A mean square formula for central values of twisted automorphic L-functions

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1. Introduction. Let k be an even positive integer and $S_k(\Gamma(1))$ be the space of all holomorphic cusp forms of weight k with respect to the full modular group. It is known that $S_k(\Gamma(1))$ has a basis $\mathcal{B}_k$ consisting of normalized cusp forms f which are simultaneously eigenforms for all Hecke operators T_n. To be specific, $T_n f = \lambda_f(n) n^{(k-1)/2} f$, and f has the Fourier series

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e(nz)$$

where $e(\alpha) = e^{2\pi i \alpha}$. Note that $\lambda_f(1) = 1$ and each $\lambda_f(n)$ is real.

Let χ (mod D) be a primitive Dirichlet character. Associated with each f, the twisted L-function is defined as

$$L(f \otimes \chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)\lambda_f(n)}{n^s} \qquad (\Re e\, s > 1). \tag{1.1}$$

This L-function possesses the usual properties of classical L-functions. Define

$$\Lambda(f \otimes \chi, s) = \left(\frac{D}{2\pi}\right)^s \Gamma(s + \frac{k-1}{2}) L(f \otimes \chi, s). \tag{1.2}$$

We have from [Iw, Theorem 7.6] that $\Lambda(f \otimes \chi, s)$ can be holomorphically continued to the whole of $\mathbb{C}$, bounded on any vertical strip, and satisfies the functional equation

$$\Lambda(f \otimes \chi, s) = \epsilon_k(\chi) \Lambda(f \otimes \overline{\chi}, 1 - s) \tag{1.3}$$

where the root factor $\epsilon_k(\chi) = i^k \tau(\chi)^2 / D$. ($\tau(\chi)$ is the Gaussian sum.)

The central values $L(f \otimes \chi, 1/2)$ are of particular importance and interests; indeed, the nonvanishing nature of these values are linked to different arithmetic problems (see [IS]). An interesting result about the central value

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is the non-negativity of $L(f \otimes \chi, 1/2)$ for any real character χ. To get non-vanishing results, we investigate the first and the second moments (with mollifiers). It is not hard to derive the formula for the first moment below for large k by using (1.3),

$$\sum_{f \in \mathcal{B}_k} w_f L(f \otimes \chi, 1/2) = 1 + \epsilon_k(\chi) + O(k^{-1}), \tag{1.4}$$

where

$$w_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1}\|f\|^2} \ll \frac{\log k}{k} \tag{1.5}$$

by [HL] or [KS, (4)]. (The O-constant is independent of k and $\chi \pmod D$.) In addition, for quadratic character $\chi \pmod D$, Kohnen and Sengupta [KS] proved that for any $\epsilon > 0$,

$$\sum_{f \in \mathcal{B}_k} L(f \otimes \chi, 1/2) \ll_D k^{1+\epsilon} \qquad \text{as } k \to \infty.$$

In particular, assuming the Lindelöf hypothesis $L(f \otimes \chi, 1/2) \ll_D k^{\epsilon_0}$, they showed that

$$\#\{f \in \mathcal{B}_k : L(f \otimes \chi, 1/2) \neq 0\} \gg_D \frac{k^{1-\epsilon_0}}{\log k} \qquad \text{as } k \to \infty. \tag{1.6}$$

Aiming at the problem of non-existence of Landau-Siegel zeros, Iwaniec and Sarnak [IS] investigated the moments (averaging over k)

$$\mathcal{A}_K[X_f] = \sum_{k \text{ even}} \frac{h(k/K)}{|\mathcal{B}_k|} \sum_{f \in \mathcal{B}_k} w_f X_f$$

where $X_f = L(f \otimes \chi, 1/2)$ or $L(f \otimes \chi, 1/2)^2$ (χ is real), and $h \in C_0^\infty(\mathbb{R}^+)$ is a test function. The role of h is to localize the weight k within an interval of length of order K. They got asymptotic results [IS, Theorem 1] as $K \to \infty$: let $H = \int_0^\infty h(t)\,dt$ and D be the modulus of the real character χ, then

$$\mathcal{A}_K[L(f \otimes \chi, 1/2)] \sim HK \quad \text{and} \quad \mathcal{A}_K[L(f \otimes \chi, 1/2)^2] \sim \frac{\phi(D)}{D} 2HK \log DK$$

where the asymptotics are uniform for $D \le K^\delta$ for some positive constant δ. (But this is not sufficient for their purpose and they considered mollified moments.)

In this paper, we establish an asymptotic formula for the second moment of $L(f\otimes\chi, 1/2)$ for all large even k for both real and complex primitive characters. As a consequence, we prove unconditionally the better lower bound $k/(\log^2 k)$ in (1.6). Moreover, our result here can be viewed as a supplement to giving an asymptotic formula for individual (large) k. Without the extra smoothing process over k, we cannot make use of the tool in [Sa, Section 3] or [Iw, Section 5.5].

Theorem 1. *Let $k \geq k_0$ be any sufficiently large even integer. Suppose that χ is a primitive Dirichlet character of conductor D, where $1 \leq D \leq k/(16\log k)$. We have the following.*

(a) If $D = 1$ (i.e. χ is the trivial character) and $k \equiv 0$ (4),

$$\sum_{f\in\mathcal{B}_k} w_f L(f, 1/2)^2 = 4\left(\frac{\Gamma'(k/2)}{\Gamma(k/2)} + \gamma - \log 2\pi\right) + O_A(k^{-A}),$$

where $A \geq 1$ is arbitrary, and the O-constant depends on A.

(b) If χ is real,

$$\begin{aligned}&\sum_{f\in\mathcal{B}_k} w_f L(f\otimes\chi, 1/2)^2 \\ &= 2(1+i^k\chi(-1))\frac{\phi(D)}{D}\Big(\log\frac{k}{2} + \gamma + \log\frac{D}{2\pi} + \sum_{p|D}\frac{\log p}{p-1}\Big) \\ &\quad + O(D^3k^{-1/2}(\log k)^4).\end{aligned}$$

(c) If χ is complex,

$$\begin{aligned}&\sum_{f\in\mathcal{B}_k} w_f L(f\otimes\chi, 1/2)^2 \\ &= 2\epsilon_k(\chi)\frac{\phi(D)}{D}(\log\frac{k}{2} + \gamma + \log\frac{D}{2\pi} + \sum_{p|D}\frac{\log p}{p-1}) \\ &\quad + (L(1,\chi^2) + \epsilon_k(\chi)^2 L(1,\overline{\chi}^2)) + O(D^3k^{-1/2}(\log k)^4).\end{aligned}$$

The O-constants are independent of D.

Theorem 2. *Suppose that* χ_1 *and* χ_2 *are primitive Dirichlet characters of conductors* D_1 *and* D_2 *respectively, and* $1 \le D_1 D_2 \le k/(16 \log k)$. *If* $\chi_1 \ne \chi_2$ *and* $\chi_1 \ne \overline{\chi}_2$, *then*

$$\begin{aligned}
&\sum_{f \in \mathcal{B}_k} w_f L(f \otimes \chi_1, 1/2) L(f \otimes \chi_2, 1/2) \\
&\quad = \; L(1, \chi_1 \chi_2) + \epsilon_k(\chi_1)\epsilon_k(\chi_2) L(1, \overline{\chi}_1 \overline{\chi}_2) \\
&\qquad + \epsilon_k(\chi_1) L(1, \overline{\chi}_1 \chi_2) + \epsilon_k(\chi_2) L(1, \chi_1 \overline{\chi}_2) + O((D_1 D_2)^{3/2} k^{-1/2} (\log k)^4).
\end{aligned}$$

Here $L(s, \psi)$ *denotes the Dirichlet* L*-function for the character* ψ.

Remark 1. For trivial character χ and $k \equiv 2 \mod 4$, the central value $L(f, 1/2)$ is zero by the functional equation (1.3).

Remark 2. A character is said to be complex when it is not a real character.

Remark 3. The error terms in the last three asymptotic formulas become prominent when $D_1 D_2 \gg k^{1/3}/(\log k)^2$. ($D_1 = D_2 = D$ in (b) and (c).)

Remark 4. Let $\mathcal{R}_T$ be the positively oriented rectangular contour with vertices at $\pm 2 \pm iT$. Taking $T \longrightarrow \infty$ and using (1.3), we have

$$\begin{aligned}
(1.7)\quad & \Lambda(f \otimes \chi, 1/2) \\
&= \frac{1}{2\pi i} \int_{\mathcal{R}_T} \Lambda(f \otimes \chi, 1/2 + w) \frac{dw}{w} \\
&= \frac{1}{2\pi i} \int_{(2)} \Lambda(f \otimes \chi, 1/2 + w) \frac{dw}{w} \\
&\qquad - \frac{\epsilon_k(\chi)}{2\pi i} \int_{(-2)} \Lambda(f \otimes \overline{\chi}, 1/2 - w) \frac{dw}{w} \\
&= \frac{1}{2\pi i} \int_{(2)} (\Lambda(f \otimes \chi, 1/2 + w) + \epsilon_k(\chi) \Lambda(f \otimes \overline{\chi}, 1/2 + w)) \frac{dw}{w}.
\end{aligned}$$

It is apparent that $\overline{\Lambda(f \otimes \chi, s)} = \Lambda(f \otimes \overline{\chi}, \overline{s})$ for $\Re e\, s > 1$. Hence

$$\overline{\Lambda(f \otimes \chi, 1/2)} = \Lambda(f \otimes \overline{\chi}, 1/2).$$

Using (1.3), we see that

$$\epsilon_k(\overline{\chi}) \Lambda(f \otimes \chi, 1/2)^2 = |\Lambda(f \otimes \chi, 1/2)|^2,$$

or equivalently, $\epsilon_k(\overline{\chi})L(f\otimes\chi,1/2)^2 = |L(f\otimes\chi,1/2)|^2$. Thus, for complex χ, Theorem 1 (c) is equivalent to

$$
\begin{aligned}
(1.8)\qquad & \sum_{f\in\mathcal{B}_k} w_f|L(f\otimes\chi,1/2)|^2 \\
= \;& 2\frac{\phi(D)}{D}(\log\frac{k}{2}+\gamma+\log\frac{D}{2\pi}+\sum_{p|D}\frac{\log p}{p-1}) \\
& + 2\Re e\,(\epsilon_k(\chi)L(1,\overline{\chi}^2)) + O(D^3k^{-1/2}(\log k)^4).
\end{aligned}
$$

since, for even k, $\epsilon_k(\overline{\chi})\epsilon_k(\chi) = |\epsilon_k(\chi)|^2 = 1$.

Remark 5. Our proof is based on the Petersson trace formula, which is different from [KS]. The approach of using this trace formula and investigating the contributions from the so-called diagonal and off-diagonal terms is explored in various articles, for example, [Du], [IS], [MV] and [Sa]. (Note that these papers do not deal with the situation of large individual weight.)

Finally we give a direct application of Theorem 1 on the non-vanishing of $L(f\otimes\chi,1/2)$.

Corollary 3. *Let k be any sufficiently large even integer. Suppose that either*

(i) χ is a real primitive character mod D where $1\le D\le k^{1/6}/(\log k)^5$, or

(ii) χ is a complex primitive character mod D with $\log D\le c_0\dfrac{\log k}{\log\log k}$ for some suitable positive constant c_0. Then we have

$$\#\{f\in\mathcal{B}_k : L(f\otimes\chi,1/2)\neq 0\} \gg |1+\epsilon_k(\chi)|^2\frac{D}{\phi(D)}\frac{k}{\log^2 k}$$

where the implied constant is independent of D but depends on c_0 in case (ii). (As $\epsilon_k(\chi)=i^k\chi(-1)$ for real χ, both sides will equal zero if $i^k\chi(-1)=-1$.)

Proof. In view of Theorem 1 (b) and (1.8) (for real and complex characters respectively), we obtain, by using the bound $L(1,\overline{\chi}^2)\ll\log D$ (as $\overline{\chi}^2$ is nonprincipal) for case (ii), that

$$\sum_{f\in\mathcal{B}_k} w_f|L(f\otimes\chi,1/2)|^2 \ll \frac{\phi(D)}{D}\log k$$

for D in the specified ranges. By the Cauchy-Schwarz inequality and (1.4),

$$\begin{aligned} |1+\epsilon_k(\chi)|^2 &\ll \Big| \sum_{f\in\mathcal{B}_k} w_f L(f\otimes\chi, 1/2)\Big|^2 \\ &\ll \sum_{f\in\mathcal{B}_k} w_f |L(f\otimes\chi,1/2)|^2 \sum_{\substack{f\in\mathcal{B}_k \\ L(f\otimes\chi,1/2)\neq 0}} \frac{\log k}{k} \end{aligned}$$

by (1.5). The result follows.

