfrak{m}} |S_2(\alpha)|^2 \int_0^1 |S_2(\alpha)|^4 d\alpha \ll N^{15/8+\varepsilon}, \tag{4.6}$$

by (4.5) and Hua's lemma. The assertion of Theorem 1.3 now follows from (4.2) and (4.6) via a standard argument. □

Acknowledgements The author is supported by Post-Doctoral Fellowship of The University of Hong Kong.

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