2. Some Preparation. The idea of our proof is to express the central value of $L(f\otimes\chi_1,s)L(f\otimes\chi_2,s)$ in terms of infinite sums via an integral analogous to (1.7). For $\Re e\, s>1$, we deduce from (1.1) and the relation $\lambda_f(m)\lambda_f(n)=\sum_{d|(m,n)}\lambda_f(mn/d^2)$ that

$$\begin{aligned} L(f\otimes\chi_1,s)L(f\otimes\chi_2,s) &= \sum_{m,n=1}^{\infty}\frac{\chi_1(m)\chi_2(n)}{(mn)^s}\sum_{d|(m,n)}\lambda_f(\frac{mn}{d^2}) \\ &= \sum_{d=1}^{\infty}\chi_1\chi_2(d)d^{-2s}\sum_{m,n=1}^{\infty}\frac{\chi_1(m)\chi_2(n)\lambda_f(mn)}{(mn)^s} \\ &= L(2s,\chi_1\chi_2)\sum_{n=1}^{\infty}\frac{\lambda_f(n)\tau_{\chi_1,\chi_2}(n)}{n^s} \qquad (2.1) \end{aligned}$$

where $L(\cdot,\chi_1\chi_2)$ is the Dirichlet L-function for the character $\chi_1\chi_2$ and

$$\tau_{\chi_1,\chi_2}(n)=\sum_{ab=n}\chi_1(a)\chi_2(b).$$

It turns out that the central value is represented by a sum of two rapidly convergent series. Averaging the series over all $f\in\mathcal{B}_k$ with the Petersson trace formula, the sum will consist of two types of terms: those coming from the Kronecker delta, called the diagonal terms, and those which involve the Kloosterman sums, the off-diagonal terms. The diagonal terms can be easily handled while for the off-diagonal terms, we open the Kloosterman sums and insert the Mellin transform for the Bessel function J_{k-1}. After rearrangements, one can find among all factors the (twisted) Dirichlet series associated with $\tau_{\chi_1,\chi_2}(n)$,

$$E_{\chi_1,\chi_2}(s,a/c)=\sum_{n=1}^{\infty}\tau_{\chi_1,\chi_2}(n)e(an/c)n^{-s} \qquad (\Re e\, s>1),$$

where $(a,c)=1$. This series will play a crucial role in our investigation. In fact, the main contribution of the off-diagonal terms comes from its pole.

The series $E_{\chi_1,\chi_2}(s,a/c)$ can be viewed as a generalization of $E(s,a/c)$ investigated by Estermann [Es] (or see [Ju]). Like $E(s,a/c)$, it possesses nice properties, as stated in Lemma 2.1 below. (The proof of this will be given in the last section.)

Lemma 2.1 *The function $E_{\chi_1,\chi_2}(s,a/c)$ can be analytically continued to a meromorphic function, which is holomorphic on $\mathbb{C}$ except possibly at $s=1$. The Laurent expansion of $E_{\chi_1,\chi_2}(s,a/c)$ at $s=1$ is of the form*

$$E_{\chi_1,\chi_2}(s,a/c) \;=\; A_{\chi_1,\chi_2}(a,c)(s-1)^{-2}+B_{\chi_1,\chi_2}(a,c)(s-1)^{-1}+\cdots .$$

When $\chi_1=\chi_2$, we put $\chi=\chi_1=\chi_2$ and $D=D_1=D_2$. For $c=D\kappa$ with $(D,\kappa)=1$,

$$\begin{aligned} A_{\chi,\chi}(a,c) &= c^{-1}\tau(\chi)\overline{\chi}(a)\chi(\kappa)\frac{\phi(D)}{D}, \qquad \textit{and}\\ B_{\chi,\chi}(a,c) &= 2c^{-1}\tau(\chi)\overline{\chi}(a)\chi(\kappa)\frac{\phi(D)}{D}(\gamma-\log\kappa+\sum_{p|D}\frac{\log p}{p-1}). \end{aligned}$$

In all other cases $A_{\chi_1,\chi_2}(a,c)=0$, and we have (for $\chi_1\neq\chi_2$)

$$\begin{aligned} B_{\chi_1,\chi_2}(a,c) &= \delta_{12}(c)c^{-1}\tau(\chi_1)\overline{\chi}_1(a)\chi_2(\frac{c}{D_1})L(1,\chi_2\overline{\chi}_1)\\ &\quad +\delta_{21}(c)c^{-1}\tau(\chi_2)\overline{\chi}_2(a)\chi_1(\frac{c}{D_2})L(1,\chi_1\overline{\chi}_2) \end{aligned}$$

where $\delta_{ij}(c)=1$ if $D_i|c$ and $(\frac{c}{D_i},D_j)=1$, and $\delta_{ij}(c)=0$ otherwise.

In addition, $E_{\chi_1,\chi_2}(s,a/c)$ satisfies the functional equation

$$\begin{aligned} &E_{\chi_1,\chi_2}(s,a/c)\\ &\quad = c_1[D_1,c]^{-s}[D_2,c]^{-s}(2\pi)^{2s-2}\Gamma(1-s)^2\sum_{\substack{u\,(D_1)\\ v\,(D_2)}}\chi_1(u)\chi_2(v)e(\frac{uva_0}{c})\\ &\qquad\times\Big(\Big(1+\chi_1\chi_2(-1)\Big)\varphi^+_{a,c}(1-s;u,-v)\\ &\qquad\qquad-\Big(e(\frac{s}{2})+\chi_1\chi_2(-1)e(-\frac{s}{2})\Big)\varphi^-_{a,c}(1-s;u,v)\Big) \end{aligned}$$

where c_1 divides c and a_0 is an integral multiple of a. When $\Re e\, s > 1$, the functions $\varphi^{\mp}_{a,c}(s;u,v)$, abbreviated for $\varphi^{\mp}_{a,c,D_1,D_2}(s;u,v)$, are given by

$$\varphi^{\mp}_{a,c}(s;u,v) = \sum_{n=1}^{\infty} n^{-s}\tau^{\mp}_{a,c}(n;u,v)e(\mp\frac{na_1}{c})$$

for some integer a_1. Also, we have $|\tau^{\mp}_{a,c}(n;u,v)| \le d(n)$. ($d(n) = \sum_{d|n} 1$ is the divisor function.)

Remark. The constants a_0 and a_1 depend only on a, c, D_1 and D_2, and the functions $\tau^{\mp}_{a,c}(n;u,v)$ also depend on D_1, D_2. When $D_1 = D_2 = 1$, we have $c_1 = c$, $h_0 \equiv 0 \mod c$ and $\varphi^{\mp}_{a,c}(s;1,1) = \sum_{n=1}^{\infty} n^{-s}d(n)e(\mp\overline{a}n/c) = E(s,\mp\overline{a}/c)$ for $\Re e\, s > 1$. Hence, the functional equation reduces to [Ju, Lemma 1]

$$E(s,a/c) = 2c^{1-2s}(2\pi)^{2s-2}\Gamma(1-s)^2(E(1-s,\overline{a}/c) - \cos(\pi s)E(1-s,-\overline{a}/c)).$$

Following from Lemma 2.1 and the Phragmén-Lindelöf Theorem, the function $E_{\chi_1,\chi_2}(s,h/k)$ satisfies the convexity bound

$$(2.2)\qquad E_{\chi_1,\chi_2}(\sigma+it,h/k) \ll_{D_1,D_2,k,C,\epsilon} (|t|+1)^{\alpha(\sigma)+\epsilon} \qquad \text{for any } \epsilon > 0,$$

where $C > 0$ is an arbitrary constant, $\alpha(\sigma) = 0$ for $\sigma \ge 1$, $\alpha(\sigma) = 1-\sigma$ for $0 \le \sigma \le 1$ and $\alpha(\sigma) = 1-2\sigma$ for $-C \le \sigma \le 0$.

In addition we need a few lemmas. We start with some results on the Bessel functions $J_n(x)$ and $Y_0(x)$, which will be used later. These two Bessel functions can be defined, for $x > 0$, as

$$(2.3)\qquad J_n(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!(n+l)!}\left(\frac{x}{2}\right)^{n+2l} \qquad (n = 0, 1, \cdots),$$

$$(2.4)\qquad Y_0(x) = \frac{2}{\pi}J_0(x)\log\frac{x}{2} - \frac{2}{\pi}\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma'(l+1)}{\Gamma(l+1)^3}\left(\frac{x}{2}\right)^{2l}.$$

For all $x \ge 1$, [Le, (5.11.6) and (5.11.7)]

$$(2.5)\qquad \begin{aligned} Y_0(x) &= \sqrt{\frac{2}{\pi x}}\sin(x-\pi/4) + O(x^{-3/2}), \\ J_n(x) &= \sqrt{\frac{2}{\pi x}}\cos(x - n\pi/2 - \pi/4) + O_n(x^{-3/2}), \end{aligned}$$

where the O-term in the second formula depends on n. Furthermore [Le, (5.10.8)], for any positive integer n,

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta - n\theta)\,d\theta = \frac{1}{\pi}\int_0^{\pi/2}\Re e\, f_n(\theta, x)\,d\theta, \tag{2.6}$$

where $f_n(\theta, x) = (e^{-in\theta} + (-1)^n e^{in\theta})e^{ix\sin\theta}$. Also, [Le, (5.10.2)]

$$\begin{aligned} J_{k-1}(x) &= \frac{1}{\sqrt{\pi}\Gamma(k-1/2)}\left(\frac{x}{2}\right)^{k-1}\int_{-1}^{1}(1-t^2)^{k-3/2}\cos(xt)\,dt \\ &\ll \left(\frac{ex}{2k}\right)^{k-1}, \end{aligned} \tag{2.7}$$

with an absolute implied constant. Finally, we notice that the functions $J_{k-1}(x)$ and $2^{s-1}\Gamma((k-1+s)/2)/\Gamma((k+1-s)/2)$ are Mellin transform pairs; that is,

$$J_{k-1}(x) = \frac{1}{2\pi i}\int_{(-1)}\frac{\Gamma((k+s)/2)}{\Gamma((k-s)/2)}2^s x^{-s-1}\,ds, \tag{2.8}$$

$$\frac{\Gamma((k+s)/2)}{\Gamma((k-s)/2)} = \int_0^{\infty} J_{k-1}(x)\left(\frac{x}{2}\right)^s dx \qquad (-k < \Re e\, s < -1/2). \tag{2.9}$$

Our first lemma below prepares an estimate of the Gamma function. The second one transforms two integrals of the Gamma functions into integrals of the Bessel functions. The third lemma gives upper bound estimates for certain integrals of the Bessel functions, which we will make use of later.

Lemma 2.2 *Let $s = \sigma + it$ and $A > 1/2$ be a fixed constant. For all sufficiently large k ($\geq k_0(A)$) and $0 \leq \sigma < A$, we have*

$$\frac{\Gamma(k-s)}{\Gamma(k+s)} \ll_A (k+|t|)^{-2\sigma}.$$

The implied constant depends on A only.

Proof. Using Stirling's formula [Le, (1.4.12)], we obtain that

$$\begin{aligned} &\Re e\,(\log\Gamma(k-s) - \log\Gamma(k+s)) \\ &= \frac{1}{2}(k-\sigma-1/2)\log\frac{(k-\sigma)^2+t^2}{(k+\sigma)^2+t^2} \\ &\qquad - \sigma\log((k+\sigma)^2+t^2) - t\tan^{-1}\frac{2\sigma t}{k^2-\sigma^2+t^2} + O(1) \\ &= -\sigma\log((k+\sigma)^2+t^2) + O(1). \end{aligned}$$

Lemma 2.3 *Let $k > 2$ be any integer and $y > 0$. Suppose that $0 < \Re e\, w < k/2 - 2$. Then,*

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{(3)} \frac{\Gamma(k/2-w-z/2)}{\Gamma(k/2+w+z/2)}\Gamma(\frac{z}{2})^2\cos(\frac{\pi z}{2})y^{-z}\,dz \\
&= -2^{1+2w}\pi\int_0^\infty J_{k-1}(x)Y_0(yx)x^{-2w}\,dx, \\
&\frac{1}{2\pi i}\int_{(3)} \frac{\Gamma(k/2-w-z/2)}{\Gamma(k/2+w+z/2)}\Gamma(\frac{z}{2})^2\sin(\frac{\pi z}{2})y^{-z}\,dz \\
&= 2^{1+2w}\pi\int_0^\infty J_{k-1}(x)J_0(yx)x^{-2w}\,dx.
\end{aligned}$$

Proof. It can be seen that these four integrals are holomorphic in w for $0 < \Re e\, w < k/2 - 2$. Thus, it suffices to show that the equalities hold in a certain set (containing an accumulation point). Suppose that $w > 1$ is real. Applying the residue theorem with (2.3) and (2.4) (or see [Ti, p.197]), we obtain, for $x > 0$,

$$2\pi(iJ_0(x) - Y_0(x)) = \frac{1}{2\pi i}\int_{(1/2)} 2^s\Gamma(\frac{s}{2})^2 e^{i\pi s/2}x^{-s}\,ds.$$

Consider the integral

$$\frac{1}{2\pi i}\int_{(3)} \frac{\Gamma(k/2-w-z/2)}{\Gamma(k/2+w+z/2)}\Gamma(\frac{z}{2})^2 e^{i\pi z/2}y^{-z}\,dz.$$

Moving the line of integration to $\Re e\, z = 1/2$ and using (2.9), it becomes

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{2\pi i}\int_{1/2-iT}^{1/2+iT}\int_0^\infty J_{k-1}(x)\left(\frac{x}{2}\right)^{-2w-z}dx\,\Gamma(\frac{z}{2})^2 e^{i\pi z/2}y^{-z}\,dz \\
&= \int_0^\infty J_{k-1}(x)\left(\frac{x}{2}\right)^{-2w}\frac{1}{2\pi i}\int_{(1/2)} 2^z\Gamma(\frac{z}{2})^2 e^{i\pi z/2}(xy)^{-z}\,dz\,dx \\
&= 2\pi\int_0^\infty J_{k-1}(x)(iJ_0(yx) - Y_0(yx))\left(\frac{x}{2}\right)^{-2w}dx.
\end{aligned}$$

This completes the proof by equating the real and imaginary parts.

Lemma 2.4 *Let $s = \sigma + it$ and $A > 1/2$ be a fixed constant. We denote*

$$B_0(x) = J_0(x) \text{ or } Y_0(x).$$

Then, for all sufficiently large k ($\geq k_0(A)$), and $1/2 \leq \sigma \leq A$, *(a) if $a > 1$,*

$$\int_0^\infty J_{k-1}(x)B_0(ax)x^{-s}\,dx \ll e^{\pi|t|/2}\frac{a^{\sigma-k}}{1-a^{-2}}.$$

(b) if $k^{-1/2} \le a \le 1$,

$$\int_0^\infty J_{k-1}(x)B_0(ax)x^{-s}\,dx \ll (|t|+1)a^{-1/2}k^{-\sigma-1/2}(\log k)^2.$$

Proof. (a) For $a > 1$, we have the formulae [Er, §6.8 (37)] and [WG, §7.15 (8)]:

$$\begin{aligned}
&\int_0^\infty J_{k-1}(x)Y_0(ax)x^{-s}\,dx \\
&= \frac{\cos(\pi s/2)}{2^s\pi a^{k-s}}\Gamma(\frac{k-s}{2})^2\Gamma(k)^{-1}F(\frac{k-s}{2},\frac{k-s}{2};k,a^{-2}),\\
&\int_0^\infty J_{k-1}(x)J_0(ax)x^{-s}\,dx \\
&= \frac{\sin(\pi(k-s)/2)}{2^s\pi a^{k-s}}\Gamma(\frac{k-s}{2})^2\Gamma(k)^{-1}F(\frac{k-s}{2},\frac{k-s}{2};k,a^{-2}),
\end{aligned}$$

where F is the hypergeometric function, defined as

$$F(\alpha,\beta;\gamma,z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{r=0}^\infty \frac{\Gamma(\alpha+r)\Gamma(\beta+r)}{\Gamma(\gamma+r)}\frac{z^r}{r!}.$$

Observing that

$$\left|\frac{\Gamma((k-s)/2+r)^2}{r!\Gamma(k+r)}\right| \le \frac{\Gamma((k-\sigma)/2+r)^2}{r!\Gamma(k+r)} \le \frac{\Gamma(k/2+r)^2}{r!\Gamma(k+r)} < 1,$$

both integrals above are

$$\begin{aligned}
&\ll \frac{e^{\pi|t|/2}}{2^\sigma a^{k-\sigma}}\sum_{r=0}^\infty \frac{\Gamma((k-\sigma)/2+r)^2}{r!\Gamma(k+r)}a^{-2r}\\
&\ll e^{\pi|t|/2}a^{\sigma-k}\sum_{r=0}^\infty a^{-2r} \ll e^{\pi|t|/2}a^{\sigma-k}(1-a^{-2})^{-1}.
\end{aligned}$$

(b) We split the range of integration as follows:

$$\int_0^\infty = \int_0^{k/4} + \sum_{\substack{r\geq 0\\ K=2^{r-2}k}} \int_K^{2K}. \tag{2.10}$$

We denote $I_K = \int_K^{2K}$. By (2.5), $B_0(ax) \ll (ax)^{-1/2}$ for $ax \gg 1$, so

$$I_K \ \ll \ a^{-1/2}\left(\int_0^\infty J_{k-1}(x)^2 x^{-1/2}\,dx\right)^{1/2}\left(\int_K^{2K} x^{-2\sigma-1/2}\,dx\right)^{1/2},$$

for $aK \gg 1$. By the formula [WG, §7.15 (11)]

$$\begin{aligned}\int_0^\infty J_{k-1}(t)^2 t^{-\lambda}\,dt \\ = \ 2^{-\lambda}\frac{\Gamma(\lambda)}{\Gamma(\frac{\lambda+1}{2})^2}\frac{\Gamma(k-1/2-\lambda/2)}{\Gamma(k-1/2+\lambda/2)} \qquad \text{for } 0<\lambda<2k-1,\end{aligned} \tag{2.11}$$

we obtain the estimate

$$I_K \ \ll \ a^{-1/2}k^{-1/4}K^{-\sigma+1/4}. \tag{2.12}$$

Replacing $B_0(x)$ by the formulae in (2.5), we can have another estimate for I_K. To this end we only need to consider

$$\sqrt{\frac{2}{\pi a}}\int_K^{2K} J_{k-1}(x)e^{\pm iax}x^{-s-1/2}\,dx + O(a^{-3/2}\int_K^{2K}|J_{k-1}(x)|x^{-\sigma-3/2}\,dx).$$

The O-term, by (2.11) and the Cauchy-Schwarz inequality, is $\ll a^{-3/2}k^{-1}K^{-\sigma}$ $\ll a^{-1/2}k^{-1}K^{-\sigma+1/2}$. Taking $\eta = 0.01\cdot k/K$, and applying the first-derivative test for exponential integrals ([Hu, Lemma 5.1.2]), we see that (from the line below (2.6)),

$$\int_{\pi/2-\eta}^{\pi/2} f_k(\theta,x)\,d\theta \ll k^{-1} \qquad (x\in[K,2K]).$$

Hence, by (2.6),

$$\begin{aligned}&\int_K^{2K} J_{k-1}(x)e^{\pm iax}x^{-s-1/2}\,dx \\ \ll\ & k^{-1}\int_K^{2K} x^{-\sigma-1/2}\,dx + \left|\int_0^{\pi/2-\eta}\int_K^{2K} \Re e\, f_k(\theta,x)e^{\pm iax}x^{-s-1/2}\,dx\,d\theta\right|.\end{aligned}$$

The first summand is $\ll k^{-1}K^{-\sigma+1/2}$. Applying integration by parts or bounding trivially, the x-integral in the second term is $\ll (1+|t|)K^{-\sigma-1/2}$ $\min(|a-\sin\theta|^{-1}, K)$. After a change of variable $u=\sin\theta$, the second summand becomes

$$\ll \quad (1+|t|)\lambda^{-1}K^{-\sigma-1/2}\int_0^1 \min(|u-a|^{-1}, K)\,du. \tag{2.13}$$

(Note that $d\theta \ll \eta^{-1}du$ for $\theta\in[0,\pi/2-\eta]$.) Thus, (2.13) is $\ll (|t|+1)k^{-1}K^{-\sigma+1/2}\log K$ and

$$I_K \ll a^{-1/2}(1+|t|)k^{-1}K^{-\sigma+1/2}\log K. \tag{2.14}$$

For the sum in (2.10), we apply the estimate (2.12) for $K\geq k^3$ and (2.14) for $k/4\leq K\leq k^3$. The overall contribution due to $\sum_K$ is

$$\ll (1+|t|)a^{-1/2}k^{-\sigma-1/2}(\log k)^2. \tag{2.15}$$

(Note that the power of $\log k$ can be reduced to 1 if $\sigma>1/2$.)

The estimation of the integral $\int_0^{k/4}$ in (2.10) is easy. From (2.7) and $B_0(x)\ll|\log x|$ by (2.3), (2.4) and (2.5),

$$\begin{aligned}
\int_0^{k/4} \quad &\ll \quad \left(\frac{e}{2k}\right)^{k-1}\int_0^{k/4}|B_0(ax)|x^{k-\sigma-1}\,dx\\
&\ll \quad \left(\frac{e}{2k}\right)^{k-1}\int_0^{k/4}|\log ax|x^{k-\sigma-1}\,dx\\
&= \quad \left(\frac{e}{2k}\right)^{k-1}a^{\sigma-k}\int_0^{ak/4}|\log x|x^{k-\sigma-1}\,dx\\
&\ll \quad \left(\frac{e}{2k}\right)^{k-1}a^{\sigma-k}\left(\frac{ak}{4}\right)^{k-\sigma}\frac{\log(ak/4)}{k-\sigma}\\
&\ll \quad \left(\frac{e}{8}\right)^{k-1}\left(\frac{4}{k}\right)^{\sigma}\log k.
\end{aligned}$$

The proof is complete with (2.10) and (2.15).

3. Proof of Theorems 1 and 2. Assume throughout k to be a sufficiently large even integer. Let

$$K(w)=\frac{\Gamma(2(A-w))\Gamma(2(A+w))}{\Gamma(2A)^2}\frac{1}{w}, \tag{3.1}$$

where $A > 2$ is an arbitrary but fixed constant. Then K is an odd function and has only a simple pole with residue 1 at $w = 0$ inside the strip $-A < \Re e\, w < A$. Following the argument in (1.7), we apply the residue theorem to $\Lambda(f \otimes \chi_1, 1/2 + w)\Lambda(f \otimes \chi_2, 1/2 + w)K(w)$ over $\mathcal{R}_T$. After taking $T \to \infty$ and using (1.3), we get

$$\begin{aligned} &\Lambda(f \otimes \chi_1, \frac{1}{2})\Lambda(f \otimes \chi_2, \frac{1}{2}) \\ &= \frac{1}{2\pi i}\int_{(2)} \Lambda(f \otimes \chi_1, 1/2 + w)\Lambda(f \otimes \chi_2, 1/2 + w)K(w)\, dw \\ &\quad + \epsilon_k(\chi_1)\epsilon_k(\chi_2)\frac{1}{2\pi i}\int_{(2)} \Lambda(f \otimes \overline{\chi}_1, 1/2 + w)\Lambda(f \otimes \overline{\chi}_2, 1/2 + w)K(w)\, dw. \end{aligned}$$

With (1.2) and (2.1),

$$\begin{aligned} &L(f \otimes \chi_1, 1/2)L(f \otimes \chi_2, 1/2) \\ &= \sum_{n=1}^{\infty} \frac{\lambda_f(n)\tau_{\chi_1,\chi_2}(n)}{\sqrt{n}} V_{\chi_1\chi_2}(\frac{D_1D_2}{4\pi^2 n}) \\ &\quad + \epsilon_k(\chi_1)\epsilon_k(\chi_2)\overline{\sum_{n=1}^{\infty} \frac{\lambda_f(n)\tau_{\chi_1,\chi_2}(n)}{\sqrt{n}} V_{\chi_1\chi_2}(\frac{D_1D_2}{4\pi^2 n})} \end{aligned}$$

where

$$V_\chi(y) = \frac{1}{2\pi i}\int_{(2)} \left(\frac{\Gamma(k/2 + w)}{\Gamma(k/2)}\right)^2 L(1 + 2w, \chi)K(w)y^w\, dw. \tag{3.2}$$

Here we have used $\overline{V_\chi(y)} = V_{\overline{\chi}}(y)$, due to the observation that

$$\overline{(2\pi i)^{-1}\int_{(2)} G(w)\, dw} = (2\pi i)^{-1}\int_{(2)} \overline{G(\overline{w})}\, dw.$$

By Petersson's trace formula

$$\sum_{f \in \mathcal{B}_k} w_f \lambda_f(n)\lambda_f(m) = \delta_{m,n} + 2\pi i^k \sum_{c \geq 1} c^{-1} S(m, n, c) J_{k-1}(\frac{4\pi\sqrt{mn}}{c})$$

($\delta_{m,n}$ is the Kronecker delta and $S(m, n, c)$ is the Kloosterman sum) and $\lambda_f(1) = 1$,

$$\begin{aligned} &\sum_{f \in \mathcal{B}_k} w_f L(f \otimes \chi_1, 1/2)L(f \otimes \chi_2, 1/2) \\ &= S(\chi_1, \chi_2) + \epsilon_k(\chi_1)\epsilon_k(\chi_2)\overline{S(\chi_1, \chi_2)} \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} &S(\chi_1,\chi_2)\\ &= \sum_{n=1}^{\infty} \frac{\tau_{\chi_1,\chi_2}(n)}{\sqrt{n}} V_{\chi_1\chi_2}(\frac{D_1D_2}{4\pi^2 n}) \sum_{f\in\mathcal{B}_k} w_f\lambda_f(n)\\ &= V_{\chi_1\chi_2}(\frac{D_1D_2}{4\pi^2})\\ &\quad + 2\pi i^k \sum_{c\geq 1}\sum_{n\geq 1} \frac{\tau_{\chi_1,\chi_2}(n)}{\sqrt{n}} \frac{S(n,1,c)}{c} J_{k-1}(\frac{4\pi\sqrt{n}}{c}) V_{\chi_1\chi_2}(\frac{D_1D_2}{4\pi^2 n})\\ &= V_{\chi_1\chi_2}(\frac{D_1D_2}{4\pi^2}) + 2\pi i^k M, \text{ say.} \end{aligned} \tag{3.4}$$

1° **Treatment of** $V_{\chi_1\chi_2}(D_1D_2/(4\pi^2))$. Moving the line of integration to $w=-A/2$, we have

$$\begin{aligned} &V_{\chi_1\chi_2}(\frac{D_1D_2}{4\pi^2})\\ &= \mathrm{Res}_{w=0}\left\{ \left(\frac{D_1D_2}{4\pi^2}\right)^w \frac{\Gamma(k/2+w)^2}{\Gamma(k/2)^2} L(1+2w,\chi_1\chi_2)K(w)\right\}\\ &\quad + \frac{1}{2\pi i}\int_{(-A/2)} \left(\frac{D_1D_2}{4\pi^2}\right)^w \left(\frac{\Gamma(k/2+w)}{\Gamma(k/2)}\right)^2 L(1+2w,\chi_1\chi_2)K(w)\,dw \end{aligned}$$

by (3.2). The last integral is $\ll_A (D_1D_2)^{(A-1)/2}k^{-A}$, which can be seen as follows.

Let the conductor of $\chi_1\chi_2$ be D, which divides D_1D_2. Then for $w=-A/2+it$,

$$\begin{aligned} L(1+2w,\chi_1\chi_2) &= L(1-A+2it,\chi_1\chi_2)\\ &\ll (D(|t|+1))^{A-1/2}\\ &\ll (D_1D_2(|t|+1))^{A-1/2}. \end{aligned}$$

As $|\Gamma(k/2+w)| \leq \Gamma(k/2+\Re e\, w) \ll k^{-A/2}\Gamma(k/2)$ by Stirling's formula, we have

$$\begin{aligned} \int_{(-A/2)} &\ll (D_1D_2)^{(A-1)/2}k^{-A}\int_0^\infty (|t|+1)^{A-1/2}|K(-A/2+it)|\,dt\\ &\ll (D_1D_2)^{(A-1)/2}k^{-A}\int_0^\infty (|t|+1)^{5(A-1/2)}e^{-2\pi|t|}\,dt\\ &\ll (D_1D_2)^{(A-1)/2}k^{-A}. \end{aligned}$$

To evaluate the residue, we compute the following series expansions:

$$\begin{aligned} K(w) &= w^{-1}+c_1w+\cdots, \\ \left(\frac{D_1D_2}{4\pi^2}\right)^w &= 1+w\log\frac{D_1D_2}{4\pi^2}+\cdots, \\ \frac{\Gamma(k/2+w)^2}{\Gamma(k/2)^2} &= 1+2w\frac{\Gamma'(k/2)}{\Gamma(k/2)}+\cdots \end{aligned}$$

and if $\chi_1\chi_2$ is the principal character, then $D_1=D_2=D$ and

$$L(1+2w,\chi_1\chi_2) = \prod_{p|D}(1-p^{-1})(1+2w\sum_{p|D}\frac{\log p}{p-1}+\cdots)(\frac{1}{2w}+\gamma+\cdots);$$

otherwise (i.e. $\chi_1\neq\overline{\chi}_2$), $L(1+2w,\chi_1\chi_2)=L(1,\chi_1\chi_2)+2L'(1,\chi_1\chi_2)w+\cdots$. Hence, for $\chi_1\neq\overline{\chi}_2$,

$$V_{\chi_1\chi_2}(\frac{D_1D_2}{4\pi^2}) = L(1,\chi_1\chi_2)+O((D_1D_2)^{(A-1)/2}k^{-A}), \tag{3.5}$$

and for $\chi=\chi_1=\overline{\chi}_2$ ($D=D_1=D_2$),

$$\begin{aligned} V_{\chi\overline{\chi}}(\frac{D^2}{4\pi^2}) &= \frac{\phi(D)}{D}\left\{\frac{\Gamma'(k/2)}{\Gamma(k/2)}+(\gamma+\log\frac{D}{2\pi}+\sum_{p|D}\frac{\log p}{p-1})\right\} \\ &\quad +O(D^{A-1}k^{-A}). \end{aligned} \tag{3.6}$$

2° **Treatment of** M. Opening the Kloostermann sum, and interchanging the sum (over n) and the integral in M (in (3.4)),

$$\begin{aligned} M &= \sum_{c\geq 1}c^{-1}\sideset{}{^*}\sum_{a\,(c)}e(\frac{\overline{a}}{c})\frac{1}{2\pi i}\int_{(2)}\left(\frac{D_1D_2}{4\pi^2}\right)^w\left(\frac{\Gamma(k/2+w)}{\Gamma(k/2)}\right)^2 \\ &\quad\times\sum_{n\geq 1}\frac{\tau_{\chi_1,\chi_2}(n)e(an/c)}{n^{1/2+w}}J_{k-1}(\frac{4\pi\sqrt{n}}{c})L(1+2w,\chi_1\chi_2)K(w)\,dw. \end{aligned}$$

Using the Mellin transform formula (2.8), we deduce that

$$\begin{aligned} M &= \frac{1}{4\pi}\sum_{c\geq 1}\sideset{}{^*}\sum_{a\,(c)}e(\frac{\overline{a}}{c})\left(\frac{1}{2\pi i}\right)^2\int_{(2)}\left(\frac{D_1D_2}{4\pi^2}\right)^w\left(\frac{\Gamma(k/2+w)}{\Gamma(k/2)}\right)^2 \\ &\quad\times L(1+2w,\chi_1\chi_2)K(w) \\ &\quad\times\int_{(-1)}\left(\frac{c}{2\pi}\right)^s\frac{\Gamma((k+s)/2)}{\Gamma((k-s)/2)}E_{\chi_1,\chi_2}(1+w+\frac{s}{2},\frac{a}{c})\,ds\,dw \end{aligned}$$

We can move the line of integration of the inner integral from $\Re e\, s = -1$ to $\Re e\, s = -7$ by Lemma 2.2 and (2.2). This yields from the possible pole of $E_{\chi_1,\chi_2}(\cdot, a/c)$ that

$$M = M_1 + M_2 \tag{3.7}$$

where

$$\begin{aligned} M_1 = \; & \frac{1}{4\pi} \sum_{c\geq 1} \sideset{}{^*}\sum_{a\,(c)} e(\frac{\overline{a}}{c}) \frac{1}{2\pi i} \int_{(2)} \left(\frac{D_1 D_2}{4\pi^2}\right)^w \left(\frac{\Gamma(k/2+w)}{\Gamma(k/2)}\right)^2 \\ & \times L(1+2w, \chi_1\chi_2) K(w) \\ & \times \mathrm{Res}_{s=-2w} \left(\left(\frac{c}{2\pi}\right)^s \frac{\Gamma((k+s)/2)}{\Gamma((k-s)/2)} E_{\chi_1,\chi_2}(1+w+\frac{s}{2}, \frac{a}{c}) \right) dw \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} M_2 = \; & \frac{1}{4\pi} \sum_{c\geq 1} \sideset{}{^*}\sum_{a\,(c)} e(\frac{\overline{a}}{c}) \left(\frac{1}{2\pi i}\right)^2 \int_{(2)} \left(\frac{D_1 D_2}{4\pi^2}\right)^w \left(\frac{\Gamma(k/2+w)}{\Gamma(k/2)}\right)^2 \\ & \times L(1+2w, \chi_1\chi_2) K(w) \\ & \times \int_{(-7)} \left(\frac{c}{2\pi}\right)^s \frac{\Gamma((k+s)/2)}{\Gamma((k-s)/2)} E_{\chi_1,\chi_2}(1+w+\frac{s}{2}, \frac{a}{c})\, ds\, dw. \end{aligned} \tag{3.9}$$

3° **Treatment of** M_1 We divide into two cases according as $\chi_1 = \chi_2$ or not.

Case 1. $\chi_1 = \chi_2 = \chi$. The residue inside M_1 can be written as

$$\left(\frac{c}{2\pi}\right)^{-2w} \mathrm{Res}_{z=0} \left(\left(\frac{c}{2\pi}\right)^z \frac{\Gamma(k/2-w+z/2)}{\Gamma(k/2+w-z/2)} E_{\chi,\chi}(1+\frac{z}{2}, \frac{a}{c}) \right).$$

By Lemma 2.1, the residue appears only when $D|c$ and $(D, c/D) = 1$; and it equals

$$\begin{aligned} & 4c^{-1} \tau(\chi) \overline{\chi}(a) \chi(\frac{c}{D}) \frac{\phi(D)}{D} \left(\frac{c}{2\pi}\right)^{-2w} \\ & \quad \times \Big\{ \frac{1}{2} \frac{\Gamma'(k/2-w)\Gamma(k/2+w) + \Gamma(k/2-w)\Gamma'(k/2+w)}{\Gamma(k/2+w)^2} \\ & \qquad + \frac{\Gamma(k/2-w)}{\Gamma(k/2+w)} (\gamma + \log\frac{D}{2\pi} + \sum_{p|D} \frac{\log p}{p-1}) \Big\}. \end{aligned} \tag{3.10}$$

Therefore, we have from (3.8) and (3.10),

$$(3.11)\quad M_1 = \frac{\tau(\chi)}{\pi}\frac{\phi(D)}{D^2}\sum_{\substack{c\ge 1\\(c,D)=1}}\ \sideset{}{^*}\sum_{a\,(cD)}\overline{\chi}(a)\chi(c)e(\frac{\overline{a}}{cD})$$
$$\times\frac{1}{2\pi i}\int_{(2)} c^{-1-2w}\left(\frac{\Gamma(k/2+w)}{\Gamma(k/2)}\right)^2 L(1+2w,\chi^2)K(w)$$
$$\times\Big\{\frac{1}{2}\frac{\Gamma'(k/2-w)\Gamma(k/2+w)+\Gamma(k/2-w)\Gamma'(k/2+w)}{\Gamma(k/2+w)^2}$$
$$+\frac{\Gamma(k/2-w)}{\Gamma(k/2+w)}(\gamma+\log\frac{D}{2\pi}+\sum_{p|D}\frac{\log p}{p-1})\Big\}\,dw.$$

Interchanging the sums and the integral, we get the sum

$$\sum_{\substack{c\ge 1\\(c,D)=1}} c^{-1-2w}\ \sideset{}{^*}\sum_{a\,(cD)}\overline{\chi}(a)\chi(c)e(\frac{\overline{a}}{cD})$$
$$=\sum_{\substack{c\ge 1\\(c,D)=1}} c^{-1-2w}\chi(c)^2\ \sideset{}{^*}\sum_{m\,(c)} e(\frac{m}{c})\sum_{n\,(D)}\chi(n)e(\frac{n}{D})$$
$$=\tau(\chi)\sum_{\substack{c\ge 1\\(c,D)=1}} c^{-1-2w}\mu(c)\chi(c)^2=\tau(\chi)L(1+2w,\chi^2)^{-1},$$

by firstly replacing $\overline{a}$ by a and then a by $mD+nc$ where $(m,c)=(n,D)=1$. (This is valid since $(c,D)=1$.) Inserting this into (3.11), M_1 is now expressed as

$$(3.12)\quad M_1 = \frac{\tau(\chi)^2}{2\pi}\frac{\phi(D)}{D^2}\Gamma(k/2)^{-2}\frac{1}{2\pi i}\int_{(2)}\Big\{\Gamma'(k/2-w)\Gamma(k/2+w)$$
$$+\Gamma(k/2-w)\Gamma'(k/2+w)$$
$$+2\Gamma(k/2-w)\Gamma(k/2+w)(\gamma+\log\frac{D}{2\pi}+\sum_{p|D}\frac{\log p}{p-1})\Big\}$$
$$\times K(w)\,dw$$
$$=\frac{\tau(\chi)^2}{2\pi}\frac{\phi(D)}{D^2}\left(\frac{\Gamma'(k/2)}{\Gamma(k/2)}+\gamma+\log\frac{D}{2\pi}+\sum_{p|D}\frac{\log p}{p-1}\right),$$

by residue theorem and the observation that the integrand is an odd function.

Case 2. $\chi_1 \neq \chi_2$. In this case, the residue in (3.8) is, by Lemma 2.1 again,

$$2\Big(\delta_{12}(c)c^{-1}\tau(\chi_1)\overline{\chi}_1(a)\chi_2(\frac{c}{D_1})L(1,\overline{\chi}_1\chi_2) + \delta_{21}(c)c^{-1}\tau(\chi_2)\overline{\chi}_2(a)\chi_1(\frac{c}{D_2})L(1,\chi_1\overline{\chi}_2)\Big)\left(\frac{c}{2\pi}\right)^{-2w}\frac{\Gamma(k/2-w)}{\Gamma(k/2+w)}.$$

We obtain from (3.8) that

(3.13)

$$\begin{aligned} M_1 &= \frac{\tau(\chi_1)}{2\pi D_1}L(1,\overline{\chi}_1\chi_2)\frac{1}{2\pi i}\int_{(2)}\frac{\Gamma(k/2+w)\Gamma(k/2-w)}{\Gamma(k/2)^2}K(w)\left(\frac{D_2}{D_1}\right)^w \\ &\quad \times L(1+2w,\chi_1\chi_2)\sum_{\substack{c\geq 1\\ (c,D_2)=1}}c^{-1-2w}\chi_2(c)\sideset{}{^*}\sum_{a\,(cD_1)}\overline{\chi}_1(a)e(\frac{\overline{a}}{cD_1})\,dw \\ &\quad + \frac{\tau(\chi_2)}{2\pi D_2}L(1,\chi_1\overline{\chi}_2)\frac{1}{2\pi i}\int_{(2)}\frac{\Gamma(k/2+w)\Gamma(k/2-w)}{\Gamma(k/2)^2}K(w)\left(\frac{D_1}{D_2}\right)^w \\ &\quad \times L(1+2w,\chi_1\chi_2)\sum_{\substack{c\geq 1\\ (c,D_1)=1}}c^{-1-2w}\chi_1(c)\sideset{}{^*}\sum_{a\,(cD_2)}\overline{\chi}_2(a)e(\frac{\overline{a}}{cD_2})\,dw. \end{aligned}$$

The sum over c equals

$$\begin{aligned} &\sum_{c\geq 1}c^{-1-2w}\chi_2(c)\sideset{}{^*}\sum_{a\,(cD_1)}\chi_1(a)e(\frac{a}{cD_1}) \\ =\ &\sum_{c\geq 1}c^{-1-2w}\chi_2(c)\sum_{d|cD_1}\mu(d)\sum_{\substack{a\,(cD_1)\\ d|a}}\chi_1(a)e(\frac{a}{cD_1}) \\ =\ &\sum_{c\geq 1}c^{-1-2w}\chi_2(c)\sum_{d|cD_1}\mu(d)\chi_1(d)\sum_{a\,(cD_1/d)}\chi_1(a)e(\frac{ad}{cD_1}) \\ =\ &\sum_{c\geq 1}c^{-1-2w}\chi_2(c)\sum_{\substack{d|c\\ (d,D_1)=1}}\mu(d)\chi_1(d)\sum_{v=1}^{D_1}\chi_1(v)e(\frac{vd}{D_1c})\sum_{u\,(c/d)}e(\frac{ud}{c}) \\ =\ &\sum_{c\geq 1}c^{-1-2w}\chi_2(c)\mu(c)\chi_1(c)\sum_{v\,(D_1)}\chi_1(v)e(\frac{v}{D_1}) \\ =\ &\tau(\chi_1)L(1+2w,\chi_1\chi_2)^{-1}. \end{aligned}$$

Similar argument works for the second sum on the right hand side of (3.13). We put these into (3.13) to get

(3.14)
$$M_1 = \frac{\tau(\chi_1)^2}{2\pi D_1} L(1, \overline{\chi}_1\chi_2) \frac{1}{2\pi i} \int_{(2)} \left(\frac{D_2}{D_1}\right)^w \frac{\Gamma(k/2+w)\Gamma(k/2-w)}{\Gamma(k/2)^2} K(w)\, dw$$
$$+ \frac{\tau(\chi_2)^2}{2\pi D_2} L(1, \chi_1\overline{\chi}_2) \frac{1}{2\pi i} \int_{(2)} \left(\frac{D_1}{D_2}\right)^w \frac{\Gamma(k/2+w)\Gamma(k/2-w)}{\Gamma(k/2)^2} K(w)\, dw.$$

4° **Treatment of M_2** Changing the variable $s = -(z + 2w)$, we have from (3.9),

$$M_2 = \frac{1}{4\pi} \sum_{c\geq 1} \sideset{}{^*}\sum_{a\,(c)} e(\frac{\overline{a}}{c})$$
$$\times \left(\frac{1}{2\pi i}\right)^2 \int_{(2)} \left(\frac{D_1 D_2}{c^2}\right)^w \left(\frac{\Gamma(k/2+w)}{\Gamma(k/2)}\right)^2 L(1+2w, \chi_1\chi_2) K(w)$$
$$\times \int_{(3)} \left(\frac{c}{2\pi}\right)^{-z} \frac{\Gamma(k/2-w-z/2)}{\Gamma(k/2+w+z/2)} E_{\chi_1,\chi_2}(1-\frac{z}{2}, \frac{a}{c})\, dz\, dw.$$

To bound M_2, we shall use the functional equation of $E_{\chi_1,\chi_2}(\cdot, a/c)$ in Lemma 2.1, and split M_2 into two parts

$$M_2 = M_2^+ + M_2^- \tag{3.15}$$

according to the functions $\varphi_{a,c}^+$ and $\varphi_{a,c}^-$. It follows that

(3.16)
$$M_2^- = (4\pi)^{-1} \sum_{c\geq 1} c_1([D_1, c][D_2, c])^{-1} \sideset{}{^*}\sum_{a\,(c)} e(\frac{\overline{a}}{c})$$
$$\times \sum_{\substack{u\,(D_1)\\ v\,(D_2)}} \chi_1(u)\chi_2(v) e(uva_0/c) \sum_{l\geq 1} \tau_{a,c}^-(l; u, v) e(-la_1/c)$$
$$\times \frac{1}{2\pi i} \int_{(2)} \left(\frac{D_1 D_2}{c^2}\right)^w \frac{\Gamma(k/2+w)^2}{\Gamma(k/2)^2} L(1+2w, \chi_1\chi_2) K(w)$$
$$\times \frac{1}{2\pi i} \int_{(3)} \frac{\Gamma(k/2-w-z/2)}{\Gamma(k/2+w+z/2)} \Gamma(\frac{z}{2})^2$$
$$\times (e(-\frac{z}{4}) + \chi_1\chi_2(-1) e(\frac{z}{4})) \left(\frac{c^2 l}{[D_1, c][D_2, c]}\right)^{-z/2} dz\, dw$$

where c_1 divides c. Writing

$$Q = \frac{c^2 l}{[D_1, c][D_2, c]}$$

for short, the inner integral over z, by Lemma 2.3, is equal to

$$\begin{aligned} & 2\pi\chi_1\chi_2(-1)\int_0^\infty J_{k-1}(x)(iJ_0(x\sqrt{Q}) - Y_0(x\sqrt{Q}))\left(\frac{x}{2}\right)^{-2w}\,dx \\ & \qquad - 2\pi\int_0^\infty J_{k-1}(x)(iJ_0(x\sqrt{Q}) + Y_0(x\sqrt{Q}))\left(\frac{x}{2}\right)^{-2w}\,dx \\ = & \quad -4\pi\int_0^\infty J_{k-1}(x)B_0(x\sqrt{Q})\left(\frac{x}{2}\right)^{-2w}\,dx, \end{aligned}$$

where $B_0(\cdot) = Y_0(\cdot)$ if $\chi_1\chi_2(-1) = 1$ and $B_0(\cdot) = iY_0(\cdot)$ if $\chi_1\chi_2(-1) = -1$. Hence by moving the line of integration to $1/2 \le \Re e\, w = A_c < A$ where A_c depends on c, we have

$$\begin{aligned} (3.17) \qquad M_2^- \ll \; & D_1D_2\sum_{c\ge 1} c^2([D_1,c][D_2,c])^{-1}\sum_{l\ge 1} d(l) \\ & \times \int_{(A_c)}\left|\frac{\Gamma(k/2+w)^2}{\Gamma(k/2)^2}L(1+2w,\chi_1\chi_2)K(w)\left(\frac{D_1D_2}{c^2}\right)^w\right| \\ & \times \left|\int_0^\infty J_{k-1}(x)B_0(x\sqrt{Q})\left(\frac{x}{2}\right)^{-2w}\,dx\right|\,|dw|. \end{aligned}$$

Now we choose $A_c = 1$ for $c > k$ and $A_c = 1/2$ for $c \le k$. Then applying Lemma 2.4 (b), the contribution from those terms satisfying $Q \le 1$ is

$$\begin{aligned} \ll \; & (D_1D_2)^{3/2}k^{-3/2}(\log k)^2\sum_{1\le c\le k} c([D_1,c][D_2,c])^{-1} \\ & \times \sum_{c^2 l \le [D_1,c][D_2,c]} d(l)\left(\frac{c^2 l}{[D_1,c][D_2,c]}\right)^{-1/4} \\ & \times \int_{(1/2)}\left|\frac{\Gamma(k/2+w)^2}{\Gamma(k/2)^2}L(1+2w,\chi_1\chi_2)K(w)\right|(1+|w|)\,|dw| \end{aligned}$$

$$
\begin{aligned}
&+ (D_1D_2)^2 k^{-5/2}(\log k)^2 \sum_{c\geq k}([D_1,c][D_2,c])^{-1} \\
&\quad \times \sum_{c^2 l\leq [D_1,c][D_2,c]} d(l)\left(\frac{c^2 l}{[D_1,c][D_2,c]}\right)^{-1/4} \\
&\quad \times \int_{(1)} \left|\frac{\Gamma(k/2+w)^2}{\Gamma(k/2)^2} L(1+2w,\chi_1\chi_2)K(w)\right| (1+|w|)\,|dw| \\
\ll\ & (D_1D_2)^{3/2}k^{-1/2}(\log k)^2 \log(D_1D_2) \\
&\quad \times \Big(\sum_{1\leq c\leq k} c^{-1} + (D_1D_2)^{1/2}\sum_{c>k} c^{-2}\Big) \\
\ll\ & (D_1D_2)^{3/2}k^{-1/2}(\log k)^3 \log(D_1D_2). \qquad (3.18)
\end{aligned}
$$

When $D_1 = D_2 = 1$, we only consider that k is divisible by 4. The condition $c^2 l \leq [D_1,c][D_2,c]$ is reduced to $l = 1$. In this case, $c_1 = c$ and $\tau^-_{a,c}(1;1,1)e(-a_1/c) = e(-\overline{a}/c)$ (see the remark after Lemma 2.1). Thus, by Lemma 2.3, the contribution of M_2^- is, by (3.16),

$$
\begin{aligned}
&-\sum_{c\geq 1} c^{-1}\phi(c)\frac{1}{2\pi i}\int_{(2)} 2^{2w}c^{-2w}\frac{\Gamma(k/2+w)^2}{\Gamma(k/2)^2}\zeta(1+2w)K(w) \qquad (3.19)\\
&\qquad \times \int_0^\infty J_{k-1}(x)Y_0(x)x^{-2w}\,dx\,dw \\
&= \frac{1}{\pi}\frac{1}{2\pi i}\int_{(2)} \frac{\Gamma(k/2-w)^2}{\Gamma(k/2)^2}\zeta(2w)K(w)\cos(\pi w)\,dw.
\end{aligned}
$$

The last line follows from [Er, §6.8 (36)]

$$
\int_0^\infty J_{k-1}(x)Y_0(x)x^{-2w}\,dx = -2^{-2w}\pi^{-1}\cos(\pi w)\frac{\Gamma(k/2-w)^2}{\Gamma(k/2+w)^2}.
$$

Moving the line of integration to $\Re e\, w = A/2$, it is apparent that the left side in (3.19) is $\ll k^{-A}$.

Now, we investigate the contribution from $c^2 l > [D_1,c][D_2,c]$ in (3.17). We have $c^2 l \geq [D_1,c][D_2,c]+c^2$ as $c^2|[D_1,c][D_2,c]$, and thus $c^2 l/([D_1,c][D_2,c]) \geq 1+(D_1D_2)^{-1}$. Taking $A_c = 1$, the contribution of this part is, by Lemma 2.4

(a) and (3.1),

$$\begin{aligned}
&\ll (D_1D_2)^2 \sum_{c\geq 1} c^2([D_1,c][D_2,c])^{-1} \\
&\quad\times \sum_{c^2l>[D_1,c][D_2,c]} ld(l)(c^2l-[D_1,c][D_2,c])^{-1}\left(\frac{c^2l}{[D_1,c][D_2,c]}\right)^{1-k/2} \\
&\quad\times \int_{(1)} \left|\frac{\Gamma(k/2+w)^2}{\Gamma(k/2)^2}L(1+2w,\chi_1\chi_2)K(w)\right| e^{|\mathrm{Im}\,w|\pi}\,|dw| \\
&\ll (D_1D_2)^2k^2(1+(D_1D_2)^{-1})^{4-k/2}\sum_{c\geq 1} c^{-6}([D_1,c][D_2,c])^2 \\
&\quad\times \sum_{c^2l>[D_1,c][D_2,c]} d(l)l^{-2} \\
&\ll (D_1D_2)^3k^2\exp\left(-\frac{k}{4D_1D_2}\right). \qquad (3.20)
\end{aligned}$$

(Note that our choice of $K(w)$ in (3.1) is sufficient to suppress the term $\exp(|\mathrm{Im}\,w|\pi)$.) This completes the evaluation of the left side of (3.17). In view of (3.18)-(3.20), under the condition that $D_1D_2 \leq k/(16\log k)$, we can write

$$M_2^- \ll (D_1D_2)^{3/2}k^{-1/2}(\log k)^3\log(D_1D_2)+k^{-A}. \qquad (3.21)$$

(Note that $\log(D_1D_2)=0$ when $D_1=D_2=1$.)

The evaluation of M_2^+ in (3.15) is much easier. As $\varphi^+_{a,c}(s;u,v)\ll 1$ for $\Re e\, s>1$, we move $\Re e\, z=3$ to $\Re e\, z=4$ and then $\Re e\, w=2$ to $\Re e\, w=1$. With Lemma 2.2, a crude estimate gives

$$\begin{aligned}
M_2^+ &\ll (D_1D_2)^2\sum_{c\geq 1} c^{-4}([D_1,c][D_2,c]) \\
&\quad\times \int_{(1)}\int_{(4)}\left|\frac{\Gamma(k/2+w)^2}{\Gamma(k/2)^2}\frac{\Gamma(k/2-w-z/2)}{\Gamma(k/2+w+z/2)}\Gamma\left(\frac{z}{2}\right)^2K(w)\right||dz||dw| \\
&\ll (D_1D_2)^3k^{-4}. \qquad (3.22)
\end{aligned}$$

Hence, for $D_1D_2\leq k/(16\log k)$, (3.15), (3.21) and (3.22) yield that

$$M_2 \ll (D_1D_2)^{3/2}k^{-1/2}(\log k)^3\log(D_1D_2)+k^{-A}. \qquad (3.23)$$

For simplicity, let us write

$$\Phi(k,D) = \frac{\phi(D)}{D}\left\{\frac{\Gamma'(k/2)}{\Gamma(k/2)} + (\gamma + \log\frac{D}{2\pi} + \sum_{p|D}\frac{\log p}{p-1})\right\},$$

$$I(D_1,D_2) = \frac{1}{2\pi i}\int_{(2)}\frac{\Gamma(k/2+w)\Gamma(k/2-w)}{\Gamma(k/2)^2}K(w)\left(\frac{D_1}{D_2}\right)^w dw,$$

and put $E(1,k) = k^{-A}$ and $E(\mathcal{D},k) = \mathcal{D}^{3/2}k^{-1/2}(\log k)^4$ for $\mathcal{D} > 1$. One can see that by residue theorem,

$$I(D_1,D_2) + I(D_2,D_1) = 1. \tag{3.24}$$

We now conclude our result. From (3.4), (3.7) and (3.23), we have

$$S(\chi_1,\chi_2) = V_{\chi_1\chi_2}(\frac{D_1D_2}{4\pi^2}) + 2\pi i^k M_1 + O(E(D_1D_2,k)).$$

1. When χ is real. Using $\tau(\chi) = \chi(-1)\overline{\tau(\chi)}$ (for real χ), we have from (3.7) and (3.12) that

   $$S(\chi,\chi) = (1 + i^k\chi(-1))\Phi(k,D) + O(E(D^2,k)).$$

   From (3.3) and $\epsilon_k(\chi)^2 = 1$, parts (a) and (b) of Theorem 1 follow.

2. When χ is complex. From (3.5) and (3.12),

   $$S(\chi,\chi) = L(1,\chi^2) + \epsilon_k(\chi)\Phi(k,D) + O(E(D^2,k)).$$

   This completes part (c) with (3.3).

3. When $\chi_1 \neq \chi_2$ and $\chi_1 \neq \overline{\chi}_2$. Then, by (3.5) and (3.14),

   $$\begin{aligned} S(\chi_1,\chi_2) &= L(1,\chi_1\chi_2) + \epsilon_k(\chi_1)L(1,\overline{\chi}_1\chi_2)I(D_2,D_1) \\ &\quad + \epsilon_k(\chi_2)L(1,\chi_1\overline{\chi}_2)I(D_1,D_2) + O(E(D_1D_2,k)). \end{aligned}$$

   By (3.3) and using $\overline{I(\cdot,\cdot)} = I(\cdot,\cdot)$, we deduce that

   $$\begin{aligned} &\sum_{f\in\mathcal{B}_k} w_f L(f\otimes\chi_1,1/2)L(f\otimes\chi_2,1/2) \\ &= L(1,\chi_1\chi_2) + \epsilon_k(\chi_1)\epsilon_k(\chi_2)L(1,\overline{\chi}_1\overline{\chi}_2) \\ &\quad + \Big(\epsilon_k(\chi_1)L(1,\overline{\chi}_1\chi_2) + \epsilon_k(\chi_2)L(1,\chi_1\overline{\chi}_2)\Big)\Big(I(D_1,D_2) + I(D_2,D_1)\Big) \\ &\quad + O(E(D_1D_2,k)). \end{aligned}$$

   This completes part (d) with (3.24).

4. Properties of $E_{\chi_1,\chi_2}(s,a/c)$. This section is independent of the previous parts. It is devoted to the study of the generalized Estermann function which is defined, for $\Re e\, s>1$, as

$$E_{\chi_1,\chi_2}(s,h/k)=\sum_{n=1}^{\infty}\tau_{\chi_1,\chi_2}(n)e(nh/k)n^{-s} \tag{4.1}$$

where $k\geq 1$ and $(h,k)=1$, and $\tau_{\chi_1,\chi_2}(n)=\sum_{ab=n}\chi_1(a)\chi_2(b)$. ($\chi_1$ and χ_2 are primitive characters.) We change here the notation a/c into h/k and clearly no confusion will be caused. To begin with, let us fix our notations: (m,n) and $[m,n]$ denote respectively the greatest common divisor and the least common multiple of the two natural numbers m and n. We also denote by $(\cdot,\cdot)$ an ordered pair when no confusion will occur. Given h, k and D_1, D_2 (the moduli of χ_1 and χ_2), we write

$$\begin{aligned}&\delta_1=(D_1,k), k=\delta_1\kappa_1, D_1=\delta_1 d_1, \Delta_1=(\delta_1,\kappa_2), \delta_1=\Delta_1\delta, \kappa_1=\Delta_2\kappa,\\ &\delta_2=(D_2,k), k=\delta_2\kappa_2, D_2=\delta_2 d_2, \Delta_2=(\delta_2,\kappa_1), \delta_2=\Delta_2\delta, \kappa_2=\Delta_1\kappa.\end{aligned} \tag{4.2}$$

Moreover, for any two coprime integers m and n, we define $\overline{m}^{(n)}$ and $\overline{n}^{(m)}$ to be a pair of integers satisfying $m\overline{m}^{(n)}+n\overline{n}^{(m)}=1$.

Theorem A *The function $E_{\chi_1,\chi_2}(s,h/k)$ can be analytically continued to a meromorphic function, which is holomorphic on $\mathbb{C}$ except possibly at $s=1$. The order of the pole is at most two. Suppose the Laurent expansion of $E_{\chi_1,\chi_2}(s,h/k)$ at $s=1$ is*

$$E_{\chi_1,\chi_2}(s,h/k) \;=\; A_{\chi_1,\chi_2}(h,k)(s-1)^{-2}+B_{\chi_1,\chi_2}(h,k)(s-1)^{-1}+\cdots.$$

When $\chi_1=\chi_2$, we put $\chi=\chi_1=\chi_2$ and $D=D_1=D_2$. For $k=D\kappa$ with $(D,\kappa)=1$,

$$\begin{aligned}A_{\chi,\chi}(h,k) &= k^{-1}\tau(\chi)\overline{\chi}(h)\chi(\kappa)\frac{\phi(D)}{D}, \qquad \textit{and}\\ B_{\chi,\chi}(h,k) &= 2k^{-1}\tau(\chi)\overline{\chi}(h)\chi(\kappa)\frac{\phi(D)}{D}(\gamma-\log\kappa+\sum_{p|D}\frac{\log p}{p-1}).\end{aligned}$$

In all other cases, we have $A_{\chi_1,\chi_2}(h,k)=0$. When $\chi_1\neq\chi_2$,

$$\begin{aligned}B_{\chi_1,\chi_2}(h,k) &= \delta_{12}(k)k^{-1}\tau(\chi_1)\overline{\chi}_1(h)\chi_2(\kappa_1)L(1,\chi_2\overline{\chi}_1)\\ &\quad+\delta_{21}(k)k^{-1}\tau(\chi_2)\overline{\chi}_2(h)\chi_1(\kappa_2)L(1,\chi_1\overline{\chi}_2)\end{aligned}$$

where $\delta_{ij}(k) = 1$ *if* $k = D_i\kappa_i$ *and* $(\kappa_i, D_j) = 1$, *and* $\delta_{ij}(k) = 0$ *otherwise. Here* $\phi(\cdot)$ *is the Euler phi function and* $L(s,\psi)$ *is the Dirichlet L-function for the character* ψ.

In addition, let $h_0 = h_0(\delta,\kappa) = h(1-\delta\overline{\delta}^{(\kappa)}h\overline{h}^{(\kappa)})$ *and*

$$C_0 = C_0(\delta, d_1, d_2, \kappa, \kappa_1, \kappa_2) = \delta\overline{\delta}^{(\kappa)}\overline{d_1}^{(\kappa_1)}\overline{d_2}^{(\kappa_2)},$$

$E_{\chi_1,\chi_2}(s, h/k)$ *satisfies the functional equation*

$$\begin{aligned} & E_{\chi_1,\chi_2}(s,h/k) \\ = \ & \Delta_1\kappa_1[D_1,k]^{-s}[D_2,k]^{-s}(2\pi)^{2s-2}\Gamma(1-s)^2 \sum_{\substack{a\,(D_1)\\ b\,(D_2)}} \chi_1(a)\chi_2(b)e(\frac{abh_0}{k}) \\ & \times \Big\{ \Big(1+\chi_1\chi_2(-1)\Big)\varphi^{+}_{h,k}(1-s;a,-b) \\ & \quad - \Big(e(\frac{s}{2}) + \chi_1\chi_2(-1)e(-\frac{s}{2})\Big)\varphi^{-}_{h,k}(1-s;a,b)\Big\}. \end{aligned}$$

The functions $\varphi^{\mp}_{h,k}(\cdot;a,b)$ *are given by the analytic continuation of the Dirichlet series*

$$\varphi^{\mp}_{h,k}(s;a,b) = \varphi^{\mp}_{h,k,D_1,D_2}(s;a,b) = \sum_{l=1}^{\infty} l^{-s}\tau^{\mp}_{h,k}(l;a,b)e(\mp\frac{l\overline{h}^{(\kappa)}}{k}C_0) \tag{4.3}$$

for $\Re\mathrm{e}\, s > 1$, *of the arithmetical functions* $\tau^{\mp}_{h,k}(l;a,b)$. *These functions are defined by*

$$\tau^{\mp}_{h,k}(l;a,b) = \tau^{\mp}_{h,k,D_1,D_2}(l;a,b) = \sum_{\substack{mn=l\\ (m,n)\in S(a,b,\mp)}} e\Big(\frac{am}{D_1\kappa_1}C_1 + \frac{bn}{D_2\kappa_2}C_2\Big) \tag{4.4}$$

where

$$S(a,b,\mp) = \{(m,n)\in\mathbb{N}\times\mathbb{N} : m \equiv \mp bhd_1\,(\Delta_2),\ n\equiv \mp ahd_2\,(\Delta_1)\}, \tag{4.5}$$

and

$$\begin{aligned} C_1 &= C_1(h,\delta,\kappa,d_1,\kappa_1) = 1-\delta\overline{\delta}^{(\kappa)}h\overline{h}^{(\kappa)}d_1\overline{d_1}^{(\kappa_1)}, \\ C_2 &= C_2(h,\delta,\kappa,d_2,\kappa_2) = 1-\delta\overline{\delta}^{(\kappa)}h\overline{h}^{(\kappa)}d_2\overline{d_2}^{(\kappa_2)}. \end{aligned}$$

Here and in the sequel, the summation $\sum_{\substack{mn=l\\(*)}}$ *runs over all positive integer pairs* (m,n) *with* $mn = l$ *and satisfying the constraint* $(*)$.

Proof of Theorem A. From (4.1),

$$(4.6)\quad E_{\chi_1,\chi_2}(s,h/k) = \sum_{m,n=1}^{\infty} \chi_1(m)\chi_2(n)e(mnh/k)(mn)^{-s} = \sum_{\substack{a\,(D_1)\\ b\,(D_2)}} \chi_1(a)\chi_2(b) \sum_{\alpha,\beta\,(k)} e(\frac{\alpha\beta h}{k}) \sum_{\substack{m\geq 1\\ m\equiv a\,(D_1),\, m\equiv\alpha\,(k)}} m^{-s} \sum_{\substack{n\geq 1\\ n\equiv b\,(D_2),\, n\equiv\beta\,(k)}} n^{-s}$$

for $\Re e\, s > 1$. The pair of congruence equations $m \equiv a\,(D_1)$ and $m \equiv \alpha\,(k)$ is solvable if and only if $\delta_1 = (D_1,k)|a-\alpha$. When $\delta_1|a-\alpha$, m lies on the arithmetic progression $\{D_1\kappa_1 l + \alpha d_1\overline{d_1}^{(\kappa_1)} + a\kappa_1\overline{\kappa_1}^{(d_1)} : l \in \mathbb{Z}\}$. Define $\lambda^{(1)}_{\alpha,a} \in (0,1]$ such that

$$(4.7)\qquad \lambda^{(1)}_{\alpha,a} \equiv \frac{\alpha d_1\overline{d_1}^{(\kappa_1)} + a\kappa_1\overline{\kappa_1}^{(d_1)}}{D_1\kappa_1} \mod 1$$

(i.e. the fractional part of the right side). We have

$$(4.8)\quad E_{\chi_1,\chi_2}(s,h/k) = [D_1,k]^{-s}[D_2,k]^{-s} \sum_{\substack{a\,(D_1)\\ b\,(D_2)}} \chi_1(a)\chi_2(b) \times \sum_{\substack{\alpha,\beta\,(k)\\ \alpha\equiv a\,(\delta_1)\ \beta\equiv b\,(\delta_2)}} e(\frac{\alpha\beta h}{k})\zeta(s,\lambda^{(1)}_{\alpha,a})\zeta(s,\lambda^{(2)}_{\beta,b})$$

where $\zeta(s,a) = \sum_{n=0}^{\infty}(n+a)^{-s}$ (for $\Re e\, s > 1$) is the Hurwitz's zeta function.

It is known that $\zeta(s,a)$ is meromorphic on $\mathbb{C}$ with a simple pole at $s=1$ of residue 1, and satisfies the functional equation

$$(4.9)\quad \zeta(s,a) = i^{-1}\Gamma(1-s)(2\pi)^{s-1}(e(s/4)\varphi(1-s,a) - e(-s/4)\varphi(1-s,-a))$$

where for $\Re e\, s > 1$, $\varphi(s,a) = \sum_{m=1}^{\infty} e(ma)m^{-s}$. From (4.8), the order of the possible pole of $E_{\chi_1,\chi_2}(s,h/k)$ at $s=1$ is at most two. Hence, $E_{\chi_1,\chi_2}(s,h/k)$ is holomorphic except possibly at $s=1$. Moreover, for $0 < \Re e\, w < \epsilon$, we have when $\alpha \equiv a\ (\delta_1)$,

$$(4.10)\quad \sum_{\substack{m\geq 1\\ m\equiv a\,(D_1),\, m\equiv\alpha\,(k)}} m^{-1-w} = [D_1,k]^{-1-w}\zeta(1+w,\lambda^{(1)}_{\alpha,a}) = [D_1,k]^{-1}w^{-1} + C_1(\alpha,a) + \cdots$$

and when $\beta \equiv b\ (\delta_2)$,

$$\sum_{\substack{n\geq 1\\ n\equiv b\,(D_2),\, n\equiv \beta\,(k)}} n^{-1-w} = [D_2,k]^{-1}w^{-1} + C_2(\beta,b) + \cdots .$$

Inserting these into (4.6) then yields

$$(4.11)\quad A_{\chi_1,\chi_2}(h,k) = [D_1,k]^{-1}[D_2,k]^{-1}\sum_{\substack{a\,(D_1)\\ b\,(D_2)}} \chi_1(a)\chi_2(b) \sum_{\substack{\alpha,\beta\,(k)\\ \alpha\equiv a\,(\delta_1)\ \beta\equiv b\,(\delta_2)}} e(\frac{\alpha\beta h}{k}),$$

$$(4.12)\quad B_{\chi_1,\chi_2}(h,k) = \sum_{\substack{a\,(D_1)\\ b\,(D_2)}} \chi_1(a)\chi_2(b) \times \sum_{\substack{\alpha,\beta\,(k)\\ \alpha\equiv a\,(\delta_1)\ \beta\equiv b\,(\delta_2)}} e(\frac{\alpha\beta h}{k})\Big([D_2,k]^{-1}C_1(\alpha,a) + [D_1,k]^{-1}C_2(\beta,b)\Big).$$

Denote the sum in $A_{\chi_1,\chi_2}(h,k)$ by Σ_A, i.e.

$$\Sigma_A = \sum_{\substack{a\,(D_1)\\ b\,(D_2)}} \chi_1(a)\chi_2(b) \sum_{\substack{\alpha,\beta\,(k)\\ \alpha\equiv a\,(\delta_1)\ \beta\equiv b\,(\delta_2)}} e(\frac{\alpha\beta h}{k}).$$

Noting the condition $(h,k)=1$, we have

$$\begin{aligned}\Sigma_A &= \sum_{\substack{a\,(D_1)\\ b\,(D_2)}} \chi_1(a)\chi_2(b) \sum_{\substack{u\,(\kappa_1)\\ v\,(\kappa_2)}} e((a+\delta_1 u)(b+\delta_2 v)\frac{h}{k}) \\ &= \sum_{\substack{a\,(D_1)\\ b\,(D_2)}} \chi_1(a)\chi_2(b)e(\frac{abh}{k}) \sum_{v\,(\kappa_2)} e(\frac{avh}{\kappa_2}) \sum_{u\,(\kappa_1)} e(u(b+\delta_2 v)\frac{h}{\kappa_1}) \\ (4.13)\quad &= \kappa_1 \sum_{\substack{a\,(D_1)\\ b\,(D_2)}} \chi_1(a)\chi_2(b)e(\frac{abh}{k}) \sum_{\substack{v\,(\kappa_2)\\ \kappa_1 | b+\delta_2 v}} e(\frac{avh}{\kappa_2}).\end{aligned}$$

The last sum is zero if $(\kappa_1,\delta_2)>1$ (as then $(b,D_2)>1$). Applying the same argument with the role of u and v reversed, we get

$$(4.14)\quad \Sigma_A = \kappa_2 \sum_{\substack{a\,(D_1)\\ b\,(D_2)}} \chi_1(a)\chi_2(b)e(\frac{abh}{k}) \sum_{\substack{u\,(\kappa_1)\\ \kappa_2 | a+\delta_1 u}} e(\frac{auh}{\kappa_1}).$$

Thus, $\Sigma_A = 0$ except possibly for $(\delta_1, \kappa_2) = (\delta_2, \kappa_1) = 1$. When $(\delta_1, \kappa_2) = (\delta_2, \kappa_1) = 1$, we have, in view of (4.2), $\kappa_1 = \kappa_2 = \kappa$, $\delta_1 = \delta_2 = \delta$ and $(\delta, \kappa) = 1$. From (4.13) and (4.14), we have

$$\begin{aligned} \Sigma_A &= \kappa \sum_{\substack{a\,(D_1)\\ b\,(D_2)}} \chi_1(a)\chi_2(b) e\left(\frac{abh}{\delta}\frac{1-\delta\overline{\delta}^{(\kappa)}}{\kappa}\right) \\ &= \kappa\tau(\chi_2)\overline{\chi}_2\left(d_2 h \frac{1-\delta\overline{\delta}^{(\kappa)}}{\kappa}\right) \sum_{a\,(D_1)} \chi_1(a)\overline{\chi}_2(a), \\ \Sigma_A &= \kappa\tau(\chi_1)\overline{\chi}_1\left(d_1 h \frac{1-\delta\overline{\delta}^{(\kappa)}}{\kappa}\right) \sum_{b\,(D_2)} \chi_2(b)\overline{\chi}_1(b), \end{aligned}$$

by using the primitivity of χ_1 and χ_2. Hence Σ_A is non-zero only when $d_1 = d_2 = 1$, $\chi_1 = \chi_2$ (so $D_1 = D_2$) and $k = D\kappa$ with $(D, \kappa) = 1$. In this case, $\Sigma_A = \kappa\tau(\chi)\overline{\chi}(h)\chi(\kappa)\phi(D)$. This completes the evaluation of $A_{\chi_1,\chi_2}(h,k)$ with (4.11).

In view of (4.12), we shall evaluate the sum $\Sigma_B(\chi_1, \chi_2)$, given by

$$\Sigma_B(\chi_1,\chi_2) = \sum_{\substack{a\,(D_1)\\ b\,(D_2)}} \chi_1(a)\chi_2(b) \sum_{\substack{\alpha,\beta\,(k)\\ \alpha\equiv a\,(\delta_1)\ \beta\equiv b\,(\delta_2)}} e(\frac{\alpha\beta h}{k}) C_1(\alpha, a), \tag{4.15}$$

and we have

$$B_{\chi_1,\chi_2}(h,k) = [D_2,k]^{-1}\Sigma_B(\chi_1,\chi_2) + [D_1,k]^{-1}\Sigma_B(\chi_2,\chi_1). \tag{4.16}$$

We define, for $\Re e\, s > 1$,

$$F(s;\chi_1,\chi_2) = \sum_{\substack{a\,(D_1)\\ b\,(D_2)}} \chi_1(a)\chi_2(b) \sum_{\substack{\alpha,\beta\,(k)\\ \alpha\equiv a\,(\delta_1)\ \beta\equiv b\,(\delta_2)}} e(\frac{\alpha\beta h}{k}) \sum_{\substack{m\geq 1\\ m\equiv a\,(D_1)\ m\equiv\alpha\,(k)}} m^{-s}.$$

From (4.10) and (4.15), $\Sigma_B(\chi_1, \chi_2)$ equals the constant term in the series expansion of $F(s;\chi_1,\chi_2)$ at $s = 1$. This function $F(\cdot;\chi_1,\chi_2)$ can be written as

$$
\begin{aligned}
&F(s;\chi_1,\chi_2)\\
&= \sum_{a\,(D_1)} \chi_1(a) \sum_{\substack{\alpha\,(k)\\ \alpha\equiv a\,(\delta_1)}} \sum_{\substack{m\geq 1\\ m\equiv a\,(D_1)\ m\equiv \alpha\,(k)}} m^{-s} \sum_{b\,(D_2)} \chi_2(b) \sum_{\substack{\beta\,(k)\\ \beta\equiv b\,(\delta_2)}} e(\frac{\alpha\beta h}{k})\\
&= \sum_{a\,(D_1)} \chi_1(a) \sum_{\substack{\alpha\,(k)\\ \alpha\equiv a\,(\delta_1)}} \sum_{\substack{m\geq 1\\ m\equiv a\,(D_1)\ m\equiv \alpha\,(k)}} m^{-s} \sum_{b\,(D_2)} \chi_2(b) \sum_{z\,(\kappa_2)} e(\frac{h\alpha(b+z\delta_2)}{k})\\
&= \kappa_2 \sum_{a\,(D_1)} \chi_1(a) \sum_{\substack{\alpha\,(k)\\ \alpha\equiv a\,(\delta_1)\ \alpha\equiv 0\,(\kappa_2)}} \sum_{\substack{m\geq 1\\ m\equiv a\,(D_1)\ m\equiv \alpha\,(k)}} m^{-s} \sum_{b\,(D_2)} \chi_2(b) e\left(\frac{bh}{\delta_2}\frac{\alpha}{\kappa_2}\right)\\
&= \kappa_2 \tau(\chi_2)\overline{\chi}_2(d_2 h) \sum_{a\,(D_1)} \chi_1(a) \sum_{\substack{\alpha\,(k)\\ \alpha\equiv a\,(\delta_1)\ \alpha\equiv 0\,(\kappa_2)}} \overline{\chi}_2(\frac{\alpha}{\kappa_2}) \sum_{\substack{m\geq 1\\ m\equiv a\,(D_1)\ m\equiv \alpha\,(k)}} m^{-s}.
\end{aligned}
$$

Thus, if $d_2 \neq 1$ or $(\kappa_2, \delta_1) > 1$, $F(s;\chi_1,\chi_2) \equiv 0$; otherwise from (4.2), we have that $\delta_2 = D_2$, $\Delta_1 = 1$ (so $k = D_2\kappa_2 = D_2\kappa$, $\delta_1 = \delta$) and

$$
\begin{aligned}
F(s;\chi_1,\chi_2) &= \kappa\tau(\chi_2)\overline{\chi}_2(h) \sum_{a\,(D_1)} \chi_1(a) \sum_{\substack{b\,(D_2)\\ \kappa b\equiv a\,(\delta)}} \overline{\chi}_2(b) \sum_{\substack{m\geq 1\\ m\equiv a\,(D_1)\ m\equiv \kappa b\,(\kappa D_2)}} m^{-s}\\
&= \kappa^{1-s}\tau(\chi_2)\overline{\chi}_2(h) \sum_{a\,(D_1)} \chi_1(a) \sum_{\substack{b\,(D_2)\\ \kappa b\equiv a\,(\delta)}} \overline{\chi}_2(b) \sum_{\substack{m\geq 1\\ \kappa m\equiv a\,(D_1)\ m\equiv b\,(D_2)}} m^{-s},
\end{aligned}
$$

after replacing m by κm. Therefore, $F(s;\chi_1,\chi_2) \equiv 0$ as well if $(\kappa, D_1) > 1$. When $(\kappa, D_1) = 1$, we see that $\delta = \delta_1 = (D_1, D_2\kappa) = (D_1, D_2)$, and by replacing a by κa,

$$
\begin{aligned}
(4.17)\quad &F(s;\chi_1,\chi_2)\\
&= \kappa^{1-s}\tau(\chi_2)\overline{\chi}_2(h)\chi_1(\kappa) \sum_{a\,(D_1)} \chi_1(a) \sum_{\substack{b\,(D_2)\\ b\equiv a\,(D_1,D_2)}} \overline{\chi}_2(b) \sum_{\substack{m\geq 1\\ m\,\equiv\, a\,(D_1)\\ m\,\equiv\, b\,(D_2)}} m^{-s}\\
&= \kappa^{1-s}\tau(\chi_2)\overline{\chi}_2(h)\chi_1(\kappa) L(s, \chi_1\overline{\chi}_2).
\end{aligned}
$$

When $\chi_1 = \chi_2 (= \chi)$ and $k = D\kappa$ with $(\kappa, D) = 1$, we have

$$
F(s;\chi,\chi) = \kappa^{1-s}\tau(\chi)\overline{\chi}(h)\chi(\kappa)\zeta(s)\prod_{p|D}(1-p^{-s})
$$

and hence the constant term in its series expansion at $s=1$ is

$$\Sigma_B(\chi_1,\chi_2)=\tau(\chi)\overline{\chi}(h)\chi(\kappa)\frac{\phi(D)}{D}(\gamma-\log\kappa+\sum_{p|D}\frac{\log p}{p-1}).$$

When $k=D_2\kappa$, $(\kappa,D_1)=1$ but $\chi_1\neq\chi_2$, we have from (4.17),

$$\Sigma_B(\chi_1,\chi_2)=\tau(\chi_2)\overline{\chi}_2(h)\chi_1(\kappa)L(1,\chi_1\overline{\chi}_2).$$

By (4.16), the evaluation of $B_{\chi_1,\chi_2}(h,k)$ is complete.

We proceed now to show the functional equation. Applying (4.9) to (4.8), we get that

$$\begin{aligned}(4.18)\quad & E_{\chi_1,\chi_2}(s,h/k)\\ &= -[D_1,k]^{-s}[D_2,k]^{-s}(2\pi)^{2s-2}\Gamma(1-s)^2\sum_{\substack{a\ (D_1)\\ b\ (D_2)}}\chi_1(a)\chi_2(b)\\ &\quad\times\sum_{\substack{\alpha,\beta\ (k)\\ \alpha\equiv a\ (\delta_1)\ \beta\equiv b\ (\delta_2)}} e(\frac{\alpha\beta h}{k})\Big(e(\frac{s}{2})\varphi(1-s,\lambda^{(1)}_{\alpha,a})\varphi(1-s,\lambda^{(2)}_{\beta,b})\\ &\quad+e(-\frac{s}{2})\varphi(1-s,-\lambda^{(1)}_{\alpha,a})\varphi(1-s,-\lambda^{(2)}_{\beta,b})\\ &\quad-\varphi(1-s,\lambda^{(1)}_{\alpha,a})\varphi(1-s,-\lambda^{(2)}_{\beta,b})\\ &\quad-\varphi(1-s,-\lambda^{(1)}_{\alpha,a})\varphi(1-s,\lambda^{(2)}_{\beta,b})\Big).\end{aligned}$$

Our task is then to simplify the last four sums which we write accordingly

$$(4.19)\quad \sum_{\substack{\alpha,\beta\ (k)\\ \alpha\equiv a\ (\delta_1)\\ \beta\equiv b\ (\delta_2)}} e(\frac{\alpha\beta h}{k})\varphi(s,\pm\lambda^{(1)}_{\alpha,a})\varphi(s,\pm\lambda^{(2)}_{\beta,b})=\sum_{m,n\geq 1}(mn)^{-s}T_{h,k}(\pm m,\pm n;a,b)$$

for $\Re e\, s>1$. For simplicity, we write $T(m,n;a,b)$ for $T_{h,k}(m,n;a,b)$, that is,

$$(4.20)\quad T(m,n;a,b)=T_{h,k}(m,n;a,b)=\sum_{\substack{\alpha,\beta\ (k)\\ \alpha\equiv a\ (\delta_1)\\ \beta\equiv b\ (\delta_2)}} e(\frac{\alpha\beta h}{k}+m\lambda^{(1)}_{\alpha,a}+n\lambda^{(2)}_{\beta,b}).$$

Let us take $\alpha = a + x\delta_1$ and $\beta = b + y\delta_2$ where x and y run over residue classes mod κ_1 and mod κ_2 respectively. From (4.7) and (4.20), a rearrangement of terms gives that

$$
\begin{aligned}
(4.21)\quad & T(m,n;a,b) \\
&= e\Big(\frac{abh}{k} + \frac{am}{D_1\kappa_1} + \frac{bn}{D_2\kappa_2}\Big) \sum_{y\,(\kappa_2)} e(\frac{y(ah + n\overline{d_2}^{(\kappa_2)})}{\kappa_2}) \\
&\quad \times \sum_{x\,(\kappa_1)} e(\frac{x(h(b+y\delta_2) + m\overline{d_1}^{(\kappa_1)})}{\kappa_1}) \\
&= \kappa_1 e\Big(\frac{abh}{k} + \frac{am}{D_1\kappa_1} + \frac{bn}{D_2\kappa_2}\Big) \sum_{\substack{y\,(\kappa_2) \\ h(b+\delta_2 y) + m\overline{d_1}^{(\kappa_1)} \equiv 0\,(\kappa_1)}} e(\frac{y(ah + n\overline{d_2}^{(\kappa_2)})}{\kappa_2}).
\end{aligned}
$$

(Recall the definition of $\overline{m}^{(n)}$ under (4.2).) The congruence $h(b + \delta_2 y) + m\overline{d_1}^{(\kappa_1)} \equiv 0$ (κ_1) is solvable if and only if $\Delta_2 | bh + m\overline{d_1}^{(\kappa_1)}$. Subject to this condition, we have

$$
y \equiv -\overline{\delta}^{(\kappa)}\overline{h}^{(\kappa)}(\frac{bh + m\overline{d_1}^{(\kappa_1)}}{\Delta_2}) \qquad (\mathrm{mod}\ \kappa),
$$

so that the sum in (4.21) equals

$$
e\left(-\frac{\overline{\delta}^{(\kappa)}\overline{h}^{(\kappa)}}{\kappa}\frac{ah + n\overline{d_2}^{(\kappa_2)}}{\Delta_1}\frac{bh + m\overline{d_1}^{(\kappa_1)}}{\Delta_2}\right) \sum_{z\,(\Delta_1)} e\left(\frac{z}{\Delta_1}(ah + n\overline{d_2}^{(\kappa_2)})\right).
$$

Hence, $T(m,n;a,b)$ is non-zero only if $\Delta_1 | ah + n\overline{d_2}^{(\kappa_2)}$ and $\Delta_2 | bh + m\overline{d_1}^{(\kappa_1)}$. In this case, these two conditions can be expressed as

$$
(4.22)\qquad m \equiv -bhd_1 \pmod{\Delta_2} \quad \text{and} \quad n \equiv -ahd_2 \pmod{\Delta_1},
$$

and then

$$
\begin{aligned}
(4.23)\quad & T(m,n;a,b) \\
&= \Delta_1\kappa_1 e\Big(\frac{abh}{k} + \frac{am}{D_1\kappa_1} + \frac{bn}{D_2\kappa_2} - \frac{\overline{\delta}^{(\kappa)}\overline{h}^{(\kappa)}}{\kappa}\frac{ah + n\overline{d_2}^{(\kappa_2)}}{\Delta_1}\frac{bh + m\overline{d_1}^{(\kappa_1)}}{\Delta_2}\Big) \\
&= \Delta_1\kappa_1 e\Big(\frac{ab}{k}h_0 + \frac{am}{D_1\kappa_1}C_1 + \frac{bn}{D_2\kappa_2}C_2 - \frac{mn\overline{h}^{(\kappa)}}{k}C_0\Big)
\end{aligned}
$$

where $h_0 = h(1 - \delta\overline{\delta}^{(\kappa)} h\overline{h}^{(\kappa)})$, $C_1 = 1 - \delta\overline{\delta}^{(\kappa)} h\overline{h}^{(\kappa)} d_1 \overline{d_1}^{(\kappa_1)}$, $C_2 = 1 - \delta\overline{\delta}^{(\kappa)} h\overline{h}^{(\kappa)} d_2 \overline{d_2}^{(\kappa_2)}$ and $C_0 = \delta\overline{\delta}^{(\kappa)} \overline{d_1}^{(\kappa_1)} \overline{d_2}^{(\kappa_2)}$.

In view of (4.4), (4.5), (4.22) and (4.23), we have

$$\sum_{mn=l} T(\pm m, \pm n; a, b) = \Delta_1 \kappa_1 \tau_{h,k}^{(\mp)}(l; \pm a, \pm b) e\Big(\frac{abh_0}{k} + (\mp)\frac{l\overline{h}^{(\kappa)}}{k} C_0\Big)$$

corresponding to the four cases in (4.19). The $\pm$ signs attached to a and b are chosen to be the same as the pair of $\pm$ signs in $\pm m, \pm n$; while $(\mp) = -$ if both signs taken are equal, and $(\mp) = +$ otherwise. From (4.19) and (4.3), we obtain that

$$\sum_{\substack{\alpha,\beta\,(k)\\ \alpha \equiv a\,(\delta_1)\\ \beta \equiv b\,(\delta_2)}} e(\frac{\alpha\beta h}{k}) \varphi(1 - s, \pm\lambda_{\alpha,a}^{(1)}) \varphi(1 - s, \pm\lambda_{\beta,b}^{(2)}) = \Delta_1 \kappa_1 e(\frac{abh_0}{k}) \varphi_{h,k}^{(\mp)}(s; \pm a, \pm b). \tag{4.24}$$

Here, again $(\mp)$ takes the $-$ or $+$ sign according as whether the two signs taken from $\pm\lambda_{\alpha,a}^{(1)}, \pm\lambda_{\beta,b}^{(2)}$ are the same or not. Inserting (4.24) into (4.18) we see that $E_{\chi_1,\chi_2}(s, h/k)$ consists of four multiple sums corresponding to the possible $\pm$ signs in the right side of (4.24). It is apparent that the left side of (4.24) is, by (4.7), independent of the choices of representatives $a \,(\mathrm{mod}\ D_1)$ and $b \,(\mathrm{mod}\ D_2)$. Replacing a, b by $-a$ and $-b$ in the two cases $(-,-)$ and $(-,+)$, we deduce the desired functional equation.

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